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8-Dimensional Compact Planes with an Automorphism Group which has a Normal Vector Subgroup

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Abstract. A connected group Δ of automorphisms of an 8-dimensional compact plane $\mathcal{P}$ fixes at most some collinear points or 2 points and 2 lines (double flag). For each possible configuration of fixed elements of a group of sufficiently large dimension the structure of Δ and its action on $\mathcal{P}$ is determined. Examples are given; in the case of a double flag, all planes are described explicitly.
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Let $\mathcal{P} = (P, \mathfrak{L})$ be a topological projective plane with a compact point space P of finite (covering) dimension $\dim P = 8$. A systematic treatment of such planes can be found in the book *Compact Projective Planes* [24]. Each line $L \in \mathfrak{L}$ is homotopy equivalent to a sphere $\mathbb{S}_4$, see [24] 54.11. In all known examples, L is in fact homeomorphic to $\mathbb{S}_4$; this is true, in particular, if there exists a 4-dimensional closed (Baer) subplane, cf. [24] 53.10. Taken with the compact-open topology, the automorphism group $\Sigma = \operatorname{Aut}\mathcal{P}$ (of all continuous collineations) is a locally compact transformation group of P with a countable basis. If $\dim\Sigma \geq 12$, then any connected closed subgroup $\Delta \leq \Sigma$ is a Lie group [17], and Δ is semi-simple, or Δ has a central torus subgroup or a normal vector subgroup, cf. [24] 94.26. The first two possibilities have been dealt with in [23]. All planes $\mathcal{P}$ such that $\dim\Sigma \geq 17$ are known explicitly, cf. [21] and [9] or [24] 82.25. In his dissertation, Boekholt [3] also determined large classes of planes admitting a 16-dimensional group which does not fix exactly one flag (= incident point-line pair). Here, more results will be obtained under the assumption that $\dim\Delta \geq 12$ and that Δ has a minimal normal subgroup $\Theta \cong \mathbb{R}^t$; in particular, we describe all planes such that $\dim\Delta \geq 14$ and Δ fixes 2 points and 2 lines. For analogous investigations of 16-dimensional planes see [10] and the literature cited there.

Notation. The configuration of all fixed points and fixed lines of a subset $\Gamma \subseteq \Delta$ will be denoted by $\mathcal{F}_\Gamma$. As customary, $\Delta_{[c,A]}$ denotes the subgroup of all axial collineations in Δ with axis A and center c. If $\mathcal{B}$ is a 4-dimensional subplane

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of $\mathcal{P}$, then each point of $\mathcal{P}$ is incident with a line of $\mathcal{B}$, and $\mathcal{B}$ is called a Baer subplane ($\mathcal{B} \lessdot \mathcal{P}$), see [22]. An element $\gamma \in \Delta$ is called *straight*, if each point orbit of the cyclic group $\langle\gamma\rangle$ is contained in some line; by a result of Baer [1], γ is then an axial collineation or $\mathcal{F}_\gamma \lessdot \mathcal{P}$; in the latter case γ is said to be *planar* or to be a Baer collineation. A one-parameter group Π is called straight, if each of its elements is straight; this implies that each orbit of Π is contained in some line (because Π has a locally cyclic dense subgroup). If Π is not straight, then Π is also referred to as being *crooked*. If a point set S contains a quadrangle, then $\langle S\rangle$ is the smallest *closed* subplane containing S. A homeomorphism of two spaces is indicated by $X \approx Y$, homotopy equivalence by $X \simeq Y$.

Δ^1 denotes the connected component of the topological group Δ. As customary, $\mathrm{Cs}_\Delta\Gamma$ or just $\mathrm{Cs}\,\Gamma$ is the centralizer of Γ in Δ; the center $\mathrm{Cs}\,\Delta$ is usually denoted by Z. We write $\Delta : \Gamma = \dim \Delta/\Gamma = \dim\Delta - \dim\Gamma$. Note that $\dim x^\Delta = \Delta : \Delta_x$ by the *dimension formula* [24] 96.10. If $M^\Gamma = M$, then $\Gamma|_M$ is the group induced by Γ on M. A *Levi complement* of the radical $\sqrt{\Delta}$ is a maximal semi-simple subgroup of Δ. The real affine group of all maps $t \mapsto at{+}b : \mathbb{R} \to \mathbb{R}$ with $a > 0$, $b \in \mathbb{R}$ will be called L_2, and P_2 is the point space of the real projective plane $\mathcal{P}_2\mathbb{R} = \mathrm{PG}_2\mathbb{R}$.

Stiffness refers to results on (the size of) groups acting trivially on some proper subplane. We collect the facts to be used from [2], [20], and [24] § 83 in the next theorem:

Stiffness. *Let Λ be a connected closed subgroup of Δ, and assume that the fixed elements of Λ form a* (non-degenerate) *subplane $\mathcal{E} = \mathcal{F}_\Lambda$. Then* $\dim\Lambda \le 4$ (see [2]). *Moreover*:

(i) *If $\mathcal{E}$ is connected, or if Λ is compact, then* $\dim\Lambda \le 3$.

(ii) *If $\mathcal{E} \le \mathcal{B} \lessdot \mathcal{P}$, then $\mathcal{E}$ is connected and Λ is compact*; *$\mathcal{B}^\Lambda = \mathcal{B}$ implies* $\dim\Lambda \le 1$.

(iii) *If Λ is compact and not commutative, then $\Lambda \cong \mathrm{SO}_3\mathbb{R}$.*

(*) *Without any additional hypothesis, the stabilizer Γ of a degenerate quadrangle* (3 *points on a line and one point outside*) *satisfies* $\dim\Gamma \le 7$, see [20] (*) or [24] 83.17. *Hence* $\dim\nabla \le 11$ *for each stabilizer ∇ of a triangle.*

Lemma 1. *If $\Theta \cong \mathbb{R}^t$ is a minimal normal subgroup of Δ, then one of the following holds*:

(i) *Θ is a group of elations with common axis or common center*,

(ii) *Θ is a one-dimensional group of homologies*,

(iii) *each one-parameter subgroup of Θ is crooked.*

Proof. (a) *Each straight one-parameter subgroup Π of Θ is contained in a group $\Delta_{[c,A]}$ for suitable c and A.* In fact, let $\mathbb{Q} \cong \mathsf{P} < \Pi$. As P is locally cyclic, either $\mathcal{F}_\mathsf{P} \lessdot \mathcal{P}$ or P is contained in some group $\Delta_{[c,A]}$. By continuity, the same is true for Π instead of P. In the first case Π would be compact.

(b) If axial one-parameter subgroups of Θ have different centers or different axes, then the center of one is on the axis of the other, because Θ is commutative.

(c) If $\Pi \le \Theta_{[c,A]}$ is a group of homologies ($c \notin A$), then $c^\Delta = c$ and $A^\Delta = A$, again because Θ is commutative. In this case, $\Theta = \Pi$ by [24] 61.2 and minimality of Θ.

(d) If $\Pi \le \Theta_{[c,A]}$ is a group of elations ($c \in A$), then $\Theta = \Theta_{[A]}$ or $\Theta = \Theta_{[c]}$ by commutativity and minimality of Θ. Hence Θ has no crooked subgroup. ■

Lemma 2. *Let σ be a reflection with axis W and center c in the connected group Δ, and let T denote the group of translations in Δ with axis W. If $W^\Delta = W$ and $\dim c^\Delta = k > 0$, then $\sigma^\Delta \sigma = \mathsf{T} \cong \mathbb{R}^k$, $\tau^\sigma = \tau^{-1}$ for each $\tau \in \mathsf{T}$, and k is even.*

Proof. From [24] 61.19b it is known that $\dim \mathsf{T} = k$. Each element $\sigma^\delta \sigma$ is a translation ([24] 23.20), and $\sigma^\Delta \sigma$ is contained in the connected component Ξ of T. The map $\mu = (\xi \mapsto \sigma^\xi \sigma) : \Xi \to \Xi$ is continuous and injective. As Ξ is a manifold, the Open Mapping Theorem ([24] 51.19) applies, and μ is an open map. Hence there is an open neighbourhood Ω of $\mathbb{1}$ in Ξ such that $\Omega \subseteq \Omega^2 \subseteq \sigma^\Xi \sigma$. Obviously, σ inverts each element in $\sigma^\Delta \sigma$, so that $(\alpha\beta)^\sigma = (\alpha\beta)^{-1} = \alpha^{-1}\beta^{-1} = (\beta\alpha)^{-1}$ for all $\alpha, \beta \in \Omega$. Consequently, $\langle \Omega \rangle = \Xi$ is commutative. Each compact subgroup of Ξ is trivial by [24] 55.28, and Ξ is a vector group. Moreover, σ inverts each element in Ξ, and $\sigma^\xi \sigma = \xi^{-2}$, so that μ is surjective. For any translation $\tau \in \mathsf{T}$ there is some $\xi \in \Xi$ such that $\sigma^\tau = \sigma^\xi$, $c^\tau = c^\xi$, and $\tau = \xi$. Therefore T is connected. Under conjugation, Δ induces on T a subgroup of $\mathrm{GL}_k\mathbb{R}$, and the set of determinants $\det \Delta|_\mathsf{T}$ is connected and hence positive; in particular, $\det \sigma|_\mathsf{T} = 1$, and this implies $k \equiv 0 \bmod 2$. ■

Richardson's classification [24] 96.34 of compact groups on the 4-sphere will be needed repeatedly:

(†) *A compact, connected, effective Lie transformation group Φ on $\mathbb{S}_4$ is equivalent to a subgroup of $\mathrm{SO}_5\mathbb{R}$ in its linear action. The only possibility besides the obvious ones is given by the irreducible representation of $\mathrm{SO}_3\mathbb{R}$ on $\mathbb{R}^5$.*

Observation. *If a group $\Phi \cong \mathrm{SO}_3\mathbb{R}$ fixes a line W, then each involution in Φ is planar. Either Φ has no fixed point on W or $\mathcal{F}_\Phi$ is a 2-dimensional subplane*, see [23].

Remark 1. *A group $\Phi \cong \mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$ does not act effectively on a line of an 8-dimensional plane.*

Proof. Suppose that Φ acts effectively on $W \in \mathfrak{L}$. The involutions in Φ' are planar (or else one of 3 commuting reflections would have axis W). Hence $W \approx \mathbb{S}_4$ and $\Phi|_W$ is induced by the standard action of Φ on $\mathbb{R}^5$. It follows that the the central involution ω of Φ has a fixed point set $\mathbb{S}_2 \subset W$ and $\mathcal{F}_\omega \lessdot \mathcal{P}$. The action of Φ' on $\mathcal{F}_\omega$ is non-trivial by Stiffness, and each involution in Φ' induces a reflection on $\mathcal{F}_\omega$ by [24] 55.21c. One of these reflections has axis $W \cap \mathcal{F}_\omega$. Thus the simple group $\Phi'|_{\mathcal{F}_\omega}$ is a group of axial collineations with axis W, but this contradicts [24] 71.3 . ■

Throughout we assume that Δ has a minimal normal subgroup $\Theta \cong \mathbb{R}^t$.

All planes admitting a group of dimension at least 17 have been determined explicitly: such a plane is either a Hughes plane or a (dual) translation plane [21]. In the first case, the group has no normal vector subgroup. Therefore we will only consider groups of dimension at most 16.

The next theorem improves Corollary 2 in [23].

Theorem 1. *If* $\mathcal{F}_\Delta = \emptyset$ *and if* $\dim \Delta > 6$*, then* Δ *is a Lie group and* $\dim \Delta \le 10$*. In the case of equality,* $\Delta = \Lambda \times \Psi$*, where* $\Lambda \cong \mathrm{L}_2$ *fixes all elements of a real subplane* $\mathcal{E}$ *and* $\Psi \cong \mathrm{SL}_3\mathbb{R}$ *induces the full collineation group on* $\mathcal{E}$*. The involutions in* Ψ *are reflections of* $\mathcal{P}$*, elations of* $\mathcal{E}$ *extend to elations of* $\mathcal{P}$*, and* Ψ *is sharply transitive on the space* A *of all points not incident with a line of* $\mathcal{E}$ *and, dually, on the space* $\mathfrak{A}$ *of all lines which do not meet the point set of* $\mathcal{E}$*.*

Proof. (a) $\mathcal{F}_\Theta \ne \emptyset$ by [6] XI.10.19(a) or the proof of [24] 71.4. Up to duality, Θ fixes some point a, and $\Theta|_{a^\Delta} = \mathbb{1}$. By assumption, a^Δ is not contained in a line. Hence $\mathcal{E} := \langle a^\Delta \rangle$ is a connected proper subplane of dimension $d \,|\, 4$, see [24] 55.1. Note that $\Theta|_\mathcal{E} = \mathbb{1}$. In the case $\mathcal{E} \mathrel{<\!\!\!\bullet} \mathcal{P}$, Stiffness would imply that Θ is compact. Therefore $\dim \mathcal{E} = 2$, $\dim \Theta \le 3$, and $\dim \Delta|_\mathcal{E} \le 8$ (see [24] 33.6). Define Λ by $\Psi := \Delta|_\mathcal{E} = \Delta/\Lambda$. The assumption $\mathcal{F}_\Delta = \emptyset$ implies that Ψ has no fixed element in $\mathcal{E}$. It follows that Ψ is a simple group of dimension 3 or 8, see [24] 33.1 and 38.2,3. Stiffness yields $\dim \Lambda \le 3$, moreover, Λ contains no involution, again by Stiffness.

(b) *If* $\dim \Delta > 6$*, then* $\Psi \cong \mathrm{SL}_3\mathbb{R}$*,* $\Theta \le \Lambda$*, and either* $\Delta \cong \Lambda \times \Psi$*, or* $\Theta \cong \mathbb{R}^3$ *and* $\Delta \cong \Theta \rtimes \Psi$. In fact, Δ contains a covering group $\widetilde{\Psi}$ of Ψ by [24] 94.27, and $\widetilde{\Psi} \cong \Psi$ (or else there is a central involution in $\sigma \in \widetilde{\Psi}$, $\mathcal{E} \le \mathcal{F}_\sigma \mathrel{<\!\!\!\bullet} \mathcal{P}$, and Θ would be compact by Stiffness). We will identify Ψ and $\widetilde{\Psi}$. In the second case, $\Delta = \Theta\Psi$ because Δ is connected and $\dim \Delta = \dim \Theta\Psi$, cf. [24] 93.12. In particular, Δ is a Lie group. If $\Theta \le \mathrm{Cs}\,\Psi$, we use the fact [24] 93.8 that Λ contains a compact 0-dimensional central subgroup N of Δ such that Λ/N is a Lie group. As $\Theta \le \Lambda$ and $\dim \Lambda \le 3$, the connected group Ψ acts trivially on Λ/N and on each coset $\lambda\mathsf{N}$. Hence $\Delta \cong \Lambda \times \Psi$.

(c) Λ *acts freely on the set of points outside* $\mathcal{E}$ *and* Ψ *is the full collineation group of the real plane* $\mathcal{E}$: if $\Lambda_z \ne \mathbb{1}$ for some point $z \notin \mathcal{E}$, then $\mathcal{F}_{\Lambda_z} = \langle \mathcal{E}, z \rangle \mathrel{<\!\!\!\bullet} \mathcal{P}$, and Λ would be compact by Stiffness. For the same reason, Λ does not contain an involution. The second statement follows from [24] 33.6.

(d) Assume that the involutions in Ψ are planar. Let $\mathbb{R} \cong \mathsf{P} \le \Theta$ and consider the central involution ω in a subgroup $\Gamma \cong \mathrm{SL}_2\mathbb{R}$ of $\mathrm{Cs}_\Psi \mathsf{P}$ and the group $\overline{\Gamma} = \Gamma|_{\mathcal{F}_\omega} \cong \mathrm{PSL}_2\mathbb{R}$. Each element $\sigma \in \Gamma$ with trace $\mathrm{tr}\,\sigma = 0$ induces a reflection $\overline{\sigma}$ on $\mathcal{F}_\omega$, see [24] 55.21c, and it is easy to verify that σ has a unique fixed point in $\mathcal{E}$. The center c of $\overline{\sigma}$ in $\mathcal{F}_\omega$ is a fixed point of P, and $c \in \mathcal{E}$ by step (c). Therefore c is the unique fixed point of σ in $\mathcal{E}$; this fixed point is the center of the reflection ω of $\mathcal{E}$. Analogously, the axis of $\overline{\sigma}$ in $\mathcal{F}_\omega$ coincides with the axis of ω in $\mathcal{E}$. Consequently, all involutions in $\overline{\Gamma}$ have the same center and axis. The simple group $\overline{\Gamma}$ is generated by its involutions. Hence $\overline{\Gamma}$ is a group of homologies of $\mathcal{F}_\omega$, but such a group is only 2-dimensional. This contradiction shows that the involutions in Ψ are reflections of $\mathcal{P}$. Each elation of $\mathcal{E}$ is a product of two reflections, hence it is induced by an elation of $\mathcal{P}$.

(e) If $\Delta \cong \Theta \rtimes \Psi$, and if σ is an involution in Ψ, then there is some $\tau \in \Theta$ such that $\sigma^\tau \neq \sigma$. The axis C of the reflection σ is a line of $\mathcal{E}$, and σ^τ has the axis $C^\tau = C$. Therefore $\sigma^\tau\sigma$ fixes C pointwise, but $\sigma^\tau\sigma = \tau^{-1}\tau^\sigma \in \Theta \setminus \{\mathbb{1}\}$ fixes also all points of $\mathcal{E}$, a contradiction. Consequently $\Delta = \Lambda \times \Psi$.

(f) *The space A is connected.* In fact, consider the line L joining 2 points $z, z' \in A$. The definition of A implies that $L \cap A \approx \mathbb{S}_4 \setminus \mathbb{S}_2$ (if L contains a point of $\mathcal{E}$) or $L \cap A \approx \mathbb{S}_4 \setminus \mathrm{P}_2$ (if $L \in \mathfrak{A}$). In both cases $L \cap A$ is connected, see [5] XVII Cor. 2.3. (In the second case, note that the natural embedding of $\mathcal{P}_2\mathbb{R}$ into the classical quaternion plane yields an embedding of P_2 into $\mathbb{S}_4$ with connected complement.)

(g) Δ *is a Lie group*: If Ψ acts freely on A, then for each $z \in A$ the orbit z^Ψ is open in P by [24] 51.12 and 96.11a, and $z^\Psi = A = z^\Delta$ is a manifold. In this case, Δ is a Lie group by Szenthe's Theorem [24] 96.14, see also [12]. Suppose now that $\Psi_z \neq \mathbb{1}$ for some point $z \in A$. We have $z^\Theta \neq z$ by step (c), and if L is any line through z, then $L^\Theta \neq L$ by the definition of A and the dual of (c). Hence $\mathcal{F} := \langle z^\Lambda \rangle$ is a subplane of $\mathcal{P}$, and $\mathcal{F} \neq \mathcal{P}$, since $\Psi \leq \mathrm{Cs}\,\Lambda$ and $\Psi_z|_{z^\Lambda} = \mathbb{1}$. As Λ acts effectively on $\mathcal{F}$ by step (c), it follows from [24] 32.21 and 71.2 that Λ is a Lie group.

(h) *If L is a line of $\mathcal{E}$, then Δ_L is not transitive on the point set $M := L \setminus \mathcal{E}$*: in fact, $\Delta_L = \Lambda \times \Psi_L \simeq \mathbb{S}_1$ because Λ does not contain an involution, and we have $M \approx \mathbb{S}_4 \setminus \mathbb{S}_1 \simeq \mathbb{S}_2$. If $x^{\Delta_L} = M$ and $\mathsf{X} = \Delta_{L,x}$, the exact homotopy sequence would imply

$$\ldots\ 0 = \pi_3\Delta_L \to \pi_3 M \cong \mathbb{Z} \to \pi_2\mathsf{X} = 0\ \ldots,$$

an obvious contradiction. The assertion (h) follows also from [24] 55.20.

(i) If $\dim\Lambda = 3$, then Λ has a normal subgroup $\mathbb{R}^2$ because $\Theta \trianglelefteq \Lambda$, see, e.g., [14] § 4. Hence it suffices to consider the case $\Lambda \cong \mathbb{R}^2$. As Δ_L is not transitive on M, there is some point $x \in M$ such that $\dim x^{\Delta_L} \leq 3$ and $\dim\Delta_x \geq \dim\Delta_L - 3 \geq 5$, note that $\Delta_x \leq \Delta_L$. We have $\Delta_x|_{x^\Lambda} = \mathbb{1}$, because $\Lambda \leq \mathrm{Cs}\,\Delta$. Choose distinct points $a, b \in \mathcal{E} \setminus L$. Then $\dim\Delta_{a,b,x} \geq 1$, and $\Delta_{a,b,x}$ fixes $\mathcal{F} := \langle a, b, x^\Lambda \rangle \mathrel{<\!\!\bullet} \mathcal{P}$ pointwise. Hence $\Delta_{a,b,x}$ is compact by Stiffness, but $\Lambda \times \Psi_{a,b}$ has no torus subgroup $\neq \mathbb{1}$. This contradiction shows that $\dim\Delta \leq 10$; in the case of equality, Λ is not commutative.

(j) Let $\Lambda \cong \mathrm{L}_2$. Then $\Theta = \Lambda'$ and $\Delta' = \Theta \times \Psi$. Consider an arbitrary point $z \in A$. As in step (g), it follows that $\mathcal{F} := \langle z^\Theta \rangle$ is a subplane of $\mathcal{P}$, and $\mathcal{F} \neq \mathcal{P}$, since $\Delta'_z|_{z^\Theta} = \mathbb{1}$ and $\dim\Delta'_z > 0$. If $\mathcal{F} \mathrel{<\!\!\bullet} \mathcal{P}$, then Δ'_z is compact and would contain a planar involution, contradicting step (d). Therefore $\mathcal{F}$ is a 2-dimensional subplane. Assume now that $\langle z^\Lambda \rangle = \mathcal{F}$. By [24] 33.1, the solvable group Λ fixes some element of $\mathcal{F}$, say a line L. Let $x \in L = L^\Lambda$ be a point of $\mathcal{F}$. By step (c) either $x^\Lambda = x \in \mathcal{E}$ or $\dim x^\Lambda = 2$, but the latter is impossible, as $x^\Lambda \subseteq L \cap \mathcal{F}$ has dimension at most 1. Therefore $\overline{\Lambda} = \Lambda|_\mathcal{F}$ is a group of axial collineations with axis $L \cap \mathcal{F} = L \cap \mathcal{E}$, and $\overline{\Lambda}$ contains elations and homologies, because $\overline{\Lambda}$ is not commutative. The centers of the collineations in $\overline{\Lambda}$ form a second Λ-invariant line $L' \neq L$ of $\mathcal{F}$, cf. [24] 33.4c. As before, $L' \cap \mathcal{F} = L' \cap \mathcal{E}$, but this would imply $\mathcal{F} = \mathcal{E}$. Consequently, $\langle z^\Theta \rangle < \langle z^\Lambda \rangle$. In the case $\langle z^\Lambda \rangle \mathrel{<\!\!\bullet} \mathcal{P}$ the group Λ'_z would be compact.

Hence $\langle z^{\Lambda}\rangle = \mathcal{P}$, and $\Psi_z = \mathbb{1}$ since $\Lambda \le \mathrm{Cs}\,\Psi$. Now $z^{\Psi} = A$ as in step (g). ∎

Remark 2. *In the classical quaternion plane $\mathcal{P}_{\mathbb{H}}$ the group $\Psi = \mathrm{SL}_3\mathbb{R}$ is in fact sharply transitive on the set A of all points not incident with a real line*: suppose that $a^{\psi} = a$ for some $a \in A$ and $\psi \in \Psi$, and put $\Gamma = \mathrm{Aut}\,\mathbb{H}$. Then Γ fixes exactly the real elements, $\Gamma \le \mathrm{Cs}\,\Psi$, and $\psi|_{a^{\Gamma}} = \mathbb{1}$. The choice $a \in A$ implies that $\langle a^{\Gamma}\rangle$ is a subplane, moreover, ψ fixes a real line L. If $\psi \ne \mathbb{1}$, then $\langle a^{\Gamma}\rangle \lessdot \mathcal{F}_{\psi} \lessdot \mathcal{P}$, and Γ is transitive on the flat plane $\langle a^{\Gamma}\rangle$. Hence $L \cap a^{\Gamma} \ne \emptyset$, and $a \in L$, a contradiction. As A and Ψ are connected 8-dimensional manifolds, $a^{\Psi} = A$. ∎

The following result improves [21] (4).

Theorem 2. *If Δ has no fixed point, and if* $\dim\Delta \ge 15$*, then Θ is contained in a group $\mathsf{T} = \Delta_{[W,W]}$ of translations with axis W. If* $\dim\Delta = 16$*, then T is transitive and Δ has a subgroup isomorphic to $\mathrm{SL}_2\mathbb{C}$; the planes with such a group are known explicitly*, cf. Hähl [8].

Proof. (a) Assume first that $\dim\Delta \ge 15$. By Theorem 1 and the assumption, there is a (unique) fixed line W. If some one-parameter group $\Pi \le \Theta$ is straight, then Lemma 1 shows that Θ is a group of translations with axis W. We may assume, therefore, that each one-parameter group of Θ is crooked.

(b) *If $\mathsf{X} \le \Delta$,* $\dim\mathsf{X} > 13$*, and if X fixes no line other than W and no point outside W, then the center Z of X is contained in T; hence X fixes some point $z \in W$ or Z is trivial.* In fact, let $\zeta \in \mathsf{Z} \setminus \{\mathbb{1}\}$. If $x^{\langle\zeta\rangle}$ and W generate a subplane $\mathcal{E}$, then $\mathsf{X}_x|_{\mathcal{E}} = \mathbb{1}$ and $\dim\mathsf{X} \le 12$ by the stiffness result [2]. Therefore, ζ is straight. In the case $\mathcal{F}_{\zeta} \lessdot \mathcal{P}$, Stiffness yields $\dim\mathsf{X} \le 3{\times}4+1$ since $\mathcal{F}_{\zeta}$ is X-invariant, see [24] 71.7. Thus ζ is an axial collineation with axis W and some center $z \in W$.

(c) Let $\mathbb{R} \cong \Pi \le \Theta$. Then Π is crooked by assumption, and there is a point a such that $\langle a^{\Pi}\rangle$ is a subplane of $\mathcal{P}$. Since Δ acts linearly on Θ, the centralizer $\Delta_{a,\varrho} = \Delta_a \cap \mathrm{Cs}\,\varrho$ of an element $\varrho \in \Pi \setminus \{\mathbb{1}\}$ fixes each point of the orbit a^{Π} and induces the identity on $\langle a^{\Pi}\rangle$. The dimension formula yields

$$7 \le \dim\Delta_a = \dim\varrho^{\Delta_a} + \dim\Delta_{a,\varrho},$$

and Stiffness implies $\dim\Delta_{a,\varrho} \le 3$ and $4 \le \dim\varrho^{\Delta_a} \le \dim\varrho^{\Delta} \le t = \dim\Theta$. Note that $\Theta_a \le \Delta_{a,\varrho}$ and that $\langle a^{\Pi}\rangle \le \langle a^{\Theta}\rangle := \mathcal{E}$.

(d) Suppose that $t = 4$. Then $\dim\varrho^{\Delta_a} = 4$ for each $\varrho \in \Theta \setminus \{\mathbb{1}\}$, and Δ_a is transitive on $\Theta \setminus \{\mathbb{1}\}$ by [24] 96.11. In particular, Δ_a has a subgroup $\Phi \cong \mathrm{Spin}_3\mathbb{R}$ such that $\Phi \cap \Delta_{a,\varrho} = \mathbb{1}$; moreover, $\dim\Delta_a = 7$, $\dim\Delta_{a,\varrho} = 3$, and $\dim\langle a^{\Pi}\rangle = 2$. The orbit a^{Δ} is open in P by [24] 96.11a. If $\mathcal{E} = \langle a^{\Pi}\rangle$, we conclude from [24] 38.2, 3 and $a^{\Phi} = a$ that $\Phi|_{\mathcal{E}} = \mathbb{1}$, but this contradicts Stiffness. In the case $\mathcal{E} \lessdot \mathcal{P}$, Stiffness would imply $\dim\Delta_{a,\varrho} \le 1$. Therefore $\mathcal{E} = \mathcal{P}$ and $\Delta_a \cap \mathrm{Cs}\,\Theta = \mathbb{1}$. It follows that Δ_a is a transitive subgroup of $\mathrm{Aut}\,\Theta$, and then Δ_a' is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or to $\mathrm{SL}_2\mathbb{C}$; in particular, Δ_a contains a central involution α, and α is either planar or a reflection ([24] 55.29). In the first case, the semi-simple group Δ_a' acts on the Baer subplane $\mathcal{F}_{\alpha}$ with a kernel of dimension at most 1, hence almost effectively. By [24] 71.8 and 72.4, this action is equivalent to the standard action of $\mathrm{SO}_3\mathbb{C}$ on the complex plane, in contradiction to the fact that $W^{\Delta} = W$. If α

has axis W, then $\alpha^\Delta\alpha = \mathsf{T}$ is a transitve translation group, $\mathsf{T}\cap\Theta = \mathbb{1}$ by the assumption made in step (a), and $\dim\Delta$ would be to large. Thus, α has a center $u\in W$ and an axis $A = av$, $v\in W$. Suppose first that $\Delta'_a = \Upsilon\cong \mathrm{SO}_4\mathbb{R}$. Then Δ_a has a subgroup $\mathrm{SO}_3\mathbb{R}$ and contains planar involutions, $W\approx\mathbb{S}_4$, and (†) applies. According to [24] 91.34, the group Υ is a maximal subgroup of $\mathrm{SL}_4\mathbb{R}$. Therefore $\dim\Delta/\mathrm{Cs}\,\Theta = 7$ and $\Delta = \Delta_a\Gamma$, where $\Gamma = (\mathrm{Cs}\,\Theta)^1$ is normal in Δ, and a^Γ is open in P. If $a^\gamma\in A$ for some $\gamma\in\Gamma$, then $a^{\gamma\alpha} = a^{\alpha\gamma}$ and $\gamma^\alpha\gamma^{-1}\in\Delta_a\cap\Gamma = \mathbb{1}$. Consequently, $\Xi = \Gamma\cap\mathrm{Cs}\,\alpha$ is a 4-dimensional subgroup, and Ξ centralizes a factor $\Phi\cong\mathrm{Spin}_3\mathbb{R}$ of Υ. The action of Υ on W shows that $u^\Delta = u^\Gamma = u^\Theta$ is open in W, and $\Xi|_{u^\Theta} = \mathbb{1}$. Similarly, $v^\Delta = v^\Theta$ is open in W. If A^Θ is contained in a pencil $\mathfrak{L}_x$, then $x^\Theta = x$ and $\Theta|_{x^\Delta} = \mathbb{1} = \Theta$. Hence $\langle A^\Theta\rangle$ is a subplane. As $\Xi|_{A^\Theta} = \mathbb{1}$, it follows that Ξ is trivial. This contradiction leaves only the possibility $\Delta'_a = \Omega\cong\mathrm{SL}_2\mathbb{C}$. A maximal semi-simple subgroup of Δ is then isomorphic to Ω or to a covering group Ψ of the symplectic group $\mathrm{Sp}_4\mathbb{R}$. In the second case, α is in the center of Ψ and $\Psi\le\Delta_{a,u,v}$. As Δ is not transitive on W, there is some point $w\in W$ such that $\dim w^\Delta < 4$ and $\dim\Psi_w = 7$, but then Ψ would contain a subgroup $\mathrm{SO}_4\mathbb{R}$ by [20] (**). Hence $\Delta = \Omega\sqrt{\Delta}$, and [20] ($^{\,*}_{**}$) implies that $\dim\mathrm{Cs}\,\Omega < 5$. The completely reducible action of Ω on the Lie algebra of $\sqrt{\Delta}$ shows that $\dim\mathrm{Cs}\,\Omega = 1$. Therefore $\dim\Delta|_\Theta = 7$; again $\Delta = \Delta_a\Gamma$ and $\Xi = \Gamma\cap\mathrm{Cs}\,\alpha$ is a 4-dimensional group. Consequently, $\dim\Omega\Xi = 10$ and Ω would be compact by [20] (**). This final contradiction shows that $t > 4$.

(e) We will show that $\Delta_x\cap\mathrm{Cs}\,\Theta = \mathbb{1}$ *for each point* $x\notin W$. By assumption, the orbit x^Δ is not contained in a line, and $\langle x^\Delta\rangle = \mathcal{D}$ is a connected subplane of dimension at least 4 (or else $\dim\mathcal{D} = 2$ and $\dim\Delta\le 8+3$). If $x^\Theta = x$, then $\Theta|_{\langle x^\Delta\rangle} = \mathbb{1}$, and Stiffness would imply $\dim\Theta\le 3$. Similarly, x^Θ is not contained in a line. Therefore $\langle x^\Theta, W\rangle = \mathcal{E}$ is a connected subplane. It suffices to show that $\mathcal{E} = \mathcal{P}$. If $\dim\mathcal{E} = 2$, then there is some $\delta\in\Delta$ such that $x^\delta = y\notin\mathcal{E}$. We have $\langle y^\Theta, W\rangle = \mathcal{E}^\delta$ and $\dim\Theta_{x,y}\ge t - \dim x^\Theta - \dim y^\Theta > 0$. Consequently, $\langle\mathcal{E}, y\rangle$ is a Baer subplane, and Stiffness implies that Θ_x is compact. The same is true if $\dim\mathcal{E} > 2$. Hence $\Theta_x = \mathbb{1}$ and $\dim\mathcal{E}\ge\dim x^\Theta > 4$.

(f) Suppose that $t = 5$. By step (a), no one-parameter subgroup of Θ is straight. If $\dim\Delta = 16$, step (c) implies that $\dim\varrho^\Delta = 5$ for each $\varrho\in\Theta\smallsetminus\{\mathbb{1}\}$. Hence Δ is transitive on $\Theta\smallsetminus\{\mathbb{1}\}$, cf. [24] 96.11. From [24] 96.19–22 it follows that a maximal compact subgroup Φ of Δ acts transitively on the space of rays in $\Theta\cong\mathbb{R}^5$ and induces on $\mathbb{R}^5$ a group $\mathrm{SO}_5\mathbb{R}$. According to [24] 55.40, $\mathrm{SO}_5\mathbb{R}$ is not a subgroup of Φ. As $\Phi\cap\Theta = \mathbb{1}$, we have $\dim\Phi\le 11$, and we conclude that $\Phi'\cong\mathrm{Spin}_5\mathbb{R}$ and that $\Phi'\Theta$ has a center of order 2. Because $\dim x^\Theta > 4$ for each $x\notin W$, the group $\Phi'\Theta$ satisfies the hypotheses of step (b). This implies that the center of $\Phi'\Theta$ consists of translations, but there is no translation of order 2, see [24] 55.28. Consequently, $\dim\Delta < 16$.

(g) Let again $t = 5$. By minimality of Θ and step (e), each stabilizer Δ_x acts faithfully on Θ and Δ induces on Θ an irreducible group of dimension at least 7. From [24] 95.6 and 10 it follows that the commutator group of $\Delta|_\Theta$ is a simple group $\overline{\Psi}\cong\mathrm{O}'_5(\mathbb{R}, r)$. According to [24] 94.27, Δ contains a covering group Ψ of $\overline{\Psi}$, and we have $\Delta = \Psi\Theta$. If Ψ is a proper covering of $\overline{\Psi}$, then there is an element $\zeta\ne\mathbb{1}$ in the center of $\Psi\Theta$, and step (b) shows that ζ is a translation in

$\mathsf{T}_{[z]}$ for some center z. As $z^\Delta \neq z$, the Lie group T has positive dimension, and T contains one-parameter subgroups with different centers. Consequently, $\dim \mathsf{T} > 1$ and $\dim \Delta > 16$ (because $\Theta \cap \mathsf{T} = \mathbb{1}$ by assumption), but we know that $\dim \Delta < 16$. Therefore $\Psi \cong \overline{\Psi}$; moreover, Ψ acts effectively on W, and $\Delta|_\Theta \cong \Delta/\Theta \cong \Psi$.

(h) Consider a maximal compact subgroup Φ of Ψ. Then $\Phi \ncong \mathrm{SO}_5\mathbb{R}$ by [24] 55.40, and $r > 0$. If $r = 1$, then $\Phi \cong \mathrm{SO}_4\mathbb{R}$, and (†) implies that Φ has exactly 2 fixed points u, v on W. Applied to the pencils $\mathfrak{L}_u$ and $\mathfrak{L}_v$, (†) yields a point $c \notin W$ such that $\Phi < \Delta_c$. As Δ_c embeds into Ψ and maximal compact subgroups of simple Lie groups are maximal subgroups ([24] 94.34), we conclude that $\Delta_c \cong \Psi$, and we may assume $\Delta_c = \Psi$, since all Levi complements are conjugate ([24] 94.28). The set $c^\Delta = c^\Theta$ has dimension 5, and there is some $\varrho \in \Theta$ such that $\mathcal{E} = \langle c, c^\varrho, u, v\rangle$ is a (connected) subplane. The group $\Psi_\varrho = \Delta_c \cap \mathrm{Cs}\,\varrho$ acts trivially on $\mathcal{E}$, and $\dim \Psi_\varrho = 10 - \dim \varrho^\Psi \geq 5$ in contradiction to Stiffness. Therefore $r = 2$ and $\Phi \cong \mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$, but this is excluded by Remark 1, because Ψ acts effectively on W.

(i) The case $t = 7$ can be dealt with very easily: Δ induces on Θ an irreducible group of dimension at least 7, and [24] 95.6 and 10 imply that the commutator group of $\Delta|_\Theta$ is an almost simple group of dimension ≥ 14, but then $\dim \Delta$ is too large.

(j) If $t = 8$, then Θ is sharply transitive on $P \setminus W$ and $\Delta = \Delta_c\Theta$. The commutator subgroup $\Psi = \Delta_c'$ is a Levi complement of $\sqrt{\Delta}$, and $\dim \Psi \in \{6, 8\}$. Suppose first that $\Delta = \Psi\Theta$. Then the list of representations [24] 95.10 shows that Ψ is one of the simple groups $\mathrm{SL}_3\mathbb{R}$ or $\mathrm{PSU}_3(\mathbb{C}, r)$ and that the representation of Ψ on Θ is equivalent to the adjoint representation of Ψ. This means that Θ can be identified with the additive group of the Lie algebra $\mathfrak{l}\Psi$ so that the action of Ψ is by inner automorphisms. The group $\mathrm{SL}_3\mathbb{R}$ has a one-parameter subgroup $\mathsf{E} = \{\mathrm{diag}(r, r, r^{-2}) \mid r > 0\}$ with a 4-dimensional centralizer isomorphic to $\mathrm{GL}_2\mathbb{R}$. Consider the one-parameter subgroup $\mathfrak{l}\mathsf{E} = \mathsf{H} < \Theta$, and assume that $\langle x^\mathsf{H}\rangle$ is a subplane. There is an element $\tau \in \Theta$ such that $x^\tau = c$. Transformation with τ yields $\langle c^\mathsf{H}\rangle \leq \mathcal{P}$, and Stiffness implies $\dim \Delta_c \cap \mathrm{Cs}\,\mathsf{H} \leq 3$, but $\Psi \cap \mathrm{Cs}\,\mathsf{H} \cong \mathrm{GL}_2\mathbb{R}$ by construction. This contradiction shows that H is straight, contrary the assumption. The same arguments (with $\mathrm{Cs}\,\mathsf{E} \cong \mathrm{U}_2\mathbb{C}$) exclude the groups $\mathrm{PSU}_3(\mathbb{C}, r)$.

(k) Let $t = 8$ and $\dim \Psi = 6$. Then Ψ acts faithfully on Θ by step (e), and Ψ contains a central involution σ or Ψ is isomorphic to the simple group $\mathrm{O}_4'(\mathbb{R}, 1) \cong \mathrm{SO}_3\mathbb{C}$. In the first case, σ induces inversion on Θ by [24] 95.6b, and σ is a reflection with center c; hence $\sigma^\Theta\sigma = \mathsf{T}$ is a transitive translation group, but $\Theta \cap \mathsf{T} = \mathbb{1}$ and Δ would be too large. In the second case, it follows from [24] 95. 6 and 10 that Ψ induces on Θ a direct sum of two equivalent representations of $\mathrm{O}_4'(\mathbb{R}, 1)$. The fixed elements of a maximal compact subgroup $\Phi < \Psi$ form a 2-dimensional subgroup $\Gamma < \Theta$, $\Phi|_{c^\Gamma} = \mathbb{1}$, and $\mathcal{F}_\Phi = \langle c^\Gamma\rangle$ is a 2-dimensional subplane $\mathcal{E}$ by the Observation. Step (e) implies that Γ acts freely on $\mathcal{E}$, and Γ fixes some point $z \in W$ by Brouwer's Theorem [24] 96.30; the action of Γ on the pencil $\mathfrak{L}_z$ shows that some one-parameter subgroup P of Γ acts on $\mathcal{E}$ as a group of translations with axis $W \cap \mathcal{E}$. Hence c^P is contained in a line, and so is each orbit $c^{\tau\mathsf{P}}$ with $\tau \in \Theta$. Thus P is straight contrary to the assumption in step (a).

(ℓ) The only remaing possibility is $t = 6$. By minimality of Θ, the group

Δ induces on Θ an irreducible subgroup $\overline{\Delta} = \Delta/\mathrm{Cs}\,\Theta$ of $\mathrm{GL}_6\mathbb{R}$. Step (e) asserts that Δ_x acts effectively on Θ. Therefore

$$\dim\Delta - 8 \le \dim\Delta_x \le \dim\overline{\Delta} = \Delta : \mathrm{Cs}\,\Theta \le \Delta : \Theta = \dim\Delta - 6.$$

Hence $\dim\mathrm{Cs}\,\Theta \le 8$, $\mathrm{Cs}\,\Theta : \Theta \le 2$ and $\mathrm{Cs}\,\Theta$ is solvable. According to [24] 95.6, the commutator subgroup $\overline{\Delta}'$ is semi-simple and acts irreducibly on $\mathbb{R}^6$ or on $\mathbb{R}^3$, moreover, $6 \le \dim\overline{\Delta}' \le 10$. A Levi complement Ω of $\sqrt{\Delta}$ induces on Θ the group Ω/K, where $\mathsf{K} = \Omega \cap \mathrm{Cs}\,\Theta$ is solvable. Therefore $\dim\mathsf{K} = 0$ and K is in the center of $\Omega\Theta$. If $\dim\Omega > 6$ and if $\mathsf{K} \ne \mathbb{1}$, then $\dim\Omega\Theta > 13$, and K contains a non-trivial translation $\tau \in \mathsf{T}_{[z]}$ by (b). As in step (g), $\dim\mathsf{T} > 1$. By assumption, no one-parameter subgroup of Θ is straight, and $\mathsf{T}^1 \cap \Theta = \mathbb{1}$. Consequently, $\dim\Omega\Theta\mathsf{T} = 16$ and $\Omega\Theta\mathsf{T} = \Delta$, but $\tau^{\Omega\Theta\mathsf{T}} = \tau$ and $z^{\Omega\Theta\mathsf{T}} = z$. This contradiction shows that $\dim\Omega = 6$ or $\mathsf{K} = \mathbb{1}$.

(m) A 10-dimensional semi-simple group has no irreducible representation in dimension 6, see [24] 95.10. If $\dim\Omega = 9$, then Clifford's Lemma [24] 95.5 would imply that Ω has a 6-dimensional semi-simple factor which acts faithfully and irreducibly on $\mathbb{R}^3$, but $\mathrm{SL}_3\mathbb{R}$ has no semi-simple subgroup of dimension 6. Hence $\dim\Omega \in \{6, 8\}$.

(n) Assume that $\dim\Omega = 8$. Then $\Omega \cong \overline{\Delta}'$ is almost simple, and Ω acts faithfully on Θ by step (ℓ). Representation theory [24] 95.10 shows that Ω is isomorphic to $\mathrm{SL}_3\mathbb{R}$ or to $\mathrm{SU}_3(\mathbb{C}, r)$. The center of Ω is trivial or isomorphic to $\mathbb{Z}_3$. If Ω would contain a reflection σ with axis W, then Ω would be generated by the conjugacy class σ^Ω and $\Omega|_W = \mathbb{1}$, which is impossible. Hence at most one of 3 pairwise commuting involutions is axial. Therefore $W \approx \mathbb{S}_4$ and (†) applies; in particular, Ω is not compact. The action of Ω on the Lie algebra of $\sqrt{\Delta}$ is completely reducible. Hence $\sqrt{\Delta} = \Gamma\Theta$ with $\Gamma = (\mathrm{Cs}\,\Omega)^1$, and $\dim\Gamma \in \{1, 2\}$. Let $\mathbb{1} \ne \gamma \in \Gamma \cap \mathrm{Cs}\,\Theta \le \mathrm{Cs}\,\Omega\Theta$. Then γ is in the center of $\langle\gamma\rangle\Omega\Theta$, and step (b) implies that $\gamma \in \mathsf{T}_{[z]}$ for some $z \in W$. This leads to a contradiction as at the end of step (ℓ). Consequently Γ acts faithfully on Θ.

(o) If $\dim\Delta = 16$, then [24] 95.6 shows that $\Gamma \cong \mathbb{C}^\times$ is the group of dilatations of $\Theta \cong \mathbb{C}^3$. Hence $\Omega \not\cong \mathrm{SL}_3\mathbb{R}$, and (†) implies that the involution $\alpha \in \Gamma$ is a reflection with axis W (because $\Delta|_W$ has torus rank at most 2), but then $\alpha^\Delta\alpha$ is a transitive translation group. Therefore we may assume that $\dim\Delta = 15$.

(p) Let Φ be a maximal compact subgroup of $\Omega \cong \mathrm{SL}_3\mathbb{R}$. Then Φ has no fixed point on W: in fact, if $z^\Phi = z \in W$, then (†) or the Observation implies that $\mathcal{F}_\Phi$ is a 2-dimensional subplane, and $\Phi < \Omega_z = \Omega$ by maximality of Φ (see [24] 94.34). The same argument, applied to the lines of $\mathcal{F}_\Phi$, shows that $\Omega|_{\mathcal{F}_\Phi} = \mathbb{1}$, contradicting Stiffness.

(q) Consider now the case $t = 6$ and $\Omega \cong \mathrm{SU}_3(\mathbb{C}, 1)$. A maximal compact subgroup Φ of Ω is isomorphic to $\mathrm{U}_2\mathbb{C}$ and acts on W in the standard way with two fixed points u and v. The central involution σ of Φ has no other fixed points on W. Hence σ is a reflection; let u be the center of σ and xv the axis. If $u^\Omega = u$, then Θ_u is Ω-invariant of dimension at least 2, and $\Theta_u = \Theta$ (because Ω acts irreducibly on Θ). It follows that $\dim u^\Delta = \Delta : \Omega\Theta = 1$, but $u^\delta \ne u, v$ implies $u^{\delta\Phi} \approx \mathbb{S}_3$. Consequently $u^\Omega \ne u$, and then u^Ω is open in W by the action of Φ, and so is v^Ω for the same reason.

(r) We will show that for each $z \in u^\Omega$ there is exactly one line $A = z^\alpha$ such that there exists a reflection σ_z in Δ with center z and axis A. Suppose that this is not true. Then the group E_u of all elations with center u is not trivial by the dual of [24] 23.20, and the action of Φ on E_u implies that $\dim \mathsf{E}_u \geq 4$. As each element in E_u is straight, $\mathsf{E}_u \cap \Theta = \mathbb{1}$ and $\dim \Delta \geq 18$ contrary to the assumption. As $\Omega_u \cup \Omega_v \neq \Omega$, there is some $\omega \in \Omega$ such that $u^\omega = z \neq u$ and $v^\omega \neq v$. Therefore the stabilizer $\Gamma = \Delta_{u,z}$ fixes the degenerate quadrangle $(u, z, u^\alpha \cap z^\alpha, u^\alpha \cap uz)$. According to [20] (**), the group Γ' is isomorphic to $\mathrm{SO}_4\mathbb{R}$, but this group is not contained in Δ.

(s) Finally, let $t = 6 = \dim \Omega$. Then $7 \leq \dim \Delta_x \leq \dim \overline{\Delta} = \dim \Delta|_\Theta$ and $\dim \overline{\Delta} \leq \dim \Omega + 2 = 8$. If $\dim \Delta = 16$, we have $\dim \Delta_x = \dim \overline{\Delta} = 8$, hence, as all Levi complements are conjugate, $\Delta_x = \Omega\Gamma$ with $\Gamma \cong \mathbb{C}^\times$. Again, Δ_x acts faithfully on W. Moreover, Ω is almost simple (or $\Omega\Gamma|_W \cong \Omega\Gamma$ would have torus rank 3 in contradiction to (†)), and representation theory [24] 95.10 shows that $\Omega \cong \mathrm{SO}_3\mathbb{C}$. The involution $\alpha \in \Gamma$ is either a reflection having a center $u \in W$ or α is planar. In the first case Δ_x has a subgroup $\mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$ acting with two fixed points on W, but this is impossible by (†). In the second case $\mathrm{SO}_3\mathbb{C}$ acts in the standard way on $\mathcal{F}_\alpha \cong \mathcal{P}_2\mathbb{C}$, see [24] 72.4 and 18.32; in particular, Ω cannot fix the line W. Therefore we may assume that $\dim \Delta = 15$.

(t) Suppose that $\dim \overline{\Delta} = 8$. The list [24] 95.10 of representations shows that $\Omega \cong \mathrm{SO}_3\mathbb{C}$ and $\overline{\Delta} \cong \mathbb{C}^\times\Omega$. Up to conjugacy, $\Omega = \Delta_x'$, and Ω fixes some point $x \notin W$. Put $\Gamma = (\mathrm{Cs}\,\Omega)^1$. The action of Ω on the Lie algebra of $\sqrt{\Delta}$ implies that $\dim \Gamma = 3$. Because $\Gamma_x \times \Omega \leq \Delta_x$, we conclude that $\dim x^\Gamma > 0$. The group Ω fixes each point x^γ with $\gamma \in \Gamma$ and each line xx^γ; in particular, a maximal compact subgroup Φ of Ω has a fixed point on W. Now $\mathcal{F}_\Phi$ is a 2-dimensional subplane (cf. the Observation), and Γ acts effectively on $\mathcal{F}_\Phi$. The orbit x^Γ is contained in a line L (or else $\langle x^\Gamma \rangle = \mathcal{F}_\Phi = \mathcal{F}_\Omega$ in contradiction to Stiffness). It follows that $\Gamma = \mathsf{A} \times \mathsf{B}$ with $\mathsf{A} \cong \mathrm{L}_2$ and $\mathsf{B} \cong \mathbb{R}$: this is obvious if $\mathcal{F}_\Phi \cong \mathcal{P}_2\mathbb{R}$, in the other case see [24] 33.9b and 37.2. The fact that $\overline{\Gamma} \cong \mathbb{C}^\times$ is commutative implies that $\mathsf{A}' \leq \mathrm{Cs}\,\Theta$. Hence A' is contained in the center of the 14-dimensional group $\Omega\mathsf{A}'\mathsf{B}\Theta$, and $\mathsf{A}' \leq \mathsf{T}_{[z]}$ for some center $z \in W$ by step (b), moreover, $\dim \mathsf{T}_{[z^\delta]} > 0$ for each $\delta \in \Delta$. Either $\dim \mathsf{T} \geq 4$ and $\dim \Delta \geq 16$, or $\dim \mathsf{T} \leq 3$, $\Omega \leq \mathrm{Cs}\,\mathsf{T}$, and Ω acts trivially on the subplane $\langle x^\mathsf{T} \rangle$. This contradiction shows that $\dim \overline{\Delta} = 7$.

(u) Consequently $\Delta_x \cong \overline{\Delta}$. Up to conjugacy, $\Omega = \Delta_x'$ acts effectively on Θ, and Clifford's Lemma [24] 95.5 implies that Ω is isomorphic to $\mathrm{SO}_3\mathbb{C}$ or the action of Ω is equivalent to that of a tensor product $\mathrm{O}_3'(\mathbb{R}, r) \otimes \mathrm{SL}_2\mathbb{R}$. Let again $\Gamma = (\mathrm{Cs}\,\Omega)^1$, and consider a maximal compact subgroup Φ of Ω. In the first case $\Phi \cong \mathrm{SO}_3\mathbb{R}$, $\dim \Gamma = 3$, $\dim \Gamma_x = 1$, and $\dim x^\Gamma = 2$. Because $\Omega|_{x^\Gamma} = \mathbb{1}$, it follows that x^Γ is contained in a fixed line L of Ω, and then $\mathcal{F}_\Phi$ is a 2-dimensional subplane (as in the previous step). Now $x^\Gamma \subseteq L \cap \mathcal{F}_\Phi$ satisfies $\dim x^\Gamma = 1$, which is absurd. Therefore $\Omega = \Upsilon \times \Psi$ with $\Upsilon \cong \mathrm{O}_3'(\mathbb{R}, r)$ and $\Psi \cong \mathrm{SL}_2\mathbb{R}$. In particular, Ω contains a unique central involution σ. From [24] 71.8 it follows that σ is a reflection in $\Omega_{[u,xv]}$ with $u, v \in W$. If Υ is compact, then $\mathcal{F}_\Upsilon$ is an Ω-invariant 2-dimensional subplane, and $\Psi|_{\mathcal{F}_\Upsilon}$ is a simple group, but the stabilizer of a triangle in $\mathcal{F}_\Upsilon$ is soluble, see [24] 33.8 or 10. Hence $\Upsilon \cong \mathrm{PSL}_2\mathbb{R}$. For each $\vartheta \in \Theta$ we have $\vartheta^\sigma = \vartheta^{-1}$ or, equivalently, $\sigma^\vartheta = \sigma\vartheta^2$. Consequently, the coset $\sigma\Theta$ consists of pairwise distinct reflections, and $\Theta_u = \sigma(\sigma\Theta_u) \le \Delta_{[u]}$ is a group of collineations with center u. This contradicts the assumption that no one-paramenter subgroup of Θ is straight.

(v) Transitivity of T in the case $\dim\Delta = 16$ has been announced with a short sketch of the proof in [21] (5); a detailed proof is given in Boekholt's dissertation [3] Satz 2.2. According to Satz 7.11 in [3], a Levi complement of $\sqrt{\Delta}$ is isomorphic to $\mathrm{SL}_2\mathbb{C}$ or Δ fixes some point on W. ■

Theorem 3. *If Δ fixes u and v but no line other than $W = uv$, and if* $\dim\Delta \ge 14$, *then Θ is a group of translations having one of the fixed points of Δ as center.*

Proof. (a) Recall that we may assume $\dim\Delta \le 16$. If Θ is a group of homologies, then Δ fixes the common center and axis of Θ, see Lemma 1. Also by Lemma 1, it suffices therefore to show that some one-parameter subgroup $\Pi \le \Theta$ is straight. Suppose that $\langle a^\Pi \rangle$ is a subplane and that no one-parameter subgroup of Θ is straight. Then $6 \le \dim\Delta_a \le t+3$ as in step (c) above, and $t \ge 3$. We show that $\Theta_a = 1\!\!1$: otherwise Θ_a acts trivially on $\mathcal{F} := \langle a^\Theta, u, v\rangle$, and $\mathcal{F}$ is a proper subplane of dimension 2 or 4. If $\dim\mathcal{F} = 2$, let $\Xi = \Delta_a\Theta|_{\mathcal{F}}$ and note that $\dim\Delta_a \ge 6$. By Stiffness $6 + t - \dim\Theta_a - 3 \le \dim\Xi \le 4$. Hence $\dim\Theta_a = t-1$ and $\dim a^\Xi = \dim a^\Theta = 1$. Now $\dim\Xi = \dim\Xi_a + 1 \le 3$ (because Ξ_a fixes a triangle in $\mathcal{F}$), and $t \le \dim\Theta_a$, a contradiction. If $\mathcal{F} \lessdot \mathcal{P}$, then Θ_a is compact, hence trivial. Now it follows that even $\mathsf{A} := \Delta_a \cap \mathrm{Cs}\,\Theta = 1\!\!1$: obviously, $\mathsf{A}|_{\mathcal{F}} = 1\!\!1$. If $\mathcal{F} \ne \mathcal{P}$, consider $\mathsf{X} := \Delta_a|_{\mathcal{F}} \cong \Delta_a/\mathsf{K}$ and note that X fixes a triangle, so that $\dim\mathsf{X} \le \dim\mathcal{F}$. Stiffness implies

$$6 \le \dim\Delta_a = \dim\mathsf{X} + \dim\mathsf{K} \le \dim\mathcal{F} + \dim\mathsf{K} \le 5\,,$$

which is impossible. Consequently Δ_a embeds into the irreducible group $\overline{\Delta} = \Delta|_\Theta = \Delta/\mathrm{Cs}\,\Theta$. In particular, $\overline{\Delta}'$ is semi-simple, $\dim\overline{\Delta}' > 3$, and then even $\dim\overline{\Delta}' \ge 6$, see [24] 95.6.

(b) Θ *acts freely on* $P \smallsetminus W$: Assume that $\Theta_x \ne 1\!\!1$ for some $x \notin W$. If $x^\Theta = x$, then Θ fixes x^Δ pointwise, $\langle x^\Delta\rangle$ is a 2-dimensional subplane and $\dim\Delta \le 7$. If $x^\Theta \subseteq L = L^\Theta \in \mathfrak{L}$, then $L \cap W$ is a fixed point of Δ, say $v \in L$ (otherwise $\langle L^\Delta\rangle$ would be a proper subplane, which is impossible). The group Θ_x acts trivially on x^Θ and Θ fixes each line in L^Δ. Thus $\langle x^\Theta, L^\Delta, u\rangle$ is a proper Δ_x-invariant subplane of dimension 2 or 4. In both cases $6 \le \dim\Delta_x \le 5$, a contradiction. If x^Θ generates a proper subplane $\mathcal{E}$, then $\mathcal{E}$ is Δ_x-invariant, and again $\dim\Delta_x \le 5$. Hence $\Theta_x = 1\!\!1$, and $x^\Theta \subseteq L = L^\Theta \in \mathfrak{L}$ or $\langle x^\Theta\rangle = \mathcal{P}$.

(c) If $t = 3$, then $\dim\Delta = 14$ by step (a). From $\dim\overline{\Delta}' \ge 6$ and $\overline{\Delta} \le \mathrm{GL}_3\mathbb{R}$ it follows that $\overline{\Delta}' \cong \mathrm{SL}_3\mathbb{R}$, and Δ contains a covering group Υ of $\overline{\Delta}'$ by [24] 94.27. Suppose that Υ is simply connected, i.e. that $\Upsilon\Theta$ contains a central involution σ. In the planar case, $\Upsilon\Theta$ fixes $u, v \in \mathcal{F}_\sigma$ and $8 \ge \dim\Upsilon\Theta|_{\mathcal{F}_\sigma} \ge 10$, a contradiction (cf. also [18], [19]). Hence σ is a reflection. If $x^\sigma = x \notin W$, then $\sigma|_{x^\Theta} = 1\!\!1$ and x^Θ is contained in the axis L of Θ, say $L \in \mathfrak{L}_u$, and σ has center v. As $L^\Delta \ne L$, the dual of Lemma 2 applies, and the connected component Ξ of the translation group $\mathsf{T}_{[v]}$ is isomorphic to $\mathbb{R}^k$ with $3 \le k \equiv 0 \bmod 2$. Consequently $\mathsf{T}_{[v]} \cong \mathbb{R}^4$. By assumption, $\Xi \cap \Theta = 1\!\!1$ because each one-parameter subgroup of Ξ is straight, but then $\dim\sqrt{\Delta} > 6$, which is impossible. Therefore $\Upsilon \cong \mathrm{SL}_3\mathbb{R}$ is strictly simple.

(d) According to the Observation, the fixed elements of a subgroup $\Phi \cong \mathrm{SO}_3\mathbb{R}$ of Υ form a 2-dimensional subplane $\mathcal{F}_\Phi$. Consider the action of Υ on the line pencils $\mathfrak{L}_z$ with $z \in \{u, v\}$ and choose a fixed point $x \notin W$ of Φ. Then $\dim \Upsilon_{xz} \geq 4$, $\Phi < \Upsilon_{xz}$, and $\Upsilon_{xz} = \Upsilon$, since $\mathrm{SO}_3\mathbb{R}$ is maximal in $\mathrm{SL}_3\mathbb{R}$, see [24] 94.34. Therefore $\Upsilon|_{\mathcal{F}_\Phi} = 1\!\!1$, but this contradicts Stiffness. Hence $t > 3$.

(e) Let $t = 4$. Then $\overline{\Delta}'$ is isomorphic to one of the 5 groups $\mathrm{Sp}_4\mathbb{R}$, $\mathrm{SL}_2\mathbb{C}$, or $\mathrm{O}'_4(\mathbb{R}, r)$, $r \in \{0, 1, 2\}$; this is a consequence of Clifford's Lemma [24] 95.5. According to [24] 94.27, the group $\overline{\Delta}'$ is covered by a semi-simple subgroup $\Upsilon < \Delta$. If Υ contains a central involution α, in particular in the first three cases or if Υ is a proper covering of the simple group $\mathrm{O}'_4(\mathbb{R}, 1) \cong \mathrm{SO}_3\mathbb{C}$ or if $\Upsilon \cong \mathrm{O}'_4(\mathbb{R}, 2)$, it follows from [24] 71.8 and 10 that α is not planar. Hence α is a reflection, either with axis W or with center u or v. Lemma 2 or its dual implies that the translation group T with axis W is a vector group of even dimension k. As α induces inversion on T, the group Υ acts non-trivially on T, so that $k \geq 4$ in the first 4 cases. The assumption in (a) yields $\Theta \cap \mathsf{T} = 1\!\!1$, and $\mathsf{T}\Theta \cong \mathbb{R}^s$ with $s \geq 8$ because $\mathsf{T}, \Theta \lhd \Delta$. Consequently $8 \leq \dim \sqrt{\Delta} \leq 10$, $\dim \Upsilon \leq 8$, and hence $\dim \Upsilon = 6$. If $\dim \mathsf{T} > 4$, then $\mathsf{T}_{[u]}$ and $\mathsf{T}_{[v]}$ are normal subgroups of positive dimension, and the action of Υ implies $\mathsf{T}_{[u]} \cong \mathsf{T}_{[v]} \cong \mathbb{R}^4$, but then $\dim \sqrt{\Delta}$ would be to large. Therefore $\dim \mathsf{T}\Theta = 8$, and we may assume that $\Delta = \Upsilon\mathsf{T}\Theta$. It follows that $\dim \overline{\Delta} = \dim \overline{\Delta}' = 6$ and $\Delta_a \cong \overline{\Delta}'$. By Levi's Theorem ([24] 94.28), $\Upsilon = \Delta_a$ up to conjugacy, and $\alpha|_\Theta \neq 1\!\!1$. Now $1\!\!1 \neq \alpha^\Theta\alpha = \{\vartheta^{-1}\vartheta^\alpha \mid \vartheta \in \Theta\} \subseteq \Theta \cap \mathsf{T} = 1\!\!1$. This contradiction shows that Υ is either the simple group $\mathrm{O}'_4(\mathbb{R}, 1)$ or a proper covering of $\mathrm{O}'_4(\mathbb{R}, 2)$. In particular $\dim \overline{\Delta} \leq 7$ (see [24] 95.10) and $\dim \Delta \leq 15$.

(f) Suppose that $\Upsilon \cong \mathrm{O}'_4(\mathbb{R}, 1)$ is strictly simple. Put $\Gamma = (\mathrm{Cs}\,\Theta)^1$ and $\mathsf{X} = \Gamma \cap \mathrm{Cs}\Upsilon$. Step (a) shows that $\Gamma_a = 1\!\!1$. Hence $\dim \overline{\Delta} = \Delta : \Gamma \leq 7 \leq \dim \Gamma \leq 8$. As $\overline{\Delta}'$ has no 5-dimensional subgroup, we have $\overline{\Delta}' \leq \overline{\Delta_a} \cong \Delta_a$, and we may assume again that $a^\Upsilon = a$. The action of Υ on the Lie algebra $\mathfrak{l}\Gamma$ is completely reducible, and $\mathbb{R}^4 \cong \Theta \lhd \Gamma$. Therefore $\dim \mathsf{X} = 0$ or $\dim \mathsf{X} \geq 3$. In the second case $\Upsilon|_{a^\mathsf{X}} = 1\!\!1$, $\Upsilon_z = 1\!\!1$ for some point $z \in W \smallsetminus \{u, v\}$, and $\dim \Upsilon \leq 4$. Consequently $\dim \mathsf{X} = 0$, $\dim \Gamma = 8$, and a^Γ is open in P by [24] 96.11a. We will prove that $\Gamma \cong \mathbb{R}^8$ and then derive a contradiction. Let Ξ be the connected component of Γ_{au} or of Γ_{av}. Then $\dim \Xi = 4$, $\Xi^\Upsilon = \Xi$, and Υ acts faithfully on Ξ. Hence $\Xi' = 1\!\!1$ and $\Xi \cong \mathbb{R}^4$; furthermore, the action of Υ implies that $\Xi \cap \Theta = 1\!\!1$. Thus indeed $\Gamma \cong \mathbb{R}^8$. Next we will show that Γ is sharply transitive on $P \smallsetminus W$. As we have seen in (b), either $\langle x^\Theta \rangle = \mathcal{P}$ and $\Gamma_x = 1\!\!1$, or x^Θ is a line L through a fixed point of Δ, say $v \in L$, and $x^\Theta = av \smallsetminus v$. If $\Gamma_x \neq 1\!\!1$, then Γ contains a homology $\xi \neq 1\!\!1$ with axis L and center u. By [24] 61.20, the translation group $\mathsf{T}_{[u]}$ has positive dimension, and the action of Υ implies $\mathsf{T}_{[u]} \cong \mathbb{R}^4$. Now $\Theta\mathsf{T}_{[u]} = \Gamma$, and $\Gamma_x = 1\!\!1$ after all. It follows that x^Γ is open in P for each $x \notin W$. Thus $a^\Gamma = P \smallsetminus W$, and Ξ fixes each line in the pencil $\mathfrak{L}_u$ or $\mathfrak{L}_v$ respectively. Hence Θ is contained in the transitive translation group $\Gamma = \Gamma_{au} \times \Gamma_{av}$. This contradicts the assumption in (a). Therefore $\Upsilon \not\cong \mathrm{O}'_4(\mathbb{R}, 1)$ and

(g) Υ is a proper covering group of $\mathrm{O}'_4(\mathbb{R}, 2)$, i.e. of a product $\mathrm{SL}_2\mathbb{R} \cdot \mathrm{SL}_2\mathbb{R}$ with amalgamated centers, and the center of Υ contains an element ι which is mapped onto the central involution of $\overline{\Delta}'$ such that $\zeta := \iota^2 \neq 1\!\!1$ and $\zeta \in \mathrm{Cs}\,\Upsilon\Theta$. Write Υ as an almost direct product $\Upsilon_1\Upsilon_2$ of two factors locally isomorphic to $\mathrm{SL}_2\mathbb{R}$, and put $\iota = \alpha_1\alpha_2$ where α_ν is in the center of Υ_ν. No proper covering of $\mathrm{SL}_2\mathbb{R}$ has a faithful linear representation. Therefore α_ν^2 acts trivially on $\sqrt{\Delta}$ and $\zeta \in \mathsf{Z} = \mathrm{Cs}\,\Delta$. If $x^\zeta = x$, then $\zeta|_{x^{\Theta\Upsilon}} = \mathbb{1}$, and step (b) implies that x^Θ is an affine line, say $x^\Theta = xv \setminus v$; in this case, ζ is a homology with center u and axis xv, but then xv would be a fixed line of Δ. This contradiction shows that ζ acts freely on $P \setminus W$. As $\dim \Delta_x \geq 6$, Stiffness implies that x, x^ζ, u, v is always a degenerate quadrangle. Consequently, ζ fixes each line through one of the points u, v, say $\zeta \in \Upsilon_{[v]}$; in particular, ζ belongs to the translation group T, and $\langle\zeta\rangle \cong \mathbb{Z}$ is infinite cyclic, see [24] 55.28. For $z \in W \setminus \{u, v\}$, the stabilizer $\Delta_{a,z}$ acts trivially on the subplane $\mathcal{E} = \langle a, a^\zeta, u, z\rangle$, and $\dim \mathcal{E} \leq 2$ because $\dim \Delta_{a,z} \geq 2$. The connected orbit a^Υ contains $a^\zeta \neq a$. Hence $\dim \Upsilon_a < \dim \Upsilon = 6$. On the other hand, $\Upsilon_a = \Delta_a \cap \Upsilon$ embeds into $\overline{\Delta}$. As $\dim \overline{\Delta} \leq 7$ and $\dim \Delta_a \geq \dim \overline{\Delta}' = 6$, we have $\dim \Upsilon_a \geq 5$. There exists a 2-torus $\Phi < \overline{\Delta}'$, and $\dim \overline{\Upsilon}_a \cap \Phi > 0$. Therefore Υ_a contains an involution σ. If σ is planar and if z is chosen in $\mathcal{F}_\sigma$, then $\mathcal{E} < \mathcal{F}_\sigma <\!\bullet\, \mathcal{P}$ and $\Lambda := (\Delta_{a,z})^1$ is compact by Stiffness; in fact $\Lambda \cong \mathbb{T}^2$, because Δ_a has no subgroup $\mathrm{SO}_3\mathbb{R}$. As Λ acts effectively on av, it follows from (†) that the action of Λ on $av \setminus \{v\}$ is equivalent to the linear action of $(\mathrm{SO}_2\mathbb{R})^2$ on $\mathbb{R}^4$. In particular, Λ cannot fix the point a^ζ. This contradiction shows that σ is a reflection with center u and axis av. The fact that $\dim \Delta_a \leq 7$ together with [24] 61.20 implies $\dim (av)^\Delta = \dim \mathsf{T}_{[u]} \geq 3$, and Δ_a acts faithfully on $\mathsf{T}_{[u]}$. If $\dim \mathsf{T}_{[u]} = 3$, Stiffness shows that Δ_a is transitive on $\mathsf{T}_{[u]}^1$, but then Δ_a would have a subgroup $\mathrm{SO}_3\mathbb{R}$. Hence $\mathsf{T}_{[u]} \cong \mathbb{R}^4$ and Δ_a acts irreducibly on $\mathsf{T}_{[u]}$. Consequently $\Upsilon \cong \Delta_a'$ is a Levi complement of $\sqrt{\Delta}$, and Υ contains a central involution after all. Therefore $t > 4$.

(h) Suppose that $t = 5$. Then a Levi complement Υ of $\sqrt{\Delta}$ covers the group $\overline{\Delta}' \cong \mathrm{O}_5'(\mathbb{R}, r)$, see [24] 95.10. We may assume that $\Delta = \Upsilon\Theta$, and then $\dim \Delta = 15$. If Υ is not strictly simple, then the center Z of Δ is not trivial and Z acts freely on $P \setminus W$ (in fact, $\mathsf{Z}_x|_{x^\Theta} = \mathbb{1}$, and (b) implies $\langle x^\Theta\rangle = \mathcal{P}$). Hence $x^\mathsf{Z} \subset L = L^\mathsf{Z} \in \mathfrak{L}$ for each $x \notin W$, and $\mathsf{Z} \leq \Delta_{[v]}$, say; as W is the only fixed line, even $\mathsf{Z} \leq \Delta_{[v,W]}$. If $\dim \Theta_{ux} = 4$, then Θ_{vx} would fix each line in $\mathfrak{L}_v$ and Θ_{vx} would consist of translations. It follows that $0 < \dim \Theta_{ux} < 4$. Let $\mathbb{R} \cong \Pi \leq \Theta_{ux}$ and $x \neq b \in x^\Pi$. Put $\Lambda = \Delta_{x,b}$ and note that Λ centralizes Π. We have $\dim \Lambda \geq 7 - \dim \Theta_{ux} \geq 4$, and $\mathsf{Z} \neq \mathbb{1}$ implies that Λ acts trivially on the connected subplane $\langle x^\mathsf{Z}, b^\Pi, u, v\rangle$, a contradiction to Stiffness. Consequently, $\mathsf{Z} = \mathbb{1}$ and $\Upsilon \cong \mathrm{O}_5'(\mathbb{R}, r)$, where $r > 0$ by [24] 55.40. A maximal compact subgroup Φ of Υ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or to $\mathrm{SO}_3\mathbb{R} \times \mathrm{SO}_2\mathbb{R}$. In any case, Φ has a subgroup $\mathrm{SO}_3\mathbb{R}$; the involutions in this subgroup are planar, cf. the Observation. Therefore (†) applies, and Φ fixes some lines au and av. The action of Φ on the 3 sides of the triangle a, u, v shows that $\Phi \cong \mathrm{SO}_4\mathbb{R}$. The central involution of $\sigma \in \Phi$ is a reflection; the axis of σ is one side of the triangle. Lemma 2 or its dual implies that the translation group T is a vector group of positive even dimension; moreover, σ induces inversion and Υ acts non-trivially on T. Hence $\dim \mathsf{T} > 5$, but then $\Theta \leq \mathsf{T}$ contrary to the assumption. Thus $t \neq 5$.

(i) If $\Gamma = (\mathrm{Cs}\,\Theta)^1$ has dimension > 6, then $\langle x^\Theta\rangle = \mathcal{P}$ by step (b) and $\Gamma_x = \mathbb{1}$. Notation can be chosen so that Γ is transitive on the pencil $\mathfrak{L}_v \setminus W$. Hence $\Theta_L \leq \Delta_{[v]}$ for any line L in this pencil, and $\Theta \cap \Delta_{[v]} = \Theta_{[v]} = \Theta_L \lhd \Delta$, contrary to the assumption.

(j) Finally, let $\mathsf{\Gamma} = \mathsf{\Theta} \cong \mathbb{R}^6$. Then $\dim \overline{\mathsf{\Delta}} \geq 8$ and $\overline{\mathsf{\Delta}}'$ is isomorphic to one of the groups $\mathrm{SO}_3\mathbb{C}$, $\mathrm{SL}_3\mathbb{R}$, or $\mathrm{SU}_3(\mathbb{C},1)$ since a 10-dimensional semi-simple group has no irreducible representation on $\mathbb{R}^6$, and $\dim \overline{\mathsf{\Delta}}' = 9$ is impossible by Clifford's Lemma [24] 95.5. As in step (c), there exists a covering group $\mathsf{\Upsilon}$ of $\overline{\mathsf{\Delta}}'$ in $\mathsf{\Delta}$. In the first two cases, $\mathsf{\Upsilon}$ is strictly simple: if not, then $\mathsf{\Upsilon}$ contains a central involution α, and α is a reflection because $\mathsf{\Upsilon\Theta}$ cannot act on a Baer subplane $\mathcal{F}_\alpha$. For each $x \notin W$, step (b) shows that $\langle x^{\mathsf{\Theta}} \rangle = \mathcal{P}$, so that $\mathsf{\Delta}_x \cap \mathrm{Cs}\,\mathsf{\Theta} = 1\!\!1$ and $\mathsf{\Delta}_x$ embeds faithfully into $\overline{\mathsf{\Delta}}$. Note that $\alpha \in \mathrm{Cs}\,\mathsf{\Theta}$ and let x be a fixed point of α; then $\alpha|_{x^{\mathsf{\Theta}}} = 1\!\!1$, a contradiction.

(k) If $\mathsf{\Upsilon} \cong \mathrm{SO}_3\mathbb{C}$, then $\dim \overline{\mathsf{\Delta}} = 8$, $\dim \mathsf{\Delta} = 14$, and $\dim \sqrt{\mathsf{\Delta}} = 8$; hence $\mathsf{H} = (\mathrm{Cs}_{\mathsf{\Delta}}\mathsf{\Upsilon})^1$ is a 2-dimensional group. Let $\mathsf{\Phi} < \mathsf{\Upsilon}$ be a maximal compact subgroup. By the Observation, the fixed points of $\mathsf{\Phi}$ form a 2-dimensional subplane $\mathcal{E}$. Choose $x \notin W$ such that $x^{\mathsf{\Phi}} = x$. Then $\overline{\mathsf{\Phi}} \leq \overline{\mathsf{\Delta}_x} \cap \overline{\mathsf{\Delta}}'$, $\dim \overline{\mathsf{\Delta}_x} \cap \overline{\mathsf{\Delta}}' \geq 4$, and $\overline{\mathsf{\Delta}}' \leq \overline{\mathsf{\Delta}_x}$ by maximality of $\mathsf{\Phi}$, see [24] 94.34. It follows that $\mathsf{\Delta}_x$ contains a Levi complement of $\sqrt{\mathsf{\Delta}}$, and we may assume that $\mathsf{\Upsilon} \leq \mathsf{\Delta}_x$ and $x^{\mathsf{\Upsilon}} = x$. If $\dim x^{\mathsf{H}} = 2$, then $\langle x^{\mathsf{H}} \rangle = \mathcal{E}$ and $\mathsf{\Upsilon}|_{\mathcal{E}} = 1\!\!1$, but this contradicts Stiffness. On the other hand, H acts effectively on $\mathcal{E}$, or else $\mathcal{F}_\eta < \mathcal{P}$ for some $\eta \in \mathsf{H}$, and $\mathsf{\Upsilon}$ cannot fix the points $u, v \in \mathcal{F}_\eta$, see [24] 72.4. If $x^{\mathsf{H}} = x$, then $\mathsf{H} \cong \mathbb{R}^2$ by [24] 33.10, but $\mathsf{H} \times \mathsf{\Upsilon} \leq \mathsf{\Delta}_x$ embeds into $\overline{\mathsf{\Delta}} \cong \mathbb{C}^\times \mathrm{SO}_3\mathbb{C}$. This being impossible, $\dim \mathsf{H}_x = 1$. The group H induces on $\mathsf{\Theta}$ the centralizer $\mathbb{C}^\times$ of $\mathrm{SO}_3\mathbb{C}$. Hence H is commutative, and H contains an involution α inducing inversion on $\mathsf{\Theta}$ or an element $\zeta \neq 1\!\!1$ in the center of $\mathsf{\Delta}$. In the first case, α is a reflection (or $\mathsf{\Upsilon}$ would act without fixed elements on $\mathcal{F}_\alpha$, again by [24] 72.4). Moreover, $\alpha^\vartheta \alpha = \vartheta^{-2}$ for each $\vartheta \in \mathsf{\Theta}$, and $\alpha^{\mathsf{\Theta}}\alpha = \mathsf{\Theta} \leq \alpha^{\mathsf{\Delta}}\alpha = \mathsf{T}$ by Lemma 2 or its dual, contrary to the assumption. In the second case, ζ acts freely on $P \setminus W$ (since $x^\zeta = x \Rightarrow \zeta|_{\langle x^{\mathsf{\Theta}} \rangle} = 1\!\!1$). Therefore $\mathsf{H}_x\mathsf{\Upsilon}$ fixes a degenerate quadrangle, and [20] (**) would imply $\mathsf{\Upsilon} \cong \mathrm{SO}_4\mathbb{R}$. This contradiction shows that $\dim \mathsf{\Upsilon} = 8$, and we may assume that $\mathsf{\Delta} = \mathsf{\Upsilon\Theta}$.

(ℓ) Let $\mathsf{\Upsilon} \cong \mathrm{SL}_3\mathbb{R}$. If $\mathsf{\Phi} \cong \mathrm{SO}_3\mathbb{R}$ is a compact subgroup of $\mathsf{\Upsilon}$, then $\mathcal{F}_{\mathsf{\Phi}}$ is a 2-dimensional subplane. Choose $a \notin W$ such that $a^{\mathsf{\Phi}} = a$. Then $\mathsf{\Phi} \leq \mathsf{\Upsilon}_{au}$, $\dim \mathsf{\Phi} < \dim \mathsf{\Upsilon}_{au}$, and [4] or [24] 94.34 implies $\mathsf{\Upsilon}_{au} = \mathsf{\Upsilon}$. Analogously $\mathsf{\Upsilon}_{av} = \mathsf{\Upsilon}$, so that $a^{\mathsf{\Upsilon}} = a$, $\mathsf{\Theta}_{au}$ and $\mathsf{\Theta}_{av}$ are $\mathsf{\Upsilon}$-invariant, and $\mathsf{\Theta}_{au} \cong \mathsf{\Theta}_{av} \cong \mathbb{R}^3$. A subgroup $\mathsf{\Lambda} \cong \mathrm{SL}_2\mathbb{R}$ of $\mathsf{\Upsilon}$ fixes a one-parameter subgroup of $\mathsf{\Theta}_{au}$ and of $\mathsf{\Theta}_{av}$; hence $\mathcal{F}_{\mathsf{\Lambda}}$ is a subplane. Let α be the central involution in $\mathsf{\Lambda}$. Then $\mathcal{F}_{\mathsf{\Lambda}} \leq \mathcal{F}_\alpha \lessdot \mathcal{P}$, and Stiffness would imply that $\mathsf{\Lambda}$ is compact, an obvious contradiction.

(m) In the only remaining case, $\mathsf{\Upsilon}$ is a covering group of $\overline{\mathsf{\Delta}}' \cong \mathrm{SU}_3(\mathbb{C},1)$, and $\mathsf{\Upsilon}$ has a compact subgroup $\mathsf{\Phi} \cong \mathrm{SU}_2\mathbb{C}$. The only involution $\omega \in \mathsf{\Phi}$ is planar (or else Lemma 2 implies that $\omega^{\mathsf{\Delta}}\omega = \mathsf{T} \cong \mathbb{R}^k$ and that ω inverts the elements of T, so that $\mathsf{\Upsilon}$ acts non-trivially on T and $k \geq 6$, but then $\mathsf{\Theta T} \leq \sqrt{\mathsf{\Delta}}$ would be too big.) Now it follows from (†) and [24] 71.10 that $\mathsf{\Phi}|_{\mathcal{F}_\omega} = 1\!\!1$. This contradicts the Stiffness Theorem. ■

Theorem 4. *Under the assumptions of Theorem* 3 *the group* $\mathsf{\Delta}$ *is not solvable. Moreover, one of the translation groups* $\mathsf{T}_{[u]}$ *or* $\mathsf{T}_{[v]}$ *is transitive, or* $\dim \mathsf{\Delta} = 14$ *and* $\mathsf{\Delta}$ *is transitive on* $P \setminus W$.

Proof. (a) *The lines of* $\mathcal{P}$ *are homeomorphic to* $\mathbb{S}_4$ *and* (†) *applies*. Otherwise each orbit of a subgroup of $\mathsf{\Delta}$ on a line or a pencil has dimension at most 3, in particular, $\dim\Theta < 4$. Let $a \notin W$, $z \in S := W \smallsetminus \{u, v\}$, $c \in a^\Theta \smallsetminus \{a\}$, and put $\Omega = (\Delta_{a,z})^1$, $\Lambda = \Omega_c^1$. Then $\dim\Omega \geq 5$, $\dim\Lambda > 1$, $c \in a^\Pi$ for some one-parameter subgroup of Θ and $\Lambda \leq \mathrm{Cs}\,\Pi$. Hence $\mathcal{E} = \mathcal{F}_\Lambda$ is a 2-dimensional subplane, and $\dim\Lambda \leq 3$. Either Ω is transitive on $\Theta \cong \mathbb{R}^3$ or there is some c such that $\dim c^\Omega = 2 \leq \dim a^\Theta$ and $\dim\Lambda = 3$. In the first case, Ω acts effectively on Θ and contains a subgroup $\Phi \cong \mathrm{SO}_3\mathbb{R}$, but $\mathrm{SO}_3\mathbb{R}$ is maximal in $\mathrm{SL}_3\mathbb{R}$ (see [4] or [24] 94.34) and $\dim\Omega \leq 6$. In the second case there exists some point $d \in a^\Theta \smallsetminus \mathcal{E}$ with $\dim\Lambda_d > 0$, $\langle\mathcal{E}, d\rangle \,{<\!\!\bullet}\, \mathcal{P}$, and Λ is compact by Stiffness. The Remark in [23] p. 691 shows that $\Lambda \not\cong \mathbb{T}^3$. Hence $\Lambda \cong \mathrm{SO}_3\mathbb{R}$, the involutions in Λ are planar, and $W \approx \mathbb{S}_4$ after all.

(b) We may assume that Θ has center v. If Δ is solvable, then $\dim\Theta \leq 2$. According to [11] Lemma 2, there are points $b \in au$ and $c \in a^\Theta \smallsetminus a$ such that $\dim b^{\Delta_a} \leq 2$, $\dim\Delta_{a,b} \geq 4$, and $\Lambda = (\Delta_{a,b,c})^1$ satisfies $4 - t \leq \dim\Lambda \leq 3$. The group $\Delta_{a,b}|_\Theta$ is contained in $\mathrm{GL}_t\mathbb{R}$. Put $\Gamma = (\mathrm{Cs}\,\Theta)_{a,b}^1$ and note that $\Gamma \leq \Lambda$. Suppose first that $\Theta \cong \mathbb{R}$. Then $\Delta_{a,b} : \Gamma \leq 1$, $\dim\Gamma = 3$, $\dim\Delta_{a,b} = 4$, and $\dim b^{\Delta_a} = 2$. Hence there is some $b' \in b^{\Delta_a}$ with $\dim\Lambda_{b'} > 0$ and $\langle a, b, b', c\rangle \,{<\!\!\bullet}\, \mathcal{P}$. Now $\Lambda \cong \mathbb{T}^3$ by Stiffness, and (†) would imply that there exists a reflection in $\Lambda_{[W]}$, which is absurd. Consequently, $\Theta \cong \mathbb{R}^2$. As $\Delta_{a,b}$ is solvable, $\Delta_{a,b} : \Gamma < 4$, $\dim\Gamma > 0$ and $\mathcal{F}_\Gamma = \langle a^\Theta, b, u\rangle \,{<\!\!\bullet}\, \mathcal{P}$. Therefore Λ is compact, $\Lambda|_\Theta = \mathbb{1}$, $\Lambda = \Gamma$, $\mathcal{F}_\Lambda \,{<\!\!\bullet}\, \mathcal{P}$, and Stiffness implies $\dim\Lambda = 1$, a contradiction.

(c) Assume from now on that $\dim\Delta > 14$ or that Δ is not transitive on $P \smallsetminus W$. Then there is some point $a \notin W$ such that $\nabla = (\Delta_a)^1$ satisfies $\dim\nabla \geq 7$. Suppose that ∇ is transitive on $S = W \smallsetminus \{u, v\} \simeq \mathbb{S}_3$ and that $\dim\Theta < 4$. Then $\dim\nabla \leq 10$ by the Stiffness result (*). Let Υ be a maximal compact subgroup of ∇. Together with [24] 94.31, the exact homotopy sequence

$$\cdots \to \mathbb{Z}_2 = \pi_4 S \to \pi_3\nabla_z \to \pi_3\nabla \cong \pi_3\Upsilon' \to \pi_3 S = \mathbb{Z} \to \pi_2\nabla_z = 0$$

shows that $\Upsilon' \neq \mathbb{1}$, and (†) implies that each almost simple factor of Υ' is 3-dimensional. If each factor of Υ' is simple, then $\Upsilon' \cong \mathrm{SO}_3\mathbb{R}$ by (†), $\mathcal{F}_{\Upsilon'}$ is a subplane, and we may assume that $\Upsilon' \leq \nabla_z$. In this case, the homotopy sequence reads $\mathbb{Z}_2 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to 0$, obviously a contradiction. Therefore some factor Φ of Υ' is isomorphic to $\mathrm{Spin}_3\mathbb{R}$, and it follows from Stiffness and [24] 71.10 that the involution $\iota \in \Phi$ is not planar. Hence ι is a reflection, its axis is av (because ι acts trivially on a^Θ). By Lemma 2 the group Φ acts effectively on $\mathsf{T}_{[u]}$, and $\mathsf{T}_{[u]} \cong \mathbb{R}^4$ as claimed. Thus we may also assume that for some $z \in S$ the stabilizer $\Omega = \nabla_z^1$ satisfies $\dim\Omega \geq 4$.

(d) Postponing the case $\Theta \cong \mathbb{R}$, suppose that $\Theta \cong \mathbb{R}^2$, let $\Lambda = \Omega_c^1$ with $a \neq c \in a^\Theta$, and note that $\dim\Lambda \geq 2$. In the case $\langle a^\Theta, u, z\rangle \,{<\!\!\bullet}\, \mathcal{P}$, Stiffness would imply $\dim\Lambda \leq 1$. Therefore $\langle a^\Theta, u, z\rangle = \mathcal{P}$ and $\Omega \leq \mathrm{Aut}\,\Theta \cong \mathrm{GL}_2\mathbb{R}$, and then $\dim\Omega = 4$, $\dim\nabla = 7$, and $\dim a^\Delta \geq 7$. If the involution $\iota \in \Omega' \cong \mathrm{SL}_2\mathbb{R}$ is planar, then $\Omega'|_{\mathcal{F}_\iota} \cong \mathrm{PSL}_2\mathbb{R}$, but the stabilizer of a degenerate quadrangle in $\mathcal{F}_\iota$ has dimension at most 2 ([24] 71.7b). It follows that ι is a reflection with axis W and center a. By Lemma 2 we have $\iota^\Delta\iota = \mathsf{T} \cong \mathbb{R}^k$ and $7 \leq \dim\iota^\Delta = k \equiv 0 \bmod 2$. Consequently T is transitive.

(e) Next, let $\Theta \cong \mathbb{R}^3$. Then Ω acts faithfully on Θ. The arguments in step (d) show that either there is an Ω-invariant one-parameter subgroup $\Pi < \Theta$, or Ω is an irreducible subgroup of $\mathrm{Aut}\,\Theta$. In the first case Ω is sharply transitive on $\Theta\setminus\Pi$ (otherwise there are points $c\in a^{\Pi}\setminus\{a\}$ and $d\in a^{\Theta\setminus\Pi}$ such that $\Omega_{c,d}\neq\mathbb{1}$ and $\langle a^{\Pi},d,u,z\rangle\lessdot\mathcal{P}$, but then $\Lambda=\Omega_c^1$ is compact by Stiffness, Λ would be equivalent to $\mathrm{SO}_3\mathbb{R}$ and would act transitively on the set of one-parameter subgroups of Θ). It follows that Λ is homeomorphic to $\mathbb{R}\times\mathbb{C}^{\times}$, and then Λ contains an involution. Stiffness implies that Λ is compact, a contradiction.

(f) Consequently, ∇ induces on Θ an irreducible group $\nabla|_{\Theta}=\nabla/\mathsf{N}$. Note that N fixes each point of the orbit a^{Θ} and acts freely on $S=W\setminus\{u,v\}$. From [24] 95.5,6,10 it follows that $\nabla'|_{\Theta}$ is isomorphic to one of the groups $\mathrm{PSL}_2\mathbb{R}\cong\mathrm{O}_3'(\mathbb{R},1)$, $\mathrm{SO}_3\mathbb{R}$, or $\mathrm{SL}_3\mathbb{R}$. In the last case, ∇ has a subgroup Ψ covering $\mathrm{SL}_3\mathbb{R}$. In fact, $\Psi\cong\mathrm{SL}_3\mathbb{R}$, or Ψ would contain a central involution, and $\mathsf{T}_{[u]}\cong\mathbb{R}^4$ as at the end of step (c). Consider a subgroup $\Phi\cong\mathrm{SO}_3\mathbb{R}$ of Ψ. By (†), the group Φ fixes some point $z\in S$, $\Phi\le\Psi_z$. As $\dim\Psi_z>3$ and Φ is maximal in Ψ, it follows that $z^{\Psi}=z$. Hence $\dim\nabla_{c,z}\ge 5$, which contradicts Stiffness.

(g) In the two other cases, $\dim\nabla'|_{\Theta}=3$, $\nabla:\mathsf{N}\le 4$ and $\dim\mathsf{N}\ge 3$. If $\dim\mathsf{N}>3$, then N is sharply transitive on S, and N would have a subgroup $\mathrm{Spin}_3\mathbb{R}$, see step (c). Therefore $\dim\mathsf{N}=3$, $\dim\nabla=7$, and $\Delta:\nabla=\dim a^{\Delta}\ge 7$. Suppose that $\nabla'|_{\Theta}$ is compact. Then ∇ has a subgroup $\Phi\cong\mathrm{SO}_3\mathbb{R}\cong\Phi|_{\Theta}$ (as in step (c), the possibility $\mathrm{Spin}_3\mathbb{R}$ can be excluded), and $\Phi\cap\mathsf{N}=\mathbb{1}$. If $\Phi\le\mathrm{Cs}\,\mathsf{N}$, then N acts faithfully on the 2-dimensional subplane $\mathcal{F}_{\Phi}$ and fixes a triangle in $\mathcal{F}_{\Phi}$, but this implies $\dim\mathsf{N}\le 2$. Consequently, Φ acts effectively on N, and $\mathsf{N}\cong\mathbb{R}^3$. It follows that $\nabla=\Phi\sqrt{\nabla}$ with $\sqrt{\nabla}=\Pi\mathsf{N}$ and $\mathbb{R}\cong\Pi=\mathrm{Cs}_{\nabla}\Phi$. By step (c), the solvable group $\sqrt{\nabla}$ is not transitive on S. If $w^{\Phi}=w\in S$, then $w^{\nabla}=w^{\sqrt{\nabla}}\neq S$. As N acts freely on S, there is some point $z\in S$ such that $\dim z^{\nabla}=3$ and $\dim\nabla_z=4$. Analogously, $\dim\nabla_b=4$ for some $b\in au\setminus\{a,u\}$. (Note that the arguments of (c) apply also to the action of ∇ on au.) Because $\nabla_z\cap\mathsf{N}=\mathbb{1}$, it follows that $\nabla_z\cong\nabla/\mathsf{N}\cong\nabla_b$, and there are Levi complements Φ and Ψ of $\sqrt{\nabla}$ such that $\nabla_z=\Pi\Phi$ and $\nabla_b=\mathsf{P}\Psi$. By [24] 94.28c, the groups Φ and Ψ are conjugate in ∇, in fact $\Psi^{\nu}=\Phi$ for some $\nu\in\mathsf{N}$. Now $\nabla_{b^{\nu}}=\mathsf{P}^{\nu}\Phi=\nabla_z$ in contradiction to Stiffness.

(h) The remaining case $\nabla'|_{\Theta}\cong\mathrm{O}_3'(\mathbb{R},1)$ can be dealt with similarly. (†) and the fact that N acts freely on S show that $\mathsf{N}\ncong\mathrm{SO}_3\mathbb{R}$. Step (c) implies that ∇ is not transitive on S. Hence again $\dim\nabla_z=4$ for some $z\in S$, and $\nabla_z\cong\nabla/\mathsf{N}$. In particular, ∇_z has a subgroup $\Omega\cong\mathrm{O}_3'(\mathbb{R},1)$. If $\Omega\le\mathrm{Cs}\,\mathsf{N}$, then $\Omega|_{z^{\mathsf{N}}}=\mathbb{1}$ and Ω would act freely on $a^{\Theta}\setminus a$ and on $\Theta\setminus\mathbb{1}$. Therefore $\mathsf{N}\cong\mathbb{R}^3$, and Ω is a Levi complement of $\sqrt{\nabla}$. As in step (g), the group Ω fixes also a point $b\in au\setminus\{a,u\}$. Because Ω contains planar involutions, Stiffness would imply that Ω is compact. Hence the theorem is true for $\Theta\cong\mathbb{R}^3$.

(i) In the case $\Theta\cong\mathbb{R}$ the group $\Xi=(\Delta_{au}\cap\mathrm{Cs}\,\Theta)^1$ acts effectively on the line $X:=au\setminus\{u\}$ and satisfies $9\le\dim\Xi\le 11$. If $\dim\Xi=11$, then Ξ is doubly transitive on X, and [24] 96.16 shows that Ξ_a is equivalent to a transitive group on $\mathbb{R}^4$. It follows that Ξ_a contains a reflection with center u and axis av, and $\mathsf{T}_{[u]}$ is transitive by Lemma 2. Now let $\dim\Xi=10$. Then Ξ is still transitive on X. If Ξ is semi-simple, then Ξ is an infinite covering group of $\mathrm{O}_5'(\mathbb{R},2)$ by [23] 4.1, and Ξ contains a central element $\zeta\neq\mathbb{1}$. Consequently $\Xi_a=\Xi_{a^{\zeta}}$ fixes a quadrangle, but this contradicts Stiffness. Hence Ξ has a minimal normal vector subgroup H

such that $a^{\mathsf{H}} \neq a$, $\dim a^{\mathsf{H}} \geq 3$, and Ξ_a acts faithfully on H. Suppose that $\mathsf{H} \cong \mathbb{R}^3$. Then Ξ_a is a transitive subgroup of $\mathrm{GL}_3\mathbb{R}$, and $6 = \dim \Xi_a \geq 8$ since $\mathrm{SO}_3\mathbb{R}$ is maximal in $\mathrm{SL}_3\mathbb{R}$.

(j) Therefore $\dim \mathsf{H} > 3$, $\mathsf{H}_a|_{a^{\mathsf{H}}} \cong \mathsf{H}_a = \mathbb{1}$, $\mathsf{H} \cong \mathbb{R}^4$, H is transitive on X, and $\Xi = \Xi_a \mathsf{H}$; moreover, Ξ_a is an irreducible subgroup of $\mathrm{Aut}\,\mathsf{H}$. By [24] 95.6b, the commutator group $\Upsilon = \Xi_a'$ is semi-simple of dimension > 3, thus $\dim \Upsilon = 6$ and $\Xi = \Upsilon \mathsf{H}$. It follows that Υ is isomorphic to one of the groups $\mathrm{SL}_2\mathbb{C}$ or $\mathrm{O}_4'(\mathbb{R}, r)$. If Υ contains a central involution σ, then σ is a reflection with axis av and $\mathsf{T}_{[u]}$ is transitive. Only one possibility remains: Υ is the simple group $\mathrm{SO}_3\mathbb{C} \cong \mathrm{O}_4'(\mathbb{R}, 1)$. In this case we consider the complement $\Gamma = \Delta_{av}$ of H. Note that $10 \leq \dim \Gamma \leq 11$ by the assumptions, step (c) and Stiffness. Let Φ be a maximal compact subgroup of Υ. The group $\mathsf{P} = \mathrm{Cs}_\Gamma \Upsilon$ contains $\Theta \cong \mathbb{R}$ and acts almost effectively on the 2-dimensional plane $\mathcal{F}_\Phi$. Hence $\dim \mathsf{P} \leq 3$ and $\Upsilon \mathsf{P} < \Gamma$. The action of Υ on the additive group of the Lie algebra $\mathfrak{l}\Gamma$ shows that $\Gamma{:}\Upsilon\mathsf{P} \geq 4$. Consequently $\dim \Gamma = 11$, $\dim \nabla = 7$, $\mathrm{Cs}_\nabla \Upsilon = \mathsf{P}_a$, and $\dim \mathsf{P}_a = 1$. Because $\Theta \leq \mathsf{P}$, we have $\dim a^{\mathsf{P}} \geq 1$ and $1 < \dim \mathsf{P}$, but then $\dim \Gamma > 11$, a contradiction.

(k) If $\dim \Xi = 9$, then $\dim \Delta = 14$. If Δ is not transitive on $P \setminus W$, then $\dim \Xi_a = 6$ for some $a \in X$, and Stiffness implies that Ξ is doubly transitive on the orbit a^Ξ. The action is also effective, and it follows from [24] 96.16 and 17 that $a^\Xi \approx \mathbb{S}_3$ and then $\dim \Xi \in \{8, 10\}$. This contradiction proves transitivity of Δ. ∎

Remark 3. Proper translation planes such that the full automorphism group Σ is 15-dimensional and fixes exactly the points u, v have been constructed by Hähl [7] under the additional assumption that Σ has a subgroup $\mathrm{Spin}_4\mathbb{R}$ acting almost effectively on the line uv. Such planes are coordinatized by generalized André systems $(\mathbb{H}, +, \bullet)$ defined as follows: let $\varphi : e^{\mathbb{R}} \to \mathbb{H}$ be any continuous map, and put $a \bullet x = a\, x^{\varphi(|a|)}$, $0 \bullet x = 0$. Then Σ contains the maps $(x, y) \mapsto (a \bullet x\, r + c,\ b \bullet y\, r + d)$ with $a\bar{a} = b\bar{b} = 1$, $r \in \mathbb{R}$, and $c, d \in \mathbb{H}$; in general, there are no other collineations. If φ is a non-trivial homomorphism into $\{c \in \mathbb{C} \mid c\bar{c} = 1\}$, then $(\mathbb{H}, +, \bullet)$ is a nearfield and $\dim \Sigma = 17$.

Theorem 5. *If* Δ *fixes* 3 *points* $u, v, w \in W$, *and if* $\dim \Delta \geq 12$, *then* Δ *is transitive on* $P \setminus W$. *If* $\dim \Delta > 12$, *then* Δ *contains a transitive translation group* T, *and* $\mathcal{P}$ *is classical or* $\dim \Delta = 13$ *and a complement of* T *is isomorphic to* $e^{\mathbb{R}} \cdot \mathrm{U}_2\mathbb{C}$.

Proof. (a) If $\dim \Delta > 12$ or if Δ is not transitive on the complement of W, then there is some point $a \notin W$ such that $\nabla = (\Delta_a)^1$ has dimension at least 5. On the other hand, $\dim \nabla \leq 7$ by the Stiffness result (*). Hence $\dim a^\Delta \geq 5$. Moreover, $a^\Theta \neq a$ (or Θ would induce the identity on $\langle a^\Delta \rangle = \mathcal{P}$). Put $\langle a^\Theta, u, v, w \rangle = \mathcal{F}$ and note that $\mathcal{F}^{\Delta_a} = \mathcal{F}$. Let $\nabla|_\mathcal{F} = \nabla/\Lambda$. If $\dim \mathcal{F} = 2$, then $\dim \Lambda \leq 3$ and $2 \leq \nabla{:}\Lambda \leq 1$; if $\mathcal{F} <\!\!\bullet\, \mathcal{P}$, then $\dim \Lambda \leq 1$ and $4 \leq \nabla{:}\Lambda \leq 2$ by [24] 71.7b; thus both cases are impossible. Therefore $\mathcal{F} = \mathcal{P}$, and Δ_a acts faithfully on Θ, equivalently, $\Delta_a \cap \mathrm{Cs}\,\Theta = \mathbb{1}$; in particular, $\Theta_a = \mathbb{1}$. Consider a minimal ∇-invariant subgroup $\widehat{\Theta} \cong \mathbb{R}^s$ of Θ. As before, $\langle a^{\widehat{\Theta}}, u, v, w \rangle = \mathcal{P}$, so that ∇ acts faithfully and irreducibly on $\widehat{\Theta}$. We will show that $s = 4$. In fact, $s > 1$; for $s = 2$, the group ∇ would be contained in $\mathrm{GL}_2\mathbb{R}$. If $s = 3$, then ∇' is a semi-simple, therefore even almost simple subgroup of $\mathrm{SL}_3\mathbb{R}$ by [24] 95.6, but such a group has dimension 3 or 8, a contradiction. For $s > 4$, on the other hand, $\widehat{\Theta}_{av}$ would be a ∇-invariant subgroup of dimension $< s$.

(b) From [24] 95.5,6b, and 10 it follows that ∇' is isomorphic to one of the groups $\mathrm{SU}_2\mathbb{C}$, $\mathrm{SL}_2\mathbb{C}$, or $\mathrm{O}'_4(\mathbb{R}, r)$, $r \in \{0,1,2\}$. If $\nabla' \cong \mathrm{O}'_4(\mathbb{R},1)$ is the hyperbolic motion group of projective 3-space, choose x on the absolute cone in $a^{\widehat{\Theta}}$, and let Φ be a maximal compact subgroup of ∇'. Then $\Lambda = \nabla'_x$ is 3-dimensional, and $\Phi_x \cong \mathbb{T}$ contains a planar involution φ. As $\mathcal{F}_\Lambda < \mathcal{F}_\varphi <\!\!\bullet\, \mathcal{P}$, Stiffness would imply that Λ is compact, a contradiction. In all other cases, ∇' contains a central involution α, and α is a reflection with axis W (or ∇' would induce a solvable group on the Baer plane $\mathcal{F}_\alpha$, see [24] 71.7b). Now Lemma 2 shows that the translation group $\mathsf{T} = a^\Delta\alpha$ has even dimension $\dim a^\Delta \geq 5$. In particular, $\dim \mathsf{T}_{[z]} > 0$ for $z \in \{u, v, w\}$. If $\widehat{\Theta}$ is chosen in one of the groups $\mathsf{T}_{[z]}$, the last part of (a) implies that $\dim \mathsf{T}_{[z]} = 4$. Consequently, T is transitive. The kernel of the translation plane yields a one-dimensional group P of homologies. We may assume that $\mathsf{P} \leq \nabla$.

(c) Either $\nabla \cong \mathbb{C}^\times \cdot \mathrm{SU}_2\mathbb{C}$ and $\dim \Delta = 13$, or $\dim \nabla' = 6$, $\dim \nabla = 7$ and $\dim \Delta = 15$. In the latter case, [20] (**) shows that $\nabla = \mathsf{P} \times \Phi$ with $\Phi \cong \mathrm{SO}_4\mathbb{R}$. The lines of a translation plane are homeomorphic to $\mathbb{S}_4$, and (†) applies. Therefore $\Phi|_W \cong \mathrm{SO}_3\mathbb{R}$, and one factor $\Phi_1 \cong \mathrm{Spin}_3\mathbb{R}$ of Φ consists of homologies with axis W. Consequently, $\mathsf{P}\Phi_1 \leq \Delta_{[a,W]}$ is a transitive group of homologies. If the affine plane $\mathcal{P} \setminus W$ is coordinatized by a quasi-field $(H, +, \ )$, then the distributive law $s(x+a) = sx+sa$ follows from the fact that the maps $(x, y) \mapsto (x+a, y)$ are translations. The homologies have the form $(x, y) \mapsto (xc, yc)$ with $c \in H^\times$. This implies that multiplication of H is associative and satisfies the other distributive law. Hence, $(H, +, \ )$ is a (skew) field, cf. [13] Th. 6.6. ∎

Remark 4. (H. Hähl) There is an abundance of non-classical planes admitting a 13-dimensional group Δ as in Theorem 5. The full automorphism group Σ of such planes can be much larger; in the most homogeneous case $\dim \Sigma = 18$. These planes have been described by Hähl [9], see also [15] or [24] 82.20. They are coordinatized by a semi-field $\mathbb{H}_a = (\mathbb{H}, +, \circ)$ defined as follows: let $a = e^{i\alpha}$ with $0 < \alpha < \frac{\pi}{2}$ and $V = \{z \in \mathbb{H} \mid z+\overline{z} = 0\}$, and note that each quaternion can be written in the form $\sigma + \mathfrak{s}a$ with unique elements $\sigma \in \mathbb{R}$ and $\mathfrak{s} \in V$. Put $(\sigma + \mathfrak{s}a) \circ x = \sigma x + \mathfrak{s}xa$. Then Σ contains a group Δ as in Theorem 5, a transitive elation group with center v, and a torus group Λ acting trivially on the complex subplane. Vast generalizations of this construction have been discussed in [9]. An explicit description of all planes as in Theorem 5 with $\dim \Sigma = 13$ does not seem feasible.

Theorem 6. *If $\mathcal{F}_\Delta = (a, W)$ with $a \notin W$, and if $\dim \Delta \geq 15$, then the plane is classical.*

Proof. (a) If Δ is transitive on W, then Δ has a subgroup $\Phi \cong \mathrm{Spin}_5\mathbb{R}$; this follows, e.g., from [24] 96.19–22 and 55.40. The connected component Γ of $\mathrm{Cs}_\Delta\Phi$ consists of homologies with axis W, since Φ_z fixes the orbit z^Γ pointwise and each $z \in W$ is an isolated fixed point of Φ_z on W. Hence $\dim \Gamma \leq 4$. The representation of Φ on the additive group of the Lie algebra $\mathfrak{l}\Delta$ is completely reducible, see [24] 95.3. Assume first that $\dim \Theta = t \leq 4$. Then $\Theta \leq \Gamma$, $\Phi\Gamma < \Delta$, and $\Delta:\Phi\Gamma \geq 5$, so that $\dim \Delta \geq 16$. For $v \in W$, the group $\Phi_v \cong (\mathrm{Spin}_3\mathbb{R})^2$ fixes a unique second point $u \in W$, the center of Φ_v contains 3 reflections (because $\Phi_v|_W \cong \mathrm{SO}_4\mathbb{R}$), and (†) implies that $\Phi_{[u,av]} \cong \mathrm{Spin}_3\mathbb{R}$. The Stiffness result (*) yields $\dim \Delta_{u,v} \leq 11 < \dim \Delta_v$, and $u^{\Delta_v} \neq u$. From the action of Φ_v on W it follows that u^{Δ_v} is open in W. By [24] 61.19b or Lemma 2, the elation group $\Delta_{[v,av]}$ is transitive, and this is true for each $v \in W$; in other words, Δ is a group of Lenz type III. Consequently, $\mathcal{P}$ is the classical quaternion plane, cf. [24] 64.18.

(b) Now let $t > 1$. Then Θ does not consist of homologies, see [24] 61.2. As $a^\Delta = a$, the group Θ does not contain translations with axis W. Therefore $v^\Theta \neq v$ for one and then for each point $v \in W$. Since $\Theta \triangleleft \Delta$, the orbit v^Θ is Φ_v-invariant, and the action of Φ_v on W shows that v^Θ is open in W. Consequently, Θ is transitive on W, but this contradicts the fact [24] 96.19 that a transitive Lie group on $\mathbb{S}_4$ has a compact transitive subgroup.

(c) If Δ is not transitive on W, then Δ has some orbit $V \subset W$ of dimension < 4, see [24] 96.11a. By assumption V is not a singleton. For $u, v, w \in V$, the Stiffness result (*) shows that $\dim \Delta_{u,v,w} \leq 7$. Consequently, $\dim V = 3$ and Δ is doubly transitive on V, moreover, $\dim w^{\Delta_{u,v}} > 1$. Put $\Delta|_V = \Delta/\mathsf{K}$. Then K acts freely on $av \setminus \{a, v\}$ and $\dim \mathsf{K} \leq 4$. We use the classification of doubly transitive groups as summarized in [24] 96.16,17. Because $\dim w^{\Delta_{u,v}} > 1$, the space V cannot be homeomorphic to $\mathbb{R}^3$ or to projective 3-space. Hence $V \approx \mathbb{S}_3$, $\Delta:\mathsf{K} \leq 10$, and $\dim \Delta < 15$. ■

Proposition. *If Δ fixes two points und two lines and if $\dim \Delta \geq 15$, then the plane is classical.*

Proof. Stiffness implies that Δ has no further fixed elements. Write $\mathcal{F}_\Delta = (u, v, av, uv)$ and put $\nabla = \Delta_a$. From Stiffness it follows that $\dim \nabla = 11$ and that Δ is transitive on $K = av \setminus v$. According to [20] ($^{*}_{**}$), the group ∇ is a product of transitive homology groups with centers a, u, v, respectively. By Lemma 2 or [24] 61.20, the translation group $\Delta_{[v,W]}$ is transitive. Therefore the plane can be coordinatized by a Cartesian field with associative multiplication satisfying both distributive laws, that is by a field, cp. [24] 23.11 and use the fact that the (v, au)-homologies have the form $(x, y) \mapsto (x, c \circ y)$. ■

Theorem 7. *If $\dim \Delta \geq 14$ and $\mathcal{F}_\Delta = (u, v, av, uv)$, then the translation group $\Delta_{[v,uv]}$ is isomorphic to $\mathbb{R}^4$ and $(\Delta_a)'$ is isomorphic to its classical counterpart.*

Proof. We may assume that $\dim \Delta = 14$; notation will be the same as in the Proposition.

(a) Δ *is transitive on* $av \setminus v$. If not, there would exist points $a, c \in K$ such that $\dim \nabla_c \geq 8$ in contradiction to [20] (**). By the same result, either (I) Δ is doubly transitive on K, or (II) there exists some $c \in K \setminus \{a\}$ such that $\dim \nabla_c = 7$, and ∇ is transitive on $S := W \setminus \{u, v\}$. In both cases $\dim \nabla = 10$.

(b) First consider case (I). The group $\mathsf{K} := \Delta_{[u,av]}$ consists of homologies with axis av, and $\widetilde{\Delta} = \Delta/\mathsf{K}$ is an extension of $\mathbb{R}^4$ by a transitive subgroup $\widetilde{\nabla}$ of $\mathrm{GL}_4\mathbb{R}$. If $\dim \mathsf{K} = 0$, then $\widetilde{\nabla} \cong \mathrm{Sp}_4\mathbb{R}$, a maximal compact subgroup $\widetilde{\Phi}$ of $\widetilde{\nabla}$ is isomorphic to $\mathrm{U}_2\mathbb{C}$, and its preimage $\Phi \leq \nabla$ has a universal covering group

$\Upsilon \cong \mathbb{R}\times\mathrm{SU}_2\mathbb{C}$. The central involution $\sigma\in\Phi'$ is a reflection with axis W or au, say $\sigma\in\nabla_{[W]}$, and some element $s\in\mathbb{R}$ is mapped onto σ. As ∇ is almost simple, Υ induces on W a 4-dimensional group; in particular, $\sqrt{\Upsilon}|_W$ is induced by $\mathbb{R}/\langle s\rangle$. Hence $\Upsilon|_W \cong \mathrm{SO}_2\mathbb{R}\times\mathrm{SO}_3\mathbb{R}$, but this is excluded by (†). Consequently, $0<\dim\mathsf{K}\le 4$ and $6\le\dim\widetilde{\nabla}\le 9$. As $\widetilde{\nabla}$ is transitive on $\mathbb{R}^4\setminus\{0\}$, $\widetilde{\nabla}'$ has a subgroup $\mathrm{Spin}_3\mathbb{R}$. With [24] 95.4 and 10 it follows that $\widetilde{\nabla}'$ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or $\mathrm{SL}_2\mathbb{C}$, and then $\dim\widetilde{\nabla}\le 8$.

(c) Suppose that $\widetilde{\nabla}'\cong\mathrm{SL}_2\mathbb{C}$. Then ∇ has a subgroup $\Psi\cong\mathrm{SL}_2\mathbb{C}$ by [24] 94.27. The central involution $\sigma\in\Psi$ is not planar; this follows, e.g., from [24] 71.7. Therefore σ is a reflection; we may assume that the axis of σ is W. It follows that $\Psi|_W\cong\mathrm{SO}_3\mathbb{C}$ has a subgroup $\Omega\cong\mathrm{SO}_3\mathbb{R}$. According to (†), the fixed points of Ω on W form a circle containing u and v. The homology group $\mathsf{K}:=\Delta_{[u,av]}$ acts freely on S, and K is a two-ended group without a pair of commuting involutions, see [24] 61.2 and 55.32. Consequently, the compact factor of K is isomorphic to a subgroup of $\mathrm{Spin}_3\mathbb{R}$, $\Psi\le\mathrm{Cs}\,\mathsf{K}$ (because each representation of Ψ in dimension <4 is trivial), and K maps the fixed point set of Ω onto itself. We conclude that $\dim\mathsf{K}=1$, $\dim\nabla\le 9$, and $\dim\Delta<14$, a contradiction.

(d) Steps (b) and (c) imply that $\widetilde{\nabla}'\cong\mathrm{SO}_4\mathbb{R}$, and $\dim\widetilde{\nabla}\le 7$. Hence $\dim\mathsf{K}\ge 3$, $\mathsf{K}'\cong\mathrm{Spin}_3\mathbb{R}$ as in step (c), and ∇ has a 9-dimensional compact subgroup Φ. Now (†) shows that each of the 3 homology groups $\Phi_{[a]}$, $\Phi_{[u]}$, and $\Phi_{[v]}$ is isomorphic to $\mathrm{Spin}_3\mathbb{R}$; moreover, $\nabla\cong e^{\mathbb{R}}\times\Phi$, and the translation group $\Delta_{[v,W]}$ is transitive by [24] 61.20 or Lemma 2.

(e) In case (II), the group ∇_c is isomorphic to $e^{\mathbb{R}}\times\mathrm{SO}_4\mathbb{R}$ by [20] (**). Write $\nabla_c=\Pi\times\Omega$, where Ω denotes the compact factor. From (†) it follows that $\Omega|_K\cong\mathrm{SO}_3\mathbb{R}$. Hence $\Omega=\Phi\Psi$ with $\Phi=\Omega_{[u,K]}$ and $\Phi\cong\mathrm{Spin}_3\mathbb{R}\cong\Psi$. Choose any $z\in S$. Stiffness implies that $\Lambda:=\Omega_z\cong\mathrm{SO}_3\mathbb{R}$, and $\mathcal{F}_\Lambda$ is a 2-dimensional subplane; moreover, Π is transitive on each of the two connected components of $S\cap\mathcal{F}_\Lambda$ (or Π would have a fixed point on one of these arcs and then Π would be compact by Stiffness). Consequently, $\Pi\Phi$ is sharply transitive on S.

(f) Assume that Ω is a Levi complement of $\mathsf{P}=\sqrt{\nabla}$. Then $\dim\mathsf{P}=4$. As $\Pi\mathsf{P}$ is solvable and Ω-invariant, $\Pi\mathsf{P}\cap\Omega$ is finite, $\dim\Pi\mathsf{P}=4$, and $\Pi\le\mathsf{P}$. It follows that $\mathsf{P}_c^1=\Pi$ and $\dim c^{\mathsf{P}}=3$. If $\Psi\le\mathrm{Cs}\,\mathsf{P}$, then $\Psi|_{c^{\mathsf{P}}}=\mathbb{1}$, but $\Psi|_K\cong\mathrm{SO}_3\mathbb{R}$ has only a 1-dimensional set of fixed points. Since $\Pi^\Omega=\Pi$, the action of Ω on the Lie algebra $\mathfrak{l}\mathsf{P}$ shows that $\Omega|_{\mathsf{P}}\cong\mathrm{SO}_3\mathbb{R}$ and $\Phi\le\mathrm{Cs}\,\mathsf{P}$. The group $\mathsf{P}|_{S/\Phi}=\mathsf{P}/\mathsf{N}$ is solvable, and Brouwer's theorem [24] 96.30 implies that $\dim\mathsf{N}\ge 2$. In fact $\mathsf{N}\cong\mathbb{R}^3$, because $\Pi\cap\mathsf{N}=\mathbb{1}$, $\mathsf{N}^\Psi=\mathsf{N}$ and $\Psi|_{\mathsf{N}}\cong\mathrm{SO}_3\mathbb{R}$. By definition $z^{\mathsf{N}}\subseteq z^\Phi$. According to [24] 96.19, the group N is not transitive on the sphere z^Φ, and [24] 96.11 implies $\dim z^{\mathsf{N}}<3$ and $\dim\mathsf{N}_z>0$. Recall from (e) that $\Pi\le\mathrm{Cs}\,\Phi\Psi$. The action of $\Pi\Psi$ on N shows that each one-parameter subgroup of N is Π-invariant. In particular, $\mathsf{E}^\pi=\mathsf{E}$ for $\mathbb{R}\cong\mathsf{E}\le\mathsf{N}_z$ and each $\pi\in\Pi$. It follows that $\mathsf{E}|_{z^{\Pi\Phi}}=\mathbb{1}=\mathsf{E}|_W$, and then $\mathsf{E}^\psi|_W=\mathbb{1}$ for $\psi\in\Psi$. Therefore $\mathsf{N}\le\Delta_{[a,W]}$ in contradiction to the fact [24] 61.2 that connected homology groups are compact or two-ended. This shows that a Levi complement Υ of P in ∇ has dimension $\dim\Upsilon\ge 8$. Because $\Phi\le\Delta_{[u]}$ and $\dim\Delta_{[u]}\le 4$, the group Υ is not almost simple. Thus $\dim\Upsilon=9$, $\dim\mathsf{P}=1$, and $\mathsf{P}=\Pi$.

(g) The group $\Phi \le \nabla_{[u]}$ is a factor of Υ, and Υ is an almost direct product $\Phi\Gamma$ with $\dim \Gamma = 6$. We will show that Γ is compact. Suppose first that Γ is almost simple. Then $\Gamma|_{S/\Phi} = \mathbb{1}$, each orbit z^Φ is Γ-invariant, $\Gamma_z|_{z^\Phi} = \mathbb{1}$, and $\Gamma_z \trianglelefteq \Gamma$. Hence $\Gamma \le \nabla_{[a,W]}$, but almost simple groups of homologies are isomorphic to $\mathrm{Spin}_3\mathbb{R}$ by [24] 61.2 and 55.32. Therefore $\Gamma = \mathsf{AB}$ is an almost direct product of two 3-dimensional factors. If Ψ is one of these factors, say if $\Psi = \mathsf{A}$, then $\mathsf{B} \le \mathrm{Cs}\,\Omega$ and B would act non-trivially on $\mathcal{F}_\Lambda$, but the stabilizer of a triangle in a 2-dimensional plane is solvable (and has dimension at most 2, cf. [24] 33.8). It follows that Ψ is a 'diagonal' in the product AB. As Ψ is simply connected and almost simple, there exist unique projections onto A and B, and both factors are compact. Again $\Gamma|_{S/\Phi} = \mathbb{1}$ and $z^{\Phi\Gamma} = z^\Phi$. Transitivity of ∇ on S implies $z^{\mathsf{P}\Phi} = S$. Hence $\Gamma_z \le \nabla_{[a,W]}$ is another factor of Υ, and $\Gamma_z \cong \mathrm{Spin}_3\mathbb{R}$. Analogously, $\mathrm{Spin}_3\mathbb{R} \cong \Gamma_b \le \nabla_{[v]}$ for some point $b \in au$. Consequently, Υ is isomorphic to the group

$$\{(x,y) \mapsto (axc, byc) : \mathbb{H}^2 \to \mathbb{H}^2 \mid a,b,c \in \mathbb{H},\ a\bar{a} = b\bar{b} = c\bar{c} = 1\}.$$

In particular, Υ contains a reflection with center a, and Lemma 2 implies that $\Delta_{[v,W]} \cong \mathbb{R}^4$. ■

Constructions. Recall from [6] XI.4.2 or [24] 24.4 that an affine plane with a transitive group T of 'vertical' translations can be coordinatized by a so-called Cartesian field $(K,+,\bullet)$. This means that $(K,+)$ is a group and that each non-vertical line is given by an equation $y = s\bullet x + t$. In particular, $(K,+) \cong \mathsf{T}$. The other algebraic properties of a Cartesian field just express the fact that the lines of the form $y = s\bullet x + t$ together with the vertical ones indeed yield an affine plane, cf. [24] 24.4.

Distorted quaternions. Let $(\mathbb{R},+,*,1)$ be a topological Cartesian field such that identically $(-r)*s = -(r*s) = r*(-s)$. For $z \ne 0$ in the quaternion field $(\mathbb{H},+,\ )$ write $z_1 = z/|z|$. Define a new multiplication on $\mathbb{H}$ by $c \circ z = (|c|*|z|)\, c_1 z_1$ and $c \circ 0 = 0 \circ z = 0$. Then the *distorted quaternions* $(\mathbb{H},+,\circ)$ form a topological Cartesian field.

Proof. Distorted *octonions* are defined in the same way, and $(\mathbb{O},+,\circ)$ is a topological Cartesian field by [10]. Hence $\mu = (z \mapsto a\circ z - b \circ z) : \mathbb{O} \to \mathbb{O}$ is a homeomorphism whenever $a \ne b$, the restriction of μ to $\mathbb{H}$ is a continuous injection. If $a\circ z - b\circ z = d$ with $a,b,d \in \mathbb{H}$, then the equation $(|a|*|z|\, a_1 - |b|*|z|\, b_1)\, z_1 = d$ shows that z is also in $\mathbb{H}$. Thus $\mu|_{\mathbb{H}}$ is a homeomorphism, and so is $z \mapsto z\circ a - z \circ b$ by symmetry of the operation $\circ$. ■

Theorem 8. *An 8-dimensional compact plane $\mathcal{P}$ can be coordinatized by distorted quaternions if, and only if, $\mathcal{P}$ has a group Δ of automorphisms such that Δ fixes two points u, v and two lines av, uv, the translation group $\Delta_{[v,uv]}$ is transitive, and Δ has a maximal compact subgroup Φ of dimension 9.*

Proof. **A)** Let $\mathcal{P}$ be coordinatized by the Cartesian field $(\mathbb{H},+,\circ)$ of distorted quaternions. Then $\mathsf{T} = \{(x,y) \mapsto (x, y{+}t) \mid t \in \mathbb{H})\}$ is a transitive translation group. Note that the commutator group $\mathbb{H}'$ coincides with the set of all quaternions c of norm $c\bar{c} = 1$. If $s \in \mathbb{H}'$ or $x \in \mathbb{H}'$, then $s\circ x = sx$, because 1 is a unit element for the multiplication $*$. Therefore $\Phi = \{(x,y) \mapsto (axc, byc) \mid a,b,c \in \mathbb{H}'\}$ is a compact group of automorphisms, and $\Delta = \mathsf{T}\Phi$ has the required properties.

B) Assume conversely that Δ is a group with the properties specified in the theorem.

(a) As all maximal compact subgroups of Δ are conjugate ([24] 93.10), we may assume that $\Phi \le \nabla = \Delta_a$. By (†), the induced group $\Phi|_{av} = \Phi/\Phi_{[u]}$ is isomorphic to a subgroup of $\mathrm{SO}_4\mathbb{R}$. The kernel $\Phi_{[u]}$ does not contain a pair of commuting involutions, see [24] 55.32. Hence $\Phi_{[u]}$ has torus rank 1, and $\Phi_{[u]} \cong \mathrm{Spin}_3\mathbb{R}$. Analogously, $\Phi_{[a]} \cong \mathrm{Spin}_3\mathbb{R} \cong \Phi_{[v]}$, and Φ is an almost direct product of these 3 factors. Moreover, $\mathsf{T} := \Delta_{[v,uv]} \cong \mathbb{R}^4$, because T is transitive and Φ acts on T.

(b) Transitivity of T means that the coordinate system with respect to any quadrangle a,u,v,e is a Cartesian field $(H,+,\circ)$ with $(H,+) \cong \mathsf{T}$. The reflections in $\Phi_{[a]}$ and $\Phi_{[v]}$ invert each translation in T. Hence they have the form $(x,y) \mapsto (\pm x, -y)$, their product is $(x,y) \mapsto (-x,y)$. For the multiplication this gives the identity $(-r)\circ s = -(r \circ s) = r \circ (-s)$.

(c) Stiffness or (†) implies $\Lambda := \Phi_e \cong \mathrm{SO}_3\mathbb{R}$, and $\mathcal{E} = \mathcal{F}_\Lambda$ is a 2-dimensional subplane having $\mathsf{T} \cap \mathrm{Cs}\,\Lambda$ as transitive group of 'vertical' translations. Therefore $\mathcal{E}$ is coordinatized by a Cartesian field $(\mathbb{R},+,*)$, in fact, by a Cartesian subfield of $(H,+,\circ)$, so that $*$ is a restriction of the multiplication $\circ$ and satisfies the identity required in the construction of distorted quaternions.

(d) As $\Phi_{[u]}|_{av} = \mathbb{1}$, the group $\Phi|_{av} \cong \mathrm{SO}_4\mathbb{R}$ is induced by $\Psi = \Phi_{[a]}\Phi_{[v]}$. The vector space $\mathbb{R}^4$ can be identified with $(\mathbb{H},+)$ in such a way that Ψ preserves the ordinary norm $|\ |$ of the quaternions, since all maximal compact subgroups of $\mathrm{SL}_n\mathbb{R}$ are conjugate, cf. [24] 95.3.

(e) By [24] 23.11, the homologies in $\Phi_{[a]}$ have the form $(x,y) \mapsto (x \circ c, y \circ c)$, where c is an arbitrary element of norm 1, because $\Phi_{[a]} \cong \mathrm{Spin}_3\mathbb{R}$ is compact. Similarly, $\Phi_{[v]} = \{(x,y) \mapsto (x, b \circ y) \mid b \in \mathbb{H}'\}$ and $\Phi_{[u]} = \{(x,y) \mapsto (a \circ x, y) \mid a \in \mathbb{H}'\}$. It follows that the product $\circ$ is associative, whenever one of the factors has norm 1.

(f) As in the construction of distorted quaternions, write each quaternion $z \in \mathbb{H} \setminus \{0\}$ in the form $z = rc$ with $r = |z|$ and $c = z_1 \in \mathbb{H}'$. By step (d), the map $\gamma = (y \mapsto y \circ c) : \mathbb{H} \to \mathbb{H}$ is orthogonal with respect to the norm, in particular, γ is linear; moreover, $1 \circ c = c$. Hence $r \circ c = rc$ for each positive $r \in \mathbb{R}$ and each $c \in \mathbb{H}'$. Analogously, $c \circ r = rc = cr$. The group $\{(y \mapsto y \circ c) : \mathbb{H}' \to \mathbb{H}' \mid c \in \mathbb{H}'\}$ is a sharply transitive normal subgroup of $\mathrm{SO}_4\mathbb{R}$; the same is true for the ordinary multiplication. Therefore $a \circ c = ac$ or $a \circ c = ca$ for $|a| = |c| = 1$. Because the quaternions with the opposite multiplication coordinatize the dual plane, we may assume that $s_1 \circ x_1 = s_1 x_1$ holds identically. Now

$$s \circ x = (s_1|s|) \circ (|x| \circ x_1) = (s_1 \circ (|s| \circ |x|)) \circ x_1 = ((|s| * |x|) \circ s_1) \circ x_1 = (|s| * |x|)\, s_1 x_1 .$$

■

Remark 5. Suppose that Δ satisfies the conditions of Theorem 7. Then Δ has the properties of Theorem 8. Let again $\Lambda = \nabla_e$ and $\mathcal{E} = \mathcal{F}_\Lambda$. Then $\Gamma = (\mathrm{Cs}\,\Lambda)^1$ acts effectively on $\mathcal{E}$, and Γ is isomorphic to $\mathbb{R}^2$ or to L_2. In the first case $\Gamma_a|_{a^\Gamma} = \mathbb{1}$, and $\mathcal{E}$ has a transitive group of homologies with center u. Equivalently, the multiplication $*$ is associative. All such planes $\mathcal{E}$ have been described, see [16] *2.7.11.3*. It follows that $\circ$ is also associative. Therefore $\Delta_{[u,av]}$ is transitive in this case. If $\Gamma \cong \mathrm{L}_2$, then either $\mathcal{F}_\Gamma = (u, v, av, uv)$ or Γ fixes further points on uv (up to duality). All the corresponding planes $\mathcal{E}$ have been determined, cf. [10] p.99 (II). In all cases, there exists a wealth of non-classical planes.

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