

A Lie-Algebraic Formulation for Triply Orthogonal and General Coordinate Systems in Three-Dimensional Euclidean and Lorentz Spaces

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Communicated by P. Olver

Abstract. We give a Lie-algebraic formulation for the interacting geometries of orthogonal families of coordinate surfaces in 3-dimensional Euclidean- and Lorentz-orthogonal coordinate systems. A study of the Gauss-Lamé equations and their variational equations in this setting leads to formulas for constructing more general 3-dimensional coordinate transformations. To motivate the general constructions, we begin with special cases of orthogonal coordinate systems in 3-dimensional Lorentz space, built from orthogonal systems in the plane.

Mathematics Subject Classification 2000: 53C21, 53C12, 53A05, 53A35, 53Z05.

Key Words and Phrases: Orthogonal coordinate systems, Gauss-Lamé equations, Lorentz space.

1. Introduction

In real dimensions two and three, there are two essentially different inner products: the positive-definite Euclidean inner product and the indefinite Lorentz inner product. For each of these inner products, a triply orthogonal coordinate transformation yields three families of coordinate surfaces such that the pairwise intersections of their tangent spaces are mutually orthogonal at every point of the system. We use these frames to give a Lie-algebraic formulation for the interplay among the three orthogonal families of coordinate surfaces in both the Euclidean and the Lorentzian metrics. The framework is motivated by a study of Fokas and Gelfand on immersions of (not necessarily orthogonally parameterized) surfaces into Lie groups and Lie algebras [4] (see also [1, 2, 5]). We give solutions to the variational equations of the Gauss-Lamé systems in the Lie algebras $so(3)$ and $so(2, 1)$, and a corresponding formula for constructing (not necessarily orthogonal) 3-dimensional coordinate systems in the Lie algebra; these generalize formulas for surfaces in [4] to families of coordinate surfaces arising from 3-dimensional coordinate transformations. We begin the paper with special cases of orthogonal coordinate systems in 3-dimensional Lorentz space, constructed from Euclidean-

* P.D.S. was supported in part by NSF grant DMS-1022635

and Lorentz-orthogonal systems in the plane, which we use to motivate the general Lie-algebraic framework.

In \mathbb{R}^n , with Euclidean inner product $\langle \cdot \rangle = \sum_{i=1}^n dx^i \otimes dx^i$, a change of coordinates $r : \Omega \subseteq \mathbb{R}^n \rightarrow \Omega' \subseteq \mathbb{R}^n$, where

$$r(u^1, \dots, u^n) = (x^1(u^1, \dots, u^n), \dots, x^n(u^1, \dots, u^n)),$$

is n -orthogonal if the columns of the Jacobian matrix Dr satisfy

$$\langle r_{u^i}, r_{u^j} \rangle = \sum_{k=1}^n \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} = 0 \text{ for } i \neq j.$$

In the Lorentz case, a change of coordinates $r : \Omega \subseteq \mathbb{R}^{n-1,1} \rightarrow \Omega' \subseteq \mathbb{R}^{n-1,1}$, from Ω with coordinates (u^1, \dots, u^n) and Lorentz inner product

$$\langle \cdot \rangle = \sum_{i=1}^{n-1} du^i \otimes du^i - du^n \otimes du^n,$$

to Ω' with coordinates (x^1, \dots, x^n) and Lorentz inner product

$$\langle \cdot \rangle = (\sum_{i=1}^{n-1} dx^i \otimes dx^i) - dx^n \otimes dx^n,$$

is n -orthogonal if

$$\langle r_{u^i}, r_{u^j} \rangle = \left(\sum_{k=1}^{n-1} \frac{\partial x^k}{\partial u^i} \frac{\partial x^k}{\partial u^j} \right) - \frac{\partial x^n}{\partial u^i} \frac{\partial x^n}{\partial u^j} = 0 \text{ for } i \neq j,$$

$\langle r_{u^i}, r_{u^i} \rangle > 0$ for $i < n$, and $\langle r_{u^n}, r_{u^n} \rangle < 0$.

For a triply orthogonal Euclidean system $r(u^1, u^2, u^3) = (x^1, x^2, x^3)$, the metric $dl^2 = \sum_{i=1}^3 (dx^i)^2$ takes the form $dl^2 = \langle r_{u^1}, r_{u^1} \rangle (du^1)^2 + \langle r_{u^2}, r_{u^2} \rangle (du^2)^2 + \langle r_{u^3}, r_{u^3} \rangle (du^3)^2$ in the orthogonal coordinates u^i . A classic problem in differential geometry asks for what functions $H_k(u^i)$ the diagonal metric

$$dl^2 = \sum_{k=1}^3 H_k^2 (du^k)^2$$

arises from an orthogonal coordinate transformation $r(u^1, u^2, u^3) = (x^1, x^2, x^3)$, so that $H_k^2 = \langle r_{u^k}, r_{u^k} \rangle$. The Gauss-Lamé equations, which are the integrability conditions for the existence of such a coordinate system, and their analog in the Lorentz setting, are at the core of the Lie-algebraic description of the geometry of the triply orthogonal families of surfaces in Section 4. This classic problem in differential geometry has been a major thrust in the work of differential geometers, particularly in the late 19th and early 20th centuries [3], and it has deep connections to the theory of integrable partial differential equations. Zakharov applied the dressing method of integrable systems to this problem [14, 15, 16]. See also [10] for a review of this method and an extension due to Krichever [8]. The aim of this paper is not to reproduce these methods for the Lorentz case, but rather to provide a Lie-algebraic framework for triply orthogonal coordinate transformations in Lorentz and Euclidean space, and to use this framework to construct more general 3-dimensional coordinate transformations via the Gauss-Lamé equations. We use explicit constructions of Lorentz-orthogonal coordinate systems to motivate and illustrate the results. Our main theorems, in Section 5, generalize the framework

of Fokas, *et. al.* for surfaces in Lie groups and Lie algebras to families of surfaces arising from 3-dimensional curvilinear (and not necessarily orthogonal) coordinate systems in $\mathbb{R}^{2,1}$ and in \mathbb{R}^3 .

After giving special cases of Lorentz-orthogonal systems in Section 2, Section 3 gives the Lie-algebraic formulation that we use in Section 4 to study the geometry of the orthogonal families of coordinate surfaces via the Gauss-Lamé equations, in both the Euclidean and Lorentzian metrics. We follow this, in Section 5, with a treatment of the variational equations to the Gauss-Lamé systems, and more general formulas for constructing 3-dimensional coordinate transformations.

2. Triply orthogonal systems in \mathbb{R}^3 and $\mathbb{R}^{2,1}$ – special cases

2.1. Constructions from orthogonal systems in the plane.

The six constructions that follow generate triply orthogonal systems in \mathbb{R}^3 and $\mathbb{R}^{2,1}$, starting with orthogonal systems in \mathbb{R}^2 or $\mathbb{R}^{1,1}$. We will use them in motivating and illustrating the general results in Sections 3, 4, and 5. In the Lorentz case, let $\alpha : \Omega \rightarrow \Omega'$, which we write as $\alpha(s, t) = (f(s, t), g(s, t))$, be a change of coordinates from a domain $\Omega \subseteq \mathbb{R}^{1,1}$ with Lorentz metric $ds^2 - dt^2$, onto a region $\Omega' \subseteq \mathbb{R}^{1,1}$ with Lorentz metric $df^2 - dg^2$. α is *Lorentz orthogonal* if the vector fields $\alpha_s = (f_s, g_s)$ and $\alpha_t = (f_t, g_t)$ satisfy

$$f_s f_t - g_s g_t = 0, \quad (1)$$

$$f_s^2 - g_s^2 > 0, \quad \text{and} \quad f_t^2 - g_t^2 < 0. \quad (2)$$

Of special interest are Lorentz-orthogonal transformations α for which α^{-1} is also Lorentz orthogonal. We study the geometry of such mappings extensively in [13]. This situation occurs when f and g satisfy the wave equation: $f_{ss} = f_{tt}$ and $g_{ss} = g_{tt}$; in particular, it occurs when $f_s = g_t$ and $f_t = g_s$. The constructions that follow may be specialized to this situation.

For triply orthogonal coordinate transformations in \mathbb{R}^3 or in $\mathbb{R}^{2,1}$, we use coordinates (u, v, w) on the domain, with metric $du^2 + dv^2 + dw^2$ in the Euclidean case and $du^2 + dv^2 - dw^2$ in the Lorentz case. On the image, the metric is $dx^2 + dy^2 + dz^2$ in the Euclidean case and $dx^2 + dy^2 - dz^2$ in the Lorentz case. For completeness, we first record the standard Euclidean constructions of translation and rotation.

Construction 1 (Translation of \mathbb{R}^2). Let $\alpha : \Omega \subseteq \mathbb{R}^2 \rightarrow \Omega' \subseteq \mathbb{R}^2$ be a Euclidean-orthogonal system, where $\alpha(s, t) = (f(s, t), g(s, t))$. Then the mapping

$$r(u, v, w) = (u, f(v, w), g(v, w))$$

defines a triply orthogonal Euclidean system $r : \mathbb{R} \times \Omega \rightarrow \mathbb{R} \times \Omega'$.

Construction 2 (Rotation of \mathbb{R}^2). Let $\alpha(s, t) = (f(s, t), g(s, t))$ be a Euclidean-orthogonal system, restricted to the region Ω of the (s, t) -plane where $f(s, t) > 0$. Then the mapping

$$r(u, v, w) = (f(v, w) \sin u, f(v, w) \cos u, g(v, w)),$$

with $(u, v, w) \in (0, 2\pi) \times \Omega$, defines a triply orthogonal Euclidean system on its image in \mathbb{R}^3 .

Constructions 3 and 4 create triply orthogonal Lorentz systems in $\mathbb{R}^{2,1}$ by translation and rotation of Lorentz-orthogonal systems in $\mathbb{R}^{1,1}$.

Construction 3 (Translation of $\mathbb{R}^{1,1}$). Let $\alpha : \Omega \subseteq \mathbb{R}^{1,1} \rightarrow \Omega' \subseteq \mathbb{R}^{1,1}$ be a Lorentz-orthogonal system, where $\alpha(s, t) = (f(s, t), g(s, t))$. Then the translation

$$r(u, v, w) = (u, f(v, w), g(v, w)) = (x, y, z)$$

defines a triply orthogonal Lorentz system $r : \mathbb{R} \times \Omega \subseteq \mathbb{R}^{2,1} \rightarrow \mathbb{R} \times \Omega' \subseteq \mathbb{R}^{2,1}$. This mapping takes the coordinate lines of the Lorentzian (v, w) -plane into the Lorentz-orthogonal system $(f(v, w), g(v, w))$ in the (y, z) -plane.

Construction 4 (Rotation of $\mathbb{R}^{1,1}$). Let $\alpha(s, t) = (f(s, t), g(s, t))$ be a Lorentz-orthogonal system, restricted to the region Ω of the (s, t) -plane where $f(s, t) > 0$. Then the mapping

$$r(u, v, w) = (f(v, w) \sin u, f(v, w) \cos u, g(v, w)) = (x, y, z)$$

with $(u, v, w) \in (0, 2\pi) \times \Omega$ defines a triply orthogonal Lorentz system on its image in $\mathbb{R}^{2,1}$. This mapping takes the coordinate lines of the Lorentzian (v, w) -plane into the Lorentz-orthogonal system $(f(v, w), g(v, w))$ in the (y, z) plane.

Constructions 5 and 6 apply “hyperbolic rotations” to create triply orthogonal systems in $\mathbb{R}^{2,1}$ from orthogonal systems in \mathbb{R}^2 and $\mathbb{R}^{1,1}$.

Construction 5 (Hyperbolic rotation of \mathbb{R}^2). Let $\alpha(s, t) = (f(s, t), g(s, t))$ be a Euclidean-orthogonal system. Then the mapping

$$r(u, v, w) = (f(u, v) \cosh w, g(u, v), f(u, v) \sinh w) = (x, y, z)$$

defines a triply orthogonal Lorentz system. Its image lies in the region $|x| > |z|$ of $\mathbb{R}^{2,1}$. This mapping sends the coordinate lines of the Euclidean (u, v) -plane into the Euclidean-orthogonal system $(f(u, v), g(u, v))$ in the (x, y) -plane. The surfaces with w constant are planes that contain the y -axis.

Construction 6 (Hyperbolic rotation of $\mathbb{R}^{1,1}$). Let $\alpha(s, t) = (f(s, t), g(s, t))$ be a Lorentz-orthogonal system. Then the mapping

$$r(u, v, w) = (f(w, v) \sinh u, g(w, v), f(w, v) \cosh u) = (x, y, z)$$

defines a triply orthogonal Lorentz system. Its image lies in the region $|x| < |z|$ of $\mathbb{R}^{2,1}$. This mapping sends the coordinate lines of the Lorentzian (v, w) -plane into the Lorentz-orthogonal system $(g(w, v), f(w, v))$ in the (y, z) -plane. The surfaces with u constant are planes that contain the y -axis.

2.2. Illustrations of triply orthogonal Lorentz systems.

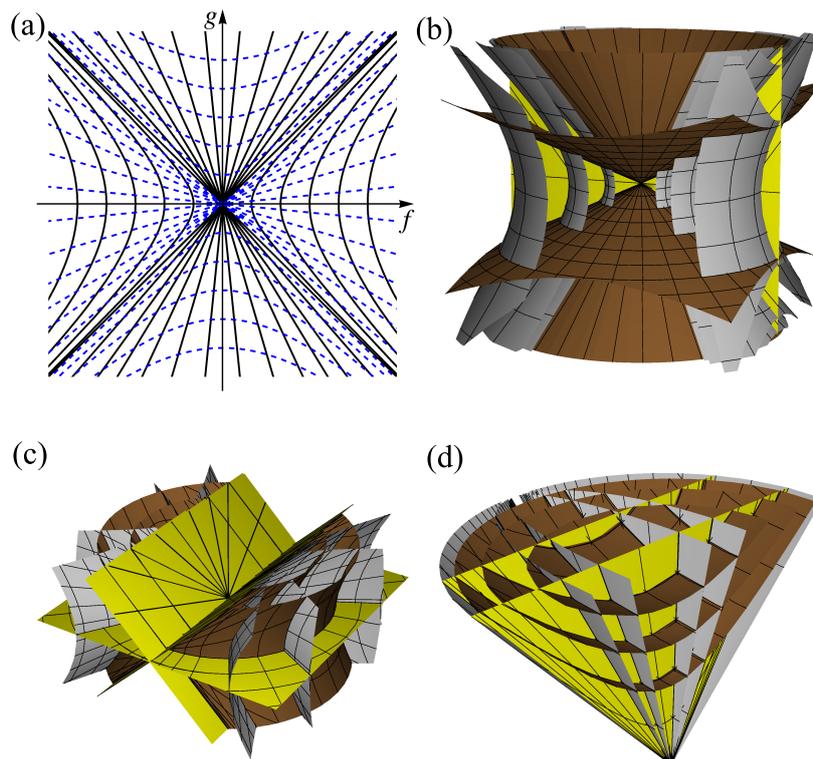


Figure 1: (a) Constant s and t coordinate lines of the systems of hyperbolas and lines in Ex. 1. (b) Hyperboloids of one sheet, cones, and planes outside the null cone in $\mathbb{R}^{2,1}$, from Const. 4 in Part (a) of Ex. 2. (c) Hyperboloids of one sheet, elliptical cones, and planes in the region $|x| > |z|$ of $\mathbb{R}^{2,1}$, from Const. 5 in Ex. 3. (d) Hyperboloids of two sheets, elliptical cones, and planes (only one sheet of each hyperboloid is shown), from Const. 6 in Ex. 5.

Example 1 (Complementary systems of hyperbolas and lines). A convenient example in the plane is the Lorentz-orthogonal analog of Euclidean-orthogonal polar coordinates. This consists in a pair of Lorentz-orthogonal systems of hyperbolas and lines in $\mathbb{R}^{1,1}$, one outside the null cone and the other inside.

(a) $\alpha(s, t) = (f(s, t), g(s, t)) = (s \cosh t, s \sinh t)$ maps the complement of the t -axis onto the outside of the null cone in $\mathbb{R}^{1,1}$, where $|f| > |g|$.

(b) $\beta(s, t) = (f(s, t), g(s, t)) = (t \sinh s, t \cosh s)$ maps the complement of the s -axis onto the inside of the null cone in $\mathbb{R}^{1,1}$, where $|f| < |g|$. (We note that for both of these transformations, the inverse is not Lorentz-orthogonal.) The two systems are depicted in Figure 1(a).

Example 2 (Hyperboloids, cones, and planes). Rotate the systems of hyperbolas and lines in Example 1, using Construction 4, to create triply orthogonal Lorentz systems in $\mathbb{R}^{2,1}$, one outside the null cone, and the other inside.

(a) Apply Construction 4 to $\alpha(s, t) = (f(s, t), g(s, t)) = (s \cosh t, s \sinh t)$. This yields the triply orthogonal Lorentz system

$$(x, y, z) = r(u, v, w) = (v \cosh w \sin u, v \cosh w \cos u, v \sinh w),$$

where $(u, v, w) \in (0, 2\pi) \times (\mathbb{R}^{1,1} \setminus w - \text{axis})$. Since $x^2 + y^2 - z^2 = v^2 > 0$, the coordinate surfaces of r lie outside of the null cone in $\mathbb{R}^{2,1}$. The surfaces are hyperboloids of one sheet (v constant), cones (w constant), and wedges of planes outside the null cone (u constant); these surfaces are illustrated in Figure 1(b).

(b) Apply Construction 4 to $\beta(s, t) = (f(s, t), g(s, t)) = (t \sinh s, t \cosh s)$. This yields the triply orthogonal Lorentz system

$$r(u, v, w) = (w \sinh v \sin u, w \sinh v \cos u, w \cosh v),$$

where $(u, v, w) \in (0, 2\pi) \times (\mathbb{R}^{1,1} \setminus v - \text{axis})$. Since $x^2 + y^2 - z^2 = v^2 < 0$, the coordinate surfaces of r lie inside the null cone in $\mathbb{R}^{2,1}$. The surfaces are hyperboloids of two sheets (w constant), cones (v constant), and wedges of planes inside the null cone (u constant).

Example 3 (Hyperbolically rotated polar coordinates). Apply Construction 5 to the Euclidean polar system $\alpha(s, t) = (s \cos t, s \sin t)$. This yields the triply orthogonal Lorentz system

$$(x, y, z) = r(u, v, w) = (u \cos v \cosh w, u \sin v, u \cos v \sinh w),$$

whose coordinate surfaces lie in the region $|x| > |z|$ of $\mathbb{R}^{2,1}$. These surfaces are hyperboloids of one sheet (u constant), elliptical cones (v constant), and planes containing the y -axis (w constant). They are shown in Figure 1(c).

Example 4 (Hyperbolically rotated ellipses and hyperbolas). Applying Construction 5 to the Euclidean system $\alpha(s, t) = (\cosh s \sin t, \sinh s \cos t)$ of ellipses and hyperbolas yields the triply orthogonal Lorentz system

$$r(u, v, w) = (\cosh u \sin v \cosh w, \sinh u \cos v, \cosh u \sin v \sinh w),$$

whose image lies in the region $|x| > |z|$ of $\mathbb{R}^{2,1}$. Its coordinate surfaces are hyperboloids of one sheet with elliptical cross-sections (u constant), hyperboloids of two sheets with elliptical cross-sections (v constant), and planes containing the y -axis (w constant).

Example 5 (Hyperbolically rotated hyperbolas and lines). Apply Construction 6 to the Lorentz orthogonal system $(s \cosh t, s \sinh t)$ from Part (a) of Example 1. This yields the triply orthogonal Lorentz system

$$r(u, v, w) = (w \cosh v \sinh u, w \sinh v, w \cosh v \cosh u),$$

whose coordinate surfaces lie in the region $|x| < |z|$ of $\mathbb{R}^{2,1}$. The surfaces, illustrated in Figure 1(d) on a restricted domain, are elliptical cones (v constant), hyperboloids of two sheets (w constant), and planes (u constant).

3. Lie-algebraic formulation

3.1. Orthonormal frames.

A coordinate transformation

$$r(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w))$$

defines a triply orthogonal system in $\mathbb{R}^{2,1}$ (respectively \mathbb{R}^3) if and only if the rows of the transposed Jacobian matrix $(Dr)^T$ of r are orthogonal in the Lorentz (respectively Euclidean) inner product. In this case, the rows of $(Dr)^T$ are an orthogonal basis of $\mathbb{R}^{2,1}$ (respectively \mathbb{R}^3) at every point of the triply orthogonal system.

To obtain a positively-oriented orthonormal frame at each point of the orthogonal system, one may scale the rows of $(Dr)^T$ by a diagonal matrix $\Lambda = \Lambda(u, v, w)$ on the left so that in the Euclidean setting,

$$\Lambda(Dr)^T = \Phi \in \text{SO}(3),$$

and in the Lorentzian setting,

$$\Lambda(Dr)^T = \Phi \in \text{SO}(2, 1).$$

In both cases, each entry of Λ is unique up to sign.

For $\Phi \in \text{SO}(3)$, the matrices

$$R = \Phi_u \Phi^{-1}, \quad S = \Phi_v \Phi^{-1}, \quad \text{and} \quad T = \Phi_w \Phi^{-1} \quad (3)$$

belong to the Lie algebra

$$\mathfrak{so}(3) = \{aL_1 + bL_2 + cL_3 : a, b, c \in \mathbb{R}\},$$

which we write in terms of the following orthonormal basis of $\mathfrak{so}(3)$:

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4)$$

For $\Phi \in \text{SO}(2, 1)$, R, S , and T in (3) belong to the Lie algebra

$$\mathfrak{so}(2, 1) = \{aL'_1 + bL'_2 + cL'_3 : a, b, c \in \mathbb{R}\},$$

written in terms of the following orthonormal basis of $\mathfrak{so}(2, 1)$:

$$L'_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L'_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L'_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (5)$$

This follows from the standard fact that if $\Phi(u, v) : \Omega \subset \mathbb{R}^2 \rightarrow G$ is a map into a Lie group G with Lie algebra \mathfrak{g} , then $\Phi_u \Phi^{-1}(u, v) \in \mathfrak{g}$ for all (u, v) .

3.2. Specialization to the six constructions.

In Examples 6 through 11, we give the matrices Φ , and R, S, T in (3), for the triply orthogonal Euclidean and Lorentz systems in Constructions 1 through 6. The dependence of the triple R, S, T in these examples yields an interesting property of the coordinate transformations in Theorem 5.2 of Section 5, which we note in closing. To avoid confusion when different variables are used in place of s and t in the planar systems $\alpha(s, t) = (f(s, t), g(s, t))$, we henceforth use subscripts 1 and 2 in place of s and t to denote partial derivatives with respect to the first and second variables, respectively. Thus, we write $\alpha_1 = (f_1, g_1)$ instead of $\alpha_s = (f_s, g_s)$ and $\alpha_2 = (f_2, g_2)$ instead of $\alpha_t = (f_t, g_t)$.

The four functions listed below appear in Examples 6 through 11; we give them special notation to simplify the examples. In the functions h_{E1} and h_{E2} , (f, g) is a Euclidean-orthogonal system in \mathbb{R}^2 . Each of these functions has two expressions in terms of partial derivatives of f and g :

$$h_{E1} = -\frac{(f_1g_2 - f_2g_1)(f_1g_{11} - g_1f_{11})}{(f_1^2 + g_1^2)^{3/2}(f_2^2 + g_2^2)^{1/2}} = -\frac{(f_1g_2 - f_2g_1)(f_2g_{12} - g_2f_{12})}{(f_1^2 + g_1^2)^{1/2}(f_2^2 + g_2^2)^{3/2}}$$

$$h_{E2} = -\frac{(f_1g_2 - f_2g_1)(f_1g_{12} - g_1f_{12})}{(f_1^2 + g_1^2)^{3/2}(f_2^2 + g_2^2)^{1/2}} = -\frac{(f_1g_2 - f_2g_1)(f_2g_{22} - g_2f_{22})}{(f_1^2 + g_1^2)^{1/2}(f_2^2 + g_2^2)^{3/2}}.$$

To verify that the two expressions are the same in each case, we use the orthogonality condition $f_1f_2 + g_1g_2 = 0$ and the equations obtained by taking the first and second partial derivatives of this condition.

In the functions h_{L1} and h_{L2} , (f, g) is a Lorentz-orthogonal system in $\mathbb{R}^{1,1}$. Each of these functions has two expressions, which are the same by the orthogonality condition (1), and the differences of squares that appear in the denominators are positive because of (2).

$$h_{L1} = \frac{(f_1g_2 - f_2g_1)(f_1g_{11} - g_1f_{11})}{(f_1^2 - g_1^2)^{3/2}(g_2^2 - f_2^2)^{1/2}} = -\frac{(f_1g_2 - f_2g_1)(f_2g_{12} - g_2f_{12})}{(f_1^2 - g_1^2)^{1/2}(g_2^2 - f_2^2)^{3/2}}$$

$$h_{L2} = \frac{(f_1g_2 - f_2g_1)(f_1g_{12} - g_1f_{12})}{(f_1^2 - g_1^2)^{3/2}(g_2^2 - f_2^2)^{1/2}} = -\frac{(f_1g_2 - f_2g_1)(f_2g_{22} - g_2f_{22})}{(f_1^2 - g_1^2)^{1/2}(g_2^2 - f_2^2)^{3/2}}$$

If we begin with a Lorentz-orthogonal system (f, g) with Lorentz-orthogonal inverse, as constructed in [13], where $f_1 = g_2$ and $f_2 = g_1$, then h_{L1} and h_{L2} specialize to

$$h_{L1} = \frac{f_1f_{12} - f_2f_{11}}{f_1^2 - f_2^2} \quad \text{and} \quad h_{L2} = \frac{f_1f_{22} - f_2f_{12}}{f_1^2 - f_2^2}.$$

Example 6 (Translation of \mathbb{R}^2). Consider the triply orthogonal Euclidean system

$$r(u, v, w) = (u, f(v, w), g(v, w)),$$

obtained from a 2-dimensional Euclidean system (f, g) as in Construction 1. For this system, $\Phi \in \text{SO}(3)$ and $R, S, T \in \text{so}(3)$ are as follows. All functions are evaluated at (v, w) , and S, T are multiples of L_1 in (4).

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{f_1}{\sqrt{f_1^2 + g_1^2}} & \frac{g_1}{\sqrt{f_1^2 + g_1^2}} \\ 0 & \frac{f_2}{\sqrt{f_2^2 + g_2^2}} & \frac{g_2}{\sqrt{f_2^2 + g_2^2}} \end{pmatrix}, \quad R = (0), \quad S = h_{E1}L_1, \quad T = h_{E2}L_1.$$

The lower right 2×2 blocks of S and T are the matrices $\hat{S} = \hat{\Phi}_s \hat{\Phi}^{-1}$ and $\hat{T} = \hat{\Phi}_t \hat{\Phi}^{-1}$ in $\text{so}(2)$, where $\hat{\Phi} \in \text{SO}(2)$ is the normalized 2×2 transposed Jacobian of $\alpha(s, t)$.

Example 7 (Rotation of \mathbb{R}^2). Consider the triply orthogonal Euclidean system

$$r(u, v, w) = (f(v, w) \sin u, f(v, w) \cos u, g(v, w)),$$

obtained by rotating a 2-dimensional Euclidean system (f, g) as in Construction 2. For this system, $\Phi \in \text{SO}(3)$ and $R, S, T \in \text{so}(3)$ are as follows. All functions are evaluated at (v, w) , and R, S, T are in terms of the basis $\{L_i\}$ of $\text{so}(3)$ in (4).

$$\Phi = \begin{pmatrix} \cos u & -\sin u & 0 \\ \frac{f_1}{\sqrt{f_1^2 + g_1^2}} \sin u & \frac{f_1}{\sqrt{f_1^2 + g_1^2}} \cos u & \frac{g_1}{\sqrt{f_1^2 + g_1^2}} \\ \frac{f_2}{\sqrt{f_2^2 + g_2^2}} \sin u & \frac{f_2}{\sqrt{f_2^2 + g_2^2}} \cos u & \frac{g_2}{\sqrt{f_2^2 + g_2^2}} \end{pmatrix},$$

$$R = \frac{f_2}{\sqrt{f_2^2 + g_2^2}} L_2 + \frac{f_1}{\sqrt{f_1^2 + g_1^2}} L_3, \quad S = h_{E1}L_1, \quad T = h_{E2}L_1.$$

Example 8 (Translation of $\mathbb{R}^{1,1}$). Consider the triply orthogonal Lorentz system

$$r(u, v, w) = (u, f(v, w), g(v, w)),$$

obtained by translating a 2-dimensional Lorentz system (f, g) as in Construction 3. The Lorentz-orthogonal frame Φ , and the matrices $R, S, T \in \text{so}(2, 1)$ are as follows, where all functions are evaluated at (v, w) and S, T are multiples of the basis element L'_1 of $\text{so}(2, 1)$ in (5).

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{f_1}{\sqrt{f_1^2 - g_1^2}} & \frac{g_1}{\sqrt{f_1^2 - g_1^2}} \\ 0 & \frac{f_2}{\sqrt{g_2^2 - f_2^2}} & \frac{g_2}{\sqrt{g_2^2 - f_2^2}} \end{pmatrix} \in \text{SO}(2, 1), \quad R = (0), \quad S = h_{L1}L'_1, \quad T = h_{L2}L'_1$$

The lower right 2×2 blocks of S and T are the matrices $\hat{S} = \hat{\Phi}_s \hat{\Phi}^{-1}$ and $\hat{T} = \hat{\Phi}_t \hat{\Phi}^{-1}$ in $\text{so}(1, 1)$, where $\hat{\Phi} \in \text{SO}(1, 1)$ is the normalized 2×2 transposed Jacobian of $\alpha(s, t)$.

Example 9 (Rotation of $\mathbb{R}^{1,1}$). Consider the triply orthogonal Lorentz system

$$r(u, v, w) = (f(v, w) \sin u, f(v, w) \cos u, g(v, w)),$$

obtained by rotating a 2-dimensional Lorentz system (f, g) as in Construction 4. For this system, $\Phi \in \text{SO}(2, 1)$ and $R \in \text{so}(2, 1)$ are as follows:

$$\Phi = \begin{pmatrix} \cos u & -\sin u & 0 \\ \frac{f_1}{\sqrt{f_1^2 - g_1^2}} \sin u & \frac{f_1}{\sqrt{f_1^2 - g_1^2}} \cos u & \frac{g_1}{\sqrt{f_1^2 - g_1^2}} \\ \frac{f_2}{\sqrt{g_2^2 - f_2^2}} \sin u & \frac{f_2}{\sqrt{g_2^2 - f_2^2}} \cos u & \frac{g_2}{\sqrt{g_2^2 - f_2^2}} \end{pmatrix}, \quad R = \frac{f_2}{\sqrt{g_2^2 - f_2^2}} L'_2 + \frac{f_1}{\sqrt{f_1^2 - g_1^2}} L'_3.$$

S and T are the same as in Example (8), the L'_i are as in (5), and all partial derivatives of f and g are evaluated at (v, w) .

Example 10 (Hyperbolic rotation of \mathbb{R}^2). Consider the triply orthogonal Lorentz system

$$r(u, v, w) = (f(u, v) \cosh w, g(u, v), f(u, v) \sinh w),$$

obtained from a 2-dimensional Euclidean system (f, g) as in Construction 5. For this system,

$$\Phi = \begin{pmatrix} \frac{f_1}{\sqrt{f_1^2 + g_1^2}} \cosh w & \frac{g_1}{\sqrt{f_1^2 + g_1^2}} & \frac{f_1}{\sqrt{f_1^2 + g_1^2}} \sinh w \\ \frac{f_2}{\sqrt{f_2^2 + g_2^2}} \cosh w & \frac{g_2}{\sqrt{f_2^2 + g_2^2}} & \frac{f_2}{\sqrt{f_2^2 + g_2^2}} \sinh w \\ \sinh w & 0 & \cosh w \end{pmatrix},$$

$$R = h_{E1} L'_3, \quad S = h_{E2} L'_3, \quad T = \frac{f_2}{\sqrt{f_2^2 + g_2^2}} L'_1 + \frac{f_1}{\sqrt{f_1^2 + g_1^2}} L'_2.$$

All partial derivatives of f and g are evaluated at (u, v) .

Example 11 (Hyperbolic rotation of $\mathbb{R}^{1,1}$). Consider the triply orthogonal Lorentz system

$$r(u, v, w) = (f(w, v) \sinh u, g(w, v), f(w, v) \cosh u),$$

obtained from a 2-dimensional Lorentz system (f, g) as in Construction 6. For this system,

$$\Phi = \begin{pmatrix} \cosh u & 0 & \sinh u \\ \frac{f_2}{\sqrt{g_2^2 - f_2^2}} \sinh u & \frac{g_2}{\sqrt{g_2^2 - f_2^2}} & \frac{f_2}{\sqrt{g_2^2 - f_2^2}} \cosh u \\ \frac{f_1}{\sqrt{f_1^2 - g_1^2}} \sinh u & \frac{g_1}{\sqrt{f_1^2 - g_1^2}} & \frac{f_1}{\sqrt{f_1^2 - g_1^2}} \cosh u \end{pmatrix},$$

$$R = \frac{f_1}{\sqrt{f_1^2 - g_1^2}} L'_2 + \frac{f_2}{\sqrt{g_2^2 - f_2^2}} L'_3, \quad S = h_{L2} L'_1, \quad T = h_{L1} L'_1.$$

All partial derivatives of f and g are evaluated at (w, v) .

4. Geometry of the triply orthogonal families of surfaces

4.1. Euclidean case.

For a triply orthogonal coordinate transformation $r(u^1, u^2, u^3) = (x^1, x^2, x^3)$ in Euclidean \mathbb{R}^3 with inner product $\langle \cdot, \cdot \rangle = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$, the metric tensor $dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ has the form $dl^2 = \langle r_{u^1}, r_{u^1} \rangle (du^1)^2 + \langle r_{u^2}, r_{u^2} \rangle (du^2)^2 + \langle r_{u^3}, r_{u^3} \rangle (du^3)^2$ in the coordinates u^i . A classic problem in differential geometry asks for what functions $H_k(u^i)$ the diagonal metric

$$dl^2 = \sum_{k=1}^3 H_k^2 (du^k)^2 \quad (6)$$

arises from an orthogonal coordinate transformation $r(u^1, u^2, u^3) = (x^1, x^2, x^3)$, so that $H_k^2 = \langle r_{u^k}, r_{u^k} \rangle$. The integrability condition for the existence of such a coordinate transformation is that the Riemann curvature tensor be identically zero. In terms of the quantities

$$Q_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial u^j}, \quad i \neq j, \quad (7)$$

(the so-called rotation coefficients [15]), the zero curvature condition $R_{ijkl} = 0$ yields the Gauss-Lamé equations

$$\begin{aligned} \frac{\partial Q_{21}}{\partial u^3} &= Q_{23} Q_{31} & \frac{\partial Q_{12}}{\partial u^3} &= Q_{13} Q_{32} & \frac{\partial Q_{13}}{\partial u^2} &= Q_{12} Q_{23} \\ \frac{\partial Q_{31}}{\partial u^2} &= Q_{32} Q_{21} & \frac{\partial Q_{32}}{\partial u^1} &= Q_{31} Q_{12} & \frac{\partial Q_{23}}{\partial u^1} &= Q_{21} Q_{13} \\ \frac{\partial Q_{23}}{\partial u^3} + \frac{\partial Q_{32}}{\partial u^2} + Q_{21} Q_{31} &= 0 & \frac{\partial Q_{13}}{\partial u^3} + \frac{\partial Q_{31}}{\partial u^1} + Q_{12} Q_{32} &= 0 & \frac{\partial Q_{12}}{\partial u^2} + \frac{\partial Q_{21}}{\partial u^1} + Q_{13} Q_{23} &= 0. \end{aligned} \quad (8)$$

If the Q_{ij} are known, the Lamé coefficients H_i may be found as solutions of the system (7). One may then find r by solving the following system of equations, where the Γ_{jk}^i are the Christoffel symbols of the metric, expressed in terms of the H_i [6, 16]:

$$\frac{\partial^2 x^i}{\partial u^k \partial u^l} = \Gamma_{kl}^k \frac{\partial x^i}{\partial u^k} + \Gamma_{lk}^l \frac{\partial x^i}{\partial u^l}, \quad \frac{\partial^2 x^i}{\partial u^l \partial u^l} = \sum_k \Gamma_{ll}^k \frac{\partial x^i}{\partial u^k}.$$

Two different solutions H_k and \tilde{H}_k with the same Q_{ij} are said to be *Combescure equivalent* ([15]). In particular, H_k and $-H_k$ are Combescure equivalent, where the ambiguity in sign comes from the two choices in square root in normalizing the rows of $(Dr)^T$ to obtain the orthogonal matrix of frames, $\Lambda(Dr)^T = \Phi \in \text{SO}(3)$.

Regarding each of the three coordinates u^j in turn as a constant parameter $u^j = c^j$ in the transformation $(x^1, x^2, x^3) = r(u^1, u^2, u^3)$, we obtain three families of surfaces, triply orthogonal to each other. Two surfaces from different families, defined by $u^i = c^i$ and $u^j = c^j$ with $i \neq j$, intersect along a (coordinate) line of principal curvature [7].

The fundamental forms I_i and II_i of the surfaces cut out by fixed values of u^i have the following form, where k_{ij} is the (principal) curvature of the i th surface in the u^j -direction:

$$\begin{aligned} I_1 &= p_1^2(du^2)^2 + q_1^2(du^3)^2 & II_1 &= p_1^2k_{12}(du^2)^2 + q_1^2k_{13}(du^3)^2 \\ I_2 &= p_2^2(du^3)^2 + q_2^2(du^1)^2 & II_2 &= p_2^2k_{23}(du^3)^2 + q_2^2k_{21}(du^1)^2 \\ I_3 &= p_3^2(du^1)^2 + q_3^2(du^2)^2 & II_3 &= p_3^2k_{31}(du^1)^2 + q_3^2k_{32}(du^2)^2. \end{aligned} \quad (9)$$

The coefficients of the first fundamental form are the corresponding coefficients H_k^2 in (6). We take the signs of p_i and q_i so that

$$p_3 = q_2 = -H_1, \quad p_1 = q_3 = -H_2, \quad p_2 = q_1 = -H_3. \quad (10)$$

Setting

$$\begin{aligned} A_1 &= p_1k_{12} & A_2 &= p_2k_{23} & A_3 &= p_3k_{31} \\ B_1 &= q_1k_{13} & B_2 &= q_2k_{21} & B_3 &= q_3k_{32}, \end{aligned}$$

the second fundamental forms appear as

$$\begin{aligned} II_1 &= p_1A_1(du^2)^2 + q_1B_1(du^3)^2 \\ II_2 &= p_2A_2(du^3)^2 + q_2B_2(du^1)^2 \\ II_3 &= p_3A_3(du^1)^2 + q_3B_3(du^2)^2. \end{aligned}$$

In terms of r and the H_i , the coefficients of the second fundamental forms are

$$\begin{aligned} p_1A_1 &= \frac{1}{H_1}\langle r_{u^2u^2}, r_{u^1} \rangle & q_1B_1 &= \frac{1}{H_1}\langle r_{u^3u^3}, r_{u^1} \rangle \\ p_2A_2 &= \frac{1}{H_2}\langle r_{u^3u^3}, r_{u^2} \rangle & q_2B_2 &= \frac{1}{H_2}\langle r_{u^1u^1}, r_{u^2} \rangle \\ p_3A_3 &= \frac{1}{H_3}\langle r_{u^1u^1}, r_{u^3} \rangle & q_3B_3 &= \frac{1}{H_3}\langle r_{u^2u^2}, r_{u^3} \rangle. \end{aligned} \quad (11)$$

The coefficients of the quadratic forms I and II satisfy the Gauss-Codazzi compatibility equations, which are identical to the Gauss-Lamé equations (8). In terms of new variables α_i and β_i defined as

$$\begin{aligned} \alpha_1 &= Q_{23} = \frac{1}{q_1} \frac{\partial p_1}{\partial u^3} & \beta_1 &= Q_{32} = \frac{1}{p_1} \frac{\partial q_1}{\partial u^2} \\ \alpha_2 &= Q_{31} = \frac{1}{q_2} \frac{\partial p_2}{\partial u^1} & \beta_2 &= Q_{13} = \frac{1}{p_2} \frac{\partial q_2}{\partial u^3} \\ \alpha_3 &= Q_{12} = \frac{1}{q_3} \frac{\partial p_3}{\partial u^2} & \beta_3 &= Q_{21} = \frac{1}{p_3} \frac{\partial q_3}{\partial u^1}, \end{aligned}$$

the Gauss-Codazzi equations read as follows, where the three equations in the i th column are expressed only in terms of the geometry of the i th family of surfaces, as indicated by the subscript i :

$$\begin{aligned} \frac{\partial A_1}{\partial u^3} - \alpha_1 B_1 &= 0 & \frac{\partial A_2}{\partial u^1} - \alpha_2 B_2 &= 0 & \frac{\partial A_3}{\partial u^2} - \alpha_3 B_3 &= 0 \\ \frac{\partial B_1}{\partial u^2} - \beta_1 A_1 &= 0 & \frac{\partial B_2}{\partial u^3} - \beta_2 A_2 &= 0 & \frac{\partial B_3}{\partial u^1} - \beta_3 A_3 &= 0 \\ \frac{\partial \beta_1}{\partial u^2} + \frac{\partial \alpha_1}{\partial u^3} + A_1 B_1 &= 0 & \frac{\partial \beta_2}{\partial u^3} + \frac{\partial \alpha_2}{\partial u^1} + A_2 B_2 &= 0 & \frac{\partial \beta_3}{\partial u^1} + \frac{\partial \alpha_3}{\partial u^2} + A_3 B_3 &= 0. \end{aligned}$$

These equations are the compatibility conditions

$$T_{u^2} - S_{u^3} + [T, S] = 0, \quad R_{u^3} - T_{u^1} + [R, T] = 0, \quad S_{u^1} - R_{u^2} + [S, R] = 0 \quad (12)$$

for the system of equations

$$\Phi_{u^1} = R\Phi, \quad \Phi_{u^2} = S\Phi, \quad \Phi_{u^3} = T\Phi,$$

where

$$\Phi = \begin{pmatrix} \frac{1}{H_1}r_{u^1} \\ \frac{1}{H_2}r_{u^2} \\ \frac{1}{H_3}r_{u^3} \end{pmatrix} \in \text{SO}(3) \quad (13)$$

and $R, S, T \in \text{so}(3)$.

Differentiating the orthogonality conditions $\langle r_{u^i}, r_{u^j} \rangle = 0$ for $i \neq j$ and $\langle r_{u^i}, r_{u^i} \rangle = H_i^2$ yields the identities

$$\langle r_{u^i u^k}, r_{u^j} \rangle = -\langle r_{u^j u^k}, r_{u^i} \rangle, \quad i \neq j$$

$$H_i \frac{\partial H_i}{\partial u^j} = \langle r_{u^i u^j}, r_{u^i} \rangle = -\langle r_{u^i u^i}, r_{u^j} \rangle, \quad i \neq j$$

$$\langle r_{u^i u^k}, r_{u^j} \rangle = 0, \quad i, j, k \text{ distinct.}$$

From these identities, together with (7) and (11), we find that

$$\begin{aligned} Q_{12} = \alpha_3 = B_2 & & Q_{23} = \alpha_1 = B_3 & & Q_{31} = \alpha_2 = B_1 \\ Q_{21} = \beta_3 = A_1 & & Q_{32} = \beta_1 = A_2 & & Q_{13} = \beta_2 = A_3, \end{aligned}$$

so that R , S , and T may be expressed in terms of the Q_{ij} or in terms of the geometry of a single family of surfaces, where the subscript i indicates the family with u^i constant:

$$\begin{aligned} R &= \begin{pmatrix} 0 & -Q_{12} & -Q_{13} \\ Q_{12} & 0 & 0 \\ Q_{13} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_3 & -A_3 \\ \alpha_3 & 0 & 0 \\ A_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B_2 & -\beta_2 \\ B_2 & 0 & 0 \\ \beta_2 & 0 & 0 \end{pmatrix}, \\ S &= \begin{pmatrix} 0 & Q_{21} & 0 \\ -Q_{21} & 0 & -Q_{23} \\ 0 & Q_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_1 & 0 \\ -A_1 & 0 & -\alpha_1 \\ 0 & \alpha_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_3 & 0 \\ -\beta_3 & 0 & -B_3 \\ 0 & B_3 & 0 \end{pmatrix}, \\ T &= \begin{pmatrix} 0 & 0 & Q_{31} \\ 0 & 0 & Q_{32} \\ -Q_{31} & -Q_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & A_2 \\ -\alpha_2 & -A_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B_1 \\ 0 & 0 & \beta_1 \\ -B_1 & -\beta_1 & 0 \end{pmatrix}. \end{aligned}$$

In terms of the basis (4) of $\text{so}(3)$,

$$\begin{aligned} R &= 0L_1 + Q_{13}L_2 + Q_{12}L_3 \\ S &= Q_{23}L_1 + 0L_2 - Q_{21}L_3 \\ T &= -Q_{32}L_1 - Q_{31}L_2 + 0L_3 \end{aligned} \quad (14)$$

Construction 7 (Euclidean systems from Euler angle coordinates on $\text{SO}(3)$). Using the geometric framework laid out above, we may construct triply orthogonal Euclidean coordinate transformations r by the following five-step procedure developed by Bonnet and studied more extensively by Darboux [3]:

1. Choose an immersion $\Phi(u^1, u^2, u^3) : \Omega \subset \mathbb{R}^3 \rightarrow \text{SO}(3)$. We may write this in terms of Euler angle coordinates as $\Phi =$

$$\begin{pmatrix} \cos \phi \cos \theta & -\cos \psi \sin \theta + \sin \psi \sin \phi \cos \theta & \sin \psi \sin \theta + \cos \psi \sin \phi \cos \theta \\ \cos \phi \sin \theta & \cos \psi \cos \theta + \sin \psi \sin \phi \sin \theta & -\sin \psi \cos \theta + \cos \psi \sin \phi \sin \theta \\ -\sin \phi & \sin \psi \cos \phi & \cos \psi \cos \phi \end{pmatrix},$$

where θ, ψ , and ϕ are functions of the u^i .

2. Form the matrices $R = \Phi_{u^1} \Phi^{-1}$, $S = \Phi_{u^2} \Phi^{-1}$, and $T = \Phi_{u^3} \Phi^{-1}$.

3. Ensure that R, S , and T have the form of Equations (14), in which each is a linear combination of only two of the L_i , by imposing the following conditions on Φ :

$$\begin{aligned} \psi_{u^1} \cos \phi \cos \theta - \phi_{u^1} \sin \theta &= 0, \\ \psi_{u^2} \cos \phi \sin \theta + \phi_{u^2} \cos \theta &= 0, \\ \psi_{u^3} \sin \phi - \theta_{u^3} &= 0. \end{aligned} \tag{15}$$

4. Solve Equations (7) for H_1, H_2 , and H_3 .

5. Take Φ satisfying Step 3, and express it as in (13). Use this expression and the H_i from Step 4 to solve for r .

Bonnet derived a single third-order equation equivalent to the system (15), and Darboux [3] introduced a change of variables leading to the compatibility equation

$$\omega_{u^1 u^2 u^3} = \omega_{u^1} \omega_{u^2 u^3} \cot \omega - \omega_{u^2} \omega_{u^1 u^3} \tan \omega,$$

for (15). This third-order equation also arises as a specialization of a (2+1)-dimensional sine-Gordon equation discovered by Konopelchenko and Rogers [9].

Example 12 (Application of Construction 7 – Fresnel functions). Denote the coordinates on \mathbb{R}^3 as (u, v, w) . To satisfy Step 3 of Construction 7, take

$$\phi = \phi(u, w), \quad \psi = \psi(v), \quad \text{and} \quad \theta = 0.$$

The equations for the H_i are then

$$\begin{aligned} \frac{\partial H_1}{\partial v} &= 0 & \frac{\partial H_3}{\partial u} &= \phi_w H_1 & \frac{\partial H_2}{\partial u} &= (\psi_v \sin \phi) H_1 \\ \frac{\partial H_1}{\partial w} &= -\phi_u H_3 & \frac{\partial H_3}{\partial v} &= 0 & \frac{\partial H_2}{\partial w} &= (\psi_v \cos \phi) H_3. \end{aligned}$$

Solving for H_1 and H_3 with $\phi(u, w) = \frac{1}{2}(u^2 + w^2)$ gives

$$H_1(u, w) = \cos(uw) \quad \text{and} \quad H_3(u, w) = \sin(uw).$$

We then find H_2 in terms of the Fresnel function $\text{FresnelS}(x) = \int_0^x \sin(\frac{\pi}{2} t^2) dt$, as

$$H_2(u, v, w) = \psi_v \frac{\sqrt{\pi}}{2} \left[\text{FresnelS}\left(\frac{u+w}{\sqrt{\pi}}\right) + \text{FresnelS}\left(\frac{u-w}{\sqrt{\pi}}\right) \right].$$

Solving for r in Step 5 of Construction 7 yields the following triply orthogonal system in terms of FresnelS and $\text{FresnelC}(x) = \int_0^x \cos(\frac{\pi}{2}t^2)dt$:

$$r(u, v, w) = \frac{\sqrt{\pi}}{2} \left(\text{FresnelC}\left(\frac{u+w}{\sqrt{\pi}}\right) + \text{FresnelC}\left(\frac{u-w}{\sqrt{\pi}}\right), \right. \\ \left. \sin \psi [\text{FresnelS}\left(\frac{u+w}{\sqrt{\pi}}\right) + \text{FresnelS}\left(\frac{u-w}{\sqrt{\pi}}\right)], \cos \psi [\text{FresnelS}\left(\frac{u+w}{\sqrt{\pi}}\right) + \text{FresnelS}\left(\frac{u-w}{\sqrt{\pi}}\right)] \right).$$

4.2. Lorentz case.

We now consider triply orthogonal families of surfaces in Lorentz space $\mathbb{R}^{2,1}$ with inner product $\langle \cdot, \cdot \rangle = dx^1 \otimes dx^1 + dx^2 \otimes dx^2 - dx^3 \otimes dx^3$. For a Lorentz-orthogonal coordinate transformation $(x^1, x^2, x^3) = r(u^1, u^2, u^3)$, we have

$$\langle r_{u^1}, r_{u^1} \rangle > 0, \quad \langle r_{u^2}, r_{u^2} \rangle > 0, \quad \text{and} \quad \langle r_{u^3}, r_{u^3} \rangle < 0.$$

In the coordinates u^i , the metric $dl^2 = (dx^1)^2 + (dx^2)^2 - (dx^3)^2$ has the form

$$dl^2 = \langle r_{u^1}, r_{u^1} \rangle (du^1)^2 + \langle r_{u^2}, r_{u^2} \rangle (du^2)^2 + \langle r_{u^3}, r_{u^3} \rangle (du^3)^2 \\ = H_1^2 (du^1)^2 + H_2^2 (du^2)^2 - H_3^2 (du^3)^2, \tag{16}$$

where the H_i are chosen such that

$$H_1^2 = \langle r_{u^1}, r_{u^1} \rangle, \quad H_2^2 = \langle r_{u^2}, r_{u^2} \rangle, \quad \text{and} \quad H_3^2 = -\langle r_{u^3}, r_{u^3} \rangle. \tag{17}$$

As in the Euclidean case, we define the quantities

$$Q_{ij} = \frac{1}{H_j} \frac{\partial H_i}{\partial u^j}, \quad i \neq j. \tag{18}$$

Differentiating the orthogonality conditions and the expressions for the H_i^2 yields the identities

$$\langle r_{u^i u^k}, r_{u^j} \rangle = -\langle r_{u^j u^k}, r_{u^i} \rangle, \quad i \neq j \\ H_i \frac{\partial H_i}{\partial u^j} = \langle r_{u^i u^j}, r_{u^i} \rangle, \quad i = 1, 2; \quad -H_3 \frac{\partial H_3}{\partial u^j} = \langle r_{u^3 u^j}, r_{u^3} \rangle \\ \langle r_{u^i u^k}, r_{u^j} \rangle = 0, \quad i, j, k \text{ distinct.} \tag{19}$$

The fundamental forms I_i and II_i of the surfaces determined by fixed values of u^i have the diagonal form

$$I_i = \langle r_{u^j}, r_{u^j} \rangle (du^j)^2 + \langle r_{u^k}, r_{u^k} \rangle (du^k)^2, \\ II_i = \frac{1}{H_i} \langle r_{u^j u^j}, r_{u^i} \rangle (du^j)^2 + \frac{1}{H_i} \langle r_{u^k u^k}, r_{u^i} \rangle (du^k)^2,$$

where $j, k \neq i$. As in the Euclidean case, we write

$$p_3 = q_2 = -H_1, \quad p_1 = q_3 = -H_2, \quad p_2 = q_1 = -H_3,$$

and

$$\begin{aligned}\alpha_1 &= Q_{23} = \frac{1}{q_1} \frac{\partial p_1}{\partial u^3} & \beta_1 &= Q_{32} = \frac{1}{p_1} \frac{\partial q_1}{\partial u^2} \\ \alpha_2 &= Q_{31} = \frac{1}{q_2} \frac{\partial p_2}{\partial u^1} & \beta_2 &= Q_{13} = \frac{1}{p_2} \frac{\partial q_2}{\partial u^3} \\ \alpha_3 &= Q_{12} = \frac{1}{q_3} \frac{\partial p_3}{\partial u^2} & \beta_3 &= Q_{21} = \frac{1}{p_3} \frac{\partial q_3}{\partial u^1}.\end{aligned}$$

Defining the variables A_i, B_i through the relations

$$\begin{aligned}p_1 A_1 &= \frac{1}{H_1} \langle r_{u^2 u^2}, r_{u^1} \rangle, & p_2 A_2 &= -\frac{1}{H_2} \langle r_{u^3 u^3}, r_{u^2} \rangle, & p_3 A_3 &= \frac{1}{H_3} \langle r_{u^1 u^1}, r_{u^3} \rangle, \\ q_1 B_1 &= -\frac{1}{H_1} \langle r_{u^3 u^3}, r_{u^1} \rangle, & q_2 B_2 &= \frac{1}{H_2} \langle r_{u^1 u^1}, r_{u^2} \rangle, & q_3 B_3 &= \frac{1}{H_3} \langle r_{u^2 u^2}, r_{u^3} \rangle,\end{aligned}$$

the fundamental forms I_i and II_i read

$$\begin{aligned}I_1 &= p_1^2 (du^2)^2 - q_1^2 (du^3)^2 & II_1 &= p_1 A_1 (du^2)^2 - q_1 B_1 (du^3)^2 \\ I_2 &= -p_2^2 (du^3)^2 + q_2^2 (du^1)^2 & II_2 &= -p_2 A_2 (du^3)^2 + q_2 B_2 (du^1)^2 \\ I_3 &= p_3^2 (du^1)^2 + q_3^2 (du^2)^2 & II_3 &= p_3 A_3 (du^1)^2 + q_3 B_3 (du^2)^2.\end{aligned}\tag{20}$$

The triply orthogonal Lorentz frame of r is

$$\Phi = \begin{pmatrix} \frac{1}{H_1} r_{u^1} \\ \frac{1}{H_2} r_{u^2} \\ \frac{1}{H_3} r_{u^3} \end{pmatrix} \in \text{SO}(2, 1).\tag{21}$$

Using the identities (19), we obtain $R = \Phi_{u^1} \Phi^{-1}$, $S = \Phi_{u^2} \Phi^{-1}$, and $T = \Phi_{u^3} \Phi^{-1}$ as

$$\begin{aligned}R &= \begin{pmatrix} 0 & -Q_{12} & Q_{13} \\ Q_{12} & 0 & 0 \\ Q_{13} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha_3 & A_3 \\ \alpha_3 & 0 & 0 \\ A_3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -B_2 & \beta_2 \\ B_2 & 0 & 0 \\ \beta_2 & 0 & 0 \end{pmatrix}, \\ S &= \begin{pmatrix} 0 & Q_{21} & 0 \\ -Q_{21} & 0 & Q_{23} \\ 0 & Q_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_1 & 0 \\ -A_1 & 0 & \alpha_1 \\ 0 & \alpha_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \beta_3 & 0 \\ -\beta_3 & 0 & B_3 \\ 0 & B_3 & 0 \end{pmatrix}, \\ T &= \begin{pmatrix} 0 & 0 & -Q_{31} \\ 0 & 0 & -Q_{32} \\ -Q_{31} & -Q_{32} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\alpha_2 \\ 0 & 0 & -A_2 \\ -\alpha_2 & -A_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -B_1 \\ 0 & 0 & -\beta_1 \\ -B_1 & -\beta_1 & 0 \end{pmatrix}.\end{aligned}$$

In terms of the orthonormal basis (5) of $so(2, 1)$, these are

$$\begin{aligned}R &= 0L'_1 + Q_{13}L'_2 + Q_{12}L'_3 \\ S &= Q_{23}L'_1 + 0L'_2 - Q_{21}L'_3 \\ T &= -Q_{32}L'_1 - Q_{31}L'_2 + 0L'_3.\end{aligned}\tag{22}$$

The compatibility conditions

$$T_{u^2} - S_{u^3} + [T, S] = 0, \quad R_{u^3} - T_{u^1} + [R, T] = 0, \quad S_{u^1} - R_{u^2} + [S, R] = 0,\tag{23}$$

for the system of equations

$$\Phi_{u^1} = R\Phi, \quad \Phi_{u^2} = S\Phi, \quad \Phi_{u^3} = T\Phi,$$

are for this case,

$$\begin{aligned}
 \frac{\partial Q_{21}}{\partial u^3} &= -Q_{23}Q_{31} & \frac{\partial Q_{12}}{\partial u^3} &= -Q_{13}Q_{32} & \frac{\partial Q_{13}}{\partial u^2} &= Q_{12}Q_{23} \\
 \frac{\partial Q_{31}}{\partial u^2} &= Q_{32}Q_{21} & \frac{\partial Q_{32}}{\partial u^1} &= -Q_{31}Q_{12} & \frac{\partial Q_{23}}{\partial u^1} &= Q_{21}Q_{13} \\
 \frac{\partial Q_{23}}{\partial u^3} + \frac{\partial Q_{32}}{\partial u^2} + Q_{21}Q_{31} &= 0 & \frac{\partial Q_{13}}{\partial u^3} + \frac{\partial Q_{31}}{\partial u^1} + Q_{12}Q_{32} &= 0 & \frac{\partial Q_{12}}{\partial u^2} + \frac{\partial Q_{21}}{\partial u^1} + Q_{13}Q_{23} &= 0.
 \end{aligned}
 \tag{24}$$

Construction 8 (Lorentz systems from angle coordinates on $SO(2, 1)$). This construction of Lorentz orthogonal systems is parallel to Construction 7.

1. Choose an immersion $\Phi(u^1, u^2, u^3) : \Omega \subset \mathbb{R}^3 \rightarrow SO(2, 1)$, where Φ is a product of matrices, in any chosen order, from the following three subgroups of $SO(2, 1)$:

$$\begin{aligned}
 P_\theta &= \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2), & P_\phi &= \begin{pmatrix} \cosh \phi & 0 & \sinh \phi \\ 0 & 1 & 0 \\ \sinh \phi & 0 & \cosh \phi \end{pmatrix} \in SO(1, 1), \\
 P_\gamma &= \begin{pmatrix} 1 & \gamma & -\gamma \\ -\gamma & 1 - \frac{\gamma^2}{2} & \frac{\gamma^2}{2} \\ -\gamma & -\frac{\gamma^2}{2} & 1 + \frac{\gamma^2}{2} \end{pmatrix} \in E(1),
 \end{aligned}$$

and θ, ϕ , and γ are functions of the u^i .

2. Form the matrices $R = \Phi_{u^1}\Phi^{-1}$, $S = \Phi_{u^2}\Phi^{-1}$, and $T = \Phi_{u^3}\Phi^{-1}$.

3. Impose conditions on the functions θ, ϕ , and γ so that R, S , and T have the form of Equations (22), where each is a linear combination of only two of the L'_i . If we choose in Step 1 the immersion $\Phi(u^1, u^2, u^3) = P_\phi P_\theta P_\gamma$, then these read

$$\begin{aligned}
 \gamma_{u^1} \cosh \phi \sin \theta + (\gamma + \theta)_{u^1} \sinh \phi &= 0, \\
 \gamma_{u^3} \sinh \phi \sin \theta + (\gamma + \theta)_{u^3} \cosh \phi &= 0, \\
 \phi_{u^2} - \gamma_{u^2} \cos \theta &= 0.
 \end{aligned}
 \tag{25}$$

4. Choose Φ to satisfy the conditions of Step 3.

5. Solve Equations (18) for H_1, H_2 , and H_3 .

6. Take Φ as in Step 3, and express it as in (21). Use this expression, and the H_i from Step 5, to solve for r .

Construction 9 (Schief’s construction of Lorentz orthogonal frames). Take $\Phi \in SO(3)$ for which R and S have the form in (14). Schief, in [12, p. 1679], gives an explicit formula for creating from Φ a Lorentz orthogonal frame $\Phi' \in SO(2, 1)$ whose R and S have the form in (22). Schief then gives one additional condition on the coefficients of Φ that guarantees that $T = \Phi'_{u^3}\Phi'^{-1}$ also has the form in (22). One may then use this Φ' in completing Steps 4 and 5 of Construction 8.

5. Variational Equations to the Gauss-Lamé System

Motivated by the framework of Fokas and Gelfand [4, 5] for surfaces in Lie groups and Lie algebras, we now consider variational equations to the Gauss-Lamé equations in $so(3)$ and $so(2, 1)$ as expressed in (12) and (23). Replacing R by $R + \epsilon A$,

S by $S + \epsilon B$ and T by $T + \epsilon C$ in equations (12) and (23), and retaining only the $O(\epsilon^1)$ term, yields a system of equations for A, B, C , namely the variational equations

$$\begin{aligned} B_w - C_v + [B, T] + [S, C] &= 0 \\ A_w - C_u + [A, T] + [R, C] &= 0 \\ A_v - B_u + [A, S] + [R, B] &= 0, \end{aligned}$$

where we write $(u^1, u^2, u^3) = (u, v, w)$. Symmetries of (12) and (23) lead to solutions of the variational equations; Theorem 5.2 in this section gives formulas for A, B, C , and a coordinate transformation F that generalize the formulas in [4], for surfaces, to families of surfaces arising from three-dimensional coordinate systems. As discussed in [1, 4, 5], the significance of this in the context of integrable PDEs that arise from Gauss-Codazzi equations for surfaces is that any of the infinitely many symmetries of the integrable PDE gives rise to a surface through a corresponding solution to the variational equation.

We begin by extending the results of the previous section in the following theorem:

Theorem 5.1. *Let $r(u, v, w) = (r_1(u, v, w), r_2(u, v, w), r_3(u, v, w))$ be a coordinate transformation, either of an open subset Ω of Euclidean space, $r : \Omega \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$, or of an open subset Ω of Lorentz space, $r : \Omega \subseteq \mathbb{R}^{2,1} \rightarrow \mathbb{R}^{2,1}$. Define the coordinate map*

$$F(u, v, w) = r_1(u, v, w)M_1 + r_2(u, v, w)M_2 + r_3(u, v, w)M_3 : \Omega \rightarrow \mathfrak{g},$$

where in the Euclidean case, $\mathfrak{g} = \mathfrak{so}(3)$, and $M_i = L_i$ as in (4), and in the Lorentz case, $\mathfrak{g} = \mathfrak{so}(2, 1)$, and $M_i = L'_i$ as in (5).

We may then form the matrix $\Phi = \Lambda(Dr)^T \in G$, where $(Dr)^T$ is the transpose of the Jacobian of r , Λ is a diagonal matrix, and G is $\text{SO}(3)$ in the Euclidean case and $\text{SO}(2, 1)$ in the Lorentz case.

Define $R \doteq \Phi_u \Phi^{-1}$, $S \doteq \Phi_v \Phi^{-1}$, and $T \doteq \Phi_w \Phi^{-1}$. Then,

$$S_w - T_v + [S, T] = 0, \quad R_w - T_u + [R, T] = 0, \quad R_v - S_u + [R, S] = 0.$$

Define $A \doteq \Phi F_u \Phi^{-1}$, $B \doteq \Phi F_v \Phi^{-1}$, and $C \doteq \Phi F_w \Phi^{-1}$. Then,

$$\begin{aligned} B_w - C_v + [B, T] + [S, C] &= 0, \\ A_w - C_u + [A, T] + [R, C] &= 0, \\ A_v - B_u + [A, S] + [R, B] &= 0. \end{aligned}$$

The metric of the coordinate system defined by r in \mathbb{R}^3 or $\mathbb{R}^{2,1}$, or by F in \mathfrak{g} , is

$$\langle A, A \rangle du^2 + \langle A, B \rangle dudv + \langle B, B \rangle dv^2 + \langle A, C \rangle dudw + \langle B, C \rangle dvdw + \langle C, C \rangle dw^2,$$

where the bracket in \mathfrak{g} is the Killing form

$$\langle M, N \rangle = \frac{1}{2} \text{tr}(MN).$$

If the coordinate system is triply orthogonal, then

$$\langle A, B \rangle = \langle A, C \rangle = \langle B, C \rangle = 0$$

so that the metric has the form

$$\langle A, A \rangle du^2 + \langle B, B \rangle dv^2 + \langle C, C \rangle dw^2. \quad \blacksquare$$

The proof of Theorem 5.1 is by direct computation. The first and second fundamental forms of the surfaces cut out by fixed u, v, w may be expressed in terms of the matrices A, B, C, R, S, T in Theorem 5.1. In the case of a triply orthogonal system, the fundamental forms given by (9) in the Euclidean case and by (20) in the Lorentz case are

$$\begin{aligned} I_u &= \langle B, B \rangle dv^2 + \langle C, C \rangle dw^2 & II_u &= \langle B_v + [B, S], \frac{\pm A}{|A|} \rangle dv^2 + \langle C_w + [C, T], \frac{\pm A}{|A|} \rangle dw^2 \\ I_v &= \langle C, C \rangle dw^2 + \langle A, A \rangle du^2 & II_v &= \langle C_w + [C, T], \frac{\pm B}{|B|} \rangle dw^2 + \langle A_u + [A, R], \frac{\pm B}{|B|} \rangle du^2 \\ I_w &= \langle A, A \rangle du^2 + \langle B, B \rangle dv^2 & II_w &= \langle A_u + [A, R], \frac{\pm C}{|C|} \rangle du^2 + \langle B_v + [B, S], \frac{\pm C}{|C|} \rangle dv^2, \end{aligned}$$

where the choice of sign in II_i is determined by the sign of H_i , where $H_1^2 = \langle A, A \rangle = |A|^2$, $H_2^2 = \langle B, B \rangle = |B|^2$, and $H_3^2 = -\langle C, C \rangle = |C|^2$.

Example 13 (Rotation of $\mathbb{R}^{1,1}$). For the system of Example (9), in terms of the basis (5) of $\mathfrak{so}(2, 1)$,

$$\begin{aligned} A &= \frac{f(f_1g_2 - g_1f_2)(2\cos^2(u) - 1)}{\sqrt{f_1^2 - g_1^2}\sqrt{g_2^2 - f_2^2}}L_1 - \frac{2fg_2\sin(u)\cos(u)}{\sqrt{g_2^2 - f_2^2}}L_2 - \frac{2fg_1\sin(u)\cos(u)}{\sqrt{f_1^2 - g_1^2}}L_3, \\ B &= \frac{2f_1(f_1g_2 - g_1f_2)\sin(u)\cos(u)}{\sqrt{f_1^2 - g_1^2}\sqrt{g_2^2 - f_2^2}}L_1 + \frac{2f_1g_2\cos^2(u) - g_1f_2 - f_1g_2}{\sqrt{g_2^2 - f_2^2}}L_2 - \frac{2f_1g_1\sin^2(u)}{\sqrt{f_1^2 - g_1^2}}L_3, \\ C &= \frac{2f_2(f_1g_2 - g_1f_2)\sin(u)\cos(u)}{\sqrt{f_1^2 - g_1^2}\sqrt{g_2^2 - f_2^2}}L_1 - \frac{2f_2g_2\sin^2(u)}{\sqrt{f_2^2 - g_2^2}}L_2 + \frac{2f_2g_1\cos^2(u) - g_1f_2 - f_1g_2}{\sqrt{g_1^2 - f_1^2}}L_3, \end{aligned}$$

where f, g , and their partial derivatives are evaluated at (v, w) .

A generalized converse of Theorem 5.1 generalizes the work of Fokas and Gelfand in [4] to obtain a formula for three-dimensional, but not necessarily orthogonal, coordinate systems:

Theorem 5.2. Let G be a Lie group with Lie algebra \mathfrak{g} . Let R, S , and T be \mathfrak{g} -valued differentiable functions of $(u, v, w) \in \Omega \subseteq \mathbb{R}^3$. Suppose that R, S , and T satisfy the PDEs

$$S_w - T_v + [S, T] = 0, \quad R_w - T_u + [R, T] = 0, \quad R_v - S_u + [R, S] = 0. \quad (26)$$

Then, there exists a G -valued differentiable function $\Phi(u, v, w)$ such that

$$\Phi_u = R\Phi, \quad \Phi_v = S\Phi, \quad \Phi_w = T\Phi. \quad (27)$$

Let A, B , and C be \mathfrak{g} -valued differentiable functions of $(u, v, w) \in \Omega$, defined as follows, where $\alpha_i \in \mathbb{R}$ and $M \in \mathfrak{g}$ are constants and λ is a parameter:

$$\begin{aligned} A &= \alpha_1 R_u + \alpha_2 R_v + \alpha_3 R_w + \alpha_4(uR_u + R) + \alpha_5 vR_v + \alpha_6 wR_w + [M, R] + \alpha_7 R_\lambda, \\ B &= \alpha_1 S_u + \alpha_2 S_v + \alpha_3 S_w + \alpha_4 uS_u + \alpha_5(vS_v + S) + \alpha_6 wS_w + [M, S] + \alpha_7 S_\lambda, \\ C &= \alpha_1 T_u + \alpha_2 T_v + \alpha_3 T_w + \alpha_4 uT_u + \alpha_5 vT_v + \alpha_6(wT_w + T) + [M, T] + \alpha_7 T_\lambda. \end{aligned} \tag{28}$$

Then A, B , and C satisfy the integrability conditions

$$\begin{aligned} B_w - C_v + [B, T] + [S, C] &= 0 \\ A_w - C_u + [A, T] + [R, C] &= 0 \\ A_v - B_u + [A, S] + [R, B] &= 0, \end{aligned} \tag{29}$$

so that there exists a \mathfrak{g} -valued immersion $F(u, v, w) : \Omega \rightarrow \mathfrak{g}$ such that

$$F_u = \Phi^{-1} A \Phi, \quad F_v = \Phi^{-1} B \Phi, \quad F_w = \Phi^{-1} C \Phi. \tag{30}$$

F has the form, up to an additive constant matrix,

$$F = \Phi^{-1}[\alpha_1 R + \alpha_2 S + \alpha_3 T + \alpha_4 uR + \alpha_5 vS + \alpha_6 wT + M + \alpha_7 \Phi_\lambda \Phi^{-1}] \Phi. \tag{31}$$

The mapping F induces the following metric on its image in \mathfrak{g} :

$$\langle A, A \rangle du^2 + \langle A, B \rangle dudv + \langle B, B \rangle dv^2 + \langle A, C \rangle dudw + \langle B, C \rangle dvdw + \langle C, C \rangle dw^2. \tag{32}$$

Proof: Equations (26) are the compatibility equations for solving for Φ in (27). Similarly, equations (29) are the compatibility equations for solving for F in (30). The formula (31) and the fact that the formulae in (28) satisfy (29) are verified by direct computation.

Remark 1. As noted above, the equations (29) are variational equations to the system (26), so that symmetries of (26) give solutions of (29). Similarly to the case of surfaces in [4], the terms $[M, \cdot]$ in the solutions (28) come from the invariance of (26) under the conjugation $R \mapsto e^{\epsilon M} R e^{-\epsilon M}$, $S \mapsto e^{\epsilon M} S e^{-\epsilon M}$, $T \mapsto e^{\epsilon M} T e^{-\epsilon M}$ for a constant matrix M . The solutions corresponding to $\alpha_1, \alpha_2, \alpha_3, \alpha_7$ come from invariance of (26) under translations of u, v, w , and λ respectively. Invariance of (26) under the pairs of dilations $u \mapsto c_1 u$ and $R \mapsto c_1 R$, $v \mapsto c_2 v$ and $S \mapsto c_2 S$, and $w \mapsto c_3 w$ and $T \mapsto c_3 T$, give rise to the solutions corresponding to $\alpha_4, \alpha_5, \alpha_6$.

Remark 2. The expressions (28) are *not* the complete general solution to the variational equations (29). For example, comparing the expressions for A, B, C given in Example (13) to the corresponding matrices R, S, T given in Example (9), we see that the former do not have the form given by equation (28) for any values of the parameters. Solutions to (29) that do not have the form (28) give rise to coordinate transformations F that are solutions to (30) but do not have the form (31).

Remark 3. Even if one starts with a solution R, S, T to the equations (26) that comes from an orthogonal coordinate system by the scheme of Theorem 5.1,

the coordinate system corresponding to the immersion F given in (31) is not, in general, orthogonal in Euclidean or Lorentz space. To see this, one may take R, S , and T from any of the examples of Section 3, insert them into the formula for F in (31), and check that the resulting F is not in general orthogonal for arbitrary values of the constants. More generally, suppose that one starts with a solution to equations (26) and produces an immersion F via the formula (31). The matrices Φ, R, S , and T as determined by Theorem 5.1 for this immersion are not in general the same as the original Φ, R, S , and T used to construct F via Theorem 5.2. That is, the formula (31) does not in general give a coordinate system with the same Φ, R, S , and T that were used to construct it.

In fact, suppose that we start with any solution R, S, T to equations (26) and the corresponding solution Φ to equations (27). Then, given *any* immersion $F : \Omega \rightarrow \mathfrak{g}$, the matrices $A = \Phi F_u \Phi^{-1}$, $B = \Phi F_v \Phi^{-1}$, $C = \Phi F_w \Phi^{-1}$ are solutions to the variational equations (29) for this R, S, T . In other words, given a solution to the Gauss-Lamé equations (26), *any* immersion $F : \Omega \rightarrow \mathfrak{g}$ satisfies equations (30) for some solution A, B, C of the variational equations (29).

Similar remarks apply to formulas analogous to those in Theorem 5.2 but restricted to surfaces, as given in [4] (rather than to families of surfaces arising from 3-dimensional coordinate transformations, as we consider here). The usefulness of the formulas in the case of surfaces is therefore not to produce 3-dimensional coordinate systems with a given scaled Jacobian Φ , but rather in the ability to produce infinitely many surfaces related to symmetries of an integrable system. Theorems 5.1 and 5.2 allow for a similar approach for integrable PDEs arising as the Gauss-Lamé equations for classes of coordinate systems, such as the (2+1)-dimensional sine-Gordon system [12]. These theorems may be extended to immersions in arbitrary Lie groups and Lie algebras, corresponding to coordinate systems in higher-dimensional inner product spaces. The approach given by these theorems has a direct connection to Cartan's method of moving frames, as described for surface immersions in the appendix of [4] and [1]. We will explore these avenues in a later publication.

We close with a note on how the results of Theorem 5.2 may be applied, making use of the following observation: In each of the coordinate systems in Examples 6 through 11 in Section 3, at least two of the rotation coefficients Q_{ij} vanish, so that two of the matrices R, S , and T are multiples of the same basis element L_i or L'_i of the corresponding Lie algebra. For example, the matrix triple of Example (9) has the form

$$R = Q_{13}L'_2 + Q_{12}L'_3, \quad S = Q_{23}L'_1, \quad T = -Q_{32}L'_1.$$

Since $\langle L'_i, L'_j \rangle = 0$ for $i \neq j$, the formulas (28) for A, B , and C in this example yield $\langle A, B \rangle = 0 = \langle A, C \rangle$ if $\alpha_7 = 0$ and $M = 0$, or if $\alpha_j = 0$ for all j and $M = m_2L'_2 + m_3L'_3$ for constants m_2, m_3 . For such choices of the α_i and M , formula (31), with either v or w equal to a fixed constant, yields a surface orthogonally parametrized in the coordinates (u, w) or (u, v) , respectively.

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Received April 29, 2013
and in final form October 1, 2013