

An Imprimitivity Theorem for Representations of a Semi-direct Product Hypergroup

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Abstract. The purpose of the present paper is to establish an imprimitivity theorem for representations of a semi-direct product hypergroup $K = H \rtimes_{\beta} G$ defined by a smooth action β of a locally compact group G on a hypergroup H . The proof of the theorem relies on a smooth irreducible absorbing action α of K on a locally compact space X and on an imprimitivity condition for the triplet $(K, C_0(X), \alpha)$.

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1. Introduction

The method of inducing representations of locally compact groups has been developed by G.W. Mackey in his seminal papers [14], [15]. Mackey was the first to prove an imprimitivity theorem and to apply it to the analysis of representations of certain classes of groups including the Heisenberg group and semi-direct products of groups where the normal factor is Abelian. The purpose of inducing representations is to reduce the problem of constructing irreducible representations of groups to subgroups for which the representations are known.

To be more precise, by inducing representations of a locally compact group G one wishes to construct unitary representations of G from those of a closed subgroup H of G . This means, induced representations are those unitary representations which arise from representations of H and the action of G on functions or sections of homogeneous bundles on the homogeneous space G/H .

Apart from the case that G/H is finite, representations of G induced from H are infinite-dimensional.

However, induced representations are an important tool in studying irreducible representations of non-compact, non-Abelian groups. In order to find all such representations Mackey provided a technique, the so-called "Mackey machine" which helps analyzing representations of G in terms of the representations of a normal subgroup H and the representations of various subgroups of G/H .

Excellent expository articles on Mackey's theory, in particular on his imprimitivity theorem for locally compact groups are published by K.R. Parthasarathy [17] and V.S. Sunder [20].

Mackey's work had a strong influence on later research on the subject; testifying sources are C.C. Moore [16] and L. Baggett [1]. These works were the starting point for the second author's research on irreducible representations of non-regular semi-direct products of groups [12].

To extend Mackey's approach to *hypergroups* many significant obstacles have to be surmounted. In the present paper the authors are offering a first attempt to achieve progress by establishing the tool for a general imprimitivity theorem for hypergroups in the sense of C.F. Dunkl, R.I. Jewett and R. Spector (see [2]) and by proving an appropriate imprimitivity statement for certain semi-direct product hypergroups.

Concerning generalized notions necessary to access the imprimitivity theorem for hypergroups cohomology arguments from previous papers [10], [11] are used. Moreover, an extended definition of an induced representation arising from an action of the hypergroup K on a locally compact space X is introduced. This definition leads beyond the one induced by P. Hermann in [7], [8] and [9]. Moreover, the corresponding imprimitivity condition in terms of continuous functions on X is specified.

This new imprimitivity condition is essentially different from the one in the group case, once one wants to reach beyond this case. The erroneous approach in [3] shows the dilemma.

Our main theorem is as follows.

Imprimitivity Theorem. *Let $K = H \rtimes_{\beta} G$ be a semi-direct product hypergroup defined by a smooth action β of a locally compact group G on a hypergroup H . Let α be a smooth irreducible absorbing action of K on a locally compact space X and let $K(x_0) = H \rtimes_{\beta} G(x_0)$ be the stability hypergroup of K at $x_0 \in X$. Then for a covariant representation (π, u) of $(K, C_0(X), \alpha)$ on a Hilbert space \mathcal{H} there exists a representation L of $K(x_0) = H \rtimes_{\beta} G(x_0)$ such that (π, u) is unitary equivalent to $(\pi^0, \text{ind}_{K(x_0)}^K L)$.*

Applying the imprimitivity theorem we have the following.

Theorem. (Mackey machine) *Let $K = H \rtimes_{\beta} G$ be a semi-direct product hypergroup defined by a smooth action β of a locally compact group G on a commutative hypergroup H . For any $\chi \in \hat{H}$ and an irreducible representation τ of $G(\chi)$ $u^{(\chi, \tau)}$ is defined by $\text{ind}_{H \rtimes_{\beta} G(\chi)}^{H \rtimes_{\beta} G}(\chi \otimes \tau)$.*

Then the following statements hold:

- (1) $u^{(\chi, \tau)}$ is an irreducible representation of K .
- (2) All irreducible representations of K are obtained in this form.
- (3) $u^{(\chi', \tau')}$ and $u^{(\chi, \tau)}$ are unitary equivalent if and only if $\text{Orb}(\chi') = \text{Orb}(\chi)$ and $\tau' \cong \tau$.

The layout of the present investigations can be described as follows. After Chapter 2 of preliminaries the authors expose the passage from groups to hypergroups which leads to a general conjecture of the imprimitivity theorem, in Chapter 3. The subsequent Chapter 4 contains the definition (and construction) of a semi-direct product hypergroup. The cohomology arguments in Chapter 5 are the technical prerequisites supporting the proof of the imprimitivity theorem given in Chapter 6. In Chapter 7 follows the extension of the "Mackey machine" to the hypergroups under discussion. Its application to classifying irreducible representations is illustrated by examples in Chapter 8.

2. Preliminaries

For a locally compact space X we shall mainly consider the subspaces $C_c(X)$ and $C_0(X)$ of the space $C(X)$ of continuous functions on X which have compact support or vanish at infinity respectively. By $M(X)$, $M^b(X)$ and $M_c(X)$ we abbreviate the spaces of all (Radon) measures on X , the bounded measures and the measures with compact support on X respectively. Let $M^1(X)$ denote the set of probability measures on X and $M_c^1(X)$ its subset $M^1(X) \cap M_c(X)$. The symbol δ_x stands for the Dirac measures in $x \in X$.

A *hypergroup* $(K, *)$ is a locally compact space K together with a *convolution* $*$ in $M^b(K)$ such that $(M^b(K), *)$ becomes a Banach algebra and that the following properties are fulfilled.

- (H1) The mapping $(\mu, \nu) \mapsto \mu * \nu$ from $M^b(K) \times M^b(K)$ into $M^b(K)$ is continuous with respect to the weak topology in $M^b(K)$.
- (H2) For $x, y \in K$ the convolution $\delta_x * \delta_y$ belongs to $M_c^1(K)$.
- (H3) There exists a *unit* element $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$, and an *involution* $x \mapsto x^-$ in K such that $\delta_{x^-} * \delta_{y^-} = (\delta_y * \delta_x)^-$ and $e \in \text{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$ whenever $x, y \in K$.
- (H4) The mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space $\mathcal{C}(K)$ of all compact subsets of K furnished with the Michael topology is continuous.

A hypergroup $(K, *)$ is said to be *commutative* if the convolution $*$ is commutative. In this case $(M^b(K), *, -)$ is a commutative Banach $*$ -algebra with identity δ_e . There is an abundance of hypergroups and there are various constructions (polynomial, Sturm-Liouville) as the reader may learn from the pioneering papers on the subject and also from the monograph [2].

Let $(K, *)$ and (L, \circ) be two hypergroups with units e_K and e_L respectively. A continuous mapping $\varphi : K \rightarrow L$ is called a hypergroup *homomorphism* if $\varphi(e_K) = e_L$ and φ is the unique linear, weakly continuous extension from $M^b(K)$ to $M^b(L)$ such that

$$\delta_{\varphi(x)} = \varphi(\delta_x), \quad \varphi(\delta_x^-) = \varphi(\delta_x)^- \quad \text{and} \quad \delta_{\varphi(x)} \circ \delta_{\varphi(y)} = \varphi(\delta_x * \delta_y)$$

whenever $x, y \in K$. If $\varphi : K \rightarrow L$ is also a homeomorphism, it will be called an isomorphism from K onto L . An isomorphism from K onto K is called an automorphism of K . We denote by $\text{Aut}(K)$ the set of all automorphisms of K . Then $\text{Aut}(K)$ becomes a topological group equipped with the weak topology of $M^b(K)$. We call β an *action* of a locally compact group G on a hypergroup H if β is a continuous homomorphism from G into $\text{Aut}(H)$. Associated with the action β of G on H one can define a semi-direct product hypergroup $K = H \rtimes_{\beta} G$ which is our model of a non-commutative hypergroup in the present paper.

Let K be a (locally compact) hypergroup and X a locally compact space. Here we introduce a notion of an action of K on X . We call α an *action* of K on X if the following conditions are satisfied.

- (1) α is a continuous homomorphism from $M^b(K)$ to $B(M^b(X))$ as Banach algebras such that $\alpha(\delta_e)$ is the identity mapping on $M^b(X)$.
- (2) For $k \in K$ and $x \in X$, $\alpha(\delta_k)\delta_x \in M^1(X)$ such that $\text{supp}(\alpha(\delta_k)\delta_x)$ is compact.

We often denote $\alpha(\delta_k)$ by $\alpha(k)$ for $k \in K$. We denote the orbit of x under the action α by $\text{Orb}(x)$, i.e. $\text{Orb}(x) = \cup_{k \in K} \text{supp}(\alpha(k)\delta_x)$. An action α of K on X is called *smooth* if each orbit $\text{Orb}(x)$ for $x \in X$ is closed in X . A subset S of X is called α -invariant if $\text{supp}(\alpha(k)\delta_x) \subset S$ for any $x \in S$ and $k \in K$. An action α of K on X is called *irreducible* if any non-empty α -invariant closed subset S of X coincides with X . We call an action α of $K = H \rtimes_{\beta} G$ on a locally compact space X *absorbing* if $K(x) \supset H$ for any $x \in X$, where $K(x)$ is the stabilizer of K at $x \in X$ under the action α of K .

Now, let \mathcal{H} denote a (separable) Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\mathcal{B}(\mathcal{H})$ be the Banach $*$ -algebra of bounded linear operators on \mathcal{H} . We refer to U as a *representation* of K on \mathcal{H} if U is a $*$ -homomorphism from the Banach $*$ -algebra $M^b(K)$ to $\mathcal{B}(\mathcal{H})$ such that $U(\delta_e) = 1$ and if for each $u, v \in \mathcal{H}$ the mapping

$$\mu \longmapsto \langle U(\mu)u, v \rangle$$

is continuous on $M^b(K)$. By $\text{Rep}(K, \mathcal{H})$ we denote the set of classes of unitary equivalent representations of K on \mathcal{H} , by $\text{Irr}(K, \mathcal{H})$ its subset of irreducible classes.

From now on our locally compact spaces K and X will be second countable, so that they carry a Borel (Polish) structure.

If the given hypergroup K is commutative, its dual \widehat{K} can be introduced as the set of all bounded continuous functions $\chi \neq 0$ on K satisfying

$$\int_K \chi(z)(\delta_x^- * \delta_y)(dz) = \overline{\chi(x)}\chi(y) \text{ for all } x, y \in K.$$

This set of characters of K becomes a locally compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup.

In the course of the exposition we shall rely on some facts from the theory of C^* -algebra the main examples appearing in the text being $C_0(X)$ for a locally

compact space X and $\mathcal{B}(\mathcal{H})$ for a Hilbert space \mathcal{H} . To supplement this knowledge the reader is referred to the books [18], [6] by G.K. Pedersen and G.B. Folland respectively. We just mention that by the Gelfand-Naimark-Segal construction every C^* -algebra is isometrically $*$ -isomorphic to a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} .

If \mathcal{A} is a C^* -algebra of operators on Hilbert space \mathcal{H} , then its commutant

$$\mathcal{A}' := \{S \in \mathcal{B}(\mathcal{H}) : ST = TS \text{ for all } T \in \mathcal{A}\}$$

is also a C^* -algebra, closed in both the norm and the weak topology. Thus the double commutant \mathcal{A}'' is a weakly closed C^* -algebra containing \mathcal{A} .

By the von Neumann density theorem the closure in the weak operator topologies of a C^* -algebra of $\mathcal{B}(\mathcal{H})$ coincide with \mathcal{A}'' .

A C^* -algebra \mathcal{A} satisfying $\mathcal{A}'' = \mathcal{A}$ is called a *von Neumann algebra*. For details on von Neumann algebra, the reader might consult the books [4] and [5] by J. Dixmier.

These sources also supply the necessary knowledge of direct integral decomposition theory which will be applied in the proof of some of our results. In this context the space $L^2(X, \mu, \mathcal{H})$ of μ -square integrable functions with values in some Hilbert space \mathcal{H} and $L^\infty(X, \mu, \mathcal{A})$ of μ -essentially bounded functions on X with values in some von Neumann algebra \mathcal{A} , will play a central role. One notes that $L^2(X, \mu, \mathcal{H})$ is a Hilbert space and $L^\infty(X, \mu, \mathcal{A})$ is a von Neumann-algebra.

3. From groups to hypergroups

For a better understanding of our extended approach to an imprimitivity theorem for representations of hypergroups, we review the classical access developed initially by G.W. Mackey [14], [15] within the framework of groups.

See also the works of K.R. Parthasarthy [17] and Sunder [20]. The following exposition has been arranged so that the significant generalization for semi-direct product hypergroups in Section 6 becomes already apparent.

Let G be a locally compact group and α a transitive action of G on a locally compact space X .

For fixed $x_0 \in X$,

$$G_0 := \{g \in G : \alpha_g(x_0) = x_0\},$$

hence $X \cong G_0 \backslash G$ such that $x = \alpha_g^{-1}(x_0)$ if and only if $x = G_0 g$.

Let $s : X \rightarrow G$ be a Borel cross section of X in the sense that

$$\pi(s(x)) = x = G_0 s(x)$$

where π denotes the canonical projection from G onto $G_0 \backslash G$. Then we have

$$x = \alpha_{s(x)}^{-1}(x_0) \text{ for all } x \in X.$$

By \mathcal{U} we denote a Borel (Polish) group. Now, let L be a Borel homomorphism $G_0 \rightarrow \mathcal{U}$. By $\text{Hom}(G_0, \mathcal{U})$ we abbreviate the set of all such Borel homomorphisms. For $L_1, L_2 \in \text{Hom}(G_0, \mathcal{U})$ we introduce the equivalence relation

$$L_1 \cong L_2$$

provided there exists $W \in \mathcal{U}$ such that

$$L_2(g) = W^{-1}L_1(g)W.$$

A further notation will be

$$H^0(G_0, \mathcal{U}) := \text{Hom}(G_0, \mathcal{U}) / \cong.$$

Let $c : G \times X \rightarrow \mathcal{U}$ be a Borel mapping satisfying the relationship

$$c(g_1g_2, x) = c(g_1, x)c(g_2, \alpha_{g_1}^{-1}(x))$$

for all $g_1, g_2 \in G$, $x \in X$. c is said to be a \mathcal{U} -valued 1-cocycle of (G, X, α) . The symbol $Z^1(G, X, \alpha, \mathcal{U})$ stands for the set of all such 1-cocycles.

Two elements c_1, c_2 of $Z^1(G, X, \alpha, \mathcal{U})$ are called *cohomologous*, in symbols

$$c_1 \cong c_2$$

if there exists a Borel mapping $A : X \rightarrow \mathcal{U}$ satisfying

$$c_2(g, x) = A(x)^{-1}c_1(g, x)A(\alpha_g^{-1}(x))$$

for all $g \in G$, $x \in X$.

Let

$$H^1(G, X, \alpha, \mathcal{U}) := Z^1(G, X, \alpha, \mathcal{U}) / \cong.$$

We are prepared to establish an auxiliary result essential for the proof of the imprimitivity theorem.

Lemma 3.1. ([12]) $H^0(G_0, \mathcal{U})$ is isomorphic to $H^1(G, X, \alpha, \mathcal{U})$ as Borel groups.

Let μ be an α -quasi-invariant measure on X with respect to the action α of G on X , namely $\mu_g(B) = 0$ if and only if $\mu(B) = 0$ where the measure μ_g on X is given by

$$\mu_g(B) := \mu(\alpha_g^{-1}(B))$$

for each Borel set B of X and $g \in G$. Let $\gamma := \gamma(g, x)$ denote the Radon-Nikodym derivative $d\mu_g/d\mu$ of μ_g with respect to μ . Then γ is a positive real-valued 1-cocycle on $G \times X$.

For a Hilbert space \mathcal{H}_0 , let $\mathcal{U}(\mathcal{H}_0)$ denote the group of all unitary operators on \mathcal{H}_0 which is a Polish group with respect to the strong operator topology. Given a $\mathcal{U}(\mathcal{H}_0)$ -valued 1-cocycle c on $G \times X$, one can define a unitary representation U^c of G on $L^2(X, \mu, \mathcal{H}_0)$ in the following way: For $\xi \in L^2(X, \mu, \mathcal{H}_0)$,

$$(U_g^c \xi)(x) := \gamma(g, x)^{\frac{1}{2}} c(g, x) (\alpha_g^{-1}(x)).$$

Lemma 3.2. ([12]) *For $c_1, c_2 \in Z^1(G, X, \alpha, \mathcal{U}(\mathcal{H}_0))$, $c_1 \cong c_2$ implies $U^{c_1} \cong U^{c_2}$.*

Let $\tilde{\alpha}$ denote the extension of α to $C_0(X)$ defined by

$$(\tilde{\alpha}_g f)(x) := f(\alpha_g^{-1}(x)) \text{ for all } f \in C_0(X).$$

Let π^0 be the $*$ -representation of $C_0(X)$ on the Hilbert space $L^2(X, \mu, \mathcal{H}_0)$ given by

$$(\pi^0(f)\xi)(x) := f(x)\xi(x) \text{ for } f \in C_0(X), \xi \in L^2(X, \mu, V) \text{ and } x \in X.$$

We remark that the following equality always holds:

$$U^c(g)\pi(f) = \pi(\tilde{\alpha}_g(f))U^c(g).$$

Definition 3.3. Given a representation U of G on a Hilbert space \mathcal{H} and a $*$ -representation π of $C_0(X)$ on \mathcal{H} , the pair (π, U) is said to be a *covariant representation* of $(G, C_0(X), \alpha)$ if the following imprimitivity condition holds :

$$U_g\pi(f) = \pi(\tilde{\alpha}_g(f))U_g \text{ for all } g \in G, f \in C_0(X).$$

Theorem 3.4. (Imprimitivity theorem for groups, [14]) *Let (π, u) be a covariant representation of $(G, C_0(X), \alpha)$ for an action α of G on X , and let*

$$G_0 := \{g \in G : \alpha_g(x_0) = x_0\}$$

be a stabilizer for some $x_0 \in X$. Then there exists a representation L of G_0 such that

$$(\pi, u) \cong (\pi^0, \text{ind}_{G_0}^G L)$$

in the sense of a unitary isomorphism.

The idea of generalizing the imprimitivity theorem to hypergroups meets still noteworthy obstacles. In the present paper we can establish an imprimitivity theorem for the case of a semi-direct product hypergroup defined by a smooth action of a locally compact group on a hypergroup H which acts on X such that the stabilizer contains H . (See the complete statement and the proof in Chapter 6.) There is also the chance of obtaining an imprimitivity theorem for an arbitrary hypergroup in the case that a stability hypergroup H is supernormal and K/H is a group.

4. Semi-direct product hypergroups

Let $H := (H, \circ)$ be a hypergroup, G a locally compact group and β an action of G on H .

Let $K = H \times G$ be the set product of H and G such that $M^b(K) = M^b(H) \otimes M^b(G)$ as Banach $*$ -algebras. Then we define a convolution $*_\beta$ in $M^b(K)$ by

$$\varepsilon_{(h_1, g_1)} *_\beta \varepsilon_{(h_2, g_2)} := (\varepsilon_{h_1} \circ \varepsilon_{\beta_{g_1}(h_2)}) \otimes \delta_{g_1 g_2}$$

with unit element

$$\varepsilon_{(e, e)} := \varepsilon_e \otimes \delta_e$$

where e denotes the unit element of H as well as that of G , and an involution by

$$(\mu \otimes \delta_g)^- := \beta_g^{-1}(\mu^-) \otimes \delta_{g^{-1}} = \beta_g^{-1}(\mu)^- \otimes \delta_{g^{-1}}$$

for all $\mu \in M^b(H)$ and $g \in G$ where the Dirac measures in $M^b(K)$ and $M^b(G)$ are denoted by ε and δ respectively. K will be written as $H \rtimes_\beta G$.

Proposition 4.1. *$H \rtimes_\beta G$ is a hypergroup.*

Proof. It is easy to check that the convolution $*_\beta$ satisfies the axiom of a hypergroup except for those concerning the involution. Now we show that the above defined mapping

$$\mu \otimes \delta_g \longmapsto (\mu \otimes \delta_g)^-$$

is in fact an involution and that this involution is consistent with the unit element $\varepsilon_{(e, e)}$.

For $(h, g) \in K = H \times G$ we compute

$$\begin{aligned} \varepsilon_{(h, g)} *_\beta \varepsilon_{(h, g)^-} &= (\varepsilon_h \otimes \delta_g) *_\beta ((\beta_g^{-1}(\varepsilon_h^-)) \otimes \delta_{g^{-1}}) \\ &= (\varepsilon_h \circ \beta_g(\beta_g^{-1}(\varepsilon_h^-))) \otimes \delta_{gg^{-1}} \\ &= (\varepsilon_h \circ \varepsilon_{h^-}) \otimes \delta_e. \end{aligned}$$

On the other hand,

$$\begin{aligned} \varepsilon_{(h, g)^-} *_\beta \varepsilon_{(h, g)} &= ((\beta_g^{-1}(\varepsilon_h^-)) \otimes \delta_{g^{-1}}) *_\beta (\varepsilon_h \otimes \delta_g) \\ &= ((\beta_g^{-1}(\varepsilon_h^-)) \circ \beta_{g^{-1}}(\varepsilon_h)) \otimes \delta_{g^{-1}g} \\ &= ((\beta_g^{-1}(\varepsilon_h^-)) \circ \beta_g^{-1}(\varepsilon_h)) \otimes \delta_e \\ &= \beta_g^{-1}(\varepsilon_h^- \circ \varepsilon_h) \otimes \delta_e \end{aligned}$$

One notes that in general

$$\varepsilon_{(h, g)} *_\beta \varepsilon_{(h, g)^-} \neq \varepsilon_{(h, g)^-} *_\beta \varepsilon_{(h, g)}.$$

Now, we see that $(e, e) \in \text{supp}(\varepsilon_{(h, g)} *_\beta \varepsilon_{(h, g)^-}) \cap \text{supp}(\varepsilon_{(h, g)^-} *_\beta \varepsilon_{(h, g)})$ for all $(h, g) \in K$.

Conversely, let $(e, e) \in \text{supp}(\varepsilon_{(h, g)} *_\beta \varepsilon_{(h_2, g_2)})$. But

$$\varepsilon_{(h, g)} *_\beta \varepsilon_{(h_2, g_2)} = (\varepsilon_h \circ \beta_g(\varepsilon_{h_2})) \otimes \delta_{gg_2},$$

hence $g_2 = g^{-1}$ and $\beta_g(\varepsilon_{h_2}) = \varepsilon_{h^-} = \varepsilon_{h^-}$ which implies

$$\varepsilon_{h_2} = \beta_g^{-1}(\varepsilon_{h^-}) = \varepsilon_{\beta_g^{-1}(h^-)}.$$

From

$$\varepsilon_{(h_2, g_2)} = \varepsilon_{\beta_g^{-1}(h^-)} \otimes \delta_{g^{-1}}$$

it follows that

$$\varepsilon_{(h_2, g_2)} = \varepsilon_{(h, g)}^-$$

which proves the assertions. ■

4.1. Examples of semi-direct product hypergroups.

Example 1. Let H be an arbitrary commutative hypergroup and $G = \mathbb{Z}_2 = \{e, g\}$ a cyclic group of order two. Considering the action β of G on H given by $\beta_g(h) := h^-$ for all $g \in G, h \in H$, where $-$ denotes the involution of H . Then $H \rtimes_{\beta} \mathbb{Z}_2$ is a semi-direct product hypergroup.

1-1. If H is chosen to be a q -deformation $\mathbb{Z}_q(3)$ of \mathbb{Z}_3 (for $0 < q \leq 1$) in the sense of [13], then $\mathbb{Z}_q(3) \rtimes_{\beta} \mathbb{Z}_2$ is a q -deformation $S_q(3)$ of the symmetric group $S_3 = \mathbb{Z}_3 \rtimes_{\beta} \mathbb{Z}_2$.

The structure of $\mathbb{Z}_q(3) = \{c_0, c_1, c_2\}$ is given by

$$\begin{aligned} \delta_{c_1} \circ \delta_{c_2} &= q\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2}, \\ \delta_{c_1} \circ \delta_{c_1} &= \frac{1-q}{2}\delta_{c_1} + \frac{1+q}{2}\delta_{c_2}, \\ \delta_{c_2} \circ \delta_{c_2} &= \frac{1+q}{2}\delta_{c_1} + \frac{1-q}{2}\delta_{c_2} \quad (c_1^- = c_2, 0 < q \leq 1). \end{aligned}$$

The character table of $\mathbb{Z}_q(3)$ is

	c_0	c_1	c_2
χ_0	1	1	1
χ_1	1	ω_q	$\overline{\omega_q}$
χ_2	1	$\overline{\omega_q}$	ω_q

where $\omega_q = \frac{-q+i\sqrt{q^2+2q}}{2}$.

1-2. If we choose $H := \mathbb{Z}_{(p,q)}(4)$ (for $0 < p \leq 1, 0 < q \leq 1$), a (p, q) -deformation of \mathbb{Z}_4 (again by [13]), then $\mathbb{Z}_{(p,q)}(4) \rtimes_{\beta} \mathbb{Z}_2$ is a (p, q) -deformation $D_{(p,q)}(4)$ of the dihedral group $D_4 = \mathbb{Z}_4 \rtimes_{\beta} \mathbb{Z}_2$.

The structure of $\mathbb{Z}_{(p,q)}(4) = \{c_0, c_1, c_2, c_3\}$ is given by

$$\begin{aligned} \delta_{c_1} \circ \delta_{c_1} &= \delta_{c_3} \circ \delta_{c_3} = \frac{1-q}{2}\delta_{c_1} + q\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\ \delta_{c_2} \circ \delta_{c_2} &= p\delta_{c_0} + (1-p)\delta_{c_2}, \quad \delta_{c_1} \circ \delta_{c_2} = \frac{1-p}{2}\delta_{c_1} + \frac{1+p}{2}\delta_{c_3}, \\ \delta_{c_1} \circ \delta_{c_3} &= \frac{2pq}{1+p}\delta_{c_0} + \frac{1-q}{2}\delta_{c_1} + \frac{q-pq}{1+p}\delta_{c_2} + \frac{1-q}{2}\delta_{c_3}, \\ \delta_{c_2} \circ \delta_{c_3} &= \frac{1+p}{2}\delta_{c_1} + \frac{1-p}{2}\delta_{c_3} \quad (c_1^- = c_3, 0 < p \leq 1, 0 < q \leq 1). \end{aligned}$$

The character table of $\mathbb{Z}_{(p,q)}(4)$ is

	c_0	c_1	c_2	c_3
χ_0	1	1	1	1
χ_1	1	$i\sqrt{pq}$	$-p$	$-i\sqrt{pq}$
χ_2	1	$-q$	1	$-q$
χ_3	1	$-i\sqrt{pq}$	$-p$	$i\sqrt{pq}$

Example 2. Let \mathbb{Z}_p be a cyclic group of prime order $p = 2m + 1$. From the fact that $\text{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1} = \mathbb{Z}_{2m}$ we obtain actions α of \mathbb{Z}_2 and β of \mathbb{Z}_m , both on \mathbb{Z}_p . Taking H to be the orbital hypergroup $K^\beta(\mathbb{Z}_p)$ of \mathbb{Z}_p under the action β of \mathbb{Z}_2 we obtain the semi-direct product hypergroup

2-1.
$$K_\beta^\alpha(p) = K^\alpha(\mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_m.$$

In the case $H := K^\beta(\mathbb{Z}_p)$ we get

2-2.
$$K_\alpha^\beta(p) = K^\beta(\mathbb{Z}_p) \rtimes_\alpha \mathbb{Z}_2.$$

Moreover, for a subgroup A of \mathbb{Z}_{2m} and a subgroup B of the quotient group \mathbb{Z}_{2m}/A we arrive at the semi-direct product $K^\alpha(\mathbb{Z}_p) \rtimes_\beta B$, where α and β are the defining actions of A and B respectively.

Example 3. Let H_0 be a commutative hypergroup and $H = H_0^n$ the n -th power direct product hypergroup of H_0 . The action β of the symmetric group S_n of order n on H is defined by permutations. Then $H_0^n \rtimes_\beta S_n$ is a semi-direct product hypergroup.

3-1. $Q_q(4) := (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_\beta \mathbb{Z}_2$ (for $0 < q \leq 1$) is a q -deformation of the quaternion group $Q_4 = (\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2$.

3-2. We take $H_0 := K^\alpha(\mathbb{T})$ to be the orbital hypergroup $([-1, 1], \circ)$ of the one-dimensional torus group \mathbb{T} under the action α of $\mathbb{Z}_2 = \{e, g\}$ given by $\alpha_g(z) = \bar{z}$ for all $z \in \mathbb{T}$, which is called Chebychev hypergroup of the first kind, see p.166, 3.3,4 in [2]. Then the resulting semi-direct product hypergroup is $(K^\alpha(\mathbb{T}) \times K^\alpha(\mathbb{T})) \rtimes_\beta \mathbb{Z}_2$.

Example 4. Let $BK(J_\alpha)$ denote the Bessel-Kingman hypergroup $([0, \infty), \circ_\alpha)$ determined by the Bessel function J_α of order $\alpha \geq 0$. In [2] $BK(J_\alpha)$ has been introduced as a Sturm-Louville hypergroup with parameter α . Let \mathbb{R}_+^\times denote the multiplicative group of strictly positive real numbers which is isomorphic to the additive group \mathbb{R} under the exponential mapping. We consider the action β of \mathbb{R}_+^\times on $BK(J_\alpha)$ given by

$$\beta_t(r) := rt$$

for all $t \in \mathbb{R}_+^\times$ and $r \in \mathbb{R}_+ = BK(J_\alpha)$. Then we obtain the semi-direct product hypergroup

$$BK(J_\alpha) \rtimes_\beta \mathbb{R}_+^\times.$$

An application of results of M. Rösler [19] yields an extension of the example to higher-rank Bessel-Kingman hypergroups.

5. Cohomology arguments

Let $K = H \rtimes_{\beta} G$ be a semi-direct product hypergroup by a smooth action β of a locally compact group G on a hypergroup H . We call an action α of $K = H \rtimes_{\beta} G$ on a locally compact space X *absorbing* if $K(x) \supset H$ for any $x \in X$, where $K(x)$ is the stabilizer of K at $x \in X$ under the action α of K , namely

$$K(x) = \{k \in K : \alpha(k)\delta_x = \delta_x\}.$$

In this case it is easy to see that $K(x)$ is given by

$$K(x) = H \rtimes_{\beta} G(x),$$

where $G(x) = \{g \in G : \alpha(g)\delta_x = \delta_x\}$.

Let s be a Borel cross section from $X \cong G(x_0) \setminus G$ to G , i.e. $p(s(x)) = x$ for $x \in X$, where p is a projection from G onto $G(x_0) \setminus G$. By the condition

$$s(x)g = a(g, x)s(\alpha_g^{-1}(x)),$$

where $a(g, x) \in G(x_0)$, we have a $G(x_0)$ -valued 1-cocycle a which satisfies

$$a(g_1g_2, x) = a(g_1, x)a(g_2, \alpha_{g_1}^{-1}(x))$$

for all $g_1, g_2 \in G$ and $x \in X$.

For a representation L of $K(x_0) = H \rtimes_{\beta} G(x_0)$ on a Hilbert space \mathcal{H} we put

$$c^L((h, g), x) := L((\beta_{s(x)}(h), a(g, x)))$$

for $(h, g) \in H \rtimes_{\beta} G = K$ and $x \in X$. Then c^L is a $B(\mathcal{H})$ -valued 1-cocycles on $K \times X$ under the action α of K on X , namely

$$c^L \in Z_{\alpha}^1(K \times X, \mathcal{H})$$

Conversely, for $c \in Z_{\alpha}^1(K \times X, \mathcal{H})$ there exists a representation L of $K(x_0) = H \rtimes_{\beta} G(x_0)$ such that

$$L(h, g) = c((h, g), x_0)$$

for $(h, g) \in H \rtimes_{\beta} G(x_0) = K(x_0)$. It is easy to check that c is cohomologous to c^L by the transitivity of the action α of G on X .

Lemma 5.1. *In the above situation the correspondence $\text{Rep}(K(x_0), \mathcal{H}) \ni L \mapsto c^L \in H_{\alpha}^1(K \times X, \mathcal{H})$ is bijective.*

Proof. For $(h, g) \in H \rtimes_{\beta} G$ and $x \in X$, put

$$b((h, g), x) = (\beta_{s(x)}(h), a(g, x)).$$

Then $b((h, g), x)$ is a $M^1(K(x_0))$ -valued 1-cocycle on X satisfying

$$(h_0, s(x))(h, g) = b((h, g), x)(h_0, s(\alpha_g^{-1}(x))).$$

For $L \in \text{Rep}K(x_0)$, the above 1-cocycle c^L defined by

$$c^L((h, g), x) := L(b((h, g), x))$$

yields the desired assertion as in the proof of Lemma 3.1. ■

6. The imprimitivity theorem for semi-direct product hypergroups

Let $K_1 = H \rtimes_{\beta} G_1$ with a subgroup G_1 of G be a closed subhypergroup of a semi-direct product hypergroup $K = H \rtimes_{\beta} G$. For a representation L of $K_1 = H \rtimes_{\beta} G_1$, we introduce an induced representation $u^L = \text{ind}_{K_1}^K L$ of $K = H \rtimes_{\beta} G$. Put $X = G_1 \setminus G$ and $\alpha_g(x) = xg$ for $g \in G$ and $x = G_1 k \in G_1 \setminus G$. As in section 3 let μ be an α -quasi-invariant measure on X with respect to the action α of G on X . We denote by $\gamma = \gamma(g, x)$ the Radon-Nikodym derivative $d\mu_g/d\mu$ of the measure μ_g with respect to μ , where μ_g is given by $\mu_g(B) = \mu(\alpha_g^{-1}(B))$ for a Borel set B of X and $g \in G$. Then γ is a positive real-valued 1-cocycle on $G \times X$.

For a representation L of $K_1 = H \rtimes_{\beta} G_1$ on a Hilbert space \mathcal{H}_0 , we define the desired induced representation $u^L := \text{ind}_{K_1}^K L$ of $K = H \rtimes_{\beta} G$ in the following way:

For $\xi \in L^2(X, \mu, \mathcal{H}_0)$,

$$(u^L(h, g)\xi)(x) := \gamma(g, x)^{\frac{1}{2}} c^L((h, g), x) \xi(\alpha_g^{-1}(x))$$

for $(h, g) \in H \rtimes_{\alpha} G = K$.

For $g \in G$ and $f \in C_0(X)$, put

$$(\tilde{\alpha}_g(f))(x) := f(\alpha_g^{-1}(x)).$$

Then we have an action α of G on the C^* -algebra $C_0(X)$. Let π^0 be a representation of $C_0(X)$ on $L^2(X, \mu, \mathcal{H}_0)$ given by

$$(\pi^0(f)\xi)(x) := f(x)\xi(x)$$

for $f \in C_0(X)$ and $\xi \in L^2(X, \mu, \mathcal{H}_0)$. Then we see that the following *imprimitivity (covariant) condition* holds:

$$u^L(h, g)\pi^0(f) = \pi^0(\tilde{\alpha}_g(f))u^L(h, g)$$

for $(h, g) \in H \rtimes_{\beta} G = K$ and $f \in C_0(X)$.

Let α be an absorbing action of $K = H \rtimes_{\alpha} G$ on an arbitrary locally compact space X . Let π be a representation of $C_0(X)$ on \mathcal{H} and u be a representation of K on the same space \mathcal{H} . We call (π, u) a *covariant representation* of $(C_0(X), K, \alpha)$ if

$$u(h, g)\pi(f) = \pi(\tilde{\alpha}_g(f))u(h, g)$$

holds for $(h, g) \in H \rtimes_{\beta} G = K$ and $f \in C_0(X)$.

Theorem 6.1. *Let $K = H \rtimes_{\beta} G$ be a semi-direct product hypergroup defined by a smooth action β of a locally compact group G on a hypergroup H . Let α be a smooth irreducible absorbing action of K on a locally compact space X and let $K(x_0) = H \rtimes_{\beta} G(x_0)$ be the stability hypergroup of K at $x_0 \in X$. Then for a covariant representation (π, u) of $(K, C_0(X), \alpha)$ on a Hilbert space \mathcal{H} there exists a representation L of $K(x_0) = H \rtimes_{\beta} G(x_0)$ such that (π, u) is unitary equivalent to $(\pi^0, \text{ind}_{K(x_0)}^K L)$.*

Proof. The von Neumann algebra $\pi(C_0(X))''$ is isomorphic to $L^\infty(X, \mu)$ for some measure μ on X .

Then we see that

$$\mathcal{H} = \int_X^\oplus \mathcal{H}_x \mu(dx)$$

and

$$\pi(C_0(X))' = \int_X^\oplus B(\mathcal{H}_x) \mu(dx).$$

For $T \in B(\mathcal{H})$ and $g \in G$ put

$$\tilde{\alpha}_g(T) = u_g T u_g^*.$$

Then the action $\tilde{\alpha}$ of G on $B(\mathcal{H})$ provides an isomorphism from $B(\mathcal{H}_x)$ to $B(\mathcal{H}_{\alpha_g^{-1}(x)})$. This implies that μ is an α -quasi-invariant measure with respect to the transitive action α of G . Moreover, we see that $\mathcal{H}_x \cong \mathcal{H}_0$ for $x \in X$ and some Hilbert space \mathcal{H}_0 . Then there exists a unitary operator $U : \mathcal{H} \rightarrow L^2(X, \mathcal{H}_0, \mu)$ such that

$$U\pi(C_0(X))U^* = L^\infty(X, \mu) \otimes \mathbb{C} \quad \text{and} \quad U\pi(f)U^* = \pi^0(f).$$

Put $u^0(h, g) = Uu(h, g)U^*$.

Let w be a unitary representation of G which is given by

$$(w_g \xi)(x) := \gamma(g, x)^{\frac{1}{2}} \xi(\alpha_g^{-1}(x))$$

for $\xi \in L^2(X, \mathcal{H}_0, \mu)$, $g \in G$ and $x \in X$. Then we have

$$w_g \pi^0(f) = \pi^0(\tilde{\alpha}_g(f)) w_g$$

for $g \in G$ and $x \in X$. Put $\rho(h, g) := u^0(h, g) w_g^*$ for $(h, g) \in H \rtimes_\alpha G = K$. Then we see that

$$\rho(h, g) \pi^0(f) = \pi^0(f) \rho(h, g)$$

for all $f \in C_0(X)$. This equality implies that

$$\rho(h, g) \in \pi^0(C_0(X))' = L^\infty(X, \mu, B(\mathcal{H}_0)).$$

Hence we get a $B(\mathcal{H}_0)$ -valued function c on $K \times X$ such that $\rho(h, g)$ coincides with the function $X \ni x \mapsto c((h, g), x)$ on X .

By the fact that $u^0(h, g) = \rho(h, g) w_g$, we obtain

$$(u^0(h, g) \xi)(x) = \gamma(g, h)^{\frac{1}{2}} c((g, h), x) \xi(\alpha_g^{-1}(x))$$

for $\xi \in L^2(X, \mathcal{H}_0, \mu)$ and $(h, g) \in H \rtimes_\beta G = K$. Since u^0 is a representation, the function c satisfies the cocycle condition. By Lemma 5.1 we see that the 1-cocycle c is cohomologous to c^L for some representation L of $K(x_0)$. Hence we obtain the desired conclusion

$$u \cong u^0 = u^c \cong u^{c^L} =: \text{ind}_{K(x_0)}^K L. \quad \blacksquare$$

7. The Mackey machine for classifying irreducible representations

Let $\text{Rep}(H, \mathcal{H})$ denote the space of all representations of a hypergroup H on a fixed Hilbert space \mathcal{H} . Let β be a smooth action of a locally compact group G on H . Then we have an action $\hat{\beta}$ defined by

$$(\hat{\beta}_g(T))(h) := T(\beta_g^{-1}(h))$$

for $T \in \text{Rep}(H, \mathcal{H})$, $g \in G$ and $h \in H$. Moreover, the action $\hat{\beta}$ of G is transitive on the orbit $\text{Orb}(T) = \{\hat{\beta}_g(T); g \in G\}$, and we denote the stability group of G by $G(T)$. For a representation τ of $G(T)$ on a Hilbert space $\mathcal{H}(\tau)$ we get a $B(\mathcal{H}(\tau))$ -valued 1-cocycle c^τ on $G \times \text{Orb}(T)$ associated with τ . Let L be a representation of $K(T) := H \rtimes_\beta G(T)$ such that $L(h, g) = T(h) \otimes \tau(g)$ for $(h, g) \in H \rtimes G(T)$. Then $u^{(T, \tau)} := \text{ind}_{K(T)}^K L$ is written in the form:

$$(u^{(T, \tau)}(h, g)\xi)(S) = \gamma(g, S)^{\frac{1}{2}} S(h) \otimes c^\tau(g, S)\xi(\hat{\beta}_g^{-1}(S))$$

for $\xi \in L^2(\text{Orb}(T), \mu, \mathcal{H}(T) \otimes \mathcal{H}(\tau))$ and $(h, g) \in H \rtimes_\beta G$.

Applying the imprimitivity theorem we have the following.

Theorem 7.1. (Mackey machine) *Let $K = H \rtimes_\beta G$ be a semi-direct product hypergroup defined by a smooth action β of a locally compact group G on a commutative hypergroup H . For any $\chi \in \hat{H}$ and an irreducible representation τ of $G(\chi)$ $u^{(\chi, \tau)}$ is defined by $\text{ind}_{H \rtimes_\beta G(\chi)}^{H \rtimes_\beta G}(\chi \otimes \tau)$. Then the following statements hold:*

- (1) $u^{(\chi, \tau)}$ is an irreducible representation of K .
- (2) All irreducible representations of K are obtained in this form.
- (3) $u^{(\chi', \tau')}$ and $u^{(\chi, \tau)}$ are unitary equivalent if and only if $\text{Orb}(\chi') = \text{Orb}(\chi)$ and $\tau' \cong \tau$.

Proof. (1) The representation $u^{(\chi, \tau)}$ is realized on the Hilbert space $\mathcal{H} = L^2(X, \mu, \mathcal{H}(\tau))$, where $X = \text{Orb}(\chi)$ in \hat{H} and μ is a $\hat{\beta}$ -quasi-invariant measure on X with respect to the action $\hat{\beta}$ of G . For $\xi(\rho) \in L^2(X, \mathcal{H}(\tau), \mu)$ ($\rho \in X$) and $(h, g) \in H \rtimes_\beta G = K$, $u^{(\chi, \tau)}$ is given by

$$(u^{(\chi, \tau)}(h, g)\xi)(\rho) = \gamma(g, \rho)^{\frac{1}{2}} \rho(h) c^\tau(g, \rho)\xi(\hat{\beta}_g^{-1}(\rho)).$$

Then it is easy to check that

$$u^{(\chi, \tau)}(H)'' = L^\infty(X, \mu) \otimes \mathbb{C} \cdot 1 \subset B(L^2(X, \mu)) \otimes B(\mathcal{H}(\tau))$$

and

$$u^{(\chi, \tau)}(G(\chi))'' = \tau((G(\chi))'') = \mathbb{C} \cdot 1 \otimes B(\mathcal{H}(\tau))$$

for $\chi \in \hat{H}$ and $\tau \in \widehat{G(\chi)}$.

Let T be an operator in $B(L^2(X, \mu)) \otimes B(\mathcal{H}(\tau))$ such that

$$Tu^{(x,\tau)}(h, g) = u^{(x,\tau)}(h, g)T$$

for all $(h, g) \in H \rtimes_{\beta} G = K$. The fact that $T \in u^{(x,\tau)}(G(\chi))'$ implies that $T = T_1 \otimes 1$, where $T_1 \in B(L^2(X, \mu))$. The second fact that $T \in u^{(x,\tau)}(H)'$ and $L^{\infty}(X, \mu)$ is the maximal Abelian subalgebra of $B(L^2(X, \mu))$ implies that $T_1 = \pi(f)$ for some $f \in L^{\infty}(X, \mu)$. By the imprimitivity condition

$$u^{(x,\tau)}(h, g)\pi(f) = \pi(\tilde{\beta}_g(f))u^{(x,\tau)}(h, g),$$

and by the commuting condition

$$u^{(x,\tau)}(h, g)\pi(f) = \pi(f)u^{(x,\tau)}(h, g)$$

we see that $\pi(f) = \pi(\tilde{\beta}_g(f))$ for all $g \in G$. Since the action $\tilde{\beta}$ of G on $L^{\infty}(X, \mu)$ is ergodic (transitive), we obtain $T_1 = \pi(f) = c1$ ($c \in \mathbb{C}$) so that $T = c1$ ($c \in \mathbb{C}$). This implies the desired conclusion that the representation $u^{(x,\tau)}$ of K is irreducible.

(2) Let u be an irreducible representation of $K = H \rtimes_{\beta} G$. Then the restriction $u|_H$ to H is decomposed as

$$u|_H \cong \int_X^{\oplus} \rho \mu(d\rho)$$

where μ is a $\hat{\beta}$ -quasi-invariant measure on \hat{H} . Since the representation u of K is irreducible, the action $\hat{\beta}$ of G on \hat{H} is transitive and $\text{supp}(\mu) = X = \text{Orb}(\chi)$ for some $\chi \in \hat{H}$. Let $G(\chi)$ denote the stability group of G at $\chi \in \hat{H}$ by the action $\hat{\beta}$ of G on \hat{H} . By the relation

$$u(h, g)u(k) = u(\beta_g(k))u(h, g)$$

there exists a representation π of $C_0(X)$ such that

$$u(h, g)\pi(f) = \pi(\tilde{\beta}_g(f))u(h, g)$$

for $f \in C_0(X)$ and $(h, g) \in H \rtimes_{\beta} G = K$. Applying the imprimitivity theorem 6.1 we obtain that there exists $\tau \in \text{Rep}G(\chi)$ such that $u \cong u^{(x,\tau)} = \text{ind}_{H \rtimes_{\beta} G(\chi)}^{H \rtimes_{\beta} G} \chi \otimes \tau$. It is easy to check that $u^{(x,\tau)}$ is not irreducible if τ is not irreducible. Then τ must be irreducible, namely, $\tau \in \widehat{G(\chi)}$.

(3) It is easy to see that $u^{(x',\tau')} \cong u^{(x,\tau)}$ if $\text{Orb}(\chi') = \text{Orb}(\chi)$ and $\tau' \cong \tau$, by Lemma 3.1 and Lemma 3.2.

Suppose that $u^{(x',\tau')}$ is unitary equivalent to $u^{(x,\tau)}$. Then the restriction $u^{(x',\tau')}|_H$ to H is also unitary equivalent to the restriction $u^{(x,\tau)}|_H$ to H . This fact implies that $\text{Orb}(\chi') = \text{Orb}(\chi)$. Put $X = \text{Orb}(\chi)$. We may assume that $\mathcal{H}(\tau') = \mathcal{H}(\tau)$. Then $u^{(x',\tau')}$ and $u^{(x,\tau)}$ are realized on the Hilbert space $L^2(X, \mu, \mathcal{H}(\tau))$.

Suppose that $Wu^{(x',\tau')}(h, g)W^* = u^{(x,\tau)}(h, g)$ for some unitary operator $W \in B(L^2(X, \mu)) \otimes B(\mathcal{H}(\tau))$. Since $Wu^{(x',\tau')}(h, g) = u^{(x,\tau)}(h, g)W$, we obtain

$W\pi(f) = \pi(f)W$ for all $f \in C_0(X)$. This fact implies that $W \in (L^\infty(X, \mu) \otimes \mathbb{C} \cdot 1)' = L^\infty(X, \mu) \otimes B(\mathcal{H}(\tau))$. Hence W is decomposed as

$$W = \int_X^\oplus W(x) \mu(dx),$$

where $W(x)$ is a unitary operator on $\mathcal{H}(\tau)$ for μ -almost all $x \in X$. Moreover we see that

$$c^{\tau'}(g, x) = W(x)c^\tau(g, x)W(\hat{\beta}_g^{-1}(x)),$$

hence that $c^{\tau'}$ is cohomologous to c^τ . By Lemma 3.2 we obtain that $\tau' \cong \tau$. ■

8. Applications and Examples

The following discussion is based on the list of Examples given in Section 4.

Example 1-1. Let $S_q(3) = \mathbb{Z}_q(3) \rtimes_\beta \mathbb{Z}_2$, where $\mathbb{Z}_q(3) = \{h_0, h_1, h_2\}$ and $\mathbb{Z}_2 = \{e, g\}$. Moreover, let $\widehat{\mathbb{Z}_q(3)} = \{\chi_0, \chi_1, \chi_2\}$ and $\widehat{\mathbb{Z}_2} = \{\rho_0, \rho_1\}$. Then

$$\widehat{S_q(3)} = \{\chi_0 \otimes \rho_0, \chi_0 \otimes \rho_1, \pi\},$$

where

$$\pi = \text{ind}_{\mathbb{Z}_q(3)}^{S_q(3)} \chi_1 \quad (\text{for } \chi_1 \in \widehat{\mathbb{Z}_q(3)}).$$

In fact,

$$\pi(h_1, e) = \begin{pmatrix} \omega_q & 0 \\ 0 & \bar{\omega}_q \end{pmatrix} \text{ and } \pi(h_1, g) = \begin{pmatrix} 0 & \omega_q \\ \bar{\omega}_q & 0 \end{pmatrix}$$

where

$$\omega_q = \frac{1}{2}(-q + \sqrt{q^2 + 2qi}),$$

in particular

$$\omega_1 = \frac{1}{2}(-1 + \sqrt{3}i) = e^{\frac{2}{3}\pi i}.$$

Example 1-2. Let $D_{(p,q)}(4) = \mathbb{Z}_{(p,q)}(4) \rtimes_\beta \mathbb{Z}_2$, where $\mathbb{Z}_{(p,q)}(4) = \{h_0, h_1, h_2, h_3\}$ with $h_1^- = h_3, h_2^- = h_2$. Clearly,

$$\widehat{\mathbb{Z}_{(p,q)}(4)} = \{\chi_0, \chi_1, \chi_2, \chi_3\} \cong \mathbb{Z}_{(q,p)}(4)$$

with $\chi_1^- = \chi_3, \chi_2^- = \chi_2$. Then

$$\widehat{D_{(p,q)}(4)} = \{\chi_0 \otimes \rho_0, \chi_0 \otimes \rho_1, \chi_2 \otimes \rho_0, \chi_2 \otimes \rho_1, \pi\},$$

where

$$\pi = \text{ind}_{\mathbb{Z}_{(p,q)}(4)}^{D_{(p,q)}(4)} \chi_1 \quad (\text{for } \chi_1 \in \widehat{\mathbb{Z}_{(p,q)}(4)}).$$

Explicitly,

$$\pi(h_1, g) = \begin{pmatrix} 0 & \sqrt{pqi} \\ -\sqrt{pqi} & 0 \end{pmatrix} \text{ and } \pi(h_2, g) = \begin{pmatrix} 0 & -p \\ -p & 0 \end{pmatrix}.$$

Example 2-1. We consider $K_\beta^\alpha(p) = K^\alpha(\mathbb{Z}_p) \rtimes_\beta \mathbb{Z}_m$, where $p = 2m + 1$ is a prime number,

$$K^\alpha(\mathbb{Z}_p) = \{h_0, h_1, \dots, h_m\}$$

with

$$\widehat{K^\alpha(\mathbb{Z}_p)} = \{\chi_0, \chi_1, \dots, \chi_m\}$$

and

$$\mathbb{Z}_m = \{e, g, g^2, \dots, g^{m-1}\}$$

with

$$\widehat{\mathbb{Z}_m} = \{\rho_0, \rho_1, \dots, \rho_{m-1}\}.$$

Then

$$\widehat{K_\beta^\alpha(p)} = \{\chi_0 \otimes \rho_0, \chi_0 \otimes \rho_1, \dots, \chi_0 \otimes \rho_{m-1}, \pi\},$$

where

$$\pi = \text{ind}_{K^\alpha(\mathbb{Z}_p)}^{K_\beta^\alpha(p)} \chi_1 \text{ for } \chi_1 \in \widehat{K^\alpha(\mathbb{Z}_p)}.$$

Specifically,

$$\pi(h_1, e) = \begin{pmatrix} c_1 & & & 0 \\ & c_2 & & \\ & & \ddots & \\ 0 & & & c_m \end{pmatrix} \text{ and } \pi(h_1, g) = \begin{pmatrix} 0 & & 0 & c_1 \\ c_2 & 0 & 0 & 0 \\ & c_3 & \ddots & \\ 0 & & \ddots & 0 \\ 0 & & & c_m & 0 \end{pmatrix}$$

where $c_k = \cos \frac{2k}{p} \pi$ ($k = 1, 2, \dots, m$).

Example 2-2. Here we consider the semi-direct product hypergroup

$$K_\alpha^\beta(p) = K^\beta(\mathbb{Z}_p) \rtimes_\alpha \mathbb{Z}_2 = \mathbb{Z}_q(3) \rtimes_\alpha \mathbb{Z}_2,$$

where $q = \frac{2}{p-1} = \frac{1}{m}$. For $K^\beta(\mathbb{Z}_p) = \{h_0, h_1, h_2\}$ and $\mathbb{Z}_2 = \{e, g\}$ the duals are given by $\widehat{K^\beta(\mathbb{Z}_p)} = \{\chi_0, \chi_1, \chi_2\}$ and $\widehat{\mathbb{Z}_2} = \{\rho_0, \rho_1\}$ respectively. Then

$$\widehat{K_\alpha^\beta(p)} = \{\chi_0 \otimes \rho_0, \chi_0 \otimes \rho_1, \pi\},$$

where

$$\pi = \text{ind}_{K^\beta(\mathbb{Z}_p)}^{K_\alpha^\beta(p)} \chi_1 \text{ for } \chi_1 \in \widehat{K^\beta(\mathbb{Z}_p)}.$$

In fact,

$$\pi(h_1, g) = \begin{pmatrix} 0 & \tau_p \\ \overline{\tau_p} & 0 \end{pmatrix}$$

where

$$\tau_p = \omega_{\frac{2}{p-1}} = \frac{p-1}{4}(1 + \sqrt{pi}).$$

Example 3-1. Let $Q_q(4) = (\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)) \rtimes_{\beta} \mathbb{Z}_2$, where $\mathbb{Z}_q(2) \times \mathbb{Z}_q(2) = \{h_0, h_1, h_2, h_3\}$, $\mathbb{Z}_2 = \{e, g\}$ and β is given by $\beta_g(h_1) = h_2$ and $\beta_g(h_3) = h_3$. Then, with

$$\widehat{\mathbb{Z}_q(2) \times \mathbb{Z}_q(2)} = \{\chi_0, \chi_1, \chi_2, \chi_3\}$$

and

$$\widehat{\mathbb{Z}_2} = \{\rho_0, \rho_1\},$$

we obtain

$$\widehat{Q_q(4)} = \{\chi_0 \otimes \rho_0, \chi_0 \otimes \rho_1, \chi_3 \otimes \rho_0, \chi_3 \otimes \rho_1, \pi\}$$

with

$$\pi = \text{ind}_H^{Q_q(4)} \chi_1 \text{ for } H = \mathbb{Z}_q(2) \times \mathbb{Z}_q(2) \ (\chi_1 \in \widehat{H}).$$

In fact

$$\pi(h_1, g) = \begin{pmatrix} 0 & -q \\ -q & 0 \end{pmatrix}.$$

We note that $\widehat{D(4)} \cong \widehat{Q(4)}$ as hypergroups and $\widehat{D_{(p,q)}(4)}$ admits a hypergroup structure. However, $\widehat{Q_q(4)}$ fails to be a hypergroup if $q \neq 1$. The latter statement follows from the formula

$$(\chi_3 \otimes \rho_0)^2 = q^2(\chi_0 \otimes \rho_0) + q(1-q)(\chi_1 \otimes \rho_0) + q(1-q)(\chi_2 \otimes \rho_0) + (1-q)^2(\chi_3 \otimes \rho_0)$$

where none of the elements $\chi_1 \otimes \rho_0$, $\chi_2 \otimes \rho_0$ and $\chi_1 \otimes \rho_0 + \chi_2 \otimes \rho_0$ belong to $\widehat{Q_q(4)}$.

Example 3-2. Consider a semi-direct hypergroup product

$$K = H \rtimes_{\beta} \mathbb{Z}_2$$

with

$$H = H_0 \times H_0 = \{(x, y) ; x, y \in H_0\} \text{ and } \beta(x, y) = (y, x)$$

for a commutative hypergroup H_0 . We write

$$\widehat{H} = \{\tau^{(\chi_1, \chi_2)} ; \chi_1, \chi_2 \in \widehat{H_0}\},$$

where $\tau^{(\chi_1, \chi_2)}(x, y) = \chi_1(x)\chi_2(y)$. Then

$$\widehat{K} = \{\tau^{(\chi, \chi)} \otimes \rho_0, \tau^{(\chi, \chi)} \otimes \rho_1, \pi^{(\chi_1, \chi_2)} ; \chi, \chi_1, \chi_2 \in \widehat{H_0}, \chi_1 \neq \chi_2\},$$

where

$$\pi^{(\chi_1, \chi_2)} = \text{ind}_H^K \tau^{(\chi_1, \chi_2)}.$$

Concretely,

$$\pi^{(\chi_1, \chi_2)}((x, y), g) = \begin{pmatrix} 0 & \chi_2(x)\chi_1(y) \\ \chi_1(x)\chi_2(y) & 0 \end{pmatrix}.$$

Example 4. Consider the semi-direct product hypergroup

$$K = BK(J_\alpha) \rtimes_\beta \mathbb{R}_+^\times$$

with

$$H := BK(J_\alpha) = \mathbb{R}_+ \cong \widehat{H} \quad \text{and} \quad \widehat{\mathbb{R}_+^\times} = \{\rho^\lambda; \lambda \in \mathbb{R}\}$$

where $\rho^\lambda(r) = e^{i\lambda \log r}$ for $r \in \mathbb{R}_+^\times$. Then

$$\widehat{K} = \{\chi_0 \otimes \rho^\lambda, \pi; \lambda \in \mathbb{R}\},$$

where

$$\pi = \text{ind}_H^K \chi \quad \text{for some } \chi \in \widehat{H} \quad (\chi \neq \chi_0).$$

π is realized on the Hilbert space $L^2(\mathbb{R}_+^\times, \frac{1}{r} dr)$ as

$$(\pi(h, t)\xi)(r) = J_\alpha(hr)\xi(rt)$$

for $(h, t) \in BK(J_\alpha) \rtimes_\beta \mathbb{R}_+^\times$ and $\xi \in L^2(\mathbb{R}_+^\times, \frac{1}{r} dr)$ ($r \in \mathbb{R}_+^\times$).

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