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Symmetries of the Poset of Abelian Ideals in a Borel Subalgebra

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Abstract. Elaborating on a paper of Suter, we provide a detailed description of the automorphism group of the poset of abelian ideals in a Borel subalgebra of a finite dimensional complex simple Lie algebra.
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1. Introduction

This paper stems out from the attempt to get a better understanding of the final part of Suter's paper [15], where the symmetries of the Hasse graph of the poset $\mathfrak{Ab}$ of abelian ideals of a Borel subalgebra in a finite dimensional complex simple Lie algebra $\mathfrak{g}$ are analyzed. After Kostant's seminal paper [10], abelian ideals of Borel subalgebras have been intensively studied. The theory of abelian ideals of Borel subalgebras offers a wide variety of applications, ranging from representation theory of Kac-Moody algebras to combinatorics and number theory. But its distinctive feature is to provide a framework linking the theory of affine Weyl groups, the structure theory for the exterior algebra of $\mathfrak{g}$ as a $\mathfrak{g}$-module, and many combinatorial aspects of the representation theory of $\mathfrak{g}$. A glimpse on these connections is given in Section 3, where we also provide a concise description of the many ways known in literature to encode abelian ideals of Borel subalgebras.

The main result of the present paper is a rigidity statement about the poset structure of $\mathfrak{Ab}$; we give a detailed proof that its symmetries are exactly the ones induced by automorphisms of the Dynkin diagram, with just one exception in type C_3. See Theorem 5.6. Our goal was to find proofs which were as far as possible independent from the inspection of the global structure of the poset: the outcome of our efforts is that proofs only require either global inspections in rank at most 4 or "local" inspections, which may be easily performed using Bourbaki's Tables.

The non trivial part in the proof of Theorem 5.6 consists in showing that an automorphism of the abstract poset $\mathfrak{Ab}$ is indeed induced by an automorphism of

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the Dynkin diagram. The proof of this fact is discussed in Section 5. The main theorem is also used in Section 6 to discuss the symmetries of the Hasse graph of $\mathfrak{Ab}$, which is the original result by Suter. In Section 4 we take the opportunity of discussing in detail some folklore results relating the automorphisms of the Dynkin diagram, the automorphisms of the extended Dynkin diagram, and the center of the connected simply connected Lie group corresponding to $\mathfrak{g}$. We also recover in our setting the dihedral symmetry of a remarkable subposet in the Young lattice discovered by Suter in [14] and further discussed in [15]; it is worthwhile to note that this symmetry recently got a renewed interest in literature: see [1], [17], [16].

2. Setup

Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. Denote by Δ the corresponding irreducible root system, and by W the Weyl group of Δ. Fix a positive system Δ^+ and let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the corresponding basis of simple roots. Recall that there are at most two possible length for roots, which are correspondingly termed as long and short. We stipulate that roots are long if just one length occurs. Let $(\cdot, \cdot)$ be *half* the Killing form of $\mathfrak{g}$. For $\alpha \in \Delta \subset \mathfrak{h}^*$, denote by $\alpha^\vee \in \mathfrak{h}$ the corresponding coroot. Let $\{\omega_1, \dots, \omega_n\} \subset \mathfrak{h}^*$, $\{\varpi_1, \dots, \varpi_n\} \subset \mathfrak{h}$ denote the dual bases of $\Pi^\vee, \Pi$, respectively, and set $\rho = \sum_{i=1}^n \omega_i$. Let P, Q be the weight and root lattices, viewed as posets via $\beta \le \gamma \iff \gamma - \beta \in \mathbb{Z}_+\Pi$. We also denote by $P^\vee, Q^\vee$ the coweight and coroot lattices.

Let F be the space of affine-linear functions on $V = \mathbb{R} \otimes_{\mathbb{Z}} Q^\vee$. Endow F with a symmetric bilinear form induced by $(\cdot, \cdot)$ on the linear part and extended by zero on the affine part.

For $\alpha \in \Delta, j \in \mathbb{Z}$, consider the following element of F:

$$a_{\alpha,j}(v) = \alpha(v) + j.$$

It is shown in [12] that the set $\widehat{\Delta} = \{a_{\alpha,j} \mid \alpha \in \Delta, j \in \mathbb{Z}\}$ is an affine root system in F. For $\alpha \in \Delta$, $j \in \mathbb{Z}$ let $s_{\alpha,j}$ be the affine reflection around the hyperplane $\alpha(x) = j$. Explicitly, $s_{\alpha,j}(v) = v - a_{\alpha,-j}(v)\alpha^\vee$. Let $\widehat{W}$ be the subgroup of $Isom(V)$ generated by $\{s_{\alpha,j} \mid a_{\alpha,j} \in \widehat{\Delta}\}$. Let t_v be the translation by v. It is well-known that $\widehat{W} = W \ltimes Q^\vee$ (where $Q^\vee$ is viewed inside $\widehat{W}$ via $\alpha^\vee \mapsto t_{\alpha^\vee}$) and that it is a Coxeter group with generating set $s_0 = s_{\theta,1} = t_{\theta^\vee} s_{\theta,0}, s_i = s_{\alpha_i,0}$, $i = 1, \dots, n$. Here $\theta = \sum_{i=1}^n m_i \alpha_i$ is the highest root of Δ.

A fundamental domain for the action of $\widehat{W}$ on V is given by

$$\{v \in V \mid \alpha(v) \ge 0 \,\forall\, \alpha \in \Delta^+, \, \theta(v) \le 1\}.$$

Identifying V and V^* by means of $(\cdot, \cdot)$, we can also define an action of $\widehat{W}$ on V^*; then

$$C_1 = \{\lambda \in V^* \mid (\alpha, \lambda) \ge 0 \,\forall\, \alpha \in \Delta^+, \, (\theta, \lambda) \le 1\}$$

is a fundamental domain for this action, called the *fundamental alcove*. We will refer to the alcoves as the $\widehat{W}$-translates of C_1.

The set

$$\widehat{\Delta}^+ = \{a_{\alpha,j} \mid \alpha \in \Delta, j > 0\} \cup \{a_{\alpha,0} \mid \alpha \in \Delta^+\}$$

can be shown to be a set of positive roots in $\widehat{\Delta}$ and the corresponding set of simple roots is $\widehat{\Pi} = \{\alpha_0, \ldots, \alpha_n\}$, where $\alpha_0 = a_{-\theta,1}$ and we identify α_i with $a_{\alpha_i,0}$, $i = 1, \ldots, n$.

Note that $\widehat{W}$ acts on F (as functions on V) and this action preserves $\widehat{\Delta}$ and fixes δ, the constant function 1. Note that if we set $m_0 = 1$ we have

$$\delta = \sum_{i=0}^{n} m_i \alpha_i. \tag{1}$$

Let $\widehat{A} = (a_{ij})_{i,j=0}^n$, $a_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}$ be the Cartan matrix associated to $\widehat{\Pi}$, so that $A = (a_{ij})_{i,j=1}^n$ is the Cartan matrix of $\mathfrak{g}$ (w.r.t. Π).

3. Abelian ideals of Borel subalgebras

Let $\mathfrak{b}$ the Borel subalgebra corresponding to our choice of $\mathfrak{h}$ and Δ^+. Let us denote by $\mathfrak{Ab}$ the poset of abelian ideals of $\mathfrak{b}$. We now sum up all the encodings of $\mathfrak{Ab}$ we shall use in the following. For $w \in \widehat{W}$, we set $N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in -\widehat{\Delta}^+\}$. Recall that if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression of w, then

$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}.$$

Also recall that, in any poset $\mathcal{P}$, a subset $I \subset \mathcal{P}$ is an order ideal (resp. dual order ideal) if $x \in I$, $y \leq x \implies y \in I$ (resp. $x \in I$, $y \geq x \implies y \in I$). Finally, an antichain $A \subset \mathcal{P}$ is a subset consisting of mutually non-comparable elements.

Spcialize now to the root poset $(\Delta^+, \leq)$. We say that a dual order ideal I is *abelian* if $\alpha, \beta \in I \implies \alpha + \beta \notin \Delta$. Let us set the following definitions.

Definition 3.1.

1. Set

$$\mathcal{W} = \{w \in \widehat{W} \mid N(w) = \delta - \Phi, \Phi \text{ abelian dual order ideal in } \Delta^+\}.$$

   These elements are called *minuscule.*

2. The ρ-points in $2C_1$ are the set of regular elements in $P \cap 2C_1$.

3. The *weight* of $\mathfrak{i} \in \mathfrak{Ab}$ is $\langle \mathfrak{i} \rangle = \sum_{\mathfrak{g}_\alpha \subset \mathfrak{i}} \alpha$.

Proposition 3.2. *The following sets are in bijection with* $\mathfrak{Ab}$*:*

1. *the set of abelian dual order ideals in* Δ^+*;*
2. *the set* $\mathcal{W}$ *of minuscule elements in* $\widehat{W}$*;*

3. *the set of alcoves in* $2C_1$*;*

4. *the set of* ρ*-points in* $2C_1$*;*

5. *the set of weights of abelian ideals.*

Proof. If $\mathfrak{i} \in \mathfrak{Ab}$, then $\mathfrak{i} = \bigoplus_{\alpha \in \Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$, where $\Phi_{\mathfrak{i}} = \{\alpha \in \Delta^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{i}\}$. The fact that $\mathfrak{i}$ is an abelian ideal of $\mathfrak{b}$ clearly translates into the fact that $\Phi_{\mathfrak{i}}$ is a dual order ideal in Δ^+ which is also abelian.

The set $\mathfrak{Ab}$ is related to $\widehat{W}$ by the following idea of Dale Peterson: if $\mathfrak{i} \in \mathfrak{Ab}$, the set $\delta - \Phi_{\mathfrak{i}} \subset \widehat{\Delta}^+$ is biconvex, hence there exists a unique element $w_{\mathfrak{i}} \in \widehat{W}$ such that $N(w_{\mathfrak{i}}) = \delta - \Phi_{\mathfrak{i}}$.

An explicit description of the set $\mathcal{W}$ of minuscule elements has been found in [2], where it has been shown that the alcoves

$$C_{\mathfrak{i}} := w_{\mathfrak{i}}(C_1) \tag{2}$$

cover $2C_1$.

Recall that we are taking as invariant form on $\mathfrak{h}$ half the Killing form so $(\cdot,\cdot)$ is twice the Killing form on $\mathfrak{h}^*$. Lemma 2.2 of [4] now shows that $P \cap C_1 = \{\rho\}$. Hence in any alcove $C_{\mathfrak{i}}$ there is just one regular element of P, which is indeed $w_{\mathfrak{i}}(\rho)$ (hence our terminology).
The fact that the map $\mathfrak{i} \mapsto \langle \mathfrak{i} \rangle$ is injective has been shown in [9, Theorem 7]. ■

Remark 3.3. We single out two more encodings of $\mathfrak{Ab}$ available in literature:

(6) the set $\{\eta \in Q^\vee \mid \eta(\alpha) \in \{-2-1,0,1\}\ \forall\, \alpha \in \Delta^+\}$;

(7) the set of antichains $A \in \Delta^+$ such that for any $\alpha, \beta \in A$ we have $\alpha + \beta \not\leq \theta$.

To get the first encoding, recall the semidirect product decomposition $\widehat{W} = Q^\vee \rtimes W$ and write $w_{\mathfrak{i}} = t_{\tau_{\mathfrak{i}}} v_{\mathfrak{i}}$ accordingly. Then the map

$$\mathcal{W} \to \{\eta \in Q^\vee \mid \eta(\alpha) \in \{-2-1,0,1\}\ \forall\, \alpha \in \Delta^+\},\ w_{\mathfrak{i}} \mapsto v_{\mathfrak{i}}^{-1}(\tau_{\mathfrak{i}})$$

is a bijection (see [2], [10]).

For the final statement, recall that any (dual) order ideal in a finite poset is determined by an antichain. It follows from [5, Theorem 1] that the antichains giving rise to abelian ideals are characterized by the property stated in (7).

Remark 3.4. The term *weight* used in item (5) has indeed a representation-theoretical meaning: one of the main results of [9] is the analysis of the structure of $\bigwedge \mathfrak{g}$ as a $\mathfrak{g}$-module. Any commutative subalgebra $\mathfrak{a} = \oplus_{i=1}^k \mathbb{C} v_i \subseteq \mathfrak{g}$ gives rise to a decomposable vector $v_{\mathfrak{a}} = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k \mathfrak{g}$. Let $\mathcal{A}$ be the span of all the vectors $v_{\mathfrak{a}}$. A key step in understanding the structure of $\bigwedge \mathfrak{g}$ consists in determining the $\mathfrak{g}$-module structure of $\mathcal{A}$. It turns out that $\mathcal{A}$ is multiplicity free and its highest weight vectors are precisely the $v_{\mathfrak{i}}$, when $\mathfrak{i}$ ranges over $\mathfrak{Ab}$. Note that the weight of $v_{\mathfrak{i}}$ is $\langle \mathfrak{i} \rangle$.

The next proposition, which is known (see [11] or [15]), shows once more that the map $\mathfrak{i} \mapsto \langle \mathfrak{i} \rangle$ is an encoding of $\mathfrak{Ab}$. For the reader's convenience we reprove it in our setting.

Proposition 3.5. $w_{\mathfrak{i}}(\rho) = \rho + \langle \mathfrak{i} \rangle$.

Proof. Recall that the action of $\widehat{W}$ on V^* is obtained by identifying V and V^* using the invariant form $(\cdot, \cdot)$. Explicitly we have that

$$s_0(\lambda) = \lambda - (\lambda(\theta^\vee) - h^\vee)\theta, \ s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i, \ (i > 0),$$

where $h^\vee = \frac{2}{(\theta,\theta)}$ is the dual Coxeter number of $\mathfrak{g}$. In particular, we have that

$$s_0(\rho) = \rho + \theta, \ s_i(\rho) = \rho - \alpha_i, \ (i > 0). \tag{3}$$

We identify V^* and $F/\mathbb{C}\delta$; let $\lambda \mapsto \bar{\lambda}$ be the projection map from F to V^*. Since δ is fixed by $\widehat{W}$, then $\lambda \mapsto \overline{w\lambda}$ defines a (linear) action of $\widehat{W}$ on V^*. Note that

$$w(\lambda) - w(\mu) = \overline{w(\lambda - \mu)}.$$

We now prove by induction on $\ell(w)$ that

$$\rho - w(\rho) = \overline{\langle N(w) \rangle}. \tag{4}$$

Indeed, if $\ell(w) = 0$ there is nothing to prove, while, if $w = vs_i$ with $\ell(w) = \ell(v)+1$, then

$$\rho - w(\rho) = \rho - v(\rho) + \overline{v(\rho - s_i(\rho))} = \overline{\langle N(v) \rangle} + \overline{v(\alpha_i)} = \overline{\langle N(w) \rangle}.$$

Now observe that, by (4), $w_{\mathfrak{i}}(\rho) - \rho = -\overline{\langle N(w_{\mathfrak{i}}) \rangle} = \langle \mathfrak{i} \rangle$. ■

Recall that W can be endowed with the following partial order, called the *left weak Bruhat order*: for $w, w' \in W$ we define

$$w \leq w' \text{ if } w' = ws_{i_1} \cdots s_{i_k}, \ \ell(ws_{i_1} \cdots s_{i_j}) = \ell(w) + j, \ j = 1, \dots, k.$$

Remark 3.6. Note that $\mathfrak{Ab}$ is a poset under inclusion, $\mathcal{W}$ is a poset under left weak Bruhat order, and the weights of ideals and the ρ-points have a poset structure induced by that of P. The maps $\mathfrak{i} \mapsto w_{\mathfrak{i}}, \mathfrak{i} \mapsto \langle \mathfrak{i} \rangle$, $\mathfrak{i} \mapsto w_{\mathfrak{i}}(\rho)$ preserve the order. Indeed, $u \leq v$ in the left weak Bruhat order if and only if $N(u) \subseteq N(w)$. Moreover if $\mathfrak{i} \subseteq \mathfrak{j}$, then $\Phi_{\mathfrak{i}} \subseteq \Phi_{\mathfrak{j}}$ and $\langle \mathfrak{i} \rangle = \sum\limits_{\alpha \in \Phi_{\mathfrak{i}}} \alpha \leq \sum\limits_{\alpha \in \Phi_{\mathfrak{j}}} \alpha = \langle \mathfrak{j} \rangle$. Finally Proposition 3.5 guarantees that $\mathfrak{i} \mapsto w_{\mathfrak{i}}(\rho)$ is order compatible.

4. *Aut*(Π), *Aut*($\widehat{\Pi}$) and dihedral symmetries in the Young lattice

Set

$$Aut(\Pi) = \{\sigma : \Pi \leftrightarrow \Pi \mid a_{ij} = a_{\sigma(i)\sigma(j)}\},$$
$$Aut(\widehat{\Pi}) = \{\sigma : \widehat{\Pi} \leftrightarrow \widehat{\Pi} \mid a_{ij} = a_{\sigma(i)\sigma(j)}\}.$$

We are identifying the action on indices with the action on simple roots.

Recall that we set $\theta = \sum_{i=1}^n m_i \alpha_i$. Define $m_i^\vee$ by setting $\theta^\vee = \sum_{i=1}^n m_i^\vee \alpha_i^\vee$. Also set $m_0 = m_0^\vee = 1$.

Lemma 4.1.

1. *If* $\sigma \in Aut(\Pi)$*, then* $m_i = m_{\sigma(i)}$ *and* $m_i^\vee = m_{\sigma(i)}^\vee$ *for all* $i = 1, \dots, n$*.*
2. *If* $\sigma \in Aut(\widehat{\Pi})$*, then* $m_i = m_{\sigma(i)}$ *and* $m_i^\vee = m_{\sigma(i)}^\vee$ *for all* $i = 0, \dots, n$*.*

Proof. (1). Extend σ to an automorphism of $\mathfrak{g}$. It preserves the Killing form, hence any bilinear invariant form, since $\mathfrak{g}$ is simple. Moreover, it induces an order preserving map on roots, hence fixes θ. In turn $m_i = m_{\sigma(i)}$ for $i = 1, \dots, n$. Since $m_i^\vee = \frac{(\alpha_i,\alpha_i)}{(\theta,\theta)} m_i$, we have $m_i^\vee = m_{\sigma(i)}^\vee$ as well.

(2). Extend σ by linearity to F. Identify $\widehat{A}$ with the operator on F whose matrix in the basis $\widehat{\Pi}$ is $\widehat{A}$. Recall that $Ker\,\widehat{A}$ is 1-dimensional generated by δ. Then we have

$$0 = \sigma(\widehat{A}\delta) = (\sigma \circ \widehat{A} \circ \sigma^{-1})(\sigma(\delta)) = \widehat{A}\sigma(\delta).$$

and in turn that $\sigma(\delta) = k\delta$. Comparing coefficients, we have that $k = 1$ and in turn that $m_i = m_{\sigma(i)}$. For the last statement recall from [8] that $(m_0^\vee, \dots, m_n^\vee)$ generates linearly the kernel of $\widehat{A}^t$, so that we can argue as above. ■

Let $Isom(V^*)$ denote the set of isometries of V^* and

$$I(C_1) = \{\phi \in Isom(V^*) \mid \phi(C_1) = C_1\}.$$

Proposition 4.2. $I(C_1) \cong Aut(\widehat{\Pi})$.

Proof. Let $\nu : V^* \to V$ be the identification via the invariant form $(\cdot,\cdot)$. Recall that $\nu(C_1)$ is the simplex with vertices $o_i, i = 0, \dots, n$, where $o_0 = 0$, $o_i = \varpi_i/m_i$. Given $\phi \in I(C_1)$, then $z = \nu \circ \phi \circ \nu^{-1}$ permutes the o_i's, hence induces a permutation of $\widehat{\Pi}$, denoted by f_ϕ. We claim that $f_\phi \in Aut(\widehat{\Pi})$. First we prove that

$$\phi(\alpha_i) = \frac{m_{f_\phi(i)}}{m_i} \alpha_{f_\phi(i)}. \tag{5}$$

Indeed $\alpha_j(o_r) = \delta_{jr}\frac{1}{m_j}$; on the other hand $\phi(\alpha_i)(o_r) = \alpha_i(z^{-1}o_r) = \delta_{rf_\phi(i)}\frac{1}{m_i} = \frac{m_{f_\phi(i)}}{m_i}\delta_{rf_\phi(i)}\frac{1}{m_{f_\phi(i)}}$.

Since ϕ is an isometry, we have, by (5),

$$||\alpha_i||^2 = ||\phi(\alpha_i)||^2 = \left(\frac{m_{f_\phi(i)}}{m_i}\right)^2 ||\alpha_{f_\phi(i)}||^2.$$

But the ratio $||\alpha_i||^2/||\alpha_{f_\phi(i)}||^2$ can be just $1, 2$ or 3 (or $1/2, 1/3$). Since $\frac{m_{f_\phi(i)}}{m_i} \in \mathbb{Q}$, the only possibility is 1, so that $m_{f_\phi(i)} = m_i$. Hence (5) simplifies to

$$\phi(\alpha_i) = \alpha_{f_\phi(i)}, \tag{6}$$

and in turn, since ϕ is an isometry, we have

$$a_{f_\phi(i)f_\phi(j)} = \frac{2(\alpha_{f_\phi(i)}, \alpha_{f_\phi(j)})}{(\alpha_{f_\phi(j)}, \alpha_{f_\phi(j)})} = \frac{2(\phi\alpha_i, \phi\alpha_j)}{(\phi\alpha_j, \phi\alpha_j)} = a_{ij}.$$

We have established a map $I(C_1) \to Aut(\widehat{\Pi})$, $\phi \mapsto f_\phi$, which is clearly a group monomorphism. To prove its surjectivity, consider $f \in Aut(\widehat{\Pi})$ and let ϕ denote the unique affine map on V^* such that $\phi(\alpha_i) = \alpha_{f(i)}$. We first check that ϕ is an isometry. By [8, (6.2.2)] there exists an invariant form $\langle \cdot, \cdot \rangle$ for which $\langle \alpha_i, \alpha_j \rangle = a_{ij} m_j (m_j^\vee)^{-1}$ so, since $f \in Aut(\widehat{\Pi})$, Lemma 4.1 and the fact that all nondegenerate invariant bilinear symmetric forms on a simple Lie algebra are proportional show that $(\phi(\alpha_i), \phi(\alpha_j)) = (\alpha_i, \alpha_j)$.

Set $z = \nu \circ \phi \circ \nu^{-1}$. Then $\alpha_j(z(o_i)) = \phi^{-1}(\alpha_j)(o_i) = \delta_{j,f(i)} \frac{1}{m_i} = \delta_{j,f(i)} \frac{1}{m_{f(i)}} = \alpha_j(o_{f(i)})$. It follows that $z(o_i) = o_{f(i)}$, hence $f_\phi = f$. ■

Let w_0 be the longest element of W and w_0^i the longest element of the parabolic subgroup generated by $s_{\alpha_j}, j \neq i$. Set $J = \{i \mid m_i = 1\}$ and let $\widehat{W}^e = P^\vee \rtimes W$ be the extended affine Weyl group. We let $\widehat{W}^e$ act on V^* via the identification $\nu : V^* \to V$. Set

$$Z = \{Id_V, t_{\varpi_i} w_0^i w_0 \mid i \in J\}.$$

It can be shown (cf. [7]) that Z is isomorphic to the center of the connected simply connected Lie group with Lie algebra $\mathfrak{g}$.

Proposition 4.3. *[7, Prop. 1.21]* $Z = \{\phi \in \widehat{W}^e \mid \phi(C_1) = C_1\}$.

Set
$$LI(C_1) = \{\phi \in Isom(V^*) \mid \phi(C_1) = C_1,\ \phi \text{ linear}\}.$$

Proposition 4.4. $LI(C_1) \cong Aut(\Pi)$ *and* $Aut(\widehat{\Pi}) \cong I(C_1) = LI(C_1) \ltimes Z$.

Proof. First remark that

$$Aut(\Pi) = \{f \in Aut(\widehat{\Pi}) \mid f(\alpha_0) = \alpha_0\}, \tag{7}$$

Indeed, it is clear that an automorphism of $\widehat{\Pi}$ fixing α_0 restricts to an automorphism of Π. Conversely an automorphism f of Π fixes θ, and in turn it fixes $\alpha_0 \in \widehat{\Pi}$. Moreover, on one hand

$$a_{0i} = \frac{2(-\theta, \alpha_i)}{(\alpha_i, \alpha_i)} = -2 \sum_{j=1}^{n} m_j \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} = -\sum_{j=1}^{n} m_j \frac{2(f(\alpha_j), f(\alpha_i))}{(f(\alpha_i), f(\alpha_i))} = \frac{2(-\theta, f(\alpha_i))}{(f(\alpha_i), f(\alpha_i))}$$
$$= a_{0f(i)},$$

on the other hand

$$a_{i0} = \frac{2(\alpha_i, -\theta)}{(\theta, \theta)} = -\sum_{j=1}^{n} m_j^\vee \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} = -\sum_{j=1}^{n} m_j^\vee \frac{2(f(\alpha_i), f(\alpha_j))}{(f(\alpha_j), f(\alpha_j))} = \frac{2(f(\alpha_i), -\theta)}{(\theta, \theta)}$$
$$= a_{f(i)0}.$$

From (7) it follows that the isometry z_f on V induced by f fixes $o_0 = 0$, hence it is linear. Thus the map $f \mapsto \nu^{-1} \circ z_f \circ \nu$ establishes a homomorphism between $Aut(\Pi)$ and $LI(C_1)$. Clearly the map $\phi \mapsto f_\phi$ is its inverse when restricted to $LI(C_1)$.

Next we prove that $LI(C_1)$ and Z generate $I(C_1)$. If $f \in Aut(\widehat{\Pi})$, let $\alpha_i = f(\alpha_0)$. Then $m_i = 1$, i.e., $i \in J$, and there exists $\phi \in Z$ such that $f_\phi(\alpha_i) = \alpha_0$, so that $f_\phi f$ fixes α_0, hence belongs to $Aut(\Pi)$. It is clear that $Z \cap LI(C_1) = \{e\}$ and that Z is normal in $I(C_1)$. ■

Set

$$Z_2 = \{Id_V, t_{2\varpi_i} w_0^i w_0 \mid i \in J\}$$

and

$$I(2C_1) = \{\phi \in Isom(V^*) \mid \phi(2C_1) = 2C_1\}.$$

From Proposition 4.4 it is clear that

$$I(2C_1) = LI(C_1) \ltimes Z_2 \cong I(C_1) \cong Aut(\widehat{\Pi}).$$

The first isomorphism is given by the identity on $LI(C_1)$ and by the map

$$t_{2\varpi_i} w_0^i w_0 \mapsto t_{\varpi_i} w_0^i w_0 \text{ on } Z_2.$$

The second isomorphism is the one we set up in Proposition 4.2.

We have therefore a natural action of $Aut(\widehat{\Pi})$ on the set of alcoves in $2C_1$. By Proposition 3.2, this action gives an action of $Aut(\widehat{\Pi})$ on $\mathfrak{Ab}$. Note that two abelian ideals $\mathfrak{i}, \mathfrak{i}'$ are connected by an edge in the Hasse diagram of $\mathfrak{Ab}$ if and only if $w_{\mathfrak{i}}(C_1)$ and $w_{\mathfrak{i}'}(C_1)$ have a face in common. Hence the action of $Aut(\widehat{\Pi})$ on $\mathfrak{Ab}$ is an automorphism of the Hasse diagram (as an abstract graph).

If $x \in Aut(\widehat{\Pi})$, let us denote by $x \cdot \mathfrak{i}$ the action of x on $\mathfrak{i} \in \mathfrak{Ab}$. On the other hand, if we identify $Aut(\widehat{\Pi})$ with $I(C_1)$ as in Proposition 4.2, then $Aut(\widehat{\Pi})$ acts naturally on V^*.

Proposition 4.5. *If $\mathfrak{i} \in \mathfrak{Ab}$ and $x \in Aut(\widehat{\Pi})$, then*

$$\langle x \cdot \mathfrak{i} \rangle = x(\langle \mathfrak{i} \rangle). \tag{8}$$

In particular, if $x = f_\phi$ with $\phi = t_{\varpi_i} w_0^i w_0$, then

$$\langle x \cdot \mathfrak{i} \rangle = w_0^i w_0(\langle \mathfrak{i} \rangle) + h^\vee \omega_i. \tag{9}$$

Proof. If $x \in Aut(\Pi)$, then $x = f_\phi$ with $\phi \in LI(C_1)$. Since $C_{x\cdot\mathfrak{i}} = \phi(C_{\mathfrak{i}})$ and $\phi(P) = P$,

$$\phi(\rho + \langle \mathfrak{i} \rangle) = \phi(w_{\mathfrak{i}}(\rho)) = w_{x\cdot\mathfrak{i}}(\rho) = \rho + \langle x \cdot \mathfrak{i} \rangle.$$

Since ϕ is linear, we have $\phi(\rho + \langle \mathfrak{i} \rangle) = \phi(\rho) + \phi(\langle \mathfrak{i} \rangle)$. Since $\phi(\rho) = \rho$, we have (8).

If $\phi = t_{\varpi_i} w_0^i w_0$ and $x = f_\phi$, then $C_{x\cdot\mathfrak{i}} = t_{2\varpi_i} w_0^i w_0(C_{\mathfrak{i}})$. As above we obtain

$$t_{2\varpi_i} w_0^i w_0(\rho + \langle \mathfrak{i} \rangle) = \rho + \langle x \cdot \mathfrak{i} \rangle.$$

Remark that, under the identification of V and V^*, $\varpi_i = \frac{2}{(\alpha_i,\alpha_i)}\omega_i = \frac{2}{(\theta,\theta)}\omega_i = h^\vee \omega_i$, hence

$$\begin{aligned} t_{2\varpi_i} w_0^i w_0(\rho + \langle \mathfrak{i} \rangle) &= w_0^i w_0(\rho) + w_0^i w_0(\langle \mathfrak{i} \rangle) + 2h^\vee \omega_i \\ &= \rho + w_0^i w_0(\rho) - \rho + w_0^i w_0(\langle \mathfrak{i} \rangle) + 2h^\vee \omega_i \\ &= \rho - \langle N(w_0^i w_0) \rangle + w_0^i w_0(\langle \mathfrak{i} \rangle) + 2h^\vee \omega_i. \end{aligned}$$

We now observe that $\langle N(w_0^i w_0)\rangle = h^\vee \omega_i$. In fact, $N(w_0^i w_0)$ is the set of roots of the nilradical $\mathfrak{n}_i$ of the parabolic subalgebra defined by ϖ_i. It follows that $\langle N(w_0^i w_0)\rangle = x\omega_i$ for some $x \in \mathbb{R}$. Moreover $\dim \mathfrak{n}_i = \langle N(w_0^i w_0)\rangle(\varpi_i) = x\omega_i(\varpi_i) = x\frac{(\alpha_i,\alpha_i)}{4} tr(\varpi_i^2) = x\frac{(\theta,\theta)}{2} \dim \mathfrak{n}_i$. It follows that $x = h^\vee$, hence

$$t_{2\varpi_i} w_0^i w_0(\rho + \langle \mathfrak{i}\rangle) = \rho + w_0^i w_0(\langle \mathfrak{i}\rangle) + h^\vee \omega_i,$$

and, in turn,

$$\langle x \cdot \mathfrak{i}\rangle = w_0^i w_0(\langle \mathfrak{i}\rangle) + h^\vee \omega_i$$

as wished. ∎

As an application, we recover a nice result by Suter on the Young lattice. Recall that the latter is the lattice of partitions of a natural number ordered by containment of the corresponding Young diagram. We display Young diagrams in the French way. Also recall that the hull of a Young diagram is the minimal rectangular diagram containing it.

For a positive integer n let Y_n be the Hasse graph for the subposet $\mathfrak{Y}_n$ of the Young lattice corresponding to those diagrams whose hulls are contained in the staircase diagram for the partition $(n-1, n-2, ..., 1)$.

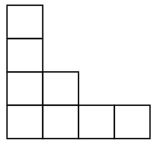
Theorem 4.6. *[14, Theorem 2.1] If $n \geq 3$, the dihedral group of order $2n$ acts faithfully on the (undirected) graph Y_n.*

Our proof of this theorem relies on the connection between the symmetries of the Young lattice and $Aut(\widehat{\Pi})$. This connection has already been observed in [15].

Specialize to $\mathfrak{g} = sl(n,\mathbb{C})$, and fix as Borel subalgebra the set of lower triangular matrices. Let e_{ij} denote the elementary matrices and set $\epsilon_i(e_{hh}) = \delta_{ih}$. Our choice of $\mathfrak{b}$ gives $\Delta^+ = \{\epsilon_i - \epsilon_j \in \mathbb{R}^n \mid i > j\}$; the corresponding simple roots are $\alpha_i = \epsilon_{i+1} - \epsilon_i$ $(i = 1, \dots, n-1)$. Moreover, the positive root spaces are $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{C}e_{ij}(i > j)$. Then abelian ideals of $\mathfrak{b}$ correspond bijectively via

$$\lambda_1 \geq \ldots \geq \lambda_k \longleftrightarrow \sum_{h=1}^{k} \sum_{j=1}^{\lambda_h} \mathbb{C}\mathfrak{g}_{\epsilon_{n-h+1} - \epsilon_j}$$

to subspaces of strictly lower triangular matrices such that their non-zero entries form a Young diagram whose hull is contained in the staircase diagram for the partition $(n-1, n-2, ..., 1)$:



$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{g}_{\epsilon_4-\epsilon_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{g}_{\epsilon_5-\epsilon_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{g}_{\epsilon_6-\epsilon_1} & \mathfrak{g}_{\epsilon_6-\epsilon_2} & 0 & 0 & 0 & 0 & 0 \\ \mathfrak{g}_{\epsilon_7-\epsilon_1} & \mathfrak{g}_{\epsilon_7-\epsilon_2} & \mathfrak{g}_{\epsilon_7-\epsilon_3} & \mathfrak{g}_{\epsilon_7-\epsilon_4} & 0 & 0 & 0 \end{pmatrix}$$

Let $\lambda(\mathfrak{i})$ be the diagram (or partition) corresponding to $\mathfrak{i} \in \mathfrak{Ab}$. Suter defines an action on $\mathfrak{Y}_n$ of two operators τ, σ_n which generate the dihedral group of order $2n$. The operator τ is the involution given by flipping the diagrams along the diagonal "South-West to North-East"; the other move is what he calls the *sliding move.* In formulas, if $\lambda = (\lambda_1, \dots \lambda_m)$ is a partition whose diagram belongs to $\mathfrak{Y}_n$ (so that $\lambda_1 \geq \ldots \geq \lambda_m$, $\lambda_1 + m \leq n$), then

$$\sigma_n(\lambda) = (\lambda_2 + 1, \dots, \lambda_m + 1, \underbrace{1, \dots, 1}_{n-m-\lambda_1}).$$

The term sliding comes form the following equivalent description: if $\mu = \lambda^t$ and $\nu = \sigma_n(\lambda)^t$, then $\nu_1 = n - \lambda_1 - 1, \nu_i = \mu_{i-1} - 1$, $i \geq 2$.

Proposition 4.7. *Set $\xi = t_{\varpi_1} w_0^1 w_0 \in I(C_1)$. We have*

$$\tau(\lambda(\mathfrak{i})) = \lambda(-w_0 \cdot \mathfrak{i}), \qquad \sigma_n(\lambda(\mathfrak{i})) = \lambda(f_\xi \cdot \mathfrak{i}).$$

In particular σ_n has order n. Hence the action of the dihedral group generated by τ, σ_n on $\mathfrak{Y}_n$ is precisely the action of $Aut(\widehat{\Pi})$ on $\mathfrak{Ab}$. In particular it is faithful.

Proof. First remark that if $\lambda(\mathfrak{i}) = (\lambda_1, \dots, \lambda_m)$, $\lambda_1 + m \leq n, \mathfrak{i} \in \mathfrak{Ab}$, and $\lambda_i^t = (\lambda'_1, \dots, \lambda'_r)$, then

$$\langle \mathfrak{i} \rangle = \sum_{i=1}^{m} \lambda_i \epsilon_{n-i+1} - \sum_{i=1}^{r} \lambda'_i \epsilon_i. \tag{10}$$

Since $-w_0(\epsilon_i) = \epsilon_{n-i+1}$, we have $-w_0(\langle \mathfrak{i} \rangle) = \sum_{i=1}^{m} \lambda_i \epsilon_i - \sum_{i=1}^{r} \lambda'_i \epsilon_{n-i+1}$, which is precisely $\langle \mathfrak{i}' \rangle$, where $\mathfrak{i}'$ is such that $\lambda(\mathfrak{i}') = \tau(\lambda(\mathfrak{i}))$. It follows from Proposition 4.5 that $\mathfrak{i}' = -w_0 \cdot \mathfrak{i}$. Next, we compute $t_{\varpi_1} w_0^1 w_0(\langle \mathfrak{i} \rangle)$. Recall that $h^\vee \omega_1 = -(n-1)\epsilon_1 + \sum_{i=2}^{n} \epsilon_i$ and that $w_0^1 w_0$ is the cycle $(1, 2, \dots, n)$. Hence,

$$\begin{aligned} t_{\varpi_1} w_0^1 w_0(\langle \mathfrak{i} \rangle) &= \lambda_1 e_1 + \sum_{i=2}^{m} \lambda_i \epsilon_{n-i+2} - \sum_{i=2}^{r+1} \lambda'_{i-1} \epsilon_i - (n-1)\epsilon_1 + \sum_{i=2}^{n} \epsilon_i \\ &= (\lambda_2 + 1)\epsilon_n + \ldots + (\lambda_m + 1)\epsilon_{n-m+2} + \sum_{i=r+2}^{n-m+1} \epsilon_i \\ &\quad - (n - 1 - \lambda_1)\epsilon_1 - (\lambda'_2 - 1)\epsilon_2 - \cdots - (\lambda'_r - 1)\epsilon_{r+1}, \end{aligned}$$

which is precisely $\langle \mathfrak{i}' \rangle$, where $\mathfrak{i}'$ is such that $\lambda(\mathfrak{i}') = \sigma_n(\lambda(\mathfrak{i}))$. It follows from Proposition 4.5 that $\mathfrak{i}' = f_\xi \cdot \mathfrak{i}$. ∎

5. Symmetries of $\mathfrak{Ab}$

This section is devoted to the proof of Theorem 5.6 below. We will exploit the poset isomorphism between $\mathcal{W}$ and $\mathfrak{Ab}$ described in Remark 3.6, so we need to translate the action of $Aut(\Pi)$ on $\mathfrak{Ab}$ into an action on $\mathcal{W}$.

Lemma 5.1. *If $\phi \in LI(C_1)$ then $w_{f_\phi \cdot \mathfrak{i}} = \phi w_{\mathfrak{i}} \phi^{-1}$. In particular, if $w_{\mathfrak{i}} = s_{i_1} \dots s_{i_r}$ is a reduced expression for $w_{\mathfrak{i}}$ and $f \in Aut(\Pi)$, then $s_{f(i_1)} \dots s_{f(i_r)}$ is a reduced expression for $w_{f \cdot \mathfrak{i}}$.*

Proof. It is enough to observe that

$$w_{f_\phi\cdot\mathfrak{i}}(C_1) = \phi(w_{\mathfrak{i}}(C_1)) = \phi w_{\mathfrak{i}}\phi^{-1}(\phi(C_1)) = \phi w_{\mathfrak{i}}\phi^{-1}(C_1).$$

■

We can define a labeling on the edges of Hasse diagram $H_{\mathfrak{Ab}}$ of $\mathfrak{Ab}$ by the following procedure: if $u, v \in \mathcal{W}, u < w$ are adjacent in $H_{\mathfrak{Ab}}$, then $v = us_i$. We assign the label i to the edge $u \to us_i$. We number diagrams as in [15].

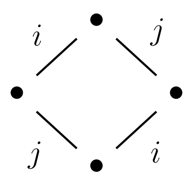
Lemma 5.2. *If $w \in \mathcal{W}$, then any reduced expression of w avoids substrings of the form $s_\alpha s_\beta s_\alpha$ except when α is a long simple root and β is a short simple root.*

Proof. In the contrary case, there exists a reduced expression of the form $w = w' s_\alpha s_\beta s_\alpha w''$, with either α, β of the same length or α short and β long. Remark that in both cases $s_\beta(\alpha) = \alpha + \beta$, so that $s_\alpha s_\beta(\alpha) = s_\alpha(\beta) - \alpha$. Then $N(w)$ contains $w'(\alpha), w'(s_\alpha(\beta)), w'(s_\alpha(\beta) - \alpha)$, against the fact that w encodes an abelian ideal. ■

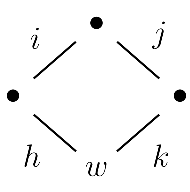
Corollary 5.3. *The only way to change a reduced expression for $w \in \mathcal{W}$ is to switch two consecutive commuting simple reflections. In particular, given $w \in \mathcal{W}$, any reduced expression of w contains the same number of occurrences of a simple reflection.*

Proof. It is well-known (see [13]) that in a Coxeter group it is possible to pass from a reduced expression of an element to another by switching commuting generators or by applying braid relations. If $w \in \mathcal{W}$, the latter moves are forbidden by the above Lemma. Indeed, let $m_{\alpha,\beta}$ be the order of $s_\alpha s_\beta$. If $m_{\alpha,\beta} = 3$, then α, β have the same length and the braid relation $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$ is forbidden by the Lemma. If $m_{\alpha,\beta} > 3$, then in a braid relation the forbidden pattern appears. ■

Remark 5.4. Observe that if v has at least two reduced expressions, then in the order ideal generated by v a diamond

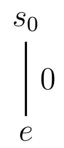


appears. We next show that diamonds in $H_{\mathfrak{Ab}}$ occur precisely in this situation. Indeed, if a diamond



occurs in $H_{\mathfrak{Ab}}$, then $i = k$, $j = h$, and $s_i s_j = s_j s_i$. This follows observing that $ws_h s_i = ws_k s_j$ obviously implies $s_h s_i = s_k s_j$ and the latter relation holds if and only if $h = j, k = i$.

Remark 5.5. The minimal abelian ideal is $\{0\}$, and there is just one 1-dimensional ideal, spanned by a highest root vector. Both are contained in any other abelian ideal in $\mathfrak{b}$. In terms of alcoves, the first corresponds to C_1 and the second to $s_0(C_1)$. Thus the Hasse diagram $H_{\mathfrak{Ab}}$ starts with a chain:



(11)

e being the neutral element of $\widehat{W}$.

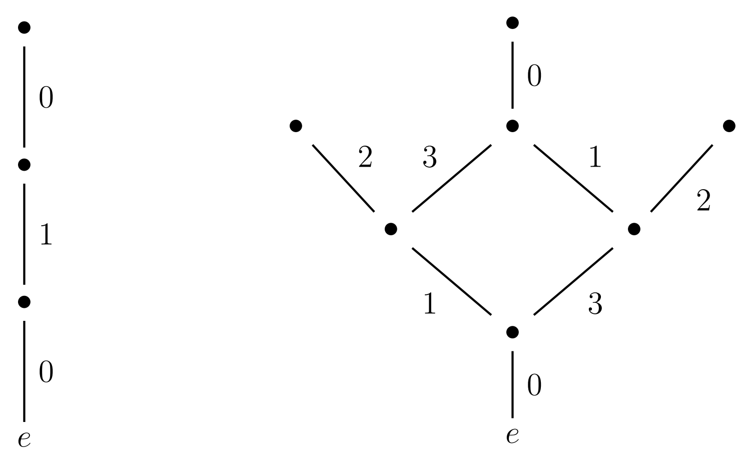
Set $\mathcal{W}_k = \{w \in \mathcal{W} \mid \ell(w) \leq k\}$. Denote by Π' the set of labels i appearing in $H_{\mathfrak{Ab}}$. Thus Π' is the set of i such that s_i occurs in a reduced expression of an element of $\mathcal{W}$. Indeed, $\Pi = \Pi'$ in any case except type C, in which the simple reflection corresponding to the long simple root in the (finite) Dynkin diagram does not appear.

Let $Aut(\mathfrak{Ab})$ be the set of poset automorphisms of $\mathfrak{Ab}$.

Theorem 5.6. *If $\mathfrak{g}$ is not of type C_3, then $Aut(\mathfrak{Ab}) \cong Aut(\Pi)$.*

Type C_3 is dealt with in [15]: the picture at p. 213 shows that $Aut(\mathfrak{Ab}) \cong \mathbb{Z}_2$; on the other hand $Aut(\Pi)$ is trivial. We exclude this case from now on.

Before tackling the proof of the Theorem, we single out the low rank cases C_2, A_3, which will be referred to in the following.



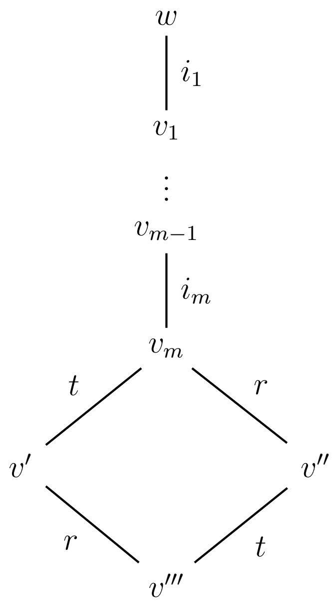
Type C_2 Type A_3

(12)

Let $\mathcal{W}'_h$ be the subset of $\mathcal{W}_h$ consisting of elements which have at least two reduced expressions.

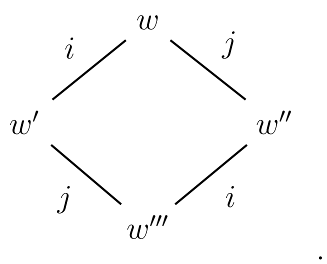
Lemma 5.7. *Let $\sigma \in Aut(\mathfrak{Ab})$ be such that $\sigma_{|\mathcal{W}_{h-1}} = Id$. Then $\sigma(w) = w$ for any $w \in \mathcal{W}'_h$.*

Proof. Since $\mathcal{W}'_1 = \emptyset$, we can clearly assume $h > 1$. Let w be a node in $\mathcal{W}'_h$. By the very definition of $\mathcal{W}'_h$, the order ideal generated by w contains a subdiagram of the form



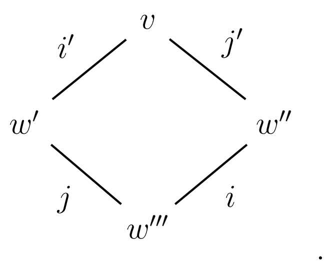
(13)

Choose a subdiagram with minimum m. If $m = 0$, we have a diamond



.

This is mapped by σ into a diamond



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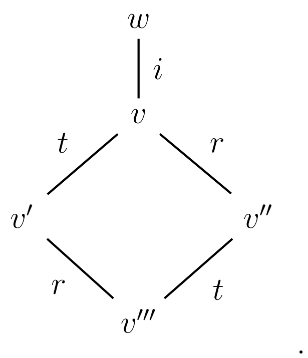
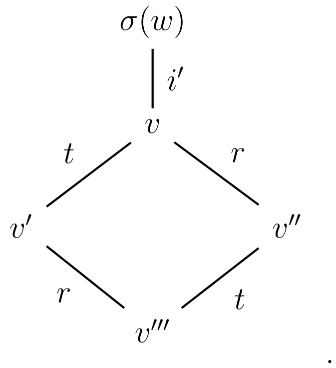
By Remark 5.4, we have $i' = i$ and $j' = j$. It follows that $v = \sigma(w) = w's_i = w$.

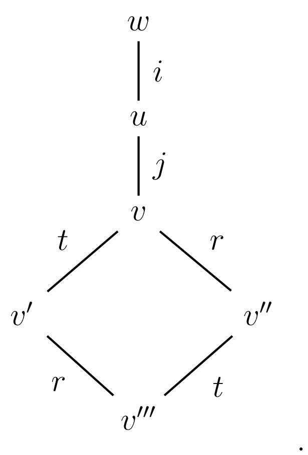
Assume now that $m = 1$, so that



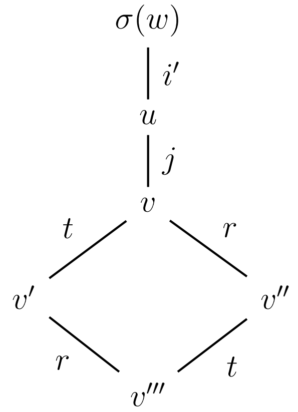
with $s_i s_r \neq s_r s_i$, and $s_i s_t \neq s_t s_i$. Thus $\alpha_i, \alpha_r, \alpha_t$ form an irreducible subsystem of $\widehat{\Delta}$ of rank 3 with α_r, α_t orthogonal. Applying σ to the above diagram we have



Thus $\alpha_{i'}$ is not orthogonal to both α_t and α_r. Except in type A_3, this implies that $i' = i$. Indeed both $\alpha_{i'}$ and α_i are connected to α_t, α_r in $\widehat{\Pi}$. Thus, if $i \neq i'$, α_t, α_r, α_i, $\alpha_{i'}$ form a cycle, hence $\widehat{\Pi}$ is of type $\widehat{A}_3$, and we are done by looking at (12). Next we assume $m = 2$:



The automorphism σ maps this configuration to

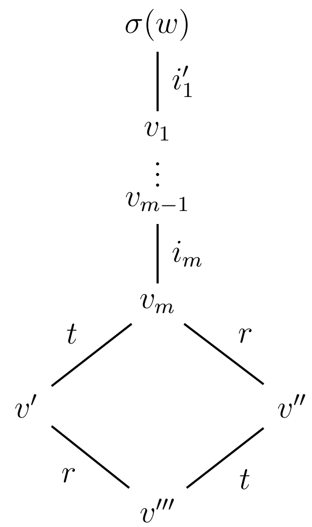


.

Assume first that $\alpha_i = \alpha_t$ or $\alpha_i = \alpha_r$. For simplicity assume $\alpha_i = \alpha_t$, then α_j must be short and α_i is long. Thus there are only two roots connected to α_j. Since $\alpha_{i'}$ is connected to α_j, we must have that either $\alpha_{i'} = \alpha_t = \alpha_i$ or $\alpha_{i'} = \alpha_r$. In the latter case α_r must be long, for, otherwise, $s_r s_j s_{i'} = s_r s_j s_r$ is forbidden braid (see Lemma 5.2). This implies that we are in type $\widehat{C}_2$, and we are done again by looking at (12).

We can therefore assume that $\alpha_i \neq \alpha_t$ and $\alpha_i \neq \alpha_r$. Since α_j is not orthogonal to α_i, α_r, α_t, there are at least three vertices stemming from α_j in $\widehat{\Pi}$. If $i' = t$ or $i' = r$, then the braid $s_r s_j s_r$ or $s_t s_j s_t$ would occur in a reduced expression for $\sigma(w)$, which is impossible by Lemma 5.2. This implies that either $i' = i$ or there are four edges stemming from α_j, i. e. we are in type D_4. This latter case is handled by a direct inspection: the Hasse diagram for type D_4 is given in [15, p. 217], and in this case one can check that $i = i' = 0$.

We now assume that $m \geq 3$. First assume α_{i_m} long and $\alpha_{i_m} \neq \alpha_{i_2}$. This implies that the roots α_t, α_r, and $\alpha_{i_{m-1}}$ are distinct and all connected to α_{i_m}. Thus α_{i_m} is a vertex of degree at least three in the Dynkin diagram. The automorphism σ maps the configuration in (13) to



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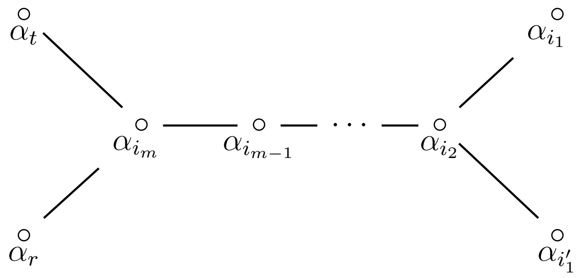
If $i_1 \neq i'_1$, then α_{i_1}, α_{i_3}, and $\alpha_{i'_1}$ are all connected to α_{i_2}. If they are not all distinct then α_{i_2} is short and α_{i_3} is long. Since there is a degree three vertex in the diagram, we are in type $\widehat{B}_n$ and both α_{i_1} and $\alpha_{i'_1}$ are connected to the unique short simple root. Hence $i_1 = i'_1$ as desired.

We can therefore assume that α_{i_1}, α_{i_3}, and $\alpha_{i'_1}$ are pairwise distinct. It follows that there are at least three vertices stemming from α_{i_2} in $\widehat{\Pi}$. Recall that we assumed that $\alpha_{i_m} \neq \alpha_{i_2}$, so there are two vertices of degree three in the Dynkin diagram. Thus we are in type $\widehat{D}_n$ with $n \geq 5$: indeed we claim that we are in the following situation,



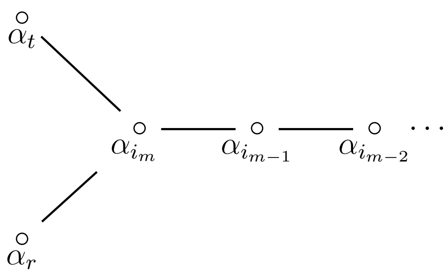
In fact, $\alpha_{i_{m-1}}$ is connected to α_{i_m} and cannot be α_t or α_r for, in such a case, the braid $s_t s_{i_m} s_t$ or $s_r s_{i_m} s_r$ would occur in a reduced expression for w. The same argument shows that $\alpha_{i_{m-j}}$ is connected to $\alpha_{i_{m-j+1}}$ and cannot be $\alpha_{i_{m-j+2}}$. We want to prove that, since $w = v''' s_t s_r s_{i_m} \dots s_{i_2} s_{i_1}$ is in $\mathcal{W}$, then $w' = v''' s_t s_r s_{i_m} \dots s_{i_2} s_{i'_1} \notin \mathcal{W}$. First observe that $v''' \neq e$. This is so because, by Remark 5.5, the Hasse diagram does not start with a diamond. Let j be the label of an edge reaching v'''. We now prove that $j = i_m$. If $j = i_a$ with $1 < a < m$, then the braid $s_{i_a} s_{i_{a+1}} s_{i_a}$ would occur in a reduced expression of w. If $j = i_1, i'_1$, then the braids $s_{i_1} s_{i_2} s_{i_1}$, $s_{i'_1} s_{i_2} s_{i'_1}$ would occur in a reduced expression of w, w' respectively. Since obviously $j \neq r, t$ we have $j = i_m$.

Repeating this argument we find that

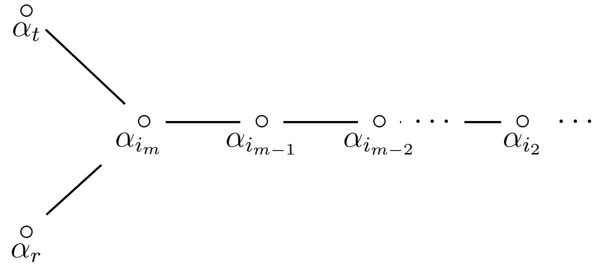
$$w = u s_{i_2} s_{i_3} \dots s_{i_m} s_r s_t s_{i_m} \dots s_{i_2} s_{i_1}, \; w' = u s_{i_2} s_{i_3} \dots s_{i_m} s_r s_t s_{i_m} \dots s_{i_2} s_{i'_1}.$$

Note that $u \neq e$, since $s_{i_2} \neq s_0$. Write $u = u' s_j$. The above argument shows that $j = i_1$ or $j = i'_1$. If $j = i'_1$, then $s_{i'_1} s_{i_2} s_{i_3} \dots s_{i_m} s_r s_t s_{i_m} \dots s_{i_2}(\alpha_{i_1}) = \delta + \alpha_{i'_1}$, thus both $u'(\alpha_{i'_1})$ and $\delta + u'(\alpha_{i'_1})$ are in $N(w)$, which is absurd, since $w \in \mathcal{W}$. It follows that $j = i_1$, but then $u'(\alpha_{i_1})$ and $\delta + u'(\alpha_{i_1})$ are both in $N(w')$ so $w' \notin \mathcal{W}$.

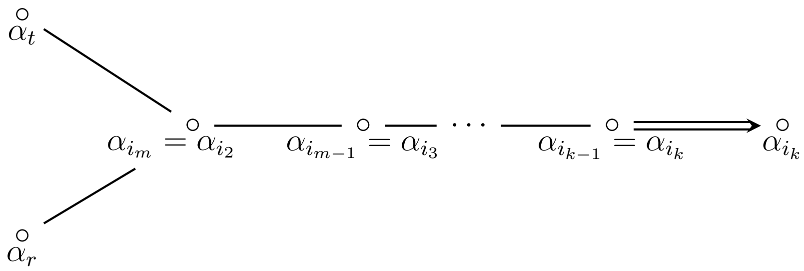
Assume now that $i_2 = i_m$. If α_{i_j} are all long roots then $i_m \neq i_{m-2}$, otherwise we would have a forbidden braid. So the Dynkin diagram is



By the same argument $i_{m-3} \neq i_{m-1}, i_{m-4} \neq i_{m-2}, \dots, i_4 \neq i_2$ so the Dynkin diagram is

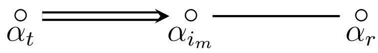


contradicting the hypothesis that $i_m = i_2$. This implies that there is k such that α_{i_k} is a short root. Thus we are in type $\widehat{B}_n$. We claim that we are in the following situation:



Indeed, let i_k be the first occurrence of the short simple root. It follows that, since $\alpha_{i_{k-1}}$ is connected to α_{i_k}, we have that $i_{k-1} = i_{k+1}$. Since $\alpha_{i_{k+2}}$ is connected to $\alpha_{i_{k+1}} = \alpha_{i_{k-1}}$ and the braid $s_{i_k} s_{i_{k+1}} s_{i_k}$ is forbidden, we see that $i_{k-2} = i_{k+2}$. The same argument shows that $i_{k+j} = i_{k-j}$ for $j = 0, \dots, k-1$. Thus $w = v''' s_r s_t s_{i_2} s_{i_3} \dots s_{i_k} \dots s_{i_2} s_{i_1}$ and $i_1 = r$ or $i_1 = t$. Assume for simplicity $i_1 = r$. Then both $v'''(\alpha_t)$ and $\delta + v'''(\alpha_t)$ are in $N(w)$ and this is impossible.

Assume now α_{i_m} short. Note that $\alpha_{i_{m-1}}$, α_r, α_t cannot be pairwise distinct for, otherwise, α_{i_m} would be a node of degree three in a non simple laced Dynkin diagram, hence α_{i_m} would be long. It follows that $\alpha_{i_{m-1}} = \alpha_t$ or $\alpha_{i_{m-1}} = \alpha_r$. Assume for simplicity that $\alpha_{i_{m-1}} = \alpha_t$. Since α_r is orthogonal to α_t we have the following situation



Now $\alpha_{i_{m-2}}$ is connected to $\alpha_{i_{m-1}}$ and it is not α_r for, otherwise, there would be a forbidden braid. It follows that we are in type $\widehat{F}_4$ and $t = i_{m-1} = 2$, $i_{m-2} = 1$, $i_m = 3$, and $r = 4$. This does not happen, as a direct inspection of the Hasse diagram shows[1] (see [15, p. 218]). ■

If $\sigma \in Aut(\mathfrak{Ab})$, let h_σ be the maximal $h \in \mathbb{N}$ such that $\sigma_{|\mathcal{W}_h} = Id$. If $\sigma = Id$ then we set $\mathcal{W}_{h_\sigma+1} = \mathcal{W}_{h_\sigma} = \mathcal{W}$.

[1]Indeed an argument avoiding the inspection could be provided, but looking at the Hasse diagram is certainly handier in this case.

Lemma 5.1 implies clearly that $Aut(\Pi)$ acts by poset automorphisms. Note also that $f \in Aut(\Pi)$ acts on $H_{\mathfrak{Ab}}$ as an automorphism of labelled graphs, and that the induced map on labels is f itself restricted to Π'. This fact will be used without comment in the proof of the following result.

Lemma 5.8. *If $\sigma \in Aut(\mathfrak{Ab})$ then $\sigma_{|\mathcal{W}_{h_\sigma+1}} \in Aut(\Pi)_{|\mathcal{W}_{h_\sigma+1}}$.*

Proof. Since the poset starts as in (11), we have that $h_\sigma \geq 1$. Clearly we can assume $\sigma \neq Id$. Let $w \in \mathcal{W}_{h_\sigma+1}$ be such that $\sigma(w) \neq w$. Then, by Lemma 5.7, we have that $w \notin \mathcal{W}'_{h_\sigma+1}$. Hence the interval from e to w is a chain:

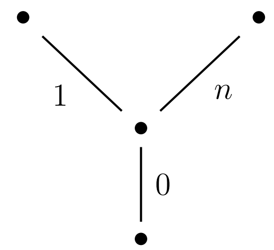
$$\begin{array}{l} \qquad w \\ \qquad \Big|\, i \\ \qquad v_1 \\ \qquad \Big|\, i_1 \\ \qquad v_2 \\ \qquad \vdots \\ v_{h_\sigma} = s_0 \\ \qquad \Big|\, i_{h_\sigma} = 0 \\ \qquad e \end{array}$$

Since $w \in \mathcal{W}_{h_\sigma+1}$, the automorphism σ maps the above chain to

$$\begin{array}{l} \qquad \sigma(w) \\ \qquad \Big|\, i' \\ \qquad v_1 \\ \qquad \Big|\, i_1 \\ \qquad v_2 \\ \qquad \vdots \\ v_{h_\sigma} = s_0 \\ \qquad \Big|\, i_{h_\sigma} = 0 \\ \qquad e \end{array}$$

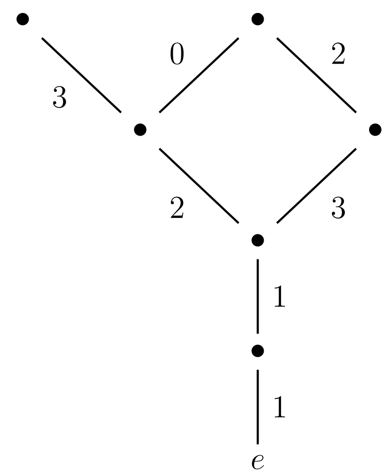
with $i \neq i'$.

Let us discuss the case $h_\sigma = 1$. If $h_\sigma = 1$, then $w = s_0 s_i$. If $\sigma(w) = s_0 s_{i'}$ with $i \neq i'$ then there are two simple roots connected to α_0. This happens only in type $\widehat{A}_n$ with $i, i' = 1, n$. In this case $\mathcal{W}_2$ is



and $Aut(\mathcal{W}_2) = Aut(\Pi)_{|\mathcal{W}_2}$. We can therefore assume that $h_\sigma \geq 2$.

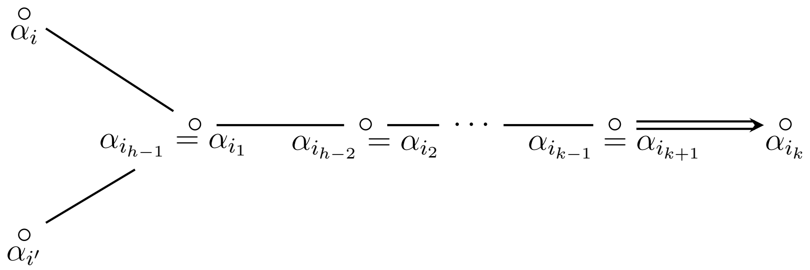
Assume first that $\alpha_{i'}$, α_{i_2}, α_i are not pairwise distinct. For simplicity assume $\alpha_i = \alpha_{i_2}$. This implies that α_{i_1} is short and α_i is long for, otherwise, we would have a forbidden braid. If $\alpha_{i'}$ is also long, then we are in type $\widehat{C}_2$. By looking at (12), we see that, in this case, $Aut(\mathfrak{Ab}) = \{Id\}$. We therefore have that $\alpha_{i'}$ short. If $h_\sigma = 2$ (so that $i = i_2 = 0$), then we are in type $\widehat{C}_n$ with $n \geq 4$. (Recall that we are excluding type $\widehat{C}_3$). In this case the Hasse diagram of $\mathcal{W}_4$ is



¿From this graph we see that $\sigma_{|\mathcal{W}_3} = Id$. Thus $h_\sigma > 2$. Since $i_3 \neq i_1$, we see that we are in type $\widehat{F}_4$, $h_\sigma = 4$, and $w = s_0s_1s_2s_3s_2$, but then $w \notin \mathcal{W}$.

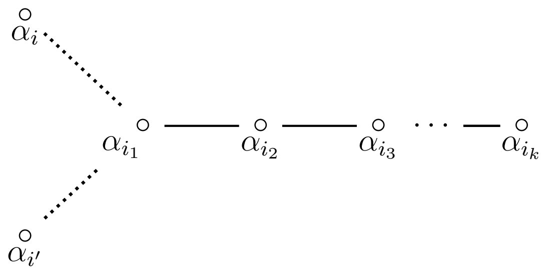
We can therefore assume that α_i, $\alpha_{i'}$, and α_{i_2} are pairwise distinct. Thus α_{i_1} is a node of degree at least three in the Dynkin diagram. In particular α_{i_1} is a long root.

Assume that there is $j > 1$ such that α_{i_j} is short. Since there is a triple node and the diagram is not simply laced, we are in type $\widehat{B}_n$: indeed we claim that we are in the following situation



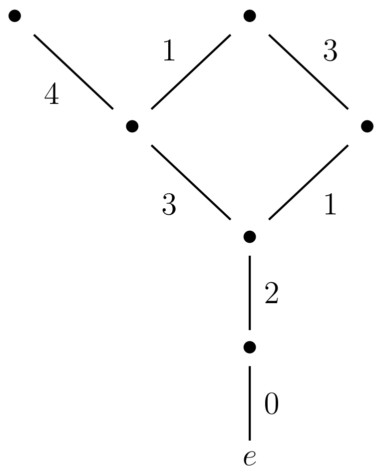
and $i = i_h = 0$ or $i' = i_h = 0$. Indeed, let i_k be the first occurrence of the short simple root. It follows: since $\alpha_{i_{k-1}}$ is connected to α_{i_k}, then $i_{k-1} = i_{k+1}$. Since $\alpha_{i_{k+2}}$ is connected to $\alpha_{i_{k+1}} = \alpha_{i_{k-1}}$ and the braid $s_{i_k}s_{i_{k+1}}s_{i_k}$ is forbidden, we see that $i_{k-2} = i_{k+2}$. The same argument shows that $i_{k+j} = i_{k-j}$ for $j = 0, \dots, k-1$. Thus $w = s_0 s_{i_1} s_{i_2} \dots s_{i_k} \dots s_{i_1} s_0$ or $w = s_0 s_{i_1} s_{i_2} \dots s_{i_k} \dots s_{i_1} s_1$. In the second case we have that $\delta + \alpha_0$ is in $N(w)$ and this is impossible. Thus $i = 0$ and $i' = 1$. But then we have $\sigma(w) = s_0 s_{i_1} s_{i_2} \dots s_{i_k} \dots s_{i_1} s_1 \notin \mathcal{W}$.

We can therefore deduce that all the roots α_{i_j} are long roots. This implies that $i_j \neq i_{j-2}$ for all $j > 2$, otherwise we would have forbidden braids. So $\{\alpha_i, \alpha_{i'}\} \cup \{\alpha_{i_j} \mid j = 1, \dots, h\}$ form a subdiagram of type



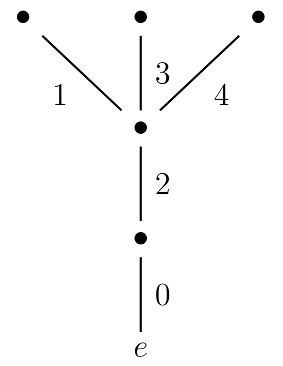
The dotted edges may be multiple. This is possible in types $\widehat{B}_n (n \geq 3), \widehat{D}_n (n \geq 4), \widehat{E}_n, (n = 6, 7, 8)$.

Case 1: type $\widehat{B}_n$. Then $h_\sigma = 2$ and $\{i, i'\} = \{1, 3\}$. The Hasse graph of $\mathcal{W}_4$ is



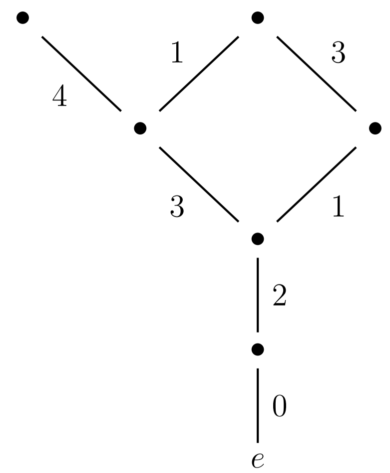
if $n \geq 4$ (if $n = 3$ just replace the label 4 by 2). From these graphs we see that $\sigma_{|\mathcal{W}_3} = Id$.

Case 2: type $\widehat{D}_n$. Either $h_\sigma = 2$ or $h_\sigma = n-2$. If $h_\sigma = 2$ then $w = s_0 s_2 s_i$ with α_i that ranges over the nodes connected to α_2 and different from α_0. If $n = 4$ we are done because $\mathcal{W}_3$ is



and $Aut(\mathcal{W}_3) = Aut(\Pi)_{|\mathcal{W}_3}$.

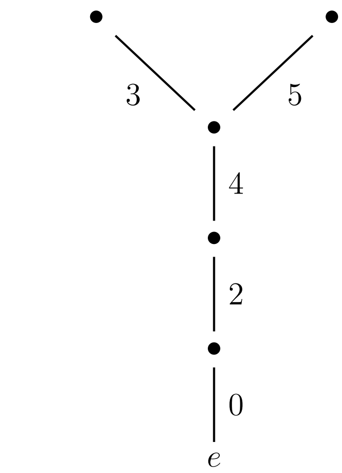
If $h_\sigma = 2$ and $n > 4$, then the Hasse graph of $\mathcal{W}_4$ is



hence $Aut(\mathfrak{Ab})_{|\mathcal{W}_3} = \{Id\}$ so this case does not occur.

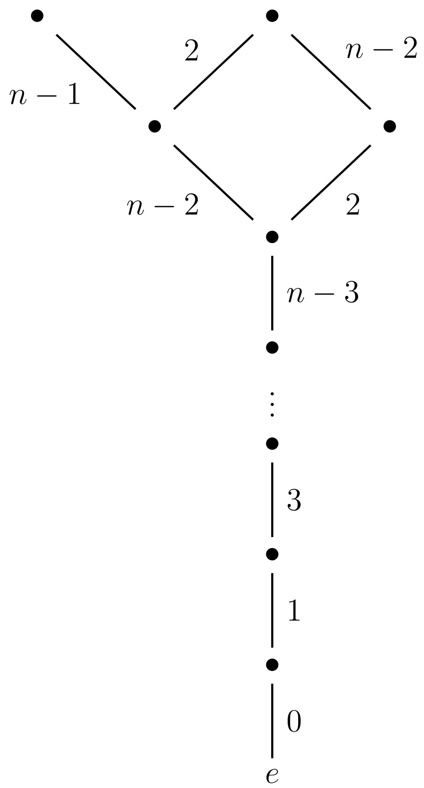
If $h_\sigma = n-2$, we may assume $n \geq 5$ and, by our analysis, there are only two elements that can be moved by σ: these are $w = s_0 s_2 \ldots s_{n-2} s_{n-1}$ and $w' = s_0 s_2 \ldots s_{n-2} s_n$, thus σ must exchange them and fix all other elements of $\mathcal{W}_{n-1}$. We need to check that $\sigma_{|\mathcal{W}_{n-1}} \in Aut(\Pi)_{|\mathcal{W}_{n-1}}$. To this end, it is enough to show that the only elements of $\mathcal{W}_{n-1}$ containing s_n or s_{n-1} in a reduced expression are w and w'. This clearly concludes the proof in this case, for, then, $\sigma_{|\mathcal{W}_{n-1}} = \sigma'_{|\mathcal{W}_{n-1}}$ with $\sigma' \in Aut(\Pi)$ exchanging α_{n-1} and α_n. Let $v \in \mathcal{W}_{n-1}$ contain s_{n-1}. Observe that any reduced expression of v starts with $s_0 s_2$. Also, the simple reflections $s_3, s_4, \ldots, s_{n-1}$ have to appear, and to appear exactly in this order. Otherwise, let i be the place where the first violation occurs: then $v = s_0 s_2 \cdots s_{i-1} s_a u, a \neq i$. If $a > i$, then $v = s_a s_0 s_2 \cdot s_{i-1} u \notin \mathcal{W}$. If $1 < a < i$ then we can move s_a to its left until we form a forbidden braid. Finally if $a = 1$, then $v = s_0 s_2 s_1 z$; repeating the above argument we see that $z = s_3s_4\cdots s_{n-1}$. But then $\ell(v) = n$, against our assumption.

Case 3: type $\widehat{E}_n$ $(n = 6, 7, 8)$. In these cases $h_\sigma = n - 3$. In type $\widehat{E}_6$, $\mathcal{W}_4$ has Hasse diagram



so, if $\sigma_{|\mathcal{W}_4} \neq Id$, then $\sigma_{|\mathcal{W}_4} = \sigma'_{|\mathcal{W}_4}$ with $\sigma' \in Aut(\Pi)$ exchanging α_3 and α_5.

In the other cases, we see that the Hasse diagram of $\mathcal{W}_{n-1}$ is



From this graph we see that $\sigma_{|\mathcal{W}_{n-2}} = Id$. Thus, in this cases, $Aut(\mathfrak{Ab}) = \{Id\}$ and there is nothing to prove. ∎

Remark 5.9. Note that the proof of Lemma 5.8 shows also that, if $\sigma \neq Id$, then

h_σ does not depend on σ.

We are now ready to prove our main Theorem:

Proof of Theorem 5.6. We will prove by induction on h that, given $\sigma \in Aut(\mathfrak{Ab})$, $\sigma_{|\mathcal{W}_h} \in Aut(\Pi)_{|\mathcal{W}_h}$ for any $h \geq 0$. If $h = 0$, there is nothing to prove. Assume $h > 0$. Then, by the induction hypothesis, there is $\tilde{\sigma} \in Aut(\Pi)$ such that $\sigma_{|\mathcal{W}_{h-1}} = \tilde{\sigma}_{|\mathcal{W}_{h-1}}$. Set $\tau = \sigma\tilde{\sigma}^{-1}$. By Lemma 5.8, there is $\sigma' \in Aut(\Pi)$ such that $\tau_{|\mathcal{W}_{h_\tau+1}} = \sigma'_{|\mathcal{W}_{h_\tau+1}}$. Clearly $h_\tau \geq h-1$, so $\tau_{|\mathcal{W}_h} = \sigma'_{|\mathcal{W}_h}$, i.e. $\sigma\tilde{\sigma}^{-1}_{|\mathcal{W}_h} = \sigma'_{|\mathcal{W}_h}$, so that $\sigma_{|\mathcal{W}_h} = (\sigma'\tilde{\sigma})_{|\mathcal{W}_h}$. ∎

6. Symmetries of the Hasse graph of $\mathfrak{Ab}$

Recall that $H_{\mathfrak{Ab}}$ is the Hasse diagram of $\mathfrak{Ab}$. We identify $\mathfrak{Ab}$ with either $\mathcal{W}$ or the set of alcoves $C_{\mathfrak{i}}$, $\mathfrak{i} \in \mathfrak{Ab}$ (cf (2)).

Lemma 6.1. *If $f \in Aut(H_{\mathfrak{Ab}})$ is such that $f(e) = e$, then $f \in Aut(\mathfrak{Ab})$.*

Proof. It suffices to prove that for $w \in \mathcal{W}$, we have $\ell(w) = \ell(f(w))$. In fact, if $v, w \in \mathcal{W}, v < w$, there exists $v = v_0 < v_1 < \cdots < v_k = w$, $v_i \in \mathcal{W}$ with $\ell(v_i) = \ell(v) + i$, hence we need to prove just that $f(v_i) < f(v_{i+1})$. Since $f(v_i)$ has to be linked in $H_{\mathfrak{Ab}}$ to $f(v_{i+1})$, the fact that $\ell(f(v_{i+1})) = \ell(f(v_i)) + 1$ implies $f(v_i) < f(v_{i+1})$.

We perform an induction on $\ell = \ell(w)$. The claim is true by assumption if $\ell = 0$ and follows from Remark 5.5 if $\ell = 1$. Now, if $w \in \mathcal{W}, \ell(w) = k, k > 1$, then w is linked to $v \in \mathcal{W}$, with $\ell(v) = k-1$. Then $\ell(f(v)) = k-1$, hence either $\ell(f(w)) = k-2$ or $\ell(f(w)) = k$; the first case can't occur, since by induction f permutes the elements of length $k-2$, hence $\ell(f(w)) = k$, as required. ∎

Lemma 6.2. *The number of edges connected to a node w in $H_{\mathfrak{Ab}}$ is equal to the number of roots in $\alpha \in \widehat{\Pi}$ such that $w(\alpha) \in \pm(\delta - \Delta^+)$.*

Proof. If v and w are connected by an edge then $vs_i = w$ with $\ell(w) = \ell(v) \pm 1$. If $\ell(w) = \ell(v) + 1$ then $v(\alpha_i) \in N(w)$, hence $v(\alpha_i) \in \delta - \Delta^+$, so $w(\alpha_i) = -v(\alpha_i) \in -(\delta - \Delta^+)$. If $\ell(w) = \ell(v) - 1$, then $ws_i = v$ hence $w(\alpha_i) \in N(v)$, so $w(\alpha_i) \in \delta - \Delta^+$. ∎

Proposition 6.3. *[15] If $\mathfrak{g}$ is not of type C_3, G_2, then $Aut(H_{\mathfrak{Ab}}) = Aut(\widehat{\Pi})$. If $\mathfrak{g}$ is of type C_3, G_2, $Aut(H_{\mathfrak{Ab}}) = Aut(\widehat{\Pi}) \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $f \in Aut(H_{\mathfrak{Ab}})$. Let $w = f(e)$. The set of faces of $w(C_1)$ is given by the hyperplanes corresponding to the roots in $w(\widehat{\Pi})$. By Remark 5.5 only one edge is connected to w. Thus, by Lemma 6.2, $w(\widehat{\Pi})$ contains exactly one root in $\pm(\delta - \Delta^+)$. Let $\alpha_j \in \widehat{\Pi}$ be the simple root such that $w(\alpha_j) \in \pm(\delta - \Delta^+)$. All other walls of $w(C_1)$ are walls of $2C_1$. Thus there is $\beta \in \Pi \cup \{\alpha_0 + \delta\}$ such that the hyperplanes corresponding to $(\Pi \cup \{\alpha_0 + \delta\}) \backslash \{\beta\}$ are walls of $w(C_1)$. It follows that there is a vertex $2o_i$ (the intersection of all the hyperplanes in $(\Pi\cup\{\alpha_0+\delta\})\backslash\{\beta\}$) that is in the closure of $w(C_1)$. It suffices to prove that, if $i\neq 0$, then $m_i=1$. Indeed, if this is the case, there exists $z\in Z_2$ such that $z\,f(C_1)=C_1$, hence we may apply Lemma 6.1 and Theorem 5.6.

We now prove that m_i is odd. Let $c_i(\alpha)$ denote the coefficient of α_i in the expansion of α in terms of simple roots. If m_i is even, then there is a root $\alpha\in\Delta^+$ such that $c_i(\alpha)=m_i/2$. Then $(\delta-\alpha)(2o_i)=0$, so the hyperplane corresponding to $\delta-\alpha$ passes through $2o_i$ and meets the interior of $2C_1$ (see [2]). Thus the hyperplanes corresponding to $(\Pi\cup\{\alpha_0+\delta\})\backslash\{\beta\}$ cannot be all walls of $w(C_1)$. This argument already finishes the proof in all classical cases, for, in these cases, we have that $m_i\leq 2$.

It remains to deal with the exceptional cases. We first prove that, letting $\mathfrak{i}\in\mathfrak{Ab}$ be the ideal corresponding to w, then

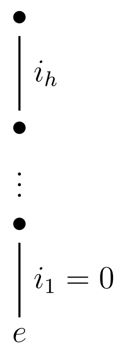
$$\Phi_{\mathfrak{i}}=\{\alpha\in\Delta^+\mid c_i(\alpha)>\frac{m_i}{2}\}.$$

Since we are assuming that $i\neq 0$, $\mathfrak{i}$ is maximal in $\mathfrak{Ab}$. Since $2o_i\in\overline{w(C_1)}$, we have $w^{-1}(2o_i)\in\overline{C_1}$. If $\alpha\in\mathfrak{i}$, then $w^{-1}(\delta-\alpha)\in-\widehat{\Delta}^+$, hence $w^{-1}(\delta-\alpha)(w^{-1}(2o_i))=(\delta-\alpha)(2o_i)\leq 0$. It follows that $1-\frac{2c_i(\alpha)}{m_i}\leq 0$, or equivalently $c_i(\alpha)\geq\frac{m_i}{2}$. Since m_i is odd, we see that

$$\Phi_{\mathfrak{i}}\subset\{\alpha\in\Delta^+\mid c_i(\alpha)>\frac{m_i}{2}\}.$$

Since $\{\alpha\in\Delta^+\mid c_i(\alpha)>\frac{m_i}{2}\}$ is clearly an abelian dual order ideal in Δ^+ hence, since $\mathfrak{i}$ is maximal, equality holds.

Our argument reduces the missing cases to a few direct inspections: the graph $H_{\mathfrak{Ab}}$ near 1 is of this type:



with $h=4$ in type F_4, $h=3,4,6$ in type E_6, E_7, E_8 respectively. Thus, the graph near $\mathfrak{i}$, has to be a chain of the same length. Using the explicit description of $\mathfrak{i}$ given above, it is easy to determine the structure of the subposet $\{\mathfrak{j}\in\mathfrak{Ab}\mid\mathfrak{j}\subset\mathfrak{i}\}$ and verify that, if $m_i>1$, then its Hasse graph near the maximum is a chain of length strictly less than h. ∎

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