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Composition Series of $\mathfrak{gl}(m)$ as a Module for its Classical Subalgebras over an Arbitrary Field

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Abstract. Let F be an arbitrary field and let $f : V \times V \to F$ be a non-degenerate symmetric or alternating bilinear form defined on a finite dimensional vector space over F. Let $L(f)$ be the subalgebra of $\mathfrak{gl}(V)$ formed by all skew-adjoint endomorphisms with respect to f. We find a composition series for the $L(f)$-module $\mathfrak{gl}(V)$ and furnish multiple identifications for its composition factors.
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1. Introduction

Let F be any field. No assumptions are made on F or its characteristic, which will be denoted by ℓ. Let V be an F-vector space of finite dimension $m \geq 2$, and let $f : V \times V \to F$ be a non-degenerate symmetric or alternating bilinear form. Consider the subalgebra $L(f)$ of $\mathfrak{gl}(V)$ defined by

$$L(f) = \{x \in \mathfrak{gl}(V) \mid f(xv, w) = -f(v, xw) \text{ for all } v, w \in V\}.$$

Thus $L(f)$ is the symplectic Lie algebra if f is alternating, or an orthogonal Lie algebra if f is symmetric and non-alternating (the last condition is only required if $\ell = 2$). Note that, in general, the isomorphism type of an orthogonal Lie algebra depends on the equivalence type of the underlying form, and we will only speak of *the* orthogonal Lie algebra when $F = F^2$, i.e., when every element of F is a square. We further let $\mathfrak{s} = Z(\mathfrak{gl}(V))$, which consists of all scalar operators.

In this paper we find a composition series for the $L(f)$-module $\mathfrak{gl}(V)$ and furnish multiple identifications for its composition factors. All possible cases are considered, without exception. Numerous cases arise, as all of ℓ, F, f and m play a role in the determination of the structure of $\mathfrak{gl}(V)$. Our main results are as follows.

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Theorem 1.1. *Suppose that $\ell = 2$, that $m = 2n$, and that f is non-degenerate and alternating. Then*

(1) If $4 \mid m$ then the $L(f)$-module $\mathfrak{gl}(V)$ has $m+6$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)} = [L(f), L(f)]$ in the series

$$0 \subset \mathfrak{s} \subset L(f)^{(2)} \subset L(f)^{(1)} \subset L(f) \subset U \subset \mathfrak{sl}(V) \subset \mathfrak{gl}(V),$$

where $U = L(f) \oplus \langle x \rangle$, $x \in \mathfrak{sl}(V)$, $[x, L(f)] \subseteq L(f)$, and

$$x = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

All composition factors are trivial, except for $L(f)^{(2)}/\mathfrak{s} \cong \mathfrak{sl}(V)/U$, which is of dimension $\binom{m}{2} - 2$. Moreover, $L(f)^{(2)}/\mathfrak{s}$ is a simple Lie algebra if and only if $m > 4$.

(2) If $m \neq 2$ and $4 \nmid m$ then the $L(f)$-module $\mathfrak{gl}(V)$ has $m+4$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)}$ in the series

$$0 \subset L(f)^{(2)} \subset L(f)^{(1)} \subset L(f) \subset \mathfrak{sl}(V) \subset \mathfrak{gl}(V).$$

All composition factors are trivial, except for $L(f)^{(2)} \cong \mathfrak{sl}(V)/L(f)$. Moreover, $L(f)^{(2)}$ is a simple Lie algebra of dimension $\binom{m}{2} - 1$.

(3) $L(f)$ is isomorphic to the symmetric square $S^2(V)$ as $L(f)$-modules, and, relative to suitable basis of V, consists of all matrices

$$\begin{pmatrix} A & B \\ C & A' \end{pmatrix}, \; A, B, C \in \mathfrak{gl}(n), \; \text{where } B, C \text{ are symmetric.} \qquad (1)$$

(4) $L(f)^{(1)}$ is isomorphic to the exterior square $\Lambda^2(V)$ as $L(f)$-modules, and consists of all matrices (1) such that B, C are alternating.

(5) $L(f)^{(2)}$ is isomorphic to the kernel of the contraction $L(f)$-epimorphism $\Lambda^2(V) \to F$, given by $v \wedge w \mapsto f(v, w)$, and consists of all matrices (1) such that B, C are alternating and $\operatorname{tr}(A) = 0$.

(6) $L(f)/L(f)^{(2)}$ is isomorphic, as Lie algebra, to $\mathfrak{h}(n)$, the Heisenberg algebra of dimension $2n+1$.

The case $m = 4$ of Theorem 1.1 is exceptional in various ways. Firstly, while $L(f)^{(2)}/\mathfrak{s}$ is an irreducible $L(f)$-module, it is not simple as a Lie algebra. A similar phenomenon occurs to $L(f)^{(1)}$ if $m = 4$, $\ell = 2$ but f is non-degenerate, symmetric and non-alternating, as discussed below. Secondly, $L(f)^{(2)}$ is also isomorphic, as Lie algebra, to $\mathfrak{h}(2)$, so $L(f)$ is an extension of $\mathfrak{h}(2)$ by $\mathfrak{h}(2)$. Thirdly, the kernel of the representation of $L(f)$ on $L(f)^{(2)}/\mathfrak{s}$ is $L(f)^{(2)}$. This gives a 4-dimensional faithful irreducible representation of $\mathfrak{h}(2)$. This phenomenon is impossible in characteristic not 2. Full details of the case $m = 4$, as well as its connections to the problem of finding the smallest dimension of a faithful module for a given Lie algebra (see [Bu] and [CR]) can be found in §12, which also treats the much easier case $m = 2$.

We remark that Bourbaki [B], Chapter I, §6, Exercise 25(b), studied the ideal structure of $L(f)$ when $\ell = 2$ and f is non-degenerate and alternating, but made mistakes involving $\mathfrak{s}$, $L(f)^{(1)}$ and $L(f)^{(2)}$ (see Note 7.9 for details). Theorem 1.1 corrects Bourbaki's information and expands it to include the structure of $\mathfrak{gl}(V)$ as $L(f)$-module, as well as providing further identifications for all composition factors.

In the case of orthogonal Lie algebras in characteristic 2, which was not considered in [B], we have the following result.

Theorem 1.2. *Suppose that $\ell = 2$, that $m \geq 2$, and that f is non-degenerate, symmetric and non-alternating. Then*

(1) The $L(f)$-module $\mathfrak{gl}(V)$ has $m+2$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)}$ in the series

$$0 \subset L(f)^{(1)} \subset L(f) \subset \mathfrak{gl}(V).$$

Moreover, if $m = 3$ or $m \geq 5$ then $L(f)^{(1)}$ is a simple Lie algebra of dimension $\binom{m}{2}$.

(2) $L(f)$ is isomorphic to the symmetric square $S^2(V)$ as $L(f)$-modules. Moreover, there is a basis of V relative to which f has Gram matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$ and, relative to this basis, $L(f)$ consists of all $A \in \mathfrak{gl}(m)$ such that $d_i A_{ij} = d_j A_{ji}$.

(3) $L(f)^{(1)}$ is isomorphic to the exterior square $\Lambda^2(V)$ as $L(f)$-modules. Moreover, relative to the above basis, $L(f)^{(1)}$ consists of all $A \in \mathfrak{gl}(m)$ such that $A_{ii} = 0$ and $d_i A_{ij} = d_j A_{ji}$.

(4) $\mathfrak{gl}(V)/L(f) \cong L(f)^{(1)}$ as $L(f)$-modules. In particular, $\mathfrak{gl}(V)$ has m trivial composition factors, and 2 composition factors isomorphic to $L(f)^{(1)} \cong \Lambda^2(V)$, which is itself the trivial module if and only if $m = 2$.

We remark that if we let $m = 4$ in Theorem 1.2 then the irreducible $L(f)$-module $L(f)^{(1)}$ need not be a simple Lie algebra. The structure of the 6-dimensional Lie algebra $L(f)^{(1)}$ depends on whether the discriminant of f is a square in F or not. This is entirely analogous to what happens to $L(f)$ itself when $\ell \neq 2$ and f is non-degenerate and symmetric, as indicated in [B], Chapter I, §6, Exercise 26(b). For a uniform treatment of both cases via current Lie algebras see [CS].

Our results in characteristic not 2 are better described by means of

$$M(f) = \{y \in \mathfrak{gl}(V) \mid f(yv, w) = f(v, yw) \text{ for all } v, w \in V\}.$$

Note that $M(f)$ is an $L(f)$-module, regardless of the nature of f and ℓ. However, if $\ell = 2$ then $M(f) = L(f)$, so $M(f)$ plays no additional role in this case.

Theorem 1.3. *Suppose that $\ell \neq 2$, that $m = 2n$, and that f is non-degenerate and skew-symmetric. Then*

(1) $M(f)$ is the orthogonal complement to $L(f)$ with respect to the bilinear form $\varphi : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to F$, given by $\varphi(x, y) = \mathrm{tr}(xy)$. Moreover, $M(f)$ consists,

relative to suitable basis of V, of all matrices

$$\begin{pmatrix} A & B \\ C & A' \end{pmatrix},\ A,B,C\in\mathfrak{gl}(n),\ \textit{where } B,C \textit{ are skew-symmetric.} \tag{2}$$

Furthermore, $M(f)$ is isomorphic to $\Lambda^2(V)$ as $L(f)$-module.

(2) $M(f)\cap\mathfrak{sl}(V)$ consists of all matrices (2) such that $\mathrm{tr}(A)=0$ and is isomorphic to the kernel of the contraction $L(f)$-epimorphism $\Lambda^2(V)\to F$ given by $v\wedge w\to f(v,w)$.

(3) If $m>2$ and $\ell\nmid m$ then $M(f)\cap\mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m}{2}-1$.

(4) If $m>2$ and $\ell\mid m$ then $M(f)\cap\mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m}{2}-2$.

(5) $L(f)$ is a simple Lie algebra, isomorphic to both $\mathfrak{gl}(V)/M(f)$ and $S^2(V)$ as $L(f)$-modules.

(6) The following are composition series of the $L(f)$-module $\mathfrak{gl}(V)$:

$$0\subset\mathfrak{s}\subset M(f)\cap\mathfrak{sl}(V)\subset M(f)\subset\mathfrak{gl}(V),\ \textit{if } m>2 \textit{ and } \ell|m,$$

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$$0\subset M(f)\subset\mathfrak{gl}(V),\ \textit{if } m=2.$$

In any case, $M(f)/M(f)\cap\mathfrak{sl}(V)$ is the trivial $L(f)$-module.

Theorem 1.4. *Suppose that $\ell\neq 2$, that $m\geq 2$, and that f is non-degenerate and symmetric. Then*

(1) $M(f)$ is the orthogonal complement to $L(f)$ with respect to the bilinear form $\varphi:\mathfrak{gl}(V)\times\mathfrak{gl}(V)\to F$, given by $\varphi(x,y)=\mathrm{tr}(xy)$. Moreover, there is a basis of V relative to which f has Gram matrix $D=\mathrm{diag}(d_1,\dots,d_n)$ and, relative to this basis, $M(f)$ consists of all $A\in\mathfrak{gl}(m)$ such that $d_iA_{ij}=d_jA_{ji}$. Furthermore, $M(f)$ is isomorphic to $S^2(V)$ as $L(f)$-module.

(2) $M(f)\cap\mathfrak{sl}(V)$ consists, relative to the above basis, of all matrices $A\in\mathfrak{gl}(m)$ such that $d_iA_{ij}=d_jA_{ji}$ and $\mathrm{tr}(A)=0$, and is isomorphic to the kernel of the contraction $L(f)$-epimorphism $S^2(V)\to F$ given by $vw\to f(v,w)$.

(3) If $m\geq 4$ and $\ell\nmid m$ then $M(f)\cap\mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2}-1$.

(4) If $m\geq 4$ and $\ell\mid m$ then $M(f)\cap\mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2}-2$.

(5) If $m=3$ or $m\geq 5$ then $L(f)$ is a simple Lie algebra, isomorphic to both $\mathfrak{gl}(V)/M(f)$ and $\Lambda^2(V)$ as $L(f)$-modules.

(6) The following are composition series of the $L(f)$-module $\mathfrak{gl}(V)$:

$$0\subset\mathfrak{s}\subset M(f)\cap\mathfrak{sl}(V)\subset M(f)\subset\mathfrak{gl}(V),\ \textit{if } m\geq 4 \textit{ and } \ell|m,$$

$$0\subset M(f)\cap\mathfrak{sl}(V)\subset M(f)\subset\mathfrak{gl}(V),\ \textit{if } m\geq 4 \textit{ and } \ell\nmid m.$$

In any case, $M(f)/M(f)\cap\mathfrak{sl}(V)$ is the trivial $L(f)$-module.

As remarked earlier, if we take $m=4$ in Theorem 1.4, the structure of Lie algebra $L(f)$ depends on the nature of the discriminant of f. On the other hand, if we take $m=2$ or $m=3$ in Theorem 1.4, the structure of the $L(f)$-module $M(f)$ depends not just on ℓ but also on F itself. See §9 for details.

Much is known about the classical Lie algebras and their representations, so a great deal of the results stated above is already known. Indeed, note that f induces an isomorphism $V\cong V^*$ of $L(f)$-modules, so $\mathfrak{gl}(V)\cong V^*\otimes V\cong V\otimes V$, where $V\otimes V/S^2(V)\cong\Lambda^2(V)$, with $V\otimes V=S^2(V)\oplus\Lambda^2(V)$ if $\ell\neq 2$. Let $\Omega: V\otimes V\to F$ be the contraction $L(f)$-epimorphism given by $v\otimes w\mapsto f(v,w)$.

Suppose $F=\mathbb{C}$. If f is skew-symmetric and $m\geq 4$ then $V\otimes V$ has the following decomposition into irreducible $L(f)$-submodules:

$$V\otimes V=S^2(V)\oplus(\ker\Omega\cap\Lambda^2(V))\oplus U,$$

where $S^2(f)\cong L(f)$ and U is trivial. If f is symmetric, with $m=3$ or $m\geq 5$, then $V\otimes V$ decomposes as follows into irreducible $L(f)$-submodules:

$$V\otimes V=\Lambda^2(V)\oplus(\ker\Omega\cap S^2(V))\oplus W,$$

where $\Lambda^2(f)\cong L(f)$ and W is trivial. We refer the reader to [FH] for these details, as well as for further information, in terms of Weyl modules, on higher tensor powers of V.

It follows from Theorems 1.1-1.4 that the above statements remain valid for any field of characteristic 0, but cease to be true if $\ell|2m$, the more substantial failure occurring when $\ell=2$. In prime characteristic, the ideal structure of $L(f)$ is described in detail in [B], Chapter I, §6, Exercises 25 and 26, although $\ell\neq 2$ is required in the orthogonal case. Given the aforementioned errors found in [B] and that full information on the $L(f)$-submodule structure of $\mathfrak{gl}(V)$, that includes all possible cases of ℓ, F, f and m, does not seem to be available in the literature, we decided to provide a self-contained account of it, including complete proofs, and requiring no prior knowledge of Lie algebras.

We begin in §2, which includes all one needs to know about Lie algebras to read this paper. This material can be covered during the first week of a course on the subject.

We now refer to [H], §1, Exercise 10, where we are required to justify the complex isomorphisms $\mathfrak{sp}(4)\cong\mathfrak{so}(5)$ and $\mathfrak{sl}(4)\cong\mathfrak{so}(6)$. How is one supposed to prove this after a single week of lecturing? Comparing multiplication tables is one option, although very tiring and time consuming, as these Lie algebras have dimensions 10 and 15, respectively. The use of Dynkin diagrams must be postponed until much later, so one is essentially led to use representations in some way or another. In the case of $\mathfrak{sl}(4)\cong\mathfrak{so}(6)$ there is the standard argument involving the action of $\mathfrak{sl}(4)$ on $\Lambda^2(W)$, where W is the natural module of $\mathfrak{sl}(4)$. From $\mathfrak{sl}(4)\cong\mathfrak{so}(6)$ one then obtains $\mathfrak{sp}(4)\cong\mathfrak{so}(5)$ by restriction to $\mathfrak{sp}(4)$. This requires prior knowledge of exterior powers, which might not be available to everyone at the beginning (or the end) of a course on Lie algebras, especially to undergraduate students.

In §3 we furnish an extremely elementary and direct proof of $\mathfrak{sp}(4)\cong\mathfrak{so}(5)$ whenever $\ell\neq 2$ and $F=F^2$ (these conditions are clearly necessary) as part of a general and canonical imbedding $\mathfrak{sp}(2n) \hookrightarrow \mathfrak{so}(2n^2-n-1)$ whenever $\ell \nmid 2n$ and $F = F^2$. The material from §3 is really a special case of our study of the $L(f)$-module $M(f)$ in the symplectic case, but we present it first to make the isomorphism $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$ available immediately after the first rudiments on Lie algebras. If we had to single out a key ingredient behind the isomorphism $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$ it would be the non-degenerate $\mathfrak{gl}(V)$-invariant symmetric bilinear form $\varphi : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to F$ used in Theorems 1.3 and 1.4.

In §5 we give an elementary proof of $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$ valid when $\ell \neq 2$ and $F = F^2$ (these conditions are, again, necessary). It uses the same idea of the isomorphism $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$, the presence of a non-degenerate invariant symmetric bilinear form, although in this case a minimal amount of calculations are needed in order to avoid the use of exterior powers. As with classical method, the simplicity of $\mathfrak{sl}(4)$ is required. For completeness, an account of the ideal structure of $\mathfrak{gl}(m)$ is given in §4. This can be found in [B], Chapter I, §6, Exercise 24.

In §6 we describe basic properties of $M(f)$ for an arbitrary bilinear form f, with emphasis on the case when f is non-degenerate, symmetric or alternating, while §7 justifies the various identifications made in Theorems 1.1-1.4 concerning $L(f)$-modules.

The last five sections are devoted to demonstrate the irreducibility aspects of Theorems 1.1-1.4, depending on whether $\ell = 2$ or not and the nature of f.

2. Preliminaries

The notation introduced in this section will be maintained throughout the entire paper.

Let F be an arbitrary field of characteristic ℓ. Thus ℓ is zero or a prime. All vector spaces are assumed to be finite dimensional over F unless otherwise mentioned. We fix a vector space V of dimension $m \geq 1$.

2.1. Lie algebras. A Lie algebra is a vector space L together with a bilinear map $[\ ,\ ] : L \times L \to L$, called bracket or commutator, satisfying:

(L1) $[x,x] = 0$ for all $x \in L$;

(L2) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in L$.

Any associative algebra A gives rise to a Lie algebra whose underlying vector space is A itself, with commutator

$$[xy] = xy - yx, \quad x, y \in A.$$

The Lie algebras corresponding to $M_m(F)$ and $\mathrm{End}(V)$ will be denoted by $\mathfrak{gl}(m)$ and $\mathfrak{gl}(V)$, respectively, and called general linear Lie algebras.

The canonical matrices e_{ij}, $1 \leq i, j \leq m$, form a basis of $M_m(F)$ and multiply as follows: $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Thus, we have the following multiplication table in $\mathfrak{gl}(m)$:

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}. \tag{3}$$

Given Lie algebras L_1 and L_2, a Lie homomorphism (resp. isomorphism) is a linear homomorphism (resp. isomorphism) $T : L_1 \to L_2$ satisfying

$$T([x,y]) = [T(x), T(y)], \quad x, y \in L_1.$$

For instance, if $\mathcal{B}$ is a basis of V then the map $M_{\mathcal{B}} : \mathfrak{gl}(V) \to \mathfrak{gl}(m)$, which sends every $x \in \mathfrak{gl}(V)$ to its matrix $M_{\mathcal{B}}(x)$ relative to $\mathcal{B}$, is a Lie isomorphism.

Given a Lie algebra L, an ideal (resp. subalgebra) is a subspace K of L satisfying $[x, y] \in K$ for all $x \in L$ and $y \in K$ (resp. all $x, y \in K$). We say that L is simple if $\dim(L) > 1$ and the only ideals of L are 0 and L.

For instance, $L^{(1)} = [L, L]$, the span of all $[x, y]$ with $x, y \in L$, is an ideal of L, as well as $L^{(2)} = [L^{(1)}, L^{(1)}]$, etc.

The special linear Lie algebra $\mathfrak{sl}(V)$, consisting of all traceless endomorphisms of V, is an ideal of $\mathfrak{gl}(V)$, corresponding to $\mathfrak{sl}(m)$, the ideal of $\mathfrak{gl}(m)$ of all traceless $m \times m$ matrices, under the isomorphism $M_{\mathcal{B}}$.

We will denote by $\mathfrak{s}$ the ideal of $\mathfrak{gl}(V)$ (resp. $\mathfrak{gl}(m)$) of all scalar endomorphisms (resp. matrices). Note that $\mathfrak{s} \subseteq \mathfrak{sl}(V)$ if and only if $\ell | m$.

2.2. Representations and modules. Let L be a Lie algebra. A representation of L on a vector space W is a Lie homomorphism $R : L \to \mathfrak{gl}(W)$, in which case we refer to W as an L-module and write $x \cdot w$ or simply xw to mean $R(x)w$. Note that the map $L \times W \to W$ is bilinear and satisfies

$$[x, y]w = xyw - yxw, \quad x, y \in L, w \in W. \tag{4}$$

Conversely, any bilinear map map $L \times W \to W$ satisfying (4) gives rise to a representation $R : L \to \mathfrak{gl}(W)$ defined by $R(x)w = xw$.

Let W be an L-module. We say that W is faithful if its associated representation is injective. An L-submodule of W is a subspace U of W such that $xu \in U$ for all $x \in L$ and $u \in U$. We refer to W as irreducible if W is non-zero and its only submodules are 0 and W. For instance, the adjoint module of a Lie algebra L is $W = L$, where $x \cdot w = [x, w]$. This is irreducible if and only if L is a simple Lie algebra or $\dim(L) = 1$.

Note that the dual space W^* becomes an L-module via

$$(x \cdot \alpha)(w) = \alpha(-x \cdot w), \quad x \in L,\ \alpha \in W^*,\ w \in W.$$

Using annihilators we easily see that W is irreducible if and only if so is W^*.

Let $R : L \to \mathfrak{gl}(W)$ and $R^* : L \to \mathfrak{gl}(W^*)$ be the representations associated to W and W^*. Let $\mathcal{C}$ be a basis of W and $\mathcal{C}^*$ is its dual basis. Then the matrix representations associated to W and W^* with respect to $\mathcal{C}$ and $\mathcal{C}^*$ are related by:

$$M_{\mathcal{C}^*}(R^*(x)) = -M_{\mathcal{C}}(R(x))', \quad x \in L,$$

where A' denotes the transpose a matrix A.

A homomorphism (resp. isomorphism) of L-modules is a linear homomorphism (resp. isomorphism) $T : W_1 \to W_2$ satisfying

$$T(xw) = xT(w), \quad x \in L, w \in W_1.$$

2.3. Classical Lie algebras. We fix throughout a bilinear form $f : V \times V \to F$ and set

$$L(f) = \{x \in \mathfrak{gl}(V) \mid f(xv, w) = -f(v, xw) \text{ for all } v, w \in V\},$$

$$M(f) = \{y \in \mathfrak{gl}(V) \mid f(yv, w) = f(v, yw) \text{ for all } v, w \in V\}.$$

Note that $L(f) = M(f)$ if $\ell = 2$.

Lemma 2.1. *$L(f)$ is a subalgebra of $\mathfrak{gl}(V)$ and $M(f)$ is an $L(f)$-submodule of $\mathfrak{gl}(V)$.*

Proof. Let $x \in L(f)$, $y \in M(f)$. We wish to see that $[xy] \in M(f)$. If $v, w \in V$ then

$$\begin{aligned} f([xy]v, w) &= f(xyv, w) - f(yxv, w) \\ &= -f(yv, xw) - f(xv, yw) \\ &= -f(v, yxw) + f(v, xyw) \\ &= f(v, [xy]w). \end{aligned}$$

The proof that $L(f)$ is a subalgebra of $\mathfrak{gl}(V)$ is entirely analogous. ∎

It will be useful to have a matrix version of $L(f)$ and $M(f)$ available. Given $A \in \mathfrak{gl}(m)$, we set

$$L(A) = \{X \in \mathfrak{gl}(m) \,|\, X'A = -AX\} \text{ and } M(A) = \{Y \in \mathfrak{gl}(m) \,|\, Y'A = AY\}. \quad (5)$$

Let $\mathcal{B} = \{v_1, \ldots, v_m\}$ be a basis of V and suppose that $A \in \mathfrak{gl}(m)$ is the Gram matrix of f relative to $\mathcal{B}$, that is,

$$A_{ij} = f(v_i, v_j), \quad 1 \leq i, j \leq m.$$

Then $M_{\mathcal{B}}$ sends $L(f)$ onto $L(A)$ and $M(f)$ onto $M(A)$.

Two matrices $A, B \in \mathfrak{gl}(m)$ are said to be congruent if there is $S \in \mathrm{GL}_m(F)$ such that

$$S'AS = B,$$

in which case the map $L(A) \to L(B)$ given by $X \mapsto S^{-1}XS$ is a Lie isomorphism.

Suppose f is non-degenerate and alternating. In this case (see [K], Theorem 19) $m = 2n$ and there is a basis $\mathcal{B}$ of V relative to which f has Gram matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \quad (6)$$

We write $\mathfrak{sp}(2n) = L(J)$ and refer to $L(f)$ as the symplectic Lie algebra. An easy computation based on (5) and (6) reveals that

$$L(J) = \left\{ \begin{pmatrix} A & B \\ C & -A' \end{pmatrix} \,\middle|\, A, B, C \in \mathfrak{gl}(n), \text{ where } B, C \text{ are symmetric} \right\}$$

and

$$M(J) = \left\{ \begin{pmatrix} A & B \\ C & A' \end{pmatrix} \,\middle|\, A, B, C \in \mathfrak{gl}(n), \text{ where } B, C \text{ are skew-symmetric} \right\}.$$

In particular,

$$\dim L(J) = \binom{m+1}{2}, \text{ and if } \ell \neq 2 \text{ then } \dim M(J) = \binom{m}{2}.$$

Suppose next that f is non-degenerate and symmetric. If f is alternating then necessarily $\ell = 2$ and $L(f)$ is the symplectic Lie algebra considered above.

If f is non-alternating then by [K], Theorems 18 and 20, there is a basis $\mathcal{B}$ of V relative to which f has diagonal Gram matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$, $d_i \neq 0$. Another calculation based on (5) shows that

$$L(D) = \{A \in \mathfrak{gl}(m) \,|\, d_i A_{ij} + d_j A_{ji} = 0 \text{ for all } 1 \leq i, j \leq m\},$$

and

$$M(D) = \{A \in \mathfrak{gl}(m) \,|\, d_i A_{ij} - d_j A_{ji} = 0 \text{ for all } 1 \leq i, j \leq m\}.$$

In particular, if f is non-degenerate, symmetric and non-alternating, then

$$\dim L(f) = \binom{m}{2} \text{ if } \ell \neq 2, \;\; \dim L(f) = \binom{m+1}{2} \text{ if } \ell = 2,$$

and

$$\dim M(f) = \binom{m+1}{2}.$$

Moreover, if $F = F^2$ (i.e., every element of F is a square) then f admits I_m as Gram matrix, in which case we refer to $L(f)$ as the orthogonal Lie algebra and write $\mathfrak{so}(m) = L(I_m)$. Clearly, $\mathfrak{so}(m)$ consists of all skew-symmetric matrices of $\mathfrak{gl}(m)$ and $M(I_m)$ of all symmetric matrices of $\mathfrak{gl}(m)$.

A matrix $A \in \mathfrak{gl}(m)$ is said to be alternating if

$$A_{ij} = -A_{ji} \text{ and } A_{ii} = 0, \quad 1 \leq i, j \leq m.$$

By above, any two invertible alternating matrices are congruent. Provided $F = F^2$, so are any two invertible symmetric non-alternating matrices.

Suppose that W is a d-dimensional module for a Lie algebra L, where $d \geq 1$. A bilinear form $\phi : W \times W \to F$ is said to be L-invariant if the representation $R : L \to \mathfrak{gl}(W)$ associated to W satisfies $R : L \to L(\phi)$, that is, if

$$\phi(x \cdot u, v) + \phi(u, x \cdot v) = 0, \quad x \in L,\, u, v \in W.$$

In this case, if the L-module W is faithful and ϕ is non-degenerate we obtain an imbedding $R : \mathfrak{g} \to \mathfrak{sp}(d)$ (resp. $R : \mathfrak{g} \to \mathfrak{so}(d)$) provided ϕ is alternating (resp. symmetric and non-alternating, and $F = F^2$).

Let $T : W \to W^*$ be a linear map. By definition, T is an L-homomorphism if and only if the associated bilinear form $\phi : V \times V \to F$, given by $\phi(u, v) = T(u)(v)$, is L-invariant, in which case T is an isomorphism if and only if ϕ is non-degenerate.

For instance, if f is non-degenerate then $V \cong V^*$ as $L(f)$-modules via the map $v \mapsto f(v, -)$, since f is $L(f)$-invariant.

2.4. A trace form. We fix throughout the bilinear form $\varphi : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to F$ given by

$$\varphi(x, y) = \mathrm{tr}(xy), \quad x, y \in \mathfrak{gl}(V). \tag{7}$$

It is well-known and easy to see that φ is symmetric and non-degenerate.

Lemma 2.2. *The bilinear form φ is $\mathfrak{gl}(V)$-invariant.*

Proof. Let $x, y, z \in \mathfrak{gl}(V)$. Then

$$\varphi(z \cdot x, y) + \varphi(x, z \cdot y) = \mathrm{tr}([zx]y) + \mathrm{tr}(x[zy]) = \mathrm{tr}(zxy - xzy) + \mathrm{tr}(xzy - xyz) = 0.$$

■

By abuse of notation we will also denote by φ the $\mathfrak{gl}(m)$-invariant non-degenerate symmetric bilinear form $\mathfrak{gl}(m) \times \mathfrak{gl}(m) \to F$ defined by $\varphi(A, B) = \mathrm{tr}(AB)$. We have

$$\mathfrak{s} = \mathfrak{sl}(m)^{\perp}, \quad \mathfrak{sl}(m) = \mathfrak{s}^{\perp}.$$

Moreover, if $\ell \nmid m$ then

$$\mathfrak{gl}(m) = \mathfrak{sl}(m) \perp \mathfrak{s}.$$

We let $\mathrm{Alt}(m)$ and $\mathrm{Sym}(m)$ stand for the spaces of all alternating and symmetric $m \times m$ matrices, respectively. Consider the linear map $\Psi : \mathfrak{gl}(m) \to \mathrm{Alt}(m)$ given by $A \mapsto A - A'$. Since $\ker(\Psi) = \mathrm{Sym}(m)$, the rank-nullity formula implies that $\mathrm{im}(\Psi) = \mathrm{Alt}(m)$, i.e., Ψ is surjective.

Observe next that if $A, B \in \mathfrak{gl}(m)$ then

$$\mathrm{tr}(AB) - \mathrm{tr}(A'B') = \mathrm{tr}(AB) - \mathrm{tr}(B'A') = \mathrm{tr}(AB) - \mathrm{tr}((AB)') = 0. \tag{8}$$

Suppose $C \in \mathrm{Alt}(m)$ and $B \in \mathrm{Sym}(m)$. Since Ψ is surjective, we have $C = A - A'$ for some $A \in \mathfrak{gl}(m)$, so by (8)

$$\varphi(C, A) = \mathrm{tr}(CB) = \mathrm{tr}((A - A')B) = \mathrm{tr}(AB - A'B') = 0. \tag{9}$$

Combining (9) with the non-degeneracy of φ, dimension considerations show that

$$\mathrm{Alt}(m)^{\perp} = \mathrm{Sym}(m), \quad \mathrm{Sym}(m)^{\perp} = \mathrm{Alt}(m).$$

Moreover, if $\ell \neq 2$ then

$$\mathfrak{gl}(m) = \mathrm{Alt}(m) \perp \mathrm{Sym}(m).$$

2.5. Weights. Suppose H and W are vector spaces and that H acts on W, i.e., there is a bilinear map $H \times W \to W$, say $(h, w) \mapsto hw$. Then every $\alpha \in H^*$ gives rise to the subspace, say W_α, of W defined by

$$W_\alpha = \{w \in W \mid hw = \alpha(h)w \text{ for all } h \in H\}.$$

We say that α is weight for the action of H on W if $W_\alpha \neq 0$. Note that if $T : W \to W'$ is an isomorphism of L-modules for a Lie algebra L and H is a subalgebra of L then $T(W_\alpha) = W'_\alpha$ for every $\alpha \in H^*$. In particular, the weights for the actions of H on W and W' are identical.

Note 2.3. As an illustration, consider the irreducible $\mathfrak{sl}(V)$-module V and the diagonal subalgebra H of $\mathfrak{sl}(V)$. The weights of H acting on V are $\varepsilon_1, \dots, \varepsilon_m$, where $\varepsilon_i : H \to F$ is the ith coordinate function, given by $\varepsilon_i(h) = h_{ii}$. The weights of H acting on V^* are $-\varepsilon_1, \dots, -\varepsilon_m$. Thus, if $m > 2$ and $\ell \neq 2$ then

$V \not\cong V^*$. If $m = 2$ then H has the same weights $\varepsilon_1, -\varepsilon_1$ acting on V and V^* and, in fact, $V \cong V^*$. If $m > 2$ but $\ell = 2$ then H has the same weights on V and V^*. However, in this case $V \not\cong V^*$, otherwise $V \otimes V \cong \mathfrak{gl}(V)$, which contradicts the $\mathfrak{sl}(V)$-submodule structures of $V \otimes V$ and $\mathfrak{gl}(V)$.

An alternative way to decide when $V \cong V^*$ is to look at the automorphism $A \mapsto -A'$ of $\mathfrak{sl}(m)$. It is given by conjugation by a fixed $S \in \mathrm{GL}_m(F)$ if and only if $m \leq 2$.

The above phenomenon when $\ell = 2$ is impossible for $F = \mathbb{C}$: an irreducible module for a complex semisimple Lie algebra is characterized by the weights of a Cartan subalgebra.

3. Viewing $\mathfrak{gl}(2n)$ as a module for $\mathfrak{sp}(2n)$

We assume throughout this section that $\ell \neq 2$ and $m = 2n$, and set $W = \mathfrak{gl}(2n)$. Recalling the matrix $J \in \mathfrak{gl}(2n)$ defined in (6), we also set $L = L(J) = \mathfrak{sp}(2n)$. Note that W is an L-module via $x \cdot w = [x, w]$. Recall, as well, the non-degenerate L-invariant symmetric bilinear form $\varphi : W \times W \to F$, defined in (7), and the L-submodule $M = M(J)$, defined in (5).

Theorem 3.1. *Suppose that $\ell \nmid 2n$. Then the L-module W has the following orthogonal decomposition into L-submodules:*

$$W = L \perp (M \cap \mathfrak{sl}(2n)) \perp \mathfrak{s}, \tag{10}$$

where $M = L^\perp$ has the matrix description given in §2.

Moreover, $M \cap \mathfrak{sl}(2n)$ is an L-module of dimension $2n^2 - n - 1$, which is faithful if $n \geq 2$. In particular, if $F = F^2$ and $n \geq 2$ then $M \cap \mathfrak{sl}(2n)$ induces an imbedding $\mathfrak{sp}(2n) \hookrightarrow \mathfrak{so}(2n^2 - n - 1)$, which is an isomorphism $\mathfrak{sp}(4) \to \mathfrak{so}(5)$ when $n = 2$.

Proof. We claim that $M = L^\perp$. Indeed, let $x \in M$ and $y \in L$. As indicated in §2, there exist $a, b, c, d, e \in \mathfrak{gl}(n)$ such that

$$x = \begin{pmatrix} a & b \\ c & a' \end{pmatrix}, \ y = \begin{pmatrix} d & e \\ f & -d' \end{pmatrix}, \text{ with } b, c \text{ skew-symmetric and } e, f \text{ symmetric.} \tag{11}$$

It follows from (8) and (9) that

$$\varphi(x, y) = \mathrm{tr}(xy) = \mathrm{tr}(ad + bf + ce - a'd') = 0.$$

This proves $M \subseteq L^\perp$. The matrix descriptions of L and M show that $W = L \oplus M$. On the other hand, the non-degeneracy of φ implies $\dim L + \dim L^\perp = \dim W$. Since $M \subseteq L^\perp$ and they have the same dimension, it follows that $M = L^\perp$. We have shown

$$W = L \perp M. \tag{12}$$

The matrix description of M makes it clear that

$$\dim M \cap \mathfrak{sl}(2n) = 2n^2 - n - 1,$$

and the condition $\ell \nmid 2n$ implies

$$M = (M \cap \mathfrak{sl}(2n)) \perp \mathfrak{s}. \tag{13}$$

Substituting (13) into (12) yields (10).

Since $\mathfrak{sl}(2n)$ and $\mathfrak{s}$ are ideals of $\mathfrak{gl}(2n)$, it follows from Lemma 2.1 that all components of (10) are L-submodules of W. Since φ is symmetric and non-degenerate on W, so its restriction to each component of (10). As explained in §2, this yields an imbedding $\mathfrak{sp}(2n) \hookrightarrow \mathfrak{so}(2n^2 - n - 1)$, provided $F = F^2$ and $M \cap \mathfrak{sl}(2n)$ is a faithful L-module. This imbedding becomes an isomorphism $\mathfrak{sp}(4) \to \mathfrak{so}(5)$ when $n = 2$, as these Lie algebras are both 10-dimensional.

It only remains to show that $M \cap \mathfrak{sl}(2n)$ is a faithful L-module whenever $n \geq 2$. For this purpose, let $y \in L$ be as in (11) and suppose that $[x, y] = 0$ for all $x \in M \cap \mathfrak{sl}(2n)$ as in (11). Setting $b = 0 = c$, it follows that $[d, a] = 0$ for all a in $\mathfrak{gl}(n)$, whence d is scalar. Letting $a = 0 = c$, we see that $2db = 0$ and $bf = 0$ for all skew-symmetric $b \in \mathfrak{gl}(n)$, so $d = 0 = f$. Finally, taking $a = 0 = b$, we get $ce = 0$ for all skew-symmetric $c \in \mathfrak{gl}(n)$, whence $e = 0$. This completes the proof. ■

Note 3.2. The irreducibility of the components of (10) is discussed in §10.

Here we sketch an indirect argument of the irreducibility and faithfulness of $M \cap \mathfrak{sl}(2n)$ when $n \geq 2$ and $F = \mathbb{C}$.

Let H be the diagonal subalgebra of L. For $1 \leq i \leq n$ consider the linear functional $\varepsilon_i : H \to F$ given by $\varepsilon_i(h) = h_{ii}$. We easily verify that the weights for the action of H on $M \cap \mathfrak{sl}(2n)$ are the sums of 2 distinct members taken from $\{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$. These are the same weights for the action of H on $V(\lambda_2)$, where λ_2 is the second fundamental module for L. But

$$\dim(M \cap \mathfrak{sl}(2n)) = 2n^2 - n - 1 = \dim(V(\lambda_2)),$$

so $M \cap \mathfrak{sl}(2n) \cong V(\lambda_2)$ is irreducible.

Since L is a simple Lie algebra and $M \cap \mathfrak{sl}(2n)$ is an irreducible L-module of dimension > 1, it follows that $M \cap \mathfrak{sl}(2n)$ is faithful.

4. Viewing $\mathfrak{gl}(m)$ as a module for $\mathfrak{sl}(m)$

We assume throughout this section that $m \geq 2$.

Theorem 4.1. *Suppose $(m, \ell) \neq (2, 2)$. Then the only composition series of $\mathfrak{gl}(m)$ as a module for $\mathfrak{gl}(m)$ or $\mathfrak{sl}(m)$ are:*

$$0 \subset \mathfrak{s} \subset \mathfrak{sl}(m) \subset \mathfrak{gl}(m), \text{ if } \ell \mid m,$$

$$0 \subset \mathfrak{sl}(m) \subset \mathfrak{gl}(m), \text{ if } \ell \nmid m.$$

In particular, $\mathfrak{sl}(m)/\mathfrak{s} \cap \mathfrak{sl}(m)$ is always simple. More explicitly,

- *if $\ell \nmid m$, then $\mathfrak{sl}(m)$ simple;*

- *if $\ell \mid m$, then the only proper non-trivial ideal of $\mathfrak{sl}(m)$ is $\mathfrak{s}$.*

Proof. Let I be a subspace of $\mathfrak{gl}(m)$ invariant under $\mathfrak{gl}(m)$ or $\mathfrak{sl}(m)$ and properly containing $\mathfrak{sl}(m) \cap \mathfrak{s}$. It suffices to show that I contains $\mathfrak{sl}(m)$.

If $e_{ij} \in I$ for some $i \neq j$ then (3) yields that all e_{kl}, with $k \neq l$, as well as all traceless diagonal matrices, are in I, so $\mathfrak{sl}(m) \subseteq I$. If some non-scalar diagonal matrix h is in I, then $h_i \neq h_j$ for some $i \neq j$, so $[h, e_{ij}] = (h_i - h_j)e_{ij} \in I$, and the first case applies. Suppose $x \in I$ and $x_{ij} \neq 0$ for some $i \neq j$. Then either $\ell \neq 2$, so $[e_{ji}, [e_{ji}, x]] = -2x_{ij}e_{ji}$, and the first case applies, or $m > 2$ and there exists $k \in \{1, \dots, n\} - \{i, j\}$, so $[e_{ki}, [e_{jk}, [e_{ji}, x]]] = x_{ij}e_{ji}$, and the first case apples. ■

Note 4.2. It is stated in [B], Chapter I, §6, Exercise 24(a), that bracketing any non-scalar element of $\mathfrak{gl}(m)$ with at most four suitable chosen elements produces a non-zero scalar multiple of one of the e_{ij}. The proof of Theorem 4.1 shows that three elements already suffice.

5. Viewing $\mathfrak{gl}(2n)$ as a module for $\mathfrak{sl}(n) \oplus \mathfrak{sl}(n)$ and $\mathfrak{sl}(n)$

If L_1, L_2 are Lie algebras then the vector space $L_1 \oplus L_2$ becomes a Lie algebra via $[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + [x_2, y_2]$ for $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$.

Let $W = \mathfrak{gl}(r + n)$ be the adjoint module for $\mathfrak{gl}(r + n)$. By means of the imbedding $\mathfrak{gl}(r) \oplus \mathfrak{gl}(n) \hookrightarrow \mathfrak{gl}(r + n)$, given by $a + b \mapsto a \oplus b$, we may view W as a module for $\mathfrak{gl}(r) \oplus \mathfrak{gl}(n)$. We have

$$\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & as - sb \\ 0 & 0 \end{pmatrix} \tag{14}$$

and

$$\left[\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ bt - ta & 0 \end{pmatrix}. \tag{15}$$

Thus $Z = M_{r \times n}(F)$ is a $\mathfrak{gl}(r) \oplus \mathfrak{gl}(n)$-module under the action

$$(a + b) \cdot s = as - sb, \quad a \in \mathfrak{gl}(r), b \in \mathfrak{gl}(n), s \in Z$$

and $A = M_{n \times r}(F)$ is a $\mathfrak{gl}(r) \oplus \mathfrak{gl}(n)$-module under the action

$$(a + b) \cdot t = bt - ta, \quad a \in \mathfrak{gl}(r), b \in \mathfrak{gl}(n), s \in A.$$

Theorem 5.1. *The map $\phi : A \to Z^*$, given by*

$$\phi_t(s) = \mathrm{tr}(ts), \quad t \in A, s \in Z,$$

is an isomorphism of $\mathfrak{gl}(r) \oplus \mathfrak{gl}(n)$-modules.

Proof. This is a linear isomorphism. Moreover, if $a + b \in \mathfrak{gl}(r) \oplus \mathfrak{gl}(n)$ then

$$\phi_{(a+b)\cdot t}(s) = \phi_{bt-ta}(s) = \mathrm{tr}(bts - tas),$$

$$((a + b) \cdot \phi_t)(s) = \phi_t(-(a + b) \cdot s) = \phi_t(-as + sb) = \mathrm{tr}(-tas + tsb). \qquad ■$$

We assume $r = n$ for the remainder of this section. By means of the imbedding $\mathfrak{gl}(n) \hookrightarrow \mathfrak{gl}(n) \oplus \mathfrak{gl}(n) \hookrightarrow \mathfrak{gl}(2n)$, given by $a \mapsto a \oplus -a'$, we may view W as a module for $\mathfrak{gl}(n)$. We have

$$\left[\begin{pmatrix} a & 0 \\ 0 & -a' \end{pmatrix}, \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & as + sa' \\ 0 & 0 \end{pmatrix} \tag{16}$$

and

$$\left[\begin{pmatrix} a & 0 \\ 0 & -a' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ -a't - ta & 0 \end{pmatrix}. \tag{17}$$

Thus $Z = \mathfrak{gl}(n)$ becomes a $\mathfrak{gl}(n)$-module under the action

$$a \cdot s = as + sa', \quad a \in \mathfrak{gl}(n),\ s \in Z$$

and $A = \mathfrak{gl}(n)$ becomes a $\mathfrak{gl}(n)$-module under the action

$$a \cdot t = -a't - ta, \quad a \in \mathfrak{gl}(n),\ t \in A.$$

This is nothing but the automorphism $a \mapsto -a'$ followed the previous action on Z.

Clearly the spaces of symmetric and alternating matrices are $\mathfrak{gl}(n)$-submodules of Z (resp. A), denoted by S and T (resp. B and C). Moreover, if $\ell \neq 2$ we have $Z = S \oplus T$ (resp. $A = B \oplus C$).

Proposition 5.2. *Suppose that $\ell \neq 2$. Then the map $\phi : A \to Z^*$ defined in Theorem 5.1 is an isomorphism of $\mathfrak{gl}(n)$-modules sending B onto S^* and C onto T^*.*

Proof. It follows from Theorem 5.1 that ϕ is an isomorphism of $\mathfrak{gl}(n)$-modules. Let $b \in B$ and suppose that $\phi_b(s) = 0$ for all $s \in S$. By (9) we also have $\phi_b(t) = 0$ for all $t \in T$. Since $Z = S \oplus T$, it follows that $b = 0$. Dimension considerations imply that ϕ sends B onto S^*. Likewise we show that ϕ sends C onto T^*. ■

Theorem 5.3. *Suppose $\ell \neq 2$ and $n = 4$. Then the map $h : T \to C$, given by*

$$s = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} \mapsto s^* = \begin{pmatrix} 0 & f & -e & d \\ -f & 0 & c & -b \\ e & -c & 0 & a \\ -d & b & -a & 0 \end{pmatrix},$$

is an isomorphism of $\mathfrak{sl}(4)$-modules. The composite map $T \to C \to T^*$, given by $s \mapsto \phi_{s^*}$, is an isomorphism of $\mathfrak{sl}(4)$-modules. The corresponding non-degenerate $\mathfrak{sl}(4)$-invariant bilinear form $g : T \times T \to F$, namely $g(s,t) = \mathrm{tr}(s^*t)$, is symmetric. Consequently, $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$ provided $F = F^2$.

Proof. Clearly h is a linear isomorphism. We easily verify that h commutes with the actions of e_{12}, e_{23}, e_{34} on T and C. In light of (3), the same happens to all e_{ij}, where $1 \leq i < j \leq 4$. Let $t \in T$. Then $t = s^*$ for $s = t^*$. If $1 \leq i < j \leq 4$

then $h(e_{ij} \cdot s) = e_{ij} \cdot h(s)$, i.e., $(e_{ij}s + se_{ji})^* = -(e_{ji}s^* + s^*e_{ij})$, which means $-(e_{ij}t^* + t^*e_{ji}) = (e_{ji}t + te_{ij})^*$, that is, $e_{ji} \cdot h(t) = h(e_{ji} \cdot t)$. Using (3) once more yields that f commutes with the action of all $x \in \mathfrak{sl}(4)$. The symmetry of g is easily verified.

We know from Theorem 4.1 that $\mathfrak{sl}(4)$ is simple and it is clear from (16) that $\mathfrak{sl}(4)$ does not act trivially on T. Thus, as explained in §2, g yields an imbedding $\mathfrak{sl}(4) \hookrightarrow \mathfrak{so}(6)$, which is an isomorphism since they are both of dimension 15. ■

Note that if $n = 1$ then $T = 0$ and if $n = 2$ then T is the trivial $\mathfrak{sl}(2)$-module. We assume $n \geq 2$ for the remainder of this section.

Theorem 5.4. *Suppose $\ell \neq 2$. If $n \neq 2, 4$ then T is not a self-dual $\mathfrak{sl}(n)$-module.*

Proof. Let H be the space of diagonal matrices $a \oplus -a'$ with $a \in \mathfrak{sl}(n)$. Let $\varepsilon_i : H \to F$ the ith coordinate function, $h \mapsto h_{ii}$, for $1 \leq i \leq n$. Observe that

$$a_1\varepsilon_1 + \cdots + a_n\varepsilon_n = 0 \Leftrightarrow a_1 = \cdots = a_n. \tag{18}$$

The eigenvalues of H acting on T can explicitly computed from (16). They are $\varepsilon_i + \varepsilon_j$, where $1 \leq i < j \leq n$. On the other hand, (17) shows that the eigenvalues of H acting on C are $-(\varepsilon_p + \varepsilon_q)$, where $1 \leq p < q \leq n$. Thus the sets of eigenvectors for the actions of H on T are C are disjoint if $n \neq 2, 4$ by (18). It follows from Proposition 5.2 that $T \ncong T^*$. ■

Theorem 5.5. *Both C and T are irreducible $\mathfrak{sl}(n)$-modules.*

Proof. Since $a \mapsto -a'$ is an automorphism of $\mathfrak{sl}(n)$, it is clear that C is irreducible if and only if so is T. We next verify that T is irreducible.

Let $a \in \mathfrak{sl}(n)$ and set $b = a'$. Suppose $t \in T$. Then

$$a \cdot t = at + ta' = at - (at)' = tb - (tb)'.$$

Thus the $\mathfrak{sl}(n)$-module generated by t contains all matrices obtained from t by arbitrary left and right multiplication by $\mathfrak{sl}(n)$ followed by "alternation". By doing this we can easily pass from any $t \neq 0$ to $e_{12} - e_{21}$ and from there to any $e_{ij} - e_{ij}$. ■

Theorem 5.6. *Suppose $\ell \neq 2$. Then both B and S are irreducible $\mathfrak{sl}(n)$-modules.*

Proof. By Proposition 5.2 we have $B \cong S^*$, so B is irreducible if and only if so is S. We next verify that S is irreducible.

Let $a \in \mathfrak{sl}(n)$ and set $b = a'$. Suppose $s \in S$. Then

$$a \cdot s = as + (as)' = sb + (sb)'.$$

Thus the $\mathfrak{sl}(n)$-module generated by s contains all matrices obtained from s by arbitrary left and right multiplication followed by "symmetrization". By doing this we can easily pass from any $s \neq 0$ to e_{11} and from there to any $e_{ij} + e_{ij}$. ■

Note 5.7. Let $\mathrm{Bil}(V)$ be the vector space all bilinear forms $\beta : V \times V \to F$. Then $\mathrm{Bil}(V)$ becomes a $\mathfrak{gl}(V)$-module via

$$(x \cdot \beta)(u,v) = -\beta(xu,v) - \beta(u,xv). \tag{19}$$

Given $\beta \in \mathrm{Bil}(V)$, the subalgebra of all $x \in \mathfrak{gl}(V)$ such that $x \cdot \beta = 0$ is just $L(\beta)$.

Let $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$ be the subspaces of symmetric and alternating bilinear forms on V. Clearly $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$ are $\mathfrak{gl}(V)$-submodules of $\mathrm{Bil}(V)$. We have canonical $\mathfrak{gl}(V)$-isomorphisms

$$\mathrm{Bil}(V) \cong (V \otimes V)^* \cong V^* \otimes V^*,$$

mapping $\mathrm{Sym}(V)$ onto the symmetric square $S^2(V^*)$ and $\mathrm{Alt}(V)$ onto the exterior square $\Lambda^2(V^*)$. Now (17) and (19) make it clear that $\mathrm{Bil}(V) \cong A$. Thus

$$B \cong S^2(V^*), C \cong \Lambda^2(V^*),$$

and, if $\ell \neq 2$, then

$$S \cong S^2(V), T \cong \Lambda^2(V).$$

Regardless of ℓ, if $n = 4$ we consider the map $\Lambda^2(V) \times \Lambda^2(V) \to \Lambda^4(V)$ given by

$$(v_1 \wedge v_2, v_3 \wedge v_4) \mapsto v_1 \wedge v_2 \wedge v_3 \wedge v_4. \tag{20}$$

Since $\dim \Lambda^4(V) = 1$, (20) yields a non-degenerate $\mathfrak{sl}(V)$-invariant symmetric bilinear form on $\Lambda^2(V)$. This form is alternating if $\ell = 2$. Thus, if $F = F^2$ we obtain an isomorphism $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$ when $\ell \neq 2$ and an imbedding $\mathfrak{sl}(4) \hookrightarrow \mathfrak{sp}(6)$ if $\ell = 2$.

6. Basic Properties of $M(f)$

Recall the definition of the bilinear form $\varphi : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to F$ given in (7).

Lemma 6.1. *If L is a Lie subalgebra of $\mathfrak{gl}(V)$ and M is an L-submodule of $\mathfrak{gl}(V)$ then*

$$M^\perp = \{x \in \mathfrak{gl}(V) \mid \varphi(M,x) = 0\}$$

is an L-submodule of $\mathfrak{gl}(V)$.

Proof. Let $z \in L$, $x \in M^\perp$ and $y \in M$. Then, by Lemma 2.2, we have

$$\varphi([zx], y) = -\varphi(x, [zy]) = 0.$$

■

Corollary 6.2. *The space $L(f)^\perp$ is an $L(f)$-submodule of $\mathfrak{gl}(V)$.*

Proof. This follows from Lemmas 2.1 and 6.1. ■

For a subspace U of V we define

$$L(U) = \{v \in V \mid f(v,U) = 0\} \text{ and } R(U) = \{v \in V \mid f(U,v) = 0\}.$$

Note that f is non-degenerate if $L(V) = 0 = R(V)$. We set $Rad(f) = L(V) \cap R(V)$.

Lemma 6.3. *(a) If $\ell = 2$ then $L(f) = M(f)$.*

(b) If $\ell \neq 2$ then $L(f) \cap M(f)$ is the ideal of $L(f)$ of all endomorphisms x of V that satisfy $xV \subseteq Rad(f)$.

(c) If $\ell \neq 2$ and $Rad(f) = 0$ then $L(f) \cap M(f) = 0$.

(d) If $\ell \neq 2$ and f is non-degenerate then $L(f) \cap M(f) = 0$.

Proof. (a) This is obvious.

(b) Let $x \in L(f) \cap M(f)$ and $v, w \in V$. Then

$$-f(v, xw) = f(xv, w) = f(v, xw) \text{ and } -f(xv, w) = f(v, xw) = f(xv, w)$$

so

$$2f(v, xw) = 0 \text{ and } 2f(xv, w) = 0.$$

If $\ell \neq 2$ then $xV \subseteq L(V) \cap R(V) = Rad(f)$. As the intersection of $L(f)$-submodules of $\mathfrak{gl}(V)$, we see that $L(f) \cap M(f)$ is an ideal of $L(f)$. Moreover, any endomorphism x of V that sends V to $Rad(f)$ is clearly in $L(f) \cap M(f)$.

(c) This follows from (b).

(d) This follows from (c). ∎

Lemma 6.4. *If f is non-degenerate and $\ell \neq 2$ then $M(f) \subseteq L(f)^{\perp}$.*

Proof. Let $\mathcal{B}$ be a basis of V and let A be the Gram matrix of f relative to $\mathcal{B}$. Let $x \in L(f)$ and $y \in M(f)$ have respective matrices $X, Y \in \mathfrak{gl}(m)$ relative to $\mathcal{B}$. Then

$$X'A = -AX, \quad Y'A = AY.$$

Since f is non-degenerate, A is invertible, whence

$$X = -A^{-1}X'A, \quad Y = A^{-1}Y'A. \tag{21}$$

It follows that

$$XY = (-A^{-1}X'A)(A^{-1}Y'A) = -A^{-1}X'Y'A.$$

Taking traces yields

$$\mathrm{tr}(XY) = -\mathrm{tr}(X'Y') = -\mathrm{tr}(Y'X') = -\mathrm{tr}((XY)') = -\mathrm{tr}(XY).$$

Therefore $2\mathrm{tr}(XY) = 0$. Since $\ell \neq 2$, we infer $\mathrm{tr}(XY) = 0$. ∎

Suppose that f is non-degenerate. Then given, $x \in \mathfrak{gl}(V)$, there exists a unique $x^* \in \mathfrak{gl}(V)$, the adjoint of x, satisfying

$$f(xv, w) = f(v, x^*w), \quad v, w \in V.$$

In matrix terms, if $\mathcal{B}$ is a basis of V and A, X, X^* are the matrices of f, x, x^*, then

$$X'A = AX^*,$$

which has the unique solution

$$X^* = A^{-1}X'A.$$

Observe that $x \in L(f) \Leftrightarrow x^* = -x$ and $y \in M(f) \Leftrightarrow y^* = y$.

Lemma 6.5. *Suppose f is non-degenerate as well as symmetric or skew-symmetric. Then*

$$x^{**} = x, \quad x \in \mathfrak{gl}(V).$$

Proof. Let $v, w \in V$. Then

$$f(xv, w) = f(v, x^*w) = \pm f(x^*w, v) = \pm f(w, x^{**}v) = (\pm 1)^2 f(x^{**}v, w). \qquad \blacksquare$$

Lemma 6.6. *Suppose f is non-degenerate, symmetric or skew-symmetric, and $\ell \neq 2$. Then*

$$\mathfrak{gl}(V) = L(f) \oplus M(f).$$

Proof. Given $z \in \mathfrak{gl}(V)$ let $x = (z - z^*)/2$ and $y = (z + z^*)/2$. Then $z = x + y$. Moreover, by Lemma 6.5, $x \in L(f)$ and $y \in M(f)$. Furthermore, $L(f) \cap M(f) = 0$ by Lemma 6.3, as required. ■

Corollary 6.7. *Suppose f is non-degenerate, symmetric or skew-symmetric, and $\ell \neq 2$. Then*

$$\mathfrak{gl}(V) = L(f) \perp M(f).$$

Proof. Since φ is non-degenerate, this follows from Lemmas 6.4 and 6.6. ■

Lemma 6.8. *Suppose f is non-degenerate and $\ell \neq 2$. Then $L(f) \subseteq \mathfrak{sl}(V)$.*

Proof. By definition $\mathfrak{s} \subseteq M(f)$ and by Lemma 6.4 we have $M(f) \subseteq L(f)^\perp$. Hence $\mathfrak{s} \subseteq L(f)^\perp$. Since φ is non-degenerate, this yields

$$L(f) = L(f)^{\perp\perp} \subseteq \mathfrak{s}^\perp = \mathfrak{sl}(V).$$

Alternatively, let $x \in L(f)$ and take traces in (21) to get $\mathrm{tr}(x) = -\mathrm{tr}(x)$. ■

Since $\mathfrak{sl}(V)$ is an ideal of $\mathfrak{gl}(V)$ it follows from Lemma 2.1 that $M(f) \cap \mathfrak{sl}(V)$ is an $L(f)$-submodule of $\mathfrak{gl}(V)$.

Corollary 6.9. *Suppose f is non-degenerate, symmetric or skew-symmetric, and $\ell \nmid 2m$. Then $\mathfrak{gl}(V)$ has the following decomposition into perpendicular $L(f)$-submodules:*

$$\mathfrak{gl}(V) = L(f) \perp (M(f) \cap \mathfrak{sl}(V)) \perp \mathfrak{s}.$$

Proof. If we had $M(f) \subseteq \mathfrak{sl}(V)$ then $\ell \neq 2$ together with Lemmas 6.6 and 6.8 would imply $\mathfrak{gl}(V) \subseteq \mathfrak{sl}(V)$, which is impossible. It follows that $M(f) \cap \mathfrak{sl}(V)$ is a hyperplane of $M(f)$. Since $\ell \nmid m$, then $\mathfrak{s} \cap M(f) \cap \mathfrak{sl}(V) = 0$, with $\mathfrak{s}$ and $M(f) \cap \mathfrak{sl}(V)$ perpendicular to each other, so $M(f) = (M(f) \cap \mathfrak{sl}(V)) \perp \mathfrak{s}$. Replacing this in Corollary 6.7 yields the desired result. ■

7. Multiple identifications

Let U and W be L-modules for a Lie algebra L. Then $U \otimes W$ becomes an L-module via

$$x \cdot (u \otimes v) = x \cdot u \otimes v + u \otimes x \cdot v.$$

We may view the symmetric and exterior squares $S^2(U)$ and $\Lambda^2(U)$ as L-submodules of $U \otimes U$. If $\ell = 2$ then $\Lambda^2(U) \subseteq S^2(U)$, while if $\ell \neq 2$ then $U \otimes U = \Lambda^2(U) \oplus S^2(U)$.

Suppose f is non-degenerate. Since f is $L(f)$-invariant, the map $\Gamma_1 : V \to V^*$ induced by f, namely

$$\Gamma_1(v) = f(v, -),$$

is an isomorphism of $L(f)$-modules. This, in turn, yields the $L(f)$-isomorphism $\Gamma_1 \otimes 1_V : V \otimes V \to V^* \otimes V$. On the other hand, we have the natural $\mathfrak{gl}(V)$-isomorphism $\Gamma_2 : V^* \otimes V \to \mathfrak{gl}(V)$, given by

$$\Gamma_2(\delta \otimes w)(u) = \delta(u)w, \quad \delta \in V^*,\ u, w \in V.$$

It follows that $\Gamma = \Gamma_2 \circ (\Gamma_1 \otimes 1_V)$ is an $L(f)$-isomorphism $V \otimes V \to \mathfrak{gl}(V)$, given by

$$\Gamma(v \otimes w)(u) = f(v, u)w, \quad u, v, w \in V.$$

We wonder what are the $L(f)$-submodules of $V \otimes V$ corresponding to $L(f)$ and $M(f)$ under Γ. If f is symmetric or skew-symmetric, the answer is as follows.

Proposition 7.1. *Suppose that f is non-degenerate.*

(a) *If $\ell \neq 2$ and f is skew-symmetric, then Γ sends $\Lambda^2(V)$ onto $M(f)$ and $S^2(V)$ onto $L(f)$.*

(b) *If $\ell \neq 2$ and f is symmetric, then Γ sends $\Lambda^2(V)$ onto $L(f)$ and $S^2(V)$ onto $M(f)$.*

(c) *If $\ell = 2$ and f is symmetric then Γ sends $S^2(V)$ onto $L(f) = M(f)$.*

Proof. The dimensions of $L(f)$ and $M(f)$ were computed in 2. On the other hand, it is well-known that

$$\dim S^2(V) = \binom{m+1}{2} \text{ and } \dim \Lambda^2(V) = \binom{m}{2}.$$

Furthermore, given $u_1, u_2, v, w \in V$, we have

$$f(\Gamma(v \otimes w)u_1, u_2) = f(v, u_1)f(w, u_2),$$

$$f(u_1, \Gamma(v \otimes w)u_2) = f(u_1, w)f(v, u_2).$$

Since $S^2(V)$ is spanned by all $v \otimes v$ and $v \otimes w + w \otimes v$, and $\Lambda^2(V)$ by all $v \otimes w - w \otimes v$, the above information combines to yield the desired result. ■

Theorem 7.2. *Suppose that $\ell \neq 2$, that $m = 2n$, and that f is non-degenerate and skew-symmetric. Then the following $L(f)$-modules are isomorphic:*

(1) $M(f) = \{y \in \mathfrak{gl}(V) \mid f(yv, w) = f(v, yw) \text{ for all } v, w \in V\}$.

(2) $\Lambda^2(V)$*, the second exterior power of the natural* $L(f)$*-module* V*.*

(3) $L(f)^{\perp}$*, the orthogonal complement of* $L(f)$ *relative to the bilinear form (7).*

(4) The space of all

$$\begin{pmatrix} a & b \\ c & a' \end{pmatrix} \in \mathfrak{gl}(2n) \tag{22}$$

such that $a \in \mathfrak{gl}(n)$ *and* $b, c \in \mathfrak{gl}(n)$ *are skew-symmetric.*

Proof. We know from Proposition 7.1 that $M(f) \cong \Lambda^2(V)$, while Corollary 6.7 shows that $M(f) = L(f)^{\perp}$ relative to φ. The matrix description of $M(f)$ is taken from §2. ∎

In a similar manner, we derive the following result.

Theorem 7.3. *Suppose that* $\ell \neq 2$*,* f *is non-degenerate and symmetric. Then the following* $L(f)$*-modules are isomorphic:*

(1) $M(f) = \{y \in \mathfrak{gl}(V) \mid f(yv, w) = f(v, yw) \text{ for all } v, w \in V\}$*.*

(2) $S^2(V)$*, the second symmetric power of the natural* $L(f)$*-module* V*.*

(3) $L(f)^{\perp}$*, the orthogonal complement of* $L(f)$ *relative to the bilinear form (7).*

(4) The space of all $A \in \mathfrak{gl}(m)$ *satisfying* $d_i A_{ij} - d_j A_{ji} = 0$ *for all* $1 \leq i, j \leq m$*, where* $D = \mathrm{diag}(d_1, \dots, d_m)$ *is the Gram matrix of* f *relative to a basis of* V*.*

In particular, if $F = F^2$ *then* $M(f)$ *is isomorphic to the* $L(f)$*-module of all symmetric* $m \times m$ *matrices.*

Lemma 7.4. *The contraction map* $\Omega : V \otimes V \to F$*, given by* $v \otimes w \mapsto f(v, w)$*, is an* $L(f)$*-homomorphism, which is surjective if and only if* $f \neq 0$*.*

Proof. This is an easy calculation. ∎

Lemma 7.5. *Suppose that* f *is non-degenerate. Then* Ω *can be identified with the trace map* $\mathrm{tr} : \mathfrak{gl}(V) \to F$*, in the sense that* $\Omega(v \otimes w) = \mathrm{tr}(\Gamma(v \otimes w))$ *for all* $v, w \in V$*.*

Proof. Let $v_1, \dots, v_m$ be a basis of V and let $w_1, \dots, w_m$ be the dual basis of V relative to f, i.e., such that $f(v_i, w_j) = \delta_{ij}$. Then $\Gamma(v_i \otimes w_j)(w_k) = \delta_{ik} w_j$, so

$$\mathrm{tr}(\Gamma(v_i \otimes w_j)) = \delta_{ij} = f(v_i, w_j) = \Omega(v_i \otimes w_j),$$

which implies the result for all $v, w \in V$. ∎

Corollary 7.6. *Suppose that* $\ell \neq 2$ *and* f *is non-degenerate.*

(a) If f *is skew-symmetric then* $\Lambda^2(V)$ *is not contained in the kernel of* Ω*, so* $\Lambda^2(V) \cap \ker \Omega$ *is an* $L(f)$*-submodule of* $\Lambda^2(V)$ *of codimension 1, which*

corresponds to $M(f)\cap\mathfrak{sl}(V)$ *under* Γ*, and in matrix form to all matrices (22) with* $a\in\mathfrak{sl}(n)$*.*

(b) If f *is symmetric then* $S^2(V)$ *is not contained in the kernel of* Ω*, therefore* $S^2(V)\cap\ker\Omega$ *is an* $L(f)$*-submodule of* $S^2(V)$ *of codimension 1, which corresponds to* $M(f)\cap\mathfrak{sl}(V)$ *under* Γ*, and in matrix form to all* $A\in\mathfrak{gl}(m)$ *as described in Theorem 7.3 satisfying* $\mathrm{tr}(A)=0$*.*

(c) If f *is symmetric or skew-symmetric then* $\mathfrak{s}$ *is contained in* $M(f)\cap\mathfrak{sl}(V)$ *if and only if* $\ell|m$*.*

Lemma 7.7. *The map* $V\otimes V\to\Lambda^2(V)$ *given by*

$$v\otimes w\mapsto v\otimes w-w\otimes v$$

is an epimorphism of $\mathfrak{gl}(V)$*-modules with kernel* $S^2(V)$*.*

Proof. This is clear. ∎

Theorem 7.8. *Suppose that* $\ell=2$ *and* f *is non-degenerate and symmetric. Then*

(1) Γ *maps the* $L(f)$*-submodule* $\Lambda^2(V)$ *of* $S^2(V)$ *onto* $L(f)^{(1)}=[L(f),L(f)]$*, an ideal of* $L(f)$ *of codimension* m*.*

(2) $\mathfrak{gl}(V)/L(f)\cong L(f)^{(1)}$ *as* $L(f)$*-modules.*

(3) Suppose f *is alternating. Then* $m=2n$ *and there is a basis* $\mathcal{B}$ *of* V *relative to which* f *has Gram matrix* J*, as defined in (6).*

Moreover, $L(J)$ *(resp.* $L(J)^{(1)}$*) consists of all matrices*

$$\begin{pmatrix} A & B\\ C & A'\end{pmatrix} \tag{23}$$

such that $A,B,C\in\mathfrak{gl}(n)$ *and* B,C *are symmetric (resp. alternating).*

Furthermore, $L(J)^{(2)}$ *consists of all matrices (23) such that* B,C *are alternating and* $\mathrm{tr}(A)=0$*. In particular,* $L(J)^{(2)}$ *has codimension 1 in* $L(J)^{(1)}$*.*

(4) Suppose f *is alternating and let* $\mathcal{B}$ *and* J *be as above. Let* $\Delta:\Lambda^2(V)\to F$ *be the* $L(f)$*-epimorphism given by*

$$v\wedge w=v\otimes w+w\otimes v\mapsto f(v,w).$$

Then $L(J)^{(2)}$ *corresponds to* $\ker\Delta$ *under the* $L(f)$*-isomorphism* $M_{\mathcal{B}}\circ\Gamma$*.*

(5) Suppose f *is non-alternating. Then there is a basis* $\mathcal{B}$ *of* V *relative to which* f *has Gram matrix*

$$D=\mathrm{diag}(d_1,\dots,d_m),\quad 0\neq d_i\in F.$$

Moreover, $L(D)$ *(resp.* $L(D)^{(1)}$*) consists of all* $A\in\mathfrak{gl}(m)$ *such that* $d_iA_{ij}=d_jA_{ji}$ *(resp.* $d_iA_{ij}=d_jA_{ji}$ *and* $A_{ii}=0$*). Furthermore,* $L(f)^{(1)}$ *is perfect provided* $m\neq 2$*. In particular, if* $D=I_m$*, which occurs when* $F=F^2$*, then* $L(D)$ *(resp.* $L(D)^{(1)}$*) consists of all symmetric (resp. alternating) matrices in* $gl(m)$*.*

(6) The space $\mathfrak{s}$ *is contained in* $L(f)^{(1)}$ *if and only if* f *is alternating, in which case* $\mathfrak{s}$ *is contained in* $L(f)^{(2)}$ *if and only if* $4|m$*.*

Proof. By Proposition 7.1, the $L(f)$-isomorphism $\Gamma : V \otimes V \to \mathfrak{gl}(V)$ sends $S^2(V)$ onto $L(f)$. Therefore,

$$\Gamma(L(f) \cdot S^2(V)) = L(f) \cdot L(f) = L(f)^{(1)}.$$

One the other hand, we easily verify that

$$L(f) \cdot S^2(V) \subseteq \Lambda^2(V),$$

which yields

$$\Gamma(\Lambda^2(V)) \supseteq L(f)^{(1)}. \tag{24}$$

Since $\Lambda^2(V)$ has codimension m in $S^2(V)$, in order to show that equality prevails in (24) it suffices to show that $L(f)^{(1)}$ has codimension m in $L(f)$.

Suppose first that f is non-alternating. By [K], Theorem 20, there is a basis $\mathcal{B}$ of V relative to which f has Gram matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$, where $0 \neq d_i \in F$. We easily see that $L(D)$ consists of all $A \in \mathfrak{gl}(m)$ such that $d_i A_{ij} = d_j A_{ji}$, with the following multiplication table:

$$[e_{ii}, d_j e_{ij} + d_i e_{ji}] = d_j e_{ij} + d_i e_{ji},$$

$$[d_j e_{ij} + d_i e_{ji}, d_k e_{ik} + d_i e_{ki}] = d_i(d_k e_{jk} + d_j e_{kj}).$$

This proves (5) and completes the verification of (24) when f is non-alternating.

Suppose next that f is alternating. By [K], Theorem 19, $m = 2n$ and there is a basis $\mathcal{B}$ of V relative to which f has Gram matrix J, as defined in (6). As seen in §2, $L(J)$ consists of the stated matrices. Moreover, we have

$$[\mathfrak{gl}(n), \mathfrak{gl}(n)] = \mathfrak{sl}(n), \tag{25}$$

$$\left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right] = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}, \tag{26}$$

$$\left[\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & AB + (AB)' \\ 0 & 0 \end{pmatrix}, \tag{27}$$

$$\left[\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 \\ CA + (CA)' & 0 \end{pmatrix}. \tag{28}$$

Combining (25) and (26) with $B = I_n$ and $C = \mathrm{diag}(1, 0, \ldots, 0)$ we deduce

$$\begin{pmatrix} A & 0 \\ 0 & A' \end{pmatrix} \in L(J)^{(1)}, \quad A \in \mathfrak{gl}(n).$$

As seen in §2, the map $\mathfrak{gl}(n) \to \mathrm{Alt}(n)$, given by $A \mapsto A + A'$, is surjective. This, together with (27) and (28) applied to the special case $B = I_n = C$, imply that for all alternating matrices $B, C \in \mathfrak{gl}(n)$, we have

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in L(J)^{(1)}.$$

It now follows from (25)-(28) that $L(J)^{(1)}$ consists of all matrices (23) such that B, C are alternating. This proves the first two statements of (3) and completes the proof of (1).

Using (27) and (28) with $A = e_{ii}$ and $B = e_{ij} + e_{ji} = C$, where $i \neq j$, we see that for all alternating matrices $B, C \in \mathfrak{gl}(n)$, we have

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \in L(J)^{(2)}.$$

Let $B, C \in \mathfrak{gl}(n)$ be alternating. We infer from (9) that

$$\mathrm{tr}(BC) = 0.$$

It now follows from (25)-(28) that $L(J)^{(2)}$ consists of all matrices (23) such that B, C are alternating and $\mathrm{tr}(A) = 0$, which completes the proof of (3).

Suppose still that f is alternating. Then f induces a linear map $\Lambda^2(V) \to F$, namely Δ. It is clear that Δ is an $L(f)$-epimorphism. Let

$$\mathcal{B} = \{v_1, \ldots, v_n, w_1, \ldots, w_n\}$$

be the given basis V, relative to which f has Gram matrix J. Then $M_{\mathcal{B}} \circ \Gamma$ satisfies:

$$v_i \wedge v_j \mapsto \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix}, w_i \wedge w_j \mapsto \begin{pmatrix} 0 & 0 \\ e_{ij} + e_{ji} & 0 \end{pmatrix}, v_i \wedge w_j \mapsto \begin{pmatrix} e_{ij} & 0 \\ 0 & e_{ji} \end{pmatrix}.$$

This, (1) and (3) show that $M_{\mathcal{B}} \circ \Gamma$ sends $\ker \Delta$ onto $L(J)^{(2)}$, which is (4).

Now (3) and (5) yield the first part of (6), while (3) gives the second. Finally, we may apply (1), Proposition 7.1 and Lemma 7.7 to derive (2). ■

Note 7.9. Suppose that $\ell = 2$ and f is non-degenerate and alternating. There are two mistakes in [B], Chapter I, §6, Exercise 25(b). In their notation, it is claimed that $\mathfrak{b} = L(f)^{(1)}$, when in fact $\mathfrak{a} = L(f)^{(1)}$ and $\mathfrak{b} = L(f)^{(2)}$. This holds for any even m, while their claim was made for $m \geq 6$. It is also claimed that if $m \geq 6$ then $\mathfrak{b}/\mathfrak{c}$ is simple, when in fact $\mathfrak{c} = \mathfrak{s}$ is only included in $\mathfrak{b} = L(f)^{(2)}$ when $4|m$.

8. The $L(f)$-module $L(f)^{(1)}$ when f is symmetric and non-alternating

We assume throughout this section that $m \geq 3$ and let $L = L(I_m) = \mathfrak{so}(m)$.

On the one hand, if $\ell \neq 2$ then L consists of all skew-symmetric matrices and $L^{(1)} = L$. On the other hand, when $\ell = 2$, L is the set of all symmetric matrices and $L^{(1)}$ consists of all alternating matrices. In any characteristic, the derived algebra $L^{(1)}$ is spanned by the matrices E_{ij} with $i < j$, where E_{ij} is defined as $e_{ij} - e_{ji}$ for any i, j. The following multiplication rules can be easily verified.

- $[E_{ij}, E_{jk}] = E_{ik}$ for $i \neq j$ and $j \neq k$;

- $[E_{ij}, E_{rs}] = 0$ if i, j, r, s are all different to each other.

Theorem 8.1. *If $m = 3$ or $m \geq 5$ then $L^{(1)}$ is a simple Lie algebra. If $m \geq 3$ and $\ell = 2$ then $L^{(1)}$ is an irreducible L-module.*

Proof. When $m = 3$, $L^{(1)}$ has basis $\{E_{12}, E_{23}, E_{31}\}$ with multiplication table given by $[E_{ij}, E_{jk}] = E_{ik}$ for $\{i, j, k\} = \{1, 2, 3\}$. So $L^{(1)}$ is 3-dimensional and perfect, and therefore simple. Assume henceforth that $m \geq 4$.

Suppose first that I is an ideal of $L^{(1)}$ such that $E_{ij} \in I$ for some $i \neq j$. Let $r \neq s$ be indices such that $\{i, j\} \neq \{r, s\}$. If $\{i, j\} \cap \{r, s\} = \emptyset$ or $j = r$, then

$$[E_{ri}, [E_{ij}, E_{js}]] = E_{rs} \in I.$$

So $I = L^{(1)}$ if I contains a basis element.

Now suppose $m \geq 5$ and let I be a nonzero ideal of $L^{(1)}$. Let $x \in I$ with $x_{ij} \neq 0$ for some $i \neq j$. As $m \geq 5$, we can pick indices r, s, t such that $|\{i, j, r, s, t\}| = 5$. Since

$$(\mathrm{ad}E_{ts} \circ \mathrm{ad}E_{rs} \circ \mathrm{ad}E_{ji} \circ \mathrm{ad}E_{rj})(x) = x_{ij}E_{tj},$$

we deduce $E_{tj} \in I$. Thus $I = L^{(1)}$.

Finally suppose $m \geq 4$ and $\ell = 2$, and let W be a nonzero L-submodule of $L^{(1)}$. Let $x \in W$ with $x_{ij} \neq 0$ for some $i \neq j$. Let r, s be indices such that $|\{i, j, r, s\}| = 4$. As

$$(\mathrm{ad}E_{rs} \circ \mathrm{ad}E_{jr} \circ \mathrm{ad}e_{ii})(x) = x_{ij}E_{is},$$

we have $E_{is} \in W$. Since W is in particular an ideal of $L^{(1)}$, we infer $W = L^{(1)}$. ∎

9. A composition series of the $\mathfrak{so}(m)$-module $\mathfrak{gl}(m)$ when $\ell \neq 2$

We suppose throughout this section that $\ell \neq 2$, that $m \geq 2$, and that f is non-degenerate and symmetric. We further assume that $L = L(I_m) = \mathfrak{so}(m)$.

Note that L consists of all skew-symmetric matrices and is spanned by the matrices E_{ij} with $i < j$, as defined in §8.

Proposition 9.1. *If $m = 3$ or $m \geq 5$ then $L(f)$ is a simple Lie algebra.*

Proof. This follows from Theorem 8.1, extending scalars if necessary. ∎

Note 9.2. Suppose that $m = 4$. Then $L(f)$ is 6-dimensional. Moreover, $L(f)$ is a simple Lie algebra if the discriminant of f is not a square in F, and the direct sum of two 3-dimensional simple ideals otherwise.

This can be found in [B], Chapter I, §6, Exercise 26(b). An alternative approach via current Lie algebras, independent of whether $\ell \neq 2$ or not, can be found in [CS].

Note 9.3. If $m = 2$ then $L(f)$ is 1-dimensional.

Consider next the L-module $M = M(I_m)$ consisting of all symmetric matrices. This module has basis $\{A_{ij} : i \le j\}$, where A_{ij} is defined as $e_{ij} + e_{ji}$ for all i, j. The matrices E_{rs} act on the A_{ij} according to the following rules.

- $[E_{ij}, A_{ij}] = A_{ii} - A_{jj}$;
- $[E_{ij}, A_{jj}] = 2A_{ij}$ for $i \neq j$;
- $[E_{ij}, A_{jk}] = A_{ik}$ if $\{i, j, k\}$ has size 3;
- $[E_{ij}, A_{rs}] = 0$ if $\{i, j, r, s\}$ has size 4.

Theorem 9.4. *Suppose $m \ge 4$. Let $M^0 = M(I_m) \cap \mathfrak{sl}(m)$. Then:*

(1) *If $\ell \nmid m$, then M^0 is an irreducible L-module.*

(2) *If $\ell | m$, then $\mathfrak{s}$ is the only non-trivial L-submodule of M^0, so $M^0/\mathfrak{s}$ is an irreducible L-submodule.*

Proof. Let $W \neq 0$ be an L-submodule of M^0. Suppose first that W consists only of diagonal matrices and let $0 \neq h \in W$. If $i \neq j$ then

$$[E_{ij}, h] = (h_{jj} - h_{ii})A_{ij} \in W,$$

whence $h_{ii} = h_{jj}$, i.e., h is scalar. This implies $W = \mathfrak{s}$ in the case $\ell | m$, and is a contradiction in the case $\ell \nmid m$.

Now suppose $A_{ij} \in W$ for some $i \neq j$. Let r, s be distinct indices such that $\{i, j\} \neq \{r, s\}$. If $j = r$ then $[E_{si}, A_{ij}] = A_{sr} \in W$, whereas $\{i, j\} \cap \{r, s\} = \emptyset$ implies $[E_{sj}, [E_{ri}, A_{ij}]] = A_{sr} \in W$. In addition $[E_{sr}, A_{sr}] = A_{ss} - A_{rr} \in W$. Therefore $W = M^0$ if W contains a basis element A_{ij} with $i \neq j$.

Finally, suppose that W contains a non-diagonal matrix x. So $x_{ij} \neq 0$ for some $i \neq j$. As $m \ge 4$, we can find indices r, s such that $\{i, j, r, s\}$ has size 4. Since

$$(\mathrm{ad}E_{ir} \circ \mathrm{ad}E_{rs} \circ \mathrm{ad}E_{ir} \circ \mathrm{ad}E_{ij} \circ \mathrm{ad}E_{rs} \circ \mathrm{ad}E_{ir})(x) = -2x_{ij}A_{ir},$$

we deduce $A_{ir} \in W$, which implies $W = M^0$. ■

Corollary 9.5. *Suppose that $m \ge 4$. Then*

(1) If ℓ does not divide m then $M(f) \cap \mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2} - 1$.

(2) If ℓ divides m then $M(f) \cap \mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2} - 2$.

Proof. The stated dimensions are clear from the matrix version of $M(f)$. Irreducibility follows from Theorem 9.4, extending scalars if necessary. ■

Note 9.6. Suppose that $\ell \nmid m$ and $m > 2$. Then Corollaries 7.6 and 9.5 show that the traceless matrices $A \in \mathfrak{gl}(m)$ described in part (4) of Theorem 7.3 form an irreducible $L(f)$-module of dimension $\binom{m+1}{2} - 1$, isomorphic to the kernel of the contraction map $S^2(V) \to F$ given by $v \otimes w + w \otimes v \mapsto f(v, w)$.

When $F=\mathbb{C}$ and $m>4$ this gives an elementary matrix description of $V(2\lambda_1)$, where λ_1 is the first fundamental weight of the orthogonal Lie algebra $\mathfrak{so}(m)$, as the space of all traceless $m\times m$ symmetric matrices.

Note 9.7. Suppose that $m=3$. If $\ell\neq 3$ then M^0 is an irreducible L-module. Now suppose $\ell=3$. If -1 is not a square in F, then $M^0/\mathfrak{s}$ is irreducible. However, if -1 is a square in F, say $i^2=-1$, the L-submodules of M^0 are X, Y and $\mathfrak{s}=X\cap Y$, where

$$X=\mathrm{span}\left\{I_3,\begin{pmatrix}0&0&0\\0&1&i\\0&i&-1\end{pmatrix},\begin{pmatrix}0&i&-1\\i&0&0\\-1&0&0\end{pmatrix}\right\},$$

$$Y=\mathrm{span}\left\{I_3,\begin{pmatrix}0&0&0\\0&-1&i\\0&i&1\end{pmatrix},\begin{pmatrix}0&i&1\\i&0&0\\1&0&0\end{pmatrix}\right\}.$$

Note 9.8. Suppose $m=2$. If -1 is not a square in F then M^0 is irreducible as a module over L. If -1 is a square in F, say $i^2=-1$, the only L-submodules of M^0 are Fx and Fy, where

$$x=\begin{pmatrix}1&i\\i&-1\end{pmatrix}\quad\text{and}\quad y=\begin{pmatrix}-1&i\\i&1\end{pmatrix}.$$

Combining the results of this section with Corollary 6.7, Proposition 7.1, Theorem 7.3, and Corollary 7.6, we obtain the following theorem.

Theorem 9.9. *Suppose that $\ell\neq 2$, that $m\geq 2$, and that f is non-degenerate and symmetric. Then*

(1) $M(f)$ is the orthogonal complement to $L(f)$ with respect to the bilinear form $\varphi:\mathfrak{gl}(V)\times\mathfrak{gl}(V)\to F$, given by $\varphi(x,y)=\mathrm{tr}(xy)$. Moreover, there is a basis of V relative to which f has Gram matrix $D=\mathrm{diag}(d_1,\dots,d_n)$ and, relative to this basis, $M(f)$ consists of all $A\in\mathfrak{gl}(m)$ such that $d_iA_{ij}=d_jA_{ji}$. Furthermore, $M(f)$ is isomorphic to $S^2(V)$ as $L(f)$-module.

(2) $M(f)\cap\mathfrak{sl}(V)$ consists, relative to the above basis, of all matrices $A\in\mathfrak{gl}(m)$ such that $d_iA_{ij}=d_jA_{ji}$ and $\mathrm{tr}(A)=0$, and is isomorphic to the kernel of the contraction $L(f)$-epimorphism $S^2(V)\to F$ given by $vw\to f(v,w)$.

(3) If $m\geq 4$ and $\ell\nmid m$ then $M(f)\cap\mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2}-1$.

(4) If $m\geq 4$ and $\ell\mid m$ then $M(f)\cap\mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m+1}{2}-2$.

(5) If $m=3$ or $m\geq 5$ then $L(f)$ is a simple Lie algebra, isomorphic to $\mathfrak{gl}(V)/M(f)$ and $\Lambda^2(V)$ as $L(f)$-modules.

(6) The following are composition series of the $L(f)$-module $\mathfrak{gl}(V)$:

$$0\subset\mathfrak{s}\subset M(f)\cap\mathfrak{sl}(V)\subset M(f)\subset\mathfrak{gl}(V),\ \text{if } m\geq 4 \text{ and } \ell|m,$$

$$0\subset M(f)\cap\mathfrak{sl}(V)\subset M(f)\subset\mathfrak{gl}(V),\ \text{if } m\geq 4 \text{ and } \ell\nmid m.$$

In any case, $M(f)/M(f)\cap\mathfrak{sl}(V)$ is the trivial $L(f)$-module.

10. A composition series of the $\mathfrak{sp}(2n)$-module $\mathfrak{gl}(2n)$ when $\ell \neq 2$

We assume throughout this section that $\ell \neq 2$, that $m = 2n$, and that f is non-degenerate and skew-symmetric.

Theorem 10.1. *The symplectic Lie algebra $L(f)$ is simple.*

Proof. We show that $L = L(J)$ is simple, where J is defined in (6). We can assume $m \geq 4$ because $\mathfrak{sp}(2) = \mathfrak{sl}(2)$. Let I be a nonzero ideal of L and suppose $0 \neq x \in I$. Write

$$x = \begin{pmatrix} a & b \\ c & -a' \end{pmatrix}.$$

with $a, b, c \in \mathfrak{gl}(n)$ and b, c symmetric. Let

$$y = \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 \\ I_n & 0 \end{pmatrix}.$$

Let S, B be the subspaces of L defined by

$$S = \left\{ \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} : s \in \mathfrak{gl}(n),\ s' = s \right\}, \quad B = \left\{ \begin{pmatrix} 0 & 0 \\ s & 0 \end{pmatrix} : s \in \mathfrak{gl}(n),\ s' = s \right\}.$$

Since $(\mathrm{ad}y \circ \mathrm{ad}z \circ \mathrm{ad}z)(x) = -2(b \oplus -b)$ and $(\mathrm{ad}z \circ \mathrm{ad}y \circ \mathrm{ad}y)(x) = 2(c \oplus -c)$, we can assume that $b = c = 0$ and $a \neq 0$. Moreover, we can assume that a is not scalar, for otherwise $[z, [e_{1,n+2} + e_{2,n+1}, a]]$ has the desired form. Then the action of $\mathfrak{gl}(n)$ on I yields $u \oplus (-u') \in I$ for all $u \in \mathfrak{sl}(n)$. Applying $\mathrm{ad}y$ we obtain $I \cap S \neq 0$, hence $S \subset I$ by Theorem 5.6. Analogously $B \subset I$. Since $[y, e_{n+1,1}] = e_{11} \oplus -e_{11}$, we conclude that $I = L$. ∎

Theorem 10.2. *Suppose that $m > 2$.*

(1) If $\ell \nmid m$ then $M(f) \cap \mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m}{2} - 1$.

(2) If $\ell \mid m$ then $M(f) \cap \mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m}{2} - 2$.

Proof. Let J like in (6). If D is an $L(J)$-submodule of $M(J) \cap \mathfrak{sl}(m)$ properly containing $M(J) \cap \mathfrak{sl}(m) \cap \mathfrak{s}$, we deduce $D = M(J) \cap \mathfrak{sl}(m)$ arguing like in (10.1). ∎

Note 10.3. Suppose that $\ell \nmid m$ and $m > 2$. Then Corollary 7.6 and Theorem 10.2 show that the subspace of $\mathfrak{gl}(2n)$ of all matrices (22) such that $b, c \in \mathfrak{gl}(n)$ are skew-symmetric and $a \in \mathfrak{sl}(n)$ is an irreducible $L(f)$-module of dimension $\binom{m}{2} - 1$, isomorphic to the kernel of the contraction map $\Lambda^2(V) \to F$, $v \wedge w \mapsto f(v, w)$.

When $F = \mathbb{C}$ this gives an elementary matrix description of $V(\lambda_2)$, the second fundamental module of the symplectic Lie algebra $\mathfrak{sp}(2n)$.

Note 10.4. If $m = 2$ then $M(f) = \mathfrak{s}$ is 1-dimensional.

Combining the results of this section with Corollary 6.7, Theorems 7.1 and 7.2, and Corollary 7.6, we obtain the following theorem.

Theorem 10.5. *Suppose that $\ell \neq 2$, that $m = 2n$, and that f is non-degenerate and skew-symmetric. Then*

(1) $M(f)$ is the orthogonal complement to $L(f)$ with respect to the bilinear form $\varphi : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to F$, given by $\varphi(x, y) = \mathrm{tr}(xy)$. Moreover, $M(f)$ consists, relative to suitable basis of V, of all matrices

$$\begin{pmatrix} A & B \\ C & A' \end{pmatrix}, \ A, B, C \in \mathfrak{gl}(n), \text{ where } B, C \text{ are skew-symmetric.} \tag{29}$$

Furthermore, $M(f)$ is isomorphic to $\Lambda^2(V)$ as $L(f)$-module.

(2) $M(f) \cap \mathfrak{sl}(V)$ consists of all matrices (29) such that $\mathrm{tr}(A) = 0$ and is isomorphic to the kernel of the contraction $L(f)$-epimorphism $\Lambda^2(V) \to F$ given by $v \wedge w \to f(v, w)$.

(3) If $m > 2$ and $\ell \nmid m$ then $M(f) \cap \mathfrak{sl}(V)$ is an irreducible $L(f)$-module of dimension $\binom{m}{2} - 1$.

(4) If $m > 2$ and $\ell \mid m$ then $M(f) \cap \mathfrak{sl}(V)/\mathfrak{s}$ is an irreducible $L(f)$-module of dimension $\binom{m}{2} - 2$.

(5) $L(f)$ is a simple Lie algebra, isomorphic to $\mathfrak{gl}(V)/M(f)$ and $S^2(V)$ as $L(f)$-modules.

(6) The following are composition series of the $L(f)$-module $\mathfrak{gl}(V)$:

$$0 \subset \mathfrak{s} \subset M(f) \cap \mathfrak{sl}(V) \subset M(f) \subset \mathfrak{gl}(V), \text{ if } m > 2 \text{ and } \ell | m,$$

$$0 \subset M(f) \cap \mathfrak{sl}(V) \subset M(f) \subset \mathfrak{gl}(V), \text{ if } m > 2 \text{ and } \ell \nmid m,$$

$$0 \subset M(f) \subset \mathfrak{gl}(V), \text{ if } m = 2.$$

In any case, $M(f)/M(f) \cap \mathfrak{sl}(V)$ is the trivial $L(f)$-module.

11. A composition series for the $\mathfrak{so}(m)$-module $\mathfrak{gl}(m)$ when $\ell = 2$

We assume throughout this section that $\ell = 2$, that $m \geq 2$, and that f is non-degenerate, symmetric and non-alternating.

Theorem 11.1. *Suppose that $m = 3$ or $m \geq 5$. Then $L(f)^{(1)}$ is a simple Lie algebra, and hence an irreducible $L(f)$-module, of dimension $\binom{m}{2}$.*

Proof. This was already proven in §9. ∎

Proposition 11.2. *Suppose that $m = 4$ and let D be the discriminant of f relative to a basis of V. Then*

(1) *$L(f)^{(1)}$ is 6-dimensional perfect Lie algebra.*

(2) *If $D \notin F^2$ then $L(f)^{(1)}$ is a simple Lie algebra.*

(3) *If $D \in F^2$ then $L(f)^{(1)} = S \ltimes R$, where S is a simple 3-dimensional subalgebra of $L(f)^{(1)}$, and R is abelian, the solvable radical of $L(f)^{(1)}$ and an irreducible $L(f)^{(1)}$-module.*

(4) *$L(f)^{(1)}$ is an irreducible $L(f)$-module.*

Proof. (1) Since $L(f)$ is 10-dimensional, Theorem 7.8 ensures that $L(f)^{(1)}$ is a 6-dimensional perfect Lie algebra.

(2) This can be found in [B], Chapter I, §6, Exercise 26(b) as well as in [CS].

(3) This can be found in [CS].

(4) It suffices to prove this when F is algebraically closed. Although the result follows from Theorem 8.1, we provide here an alternative argument. As seen in Theorem 7.8, f admits I_4 as Gram matrix and $L = L(I_m)$ (resp. $M = L(I_m)^{(1)}$) consists of all symmetric (resp. alternating) matrices. Thus a basis of L is formed by all e_{ii} and all $e_{ij} + e_{ji}$, $1 \leq i \neq j \leq 4$, and the latter form a basis of M. Set

$$f_1 = e_{12} + e_{21},\ f_2 = e_{23} + e_{32},\ f_3 = e_{13} + e_{31},$$

$$h_1 = e_{34} + e_{43},\ h_2 = e_{14} + e_{41},\ h_3 = e_{42} + e_{24},$$

$$g_1 = f_1 + h_1,\ g_2 = f_2 + h_2,\ g_3 = f_3 + h_3,$$

$$S = \langle f_1, f_2, f_3\rangle,\ R = \langle g_1, g_2, g_3\rangle.$$

Then $M = S \ltimes R$, where S is a simple Lie algebra, and R is an abelian ideal of M and an irreducible S-module (isomorphic to the adjoint module of S). It follows that S is the only non-zero proper M-submodule of M. Since

$$[e_{11}, g_1] = f_1 \notin R,$$

M is irreducible as L-module. ■

Note 11.3. If $m = 2$ then $L = L(f)$ is solvable of class 2, but not nilpotent, with

$$\dim L = 3,\ \dim L^{(1)} = 1,\ \dim L^{(2)} = 0.$$

Combining the results of this section with Proposition 7.1 and Theorem 7.8 we obtain the following theorem.

Theorem 11.4. *Suppose that $\ell = 2$, that $m \geq 2$, and that f is non-degenerate, symmetric and non-alternating. Then*

(1) The $L(f)$-module $\mathfrak{gl}(V)$ has $m+2$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)}$ in the series

$$0 \subset L(f)^{(1)} \subset L(f) \subset \mathfrak{gl}(V).$$

Moreover, if $m = 3$ or $m \geq 5$ then $L(f)^{(1)}$ is a simple Lie algebra of dimension $\binom{m}{2}$.

(2) $L(f)$ is isomorphic to the symmetric square $S^2(V)$ as $L(f)$-modules. Moreover, there is a basis of V relative to which f has Gram matrix $D = \mathrm{diag}(d_1, \ldots, d_m)$ and, relative to this basis, $L(f)$ consists of all $A \in \mathfrak{gl}(m)$ such that $d_i A_{ij} = d_j A_{ji}$.

(3) $L(f)^{(1)}$ is isomorphic to the exterior square $\Lambda^2(V)$ as $L(f)$-modules. Moreover, relative to the above basis, $L(f)^{(1)}$ consists of all $A \in \mathfrak{gl}(m)$ such that $A_{ii} = 0$ and $d_i A_{ij} = d_j A_{ji}$.

(4) $\mathfrak{gl}(V)/L(f) \cong L(f)^{(1)}$ *as* $L(f)$*-modules. In particular,* $\mathfrak{gl}(V)$ *has* m *trivial composition factors, and 2 composition factors isomorphic to* $L(f)^{(1)} \cong \Lambda^2(V)$*, which is itself the trivial module if and only if* $m=2$*.*

12. A composition series for the $\mathfrak{sp}(2n)$-module $\mathfrak{gl}(2n)$ when $\ell = 2$

We assume throughout this section that $m = 2n$ and that f is non-degenerate and alternating. We also assume that $\ell = 2$, except in Proposition 12.2 and Note 12.5, where ℓ is arbitrary.

Theorem 12.1. *Suppose* $m > 4$*. If* $4|m$ *then* $L(f)^{(2)}/\mathfrak{s}$ *is a simple Lie algebra. If* $4 \nmid m$ *then* $L(f)^{(2)}$ *is a simple Lie algebra.*

Proof. Let J as defined in (6) and let I be a nonzero ideal of $L = L(J)^{(2)}$. Let T and C be the subspaces of L defined by

$$T = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} : t \text{ alternating} \right\}, \quad C = \left\{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} : t \text{ alternating} \right\}.$$

Suppose first that I consists only of diagonal matrices. If I contains a non-scalar matrix, the action of $\mathfrak{sl}(n)$ on I yields $u \oplus u' \in I$ for all $u \in \mathfrak{sl}(n)$, against the assumption. So we have $I = \mathfrak{s}$ if $4|m$, and a contradiction if $4 \nmid m$.

Suppose next that I contains a non-diagonal matrix x. Let i, j, k be distinct indices between 1 and n. Since

$$[e_{n+j,k} + e_{e_{n+k,j}}, x]_{ik} = x_{i,n+j} \quad \text{and} \quad [e_{k,n+j} + e_{j,n+k}, x]_{ki} = x_{n+j,i},$$

we can assume that $x_{ij} \neq 0$. Since

$$\mathrm{ad}(e_{jk} + e_{n+k,n+j}) \circ \mathrm{ad}(e_{ji} + e_{n+i,n+j}) \circ \mathrm{ad}(e_{ki} + e_{n+i,n+k})(x) = x_{ij}(e_{ji} + e_{n+i,n+j}),$$

it follows $e_{ji} + e_{n+i,n+j} \in I$. So the action of $\mathfrak{sl}(n)$ on I shows that $u \oplus u' \in I$ for all $u \in \mathfrak{sl}(n)$. Taking $A = e_{31}$ and $B = e_{12} + e_{21}$ in (27) we obtain a nonzero element of T, hence $T \subset I$ by Theorem 5.5. Similarly we see that $C \subset I$. Therefore $I = L$. ∎

Proposition 12.2. *Let* $\mathfrak{h}(n)$ *be the Heisenberg algebra of dimension* $2n+1$*, whose underlying vector space is* $V \oplus F$*, with bracket*

$$[u + \beta, v + \gamma] = f(u, v).$$

Let $u_1, \dots, u_n, v_1, \dots, v_n$ *be a basis of* V *relative to which* f *has Gram matrix* J*, as defined in (6). Let* $z = 1 \in \mathfrak{h}(n)$*. Let* $F[X_1, \dots, X_n]$ *be the polynomial algebra over* F *in* n *commuting variables* $X_1, \dots, X_n$*. For* $q \in F[X_1, \dots, X_n]$ *let* m_q *be the linear endomorphism "multiplication by* q*" of* $F[X_1, \dots, X_n]$*. Let* $0 \neq \alpha \in F$*. Then* $F[X_1, \dots, X_n]$ *becomes a faithful* $\mathfrak{h}(n)$*-module via*

$$u_i \mapsto \partial/\partial X_i,\ v_i \mapsto \alpha \cdot m_{X_i}, z \mapsto \alpha \cdot I_{F[X_1,\dots,X_n]}.$$

Moreover, $F[X_1, \dots, X_n]$ *is irreducible if and only if* $\ell = 0$*. Furthermore, if* ℓ *is prime then* $(X_1^\ell, \dots, X_n^\ell)$ *is an* $\mathfrak{h}(n)$*-invariant subspace of* $F[X_1, \dots, X_n]$ *and* $F[X_1, \dots, X_n]/(X_1^\ell, \dots, X_n^\ell)$ *is a faithful irreducible* $\mathfrak{h}(n)$*-module of dimension* ℓ^n*.*

Proof. This is clear. ■

Lemma 12.3. *$L(f)/L(f)^{(2)} \cong \mathfrak{h}(n)$ as Lie algebras.*

Proof. Let $J \in \mathfrak{gl}(4)$ be defined as in (6) and identify L with $M(J)$. Consider the elements $a, b_1, \ldots, b_n, c_1, \ldots, c_n$ of $L(f)$ defined as follows in terms of $n \times n$ blocks:

$$a = \begin{pmatrix} e_{11} & 0 \\ 0 & e_{11} \end{pmatrix}, b_i = \begin{pmatrix} 0 & e_{ii} \\ 0 & 0 \end{pmatrix}, c_i = \begin{pmatrix} 0 & 0 \\ e_{ii} & 0 \end{pmatrix}.$$

The canonical projection of these elements produces a basis $L(f)/L(f)^{(2)}$, with multiplication table:

$$[b_i, c_i] = a. \tag{30}$$

Note the use of the matrix description of $L(f)^{(2)}$, given in Theorem 7.8, for the computation of this table. It follows from (30) that $L(f)/L(f)^{(2)} \cong \mathfrak{h}(n)$. ■

Proposition 12.4. *Suppose $m = 4$. Then the derived series of $L = L(f)$ satisfies*

$$\dim L = 10, \dim L^{(1)} = 6, \dim L^{(2)} = 5, \dim L^{(3)} = 1, \dim L^{(4)} = 0.$$

Here $L^{(3)} = \mathfrak{s}$, $L^{(2)} \cong \mathfrak{h}(2) \cong L/L^{(2)}$, and $U = L^{(2)}/L^{(3)}$ is a 4-dimensional abelian Lie algebra that is irreducible as L-module. More precisely, the action of L on U leaves invariant a non-degenerate alternating bilinear form $g : U \times U \to F$, the kernel of the associated representation $R : L \to L(g)$ is $L^{(2)}$, the corresponding 4-dimensional faithful representation $S : L/L^{(2)} \to \mathfrak{gl}(U)$ is irreducible, and $S(L/L^{(2)}) = R(L)$ is a 5-dimensional subalgebra of the symplectic Lie algebra $L(g)$ for which the natural module U is irreducible. Moreover, if we identify $L/L^{(2)}$ with $\mathfrak{h}(2)$ then U is isomorphic to the module $F[X_1, X_2]/(X_1^2, X_2^2)$ of Proposition 12.2, where $\alpha = 1$.

Proof. The dimensions of the terms of the derived series as well as the fact that $L^{(3)} = \mathfrak{s}$ follow from Theorem 7.8. Let $J \in \mathfrak{gl}(4)$ be defined as in (6) and identify L with $M(J)$. Consider the following basis elements of $L^{(2)}$, described in terms of 2×2 blocks:

$$x = \begin{pmatrix} 0 & e_{12} + e_{21} \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ e_{12} + e_{21} & 0 \end{pmatrix},$$

$$e = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{21} \end{pmatrix}, f = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{12} \end{pmatrix}, z = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

Their multiplication table is

$$[x, y] = z = [e, f].$$

Thus $L^{(2)}$ is a 5-dimensional Heisenberg algebra and $L^{(2)}/L^{(3)}$ is a 4-dimensional abelian Lie algebra.

The bracket $[\ ,\ ] : L^{(2)} \times L^{(2)} \to L^{(3)}$ is L-invariant by the Jacobi identity and the fact that $L^{(4)} = 0$. Since $L^{(2)} \cong \mathfrak{h}(5)$, the radical of the alternating form $[\ ,\ ]$ is $L^{(3)}$. This induces a non-degenerate L-invariant alternating form, say g, on $U = L^{(2)}/L^{(3)}$. The matrix of g with respect to the basis $\mathcal{B} = \{e+\mathfrak{s}, x+\mathfrak{s}, f+\mathfrak{s}, y+\mathfrak{s}\}$ of U is simply J. By definition of $L^{(3)}$, it follows that $L^{(2)}$ is in the kernel of the representation $R: L \to L(g) \subset \mathfrak{gl}(U)$, which gives rise to a representation $S: L/L^{(2)} \to \mathfrak{gl}(U)$. Let a, b_1, b_2, c_1, c_2 be the elements of L defined in Lemma 12.3. Then $\mathcal{C} = \{a+L^{(2)}, b_1+L^{(2)}, b_2+L^{(2)}, c_1+L^{(2)}, c_2+L^{(2)}\}$ is a basis of $L/L^{(2)} \cong \mathfrak{h}(2)$. The matrices corresponding to these basis vectors via S in $L(g)$ (identified with $M(J)$) are given in terms of 2×2 blocks as follows:

$$R(a) = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}, R(b_1) = \begin{pmatrix} 0 & e_{12}+e_{21} \\ 0 & 0 \end{pmatrix}, R(b_2) = \begin{pmatrix} e_{21} & 0 \\ 0 & e_{12} \end{pmatrix},$$

$$R(c_1) = \begin{pmatrix} 0 & 0 \\ e_{12}+e_{21} & 0 \end{pmatrix}, R(c_2) = \begin{pmatrix} e_{12} & 0 \\ 0 & e_{21} \end{pmatrix}.$$

Since these matrices are linearly independent, it follows that the kernel of R is precisely $L^{(2)}$.

Let W be a non-zero subspace of U is invariant under these matrices. We claim that $W = U$. Indeed, if W contains any vector from $\mathcal{B}$ then, acting through b_1, b_2, c_1, c_2, we see that it contains them all. Suppose, if possible, that W contains no vectors from $\mathcal{B}$. Since $R(b_1)$ is nilpotent, W must contain a non-zero vector from the nullspace of $R(b_1)$, which is 2-dimensional and spanned by $e+\mathfrak{s}, x+\mathfrak{s}$. Thus W must have a vector of the form $(e+\mathfrak{s})+\alpha(x+\mathfrak{s})$ for some $0 \neq \alpha \in F$. The same reasoning applied to $R(b_2)$ shows that W must have a vector of the form $\alpha(x+\mathfrak{s})+\beta(f+\mathfrak{s})$ for some $\beta \in F$, so $(e+\mathfrak{s})+\beta(f+\mathfrak{s}) \in W$. Applying b_2, it follows that $x+\mathfrak{s} \in W$, a contradiction. This proves our claim.

Consider the basis of $F[X_1, X_2]/(X_1^2, X_2^2)$ associated to $X_2, 1, X_1, X_1X_2$. Relative to this basis the matrices that the endomorphisms $\partial/\partial X_1$, $\partial/\partial X_2$, m_{X_1} and m_{X_2} induce on $F[X_1, X_2]/(X_1^2, X_2^2)$ are exactly $R(b_1), R(b_2), R(c_1), R(c_2)$. This shows that U arises from the $\mathfrak{h}(2)$-module of Proposition 12.2 by taking $\alpha = 1$. ■

Note 12.5. Let $d(n)$ be the smallest dimension of a faithful $\mathfrak{h}(n)$-module. It is well-known that $d(n) \leq n+2$. In fact, [Bu] proves $d(n) = n+2$ if $\ell = 0$. Note that if $\ell = 2$ then $\mathfrak{h}(1) \cong \mathfrak{sl}(2)$, so $d(1) = 2$ in this case. In all other cases, i.e., if $(\ell, n) \neq (2, 1)$, then $d(n) = n+2$, as shown in [S].

We refer the reader to [Bu] and [CR] for the problem of finding the smallest dimension of a faithful module for various Lie algebras in characteristic 0.

Note, on the other hand, that $\mathfrak{h}(n)$ has no faithful *irreducible* module if $\ell = 0$.

More generally, let L be a Lie algebra such that $[L, L] \cap Z(L) \neq 0$ and suppose that $T: L \to \mathfrak{gl}(U)$ is a faithful irreducible representation. Then $\ell | \dim(U)$.

Indeed, let z be a non-zero central element of $[L, L] \cap Z(L)$ and let $q \in F[X]$ be the minimal polynomial of $T(z)$. Since U is irreducible and z is central, we see that $q(X)$ is irreducible. Let W be a composition factor of the $L \otimes K$-module $U \otimes K$, where K is an algebraic closure of F. Since z is central, Schur's Lemma implies that z acts through a scalar operator on W. But $z \in [L, L]$, so the trace of this operator is 0. Thus, the scalar is 0 or $\ell \mid \dim_K(W)$. The first case is impossible, for otherwise 0 is a root of q, whence $q = X$, i.e. $T(z) = 0$, against the fact that T is faithful. It follows that ℓ divides the dimension over K of all composition factors of $U \otimes K$, whence $\ell | \dim(U)$.

When ℓ is prime then $\mathfrak{h}(n)$ certainly has faithful irreducible modules. The smallest dimension for such module is p^n. This is actually the only possible dimension when F is algebraically closed (but not otherwise). In fact, if F is algebraically closed there is only one such module, up to isomorphism and an automorphism of $\mathfrak{h}(n)$, namely the one described in Proposition 12.2 with $\alpha = 1$. See [S] for details.

Combining Theorem 12.1, Lemma 12.3 and Proposition 12.4 with Proposition 7.1 and Theorem 7.8 we obtain the following theorem.

Theorem 12.6. *Suppose that $\ell = 2$, that $m = 2n$, and that f is non-degenerate and alternating. Then*

(1) If $4 \mid m$ then the $L(f)$-module $\mathfrak{gl}(V)$ has $m+6$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)}$ in the series

$$0 \subset \mathfrak{s} \subset L(f)^{(2)} \subset L(f)^{(1)} \subset L(f) \subset U \subset \mathfrak{sl}(V) \subset \mathfrak{gl}(V),$$

where $U = L(f) \oplus \langle x \rangle$, $x \in \mathfrak{sl}(V)$, $[x, L(f)] \subseteq L(f)$, and

$$x = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}.$$

All composition factors are trivial, except for $L(f)^{(2)}/\mathfrak{s}/ \cong \mathfrak{sl}(V)/U$, which has dimension $\binom{m}{2} - 2$. Moreover, $L(f)^{(2)}/\mathfrak{s}$ is a simple Lie algebra if and only if $m > 4$.

(2) If $m \neq 2$ and $4 \nmid m$ then the $L(f)$-module $\mathfrak{gl}(V)$ has $m+4$ composition factors. A composition series can be obtained by inserting $m-1$ arbitrary subspaces between $L(f)$ and $L(f)^{(1)}$ in the series

$$0 \subset L(f)^{(2)} \subset L(f)^{(1)} \subset L(f) \subset \mathfrak{sl}(V) \subset \mathfrak{gl}(V).$$

All composition factors are trivial, except for $L(f)^{(2)} \cong \mathfrak{sl}(V)/L(f)$.

Moreover, $L(f)^{(2)}$ is a simple Lie algebra of dimension $\binom{m}{2} - 1$.

(3) $L(f)$ is isomorphic to the symmetric square $S^2(V)$ as $L(f)$-modules, and, relative to suitable basis of V, consists of all matrices

$$\begin{pmatrix} A & B \\ C & A' \end{pmatrix}, \; A, B, C \in \mathfrak{gl}(n), \text{ where } B, C \text{ are symmetric.} \tag{31}$$

(4) $L(f)^{(1)}$ is isomorphic to the exterior square $\Lambda^2(V)$ as $L(f)$-modules, and consists of all matrices (31) such that B, C are alternating.

(5) $L(f)^{(2)}$ is isomorphic to the kernel of the contraction $L(f)$-epimorphism $\Lambda^2(V) \to F$, given by $v \wedge w \mapsto f(v, w)$, and consists of all matrices (31) such that B, C are alternating and $\mathrm{tr}(A) = 0$.

(6) $L(f)/L(f)^{(2)}$ is isomorphic, as Lie algebra, to $\mathfrak{h}(n)$, the Heisenberg algebra of dimension $2n+1$.

Note 12.7. If $m = 2$ then $L = L(f)$ is nilpotent of class 2, where

$$\dim L = 3, \dim L^1 = 1, \dim L^2 = 0.$$

In particular, $\mathfrak{gl}(V)$ has 4 trivial composition factors as $L(f)$-module.

Note that $L \cong L/L^{(2)} \cong \mathfrak{h}(1)$ and that through this identification the natural module for L becomes the module $F[X_1]/(X_1^2)$ of Proposition 12.2 with $\alpha = 1$.

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