

Integral Formula and Upper Estimate of I and J-Bessel Functions on Jordan Algebras

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Abstract. In this paper we give a new integral expression of I and J-Bessel functions on simple Euclidean Jordan algebras, integrating on a bounded symmetric domain. From this we easily get the upper estimate of Bessel functions. As an application we give an upper estimate of the integral kernel function of the holomorphic 1-dimensional semi-group acting on the space of square integrable functions on symmetric cones.

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1. Introduction and main results

In this paper we find in Theorem 3.1 a new integral expression of I and J-Bessel functions $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ on a Jordan algebra V . J-Bessel functions are first introduced by Faraut and Travaglini [10] for special cases, associating to self-adjoint representations of Jordan algebras (see also (4.13)), and generalized by Dib [5] (for $V = \text{Sym}(r, \mathbb{R})$ case see also [12] and [18]). It is well-known that $\mathcal{I}_\lambda(x)$, $\mathcal{J}_\lambda(x)$ are the holomorphic functions on $V^\mathbb{C}$ for λ in open dense subset of \mathbb{C} . On the other hand, for countable singular λ they are still well-defined on certain subvarieties. These are defined by the series expansion (see Section 3), and satisfy the following differential equation

$$\mathcal{B}_\lambda \mathcal{I}_\lambda - e \mathcal{I}_\lambda = 0, \quad \mathcal{B}_\lambda \mathcal{J}_\lambda + e \mathcal{J}_\lambda = 0$$

where $\mathcal{B}_\lambda : C^2(V) \rightarrow C(V) \otimes V^\mathbb{C}$ is the $V^\mathbb{C}$ -valued 2nd order differential operator defined in [8, Section XV.2], and e is the unit element on V (see [5, Proposition 1.7] or [8, Theorem XV.2.6]). Also \mathcal{I}_λ and \mathcal{J}_λ have the following integral expression

$$\mathcal{I}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n} \int_{e+iV} e^{\text{tr } w} e^{(w^{-1}|x)} \Delta(w)^{-\lambda} dw, \quad (1.1)$$

$$\mathcal{J}_\lambda(x) = \frac{\Gamma_\Omega(\lambda)}{(2i\pi)^n} \int_{e+iV} e^{\text{tr } w} e^{-(w^{-1}|x)} \Delta(w)^{-\lambda} dw \quad (1.2)$$

(see [5, Définition 1.2] or [8, Theorem XV.2.2]. For notations tr , $(\cdot|\cdot)$, Δ and $\Gamma_\Omega(\lambda)$ see Section 2 and (2.5)). There are some attempts to generalize these Bessel functions to operator-valued ones (see *e.g.* [6] and references therein), but it is still not very well-understood. In this paper we only treat scalar-valued ones.

Now we briefly state our theorem. Let V be a simple Euclidean Jordan algebra (*i.e.*, V is one of the $\text{Sym}(r, \mathbb{R})$, $\text{Herm}(r, \mathbb{C})$, $\text{Herm}(r, \mathbb{H})$, $\mathbb{R}^{1, n-1}$ or $\text{Herm}(3, \mathbb{O})$). We assume $\dim V = n$, $\text{rank } V = r$. We prove

Theorem 1.1. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\text{rank } \lambda}}$ (see (2.3) and (2.8)), take $k \in \mathbb{Z}_{\geq 0}$ such that $\text{Re } \lambda + k > \frac{2n}{r} - 1$. Then, we have the integral expressions*

$$\begin{aligned} \mathcal{I}_\lambda(x^2) &= c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -x, w) e^{2(x|\text{Re } w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw, \\ \mathcal{J}_\lambda(x^2) &= c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -ix, w) e^{2i(x|\text{Re } w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw. \end{aligned}$$

where c_λ is a constant and ${}_1F_1(-k, \lambda; x, w)$ is a polynomial of degree rk with respect to both x and w .

Here \mathcal{X}_l are the $L = \text{Str}(V^\mathbb{C})_0$ -orbits. $\overline{\mathcal{X}_l}$ are also characterized as the supports of some distributions on $V^\mathbb{C}$ (see [3] and (2.4)). $D \subset V^\mathbb{C}$ is the *bounded symmetric domain* and $h(w, w)$ is the *generic norm* on $V^\mathbb{C}$ (see Section 2). For the explicit forms of c_λ and ${}_1F_1(-k, \lambda; x, w)$ see Theorem 3.1. Especially if $\text{Re } \lambda > \frac{2n}{r} - 1$ we can take $k = 0$ and

$$\mathcal{I}_\lambda(x^2) = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - \frac{n}{r})} \int_D e^{2(x|\text{Re } w)} h(w, w)^{\lambda - \frac{2n}{r}} dw$$

and \mathcal{J}_λ is similar.

Now D is naturally identified with

$$G/K = \text{Bihol}(D)/\text{Stab}(0) = \text{Co}(V)_0/\text{Aut}_{\text{JTS}}(V)_0.$$

For $\lambda > \frac{2n}{r} - 1$, the universal covering group \tilde{G} acts unitarily on

$$\mathcal{O}(D) \cap L^2(D, h(w, w)^{\lambda - \frac{2n}{r}} dw)$$

by left translation. This defines the holomorphic discrete series representation of \tilde{G} . This is analytically continued with respect to $\lambda \in \mathbb{C}$, and become unitary when $\lambda \in \mathcal{W}$, the *(Berezin-)Wallach set* (see (2.9) and [20], [4]). The trivial representation corresponds to $\lambda = 0$.

From now we set $V = \mathbb{R}$. Let $I_\lambda(x)$ be the classical I-Bessel function (see [2, (4.12.2)]), and we set $\tilde{I}_\lambda(x) = (\frac{x}{2})^{-\lambda} I_\lambda(x)$. Then \tilde{I}_λ and \mathcal{I}_λ on \mathbb{R} are related as

$$\tilde{I}_\lambda(x) = \frac{1}{\Gamma(\lambda + 1)} \mathcal{I}_{\lambda+1}\left(\frac{x^2}{4}\right).$$

Therefore the above theorem is rewritten as

$$\tilde{I}_\lambda(x) = \frac{\lambda + k}{\pi \Gamma(\lambda + 1)} \int_{|w| < 1} {}_1F_1(-k, \lambda + 1; -xw) e^{x \text{Re } w} (1 - |w|^2)^{\lambda+k-1} dw,$$

where ${}_1F_1(-k, \lambda + 1; x)$ is the classical hypergeometric polynomial. This formula seems to be new even for $V = \mathbb{R}$ case. On the other hand, the formula (1.1) is rewritten as

$$\tilde{I}_\lambda(x) = \frac{1}{2i\pi\lambda} \int_{1+i\mathbb{R}} e^{w+\frac{x^2}{w}} w^{-\lambda-1} dw.$$

These two integral formulas are mutually independent, and cannot easily deduce one from another.

Again let V be a general Jordan algebra. Since D is bounded, we can prove from this formula the following corollary.

Corollary 1.2. *For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\text{rank } \lambda}}$, if $\text{Re } \lambda + k > \frac{2n}{r} - 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k} > 0$ such that*

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Re } x|_1}, \quad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Im } x|_1}$$

where $|x|_1$ is the norm defined in Definition 2.1.

In [17, Lemma 3.1] an upper estimate of $\mathcal{J}_\lambda(x)$ is given by another method, but our estimate is sharper. For detail see Remark 3.3. When $V = \mathbb{R}$, this corollary implies that if $\text{Re } \lambda > -k$ for some $k \in \mathbb{Z}_{\geq 0}$,

$$|\tilde{I}_\lambda(x)| = \frac{1}{|\Gamma(\lambda + 1)|} \left| \mathcal{I}_{\lambda+1} \left(\frac{x^2}{4} \right) \right| \leq C'_{\lambda,k} (1 + |x|^k) e^{|\text{Re } x|}.$$

On the other hand, we have the asymptotic expansion

$$\tilde{I}_\lambda(x) \sim \frac{\left(\frac{x}{2}\right)^{-\lambda}}{\sqrt{2\pi x}} \left(e^x \sum_{m=0}^{\infty} \frac{(-1)^m (\lambda, m)}{(2x)^m} + e^{-x+(\lambda+\frac{1}{2})\pi i} \sum_{m=0}^{\infty} \frac{(\lambda, m)}{(2x)^m} \right)$$

where (λ, m) are some numbers (see [2, (4.12.7)]), and this implies that

$$|\tilde{I}_\lambda(x)| \leq C''_\lambda \left(1 + |x|^{\max\{-\lambda-\frac{1}{2}, 0\}} \right) e^{|\text{Re } x|}.$$

Therefore our result is not the sharpest when $\text{Re } \lambda \leq 0$, but it still seems to be sufficiently sharp.

This paper is organized as follows: In Section 2, we recall some notations and facts about Euclidean Jordan algebras. In Section 3 we prove our main theorem, the integral formula and upper estimates. In Section 4, as an application of the inequality (Corollary 1.2), we give an upper estimate of the integral kernel function of the 1-dimensional semigroup on the functions on the symmetric cones.

2. Preliminaries

2.1. Simple Euclidean Jordan algebras. Let V be a simple Euclidean Jordan algebra of dimension n , rank r . We denote the unit element by e . Also let $V^\mathbb{C}$

be its complexification. For $x, y, z \in V^{\mathbb{C}}$, we write

$$\begin{aligned} L(x)y &:= xy, \\ x \square y &:= L(xy) + [L(x), L(y)], \\ P(x, z) &:= L(x)L(z) + L(z)L(x) - L(xz), \\ P(x) &:= P(x, x) = 2L(x)^2 - L(x^2), \\ B(x, y) &:= I_{V^{\mathbb{C}}} - 2x \square \bar{y} + P(x)P(\bar{y}) \end{aligned}$$

where $y \mapsto \bar{y}$ is the complex conjugation with respect to the real form V . Also, we write

$$\{x, y, z\} := (x \square \bar{y})z = P(x, z)\bar{y} = (x\bar{y})z + x(\bar{y}z) - (xz)\bar{y}.$$

Then $V^{\mathbb{C}}$ becomes a positive Hermitian Jordan triple system with this triple product.

We denote the *Jordan trace* and the *Jordan determinant* of the complex Jordan algebra $V^{\mathbb{C}}$ by $\text{tr}(x)$ and $\Delta(x)$ respectively. Also let $h(x, y)$ be the *generic norm* of the Jordan triple system $V^{\mathbb{C}}$. These can be expressed by $L(x)$, $P(x)$, and $B(x, y)$ (see [8, Proposition III.4.2], [9, Part V, Proposition VI.3.6]):

$$\begin{aligned} \text{Tr } L(x) &= \frac{n}{r} \text{tr}(x), \\ \text{Det } P(x) &= \Delta(x)^{\frac{2n}{r}}, \\ \text{Det } B(x, y) &= h(x, y)^{\frac{2n}{r}} \end{aligned}$$

where Tr and Det stand for the usual trace and determinant of complex linear operators on $V^{\mathbb{C}}$. Using the Jordan trace we define the inner product on $V^{\mathbb{C}}$:

$$(x|y) := \text{tr}(x\bar{y}), \quad x, y \in V^{\mathbb{C}}.$$

Then this is positive definite since V is Euclidean. Also we define the *symmetric cone* Ω and the *bounded symmetric domain* D by

$$\Omega := \{x^2 : x \in V, \Delta(x) \neq 0\},$$

$$D := (\text{connected component of } \{w \in V^{\mathbb{C}} : h(w, w) > 0\} \text{ which contains } 0).$$

Then Ω is self-dual, *i.e.*,

$$\Omega = \{x \in V : (x|y) > 0 \text{ for any } y \in \Omega\},$$

and D is biholomorphically equivalent to $V + \sqrt{-1}\Omega \subset V^{\mathbb{C}}$.

Let K_L and K be the identity components of *automorphism groups* of the Jordan algebra V and the Jordan triple system $V^{\mathbb{C}}$. Similarly let L and $L^{\mathbb{C}}$ be the identity components of *structure groups* of V and $V^{\mathbb{C}}$. Also let G be the identity component of *conformal group* of V :

$$\begin{aligned} K_L &:= \text{Aut}_{\text{J.Alg}}(V)_0 = \{k \in GL(V) : k(xy) = kx \cdot ky, \forall x, y \in V\}_0, \\ K &:= \text{Aut}_{\text{JTS}}(V^{\mathbb{C}})_0 = \{k \in GL(V^{\mathbb{C}}) : k\{x, y, z\} = \{kx, ky, kz\}, \forall x, y, z \in V^{\mathbb{C}}\}_0, \\ L &:= \text{Str}(V)_0 = \{l \in GL(V) : l\{x, y, z\} = \{lx, {}^t l^{-1}y, lz\}, \forall x, y, z \in V\}_0, \\ L^{\mathbb{C}} &:= \text{Str}(V^{\mathbb{C}})_0 = \{l \in GL(V^{\mathbb{C}}) : l\{x, y, z\} = \{lx, (l^*)^{-1}y, lz\}, \forall x, y, z \in V^{\mathbb{C}}\}_0, \\ G &:= \text{Co}(V)_0 = \text{Bihol}(D)_0 \simeq \text{Bihol}(V + \sqrt{-1}\Omega)_0 \end{aligned}$$

where ${}^t l$ and l^* stand for the transpose with respect to the bilinear form $\text{tr}(xy)$ and the sesquilinear form $\text{tr}(x\bar{y}) = (x|y)$. Then Ω and D are naturally identified with L/K_L and G/K respectively. For the classification of these groups see [13, Table 1] or [17, Table 1].

2.2. Spectral decomposition and some norms on $V^{\mathbb{C}}$. From now on we fix a *Jordan frame* $\{c_1, \dots, c_r\} \subset V$, i.e.,

$$c_j c_k = \delta_{jk} c_j, \quad \sum_{j=1}^r c_j = e,$$

and if $d_{j1}, d_{j2} \in V$ satisfy $c_j = d_{j1} + d_{j2}$, $d_{jk} d_{jl} = \delta_{kl} d_{jk}$, then $d_{j1} = 0$ or $d_{j2} = 0$.

Then for any $x \in V^{\mathbb{C}}$ there exist the unique numbers $t_1 \geq \dots \geq t_r \geq 0$ and the element $k \in K$ such that $x = k \sum_{j=1}^r t_j c_j$ ([8, Proposition X.3.2]). Using this, we define the p -norm on $V^{\mathbb{C}}$.

Definition 2.1. For $1 \leq p \leq \infty$ and for $x = k \sum_{j=1}^r t_j c_j \in V^{\mathbb{C}}$, we define

$$|x|_p := \begin{cases} \left(\sum_{j=1}^r |t_j|^p \right)^{\frac{1}{p}} & (1 \leq p < \infty), \\ \max_{j \in \{1, \dots, r\}} |t_j| & (p = \infty). \end{cases}$$

For example, we have $(x|x) = |x|_2^2$. Also if $x \in \Omega$ then all eigenvalues (in the sense of Jordan algebras. For $V = \text{Sym}(r, \mathbb{R})$ or $\text{Herm}(r, \mathbb{C})$ this coincides with the usual one) are positive and $|x|_1 = \text{tr } x$ holds. In addition, we can define D by $D = \{w \in V^{\mathbb{C}} : |w|_{\infty} < 1\}$. This norm satisfies the following properties.

Proposition 2.2 ([19, Theorem V.4, V.5] for $V = \text{Herm}(r, \mathbb{C})$ case). *Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then the following statements hold.*

- (1) For $x, y \in V^{\mathbb{C}}$, $|(x|y)| \leq |x|_p |y|_q$.
- (2) For $x \in V^{\mathbb{C}}$, $|x|_p = \max_{y \in V^{\mathbb{C}} \setminus \{0\}} \frac{|(x|y)|}{|y|_q}$.
- (3) $x \mapsto |x|_p$ is a norm on $V^{\mathbb{C}}$.

To prove this, we quote the following lemma (see [9, Part V, Proposition VI.2.1]):

Lemma 2.3. *For $x, y \in V^{\mathbb{C}}$, if $x \square \bar{y} = y \square \bar{x}$, then there exists an element $k \in K$ such that both x and y belong to $\mathbb{R}\text{-span}\{kc_1, \dots, kc_r\}$.*

Proof. [Proof of Proposition 2.2] (1) We note that $|(x|y)| \leq \max_{k \in K} |(kx|y)| = \max_{k \in K} \text{Re}(kx|y)$ since $e^{i\theta} I_{V^{\mathbb{C}}} \in K$ for any $\theta \in \mathbb{R}$. We take $k_0 \in K$ such that

$\operatorname{Re}(kx|y)$ ($k \in K$) attains its maximum at $k = k_0 \in K$. We put $k_0x =: x_0$. Then for any $D \in \mathfrak{k} = \operatorname{Lie}(K)$,

$$\left. \frac{d}{dt} \right|_{t=0} \operatorname{Re}(e^{tD}x_0|y) = \operatorname{Re}(Dx_0|y) = 0.$$

In the case when $D = u \square \bar{v} - v \square \bar{u}$ with $u, v \in V^{\mathbb{C}}$,

$$\begin{aligned} 0 &= \operatorname{Re}((u \square \bar{v})x_0|y) - \operatorname{Re}((v \square \bar{u})x_0|y) = \operatorname{Re}((x_0 \square \bar{v})u|y) - \operatorname{Re}((x_0 \square \bar{u})v|y) \\ &= \operatorname{Re}(u|(v \square \bar{x}_0)y) - \operatorname{Re}(v|(u \square \bar{x}_0)y) = \operatorname{Re}(u|(y \square \bar{x}_0)v) - \operatorname{Re}(v|(y \square \bar{x}_0)u) \\ &= \operatorname{Re}((x_0 \square \bar{y})u|v) - \operatorname{Re}(v|(y \square \bar{x}_0)u) = \operatorname{Re}((x_0 \square \bar{y} - y \square \bar{x}_0)u|v). \end{aligned}$$

Since $u, v \in V^{\mathbb{C}}$ are arbitrary and $(\cdot|\cdot)$ is non-degenerate, $x_0 \square \bar{y} = y \square \bar{x}_0$. Therefore by Lemma 2.3 there exists $k \in K$ such that $x_0, y \in \mathbb{R}\text{-span}\{kc_1, \dots, kc_r\}$. Let $x = k' \sum_{j=1}^r t_j c_j$, $y = k \sum_{j=1}^r s_j c_j$. Then

$$\begin{aligned} |(x|y)| &\leq \max_{k \in K} \operatorname{Re}(kx|y) = \operatorname{Re}(x_0|y) = \operatorname{Re} \left(k \sum_{j=1}^r t_j c_j \left| k \sum_{j=1}^r s_j c_j \right. \right) \\ &= \sum_{j=1}^r t_j s_j \leq \left(\sum_{j=1}^r |t_j|^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^r |s_j|^q \right)^{\frac{1}{q}} = |x|_p |y|_q. \end{aligned}$$

(2) (\geq) Clear from (1).

(\leq) For $x = k \sum_{j=1}^r t_j c_j \in V^{\mathbb{C}}$ ($t_1 \geq \dots \geq t_r \geq 0$), we find a $y \in V^{\mathbb{C}}$ which attains the equality. We set

$$y := \begin{cases} k \sum_{j=1}^r t_j^{p-1} c_j & (1 \leq p < \infty), \\ kc_1 & (p = \infty). \end{cases}$$

Then,

$$|y|_q = \begin{cases} \left(\sum_{j=1}^r t_j^{(p-1)q} \right)^{\frac{1}{q}} = \left(\sum_{j=1}^r t_j^p \right)^{\frac{p-1}{p}} = |x|_p^{p-1} & (1 < p < \infty), \\ 1 & (p = 1, \infty), \end{cases}$$

and

$$(x|y) = \begin{cases} \sum_{j=1}^r t_j^p = |x|_p^p = |x|_p |x|_p^{p-1} = |x|_p |y|_q & (1 \leq p < \infty), \\ t_1 = |x|_{\infty} = |x|_{\infty} |y|_1 & (p = \infty). \end{cases}$$

(3) Positivity and homogeneity are clear. For triangle inequality, by (2), for $x, y \in V^{\mathbb{C}}$,

$$|x + y|_p = \max_{|z|_q=1} |(x + y)z| \leq \max_{|z|_q=1} |(x|z)| + \max_{|z|_q=1} |(y|z)| = |x|_p + |y|_p$$

and this completes the proof. ■

We set

$$\mathcal{X}_l := \left\{ k \sum_{j=1}^l t_j c_j : k \in K, t_j > 0 \right\} = L^{\mathbb{C}} \cdot \sum_{j=1}^l e_j \subset V^{\mathbb{C}} \quad (l = 0, \dots, r). \quad (2.3)$$

Then $\overline{\mathcal{X}}_l = \mathcal{X}_0 \cup \mathcal{X}_1 \cup \dots \cup \mathcal{X}_l$ holds. $\overline{\mathcal{X}}_l$ are also characterized as the supports of the distributions which are the analytic continuation of $|\Delta(x)|^{2(\lambda - \frac{n}{r})} dx$:

$$\text{supp} \left(|\Delta(x)|^{2(\lambda - \frac{n}{r})} dx \Big|_{\lambda = l \frac{d}{2}} \right) = \overline{\mathcal{X}}_l, \quad l = 0, 1, \dots, r - 1 \quad (2.4)$$

(see [3, Proposition 5.5]).

2.3. Peirce decomposition and generalized power function. As before we fix a Jordan frame $\{c_1, \dots, c_r\} \subset V$. Then V is decomposed as

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{jk} \quad \text{where} \quad V_{jk} = \left\{ x \in V : L(c_l)x = \frac{\delta_{jl} + \delta_{kl}}{2} x \right\}.$$

Moreover $V_{jj} = \mathbb{R}c_j$ holds, and all V_{jk} 's ($j \neq k$) have the same dimension (see [8, Theorem IV.2.1, Corollary IV.2.6]). We write $\dim V_{jk} = d$. Then $\dim V = n = r + \frac{1}{2}r(r - 1)d$ holds.

Let $V_{(l)}^{\mathbb{C}} := \bigoplus_{1 \leq j \leq k \leq l} V_{jk}^{\mathbb{C}}$ ($l = 1, \dots, r$) and $P_{(l)}$ be the orthogonal projection on $V_{(l)}^{\mathbb{C}}$. We denote by $\det_{(l)}(x)$ the Jordan determinant on the Jordan algebra $V_{(l)}^{\mathbb{C}}$. We set $\Delta_l(x) := \det_{(l)}(P_{(l)}(x))$ for $x \in V^{\mathbb{C}}$. For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, the *generalized power function* on $V^{\mathbb{C}}$ is defined by

$$\Delta_{\mathbf{s}}(x) := \Delta_1^{s_1 - s_2}(x) \Delta_2^{s_2 - s_3}(x) \cdots \Delta_{r-1}^{s_{r-1} - s_r}(x) \Delta_r^{s_r}(x).$$

Then, the *Gindikin Gamma function* and *Pochhammer symbol* are defined as follows: for $\mathbf{s} \in \mathbb{C}^r$ and $\mathbf{m} \in (\mathbb{Z}_{\geq 0})^r$,

$$\Gamma_{\Omega}(\mathbf{s}) := \int_{\Omega} e^{-\text{tr}(x)} \Delta_{\mathbf{s}}(x) \Delta(x)^{-\frac{n}{r}} dx, \quad (\mathbf{s})_{\mathbf{m}} := \frac{\Gamma_{\Omega}(\mathbf{s} + \mathbf{m})}{\Gamma_{\Omega}(\mathbf{s})}. \quad (2.5)$$

This integral converges for $\text{Re } s_j > (j - 1)\frac{d}{2}$, and both functions are extended meromorphically on \mathbb{C}^r (see [8, Theorem VII.1.1] or [11, Theorem 2.1]). Moreover, we have

$$(\mathbf{s})_{\mathbf{m}} = \prod_{j=1}^r \left(s_j - (j - 1)\frac{d}{2} \right)_{m_j} \quad \text{where} \quad (s)_m = s(s + 1) \cdots (s + m - 1).$$

For $\mathbf{s} = (s_1, \dots, s_r) \in \mathbb{C}^r$, we set $\mathbf{s}^* = (s_r, \dots, s_1)$. Then we can prove easily

$$(\mathbf{s})_{\mathbf{m} + \mathbf{n}} = (\mathbf{s})_{\mathbf{m}} (\mathbf{s} + \mathbf{m})_{\mathbf{n}}, \quad (-\mathbf{s}^*)_{\mathbf{m}} = (-1)^{|\mathbf{m}|} \left(\mathbf{s} - \mathbf{m}^* + \frac{\mathbf{n}}{r} \right)_{\mathbf{m}^*} \quad (2.6)$$

where $|\mathbf{m}| = m_1 + \dots + m_r$. Here we identify $\lambda \in \mathbb{C}$ and $(\lambda, \dots, \lambda) \in \mathbb{C}^r$.

2.4. Polynomials on $V^{\mathbb{C}}$. We set $\mathbb{Z}_{++}^r := \{\mathbf{m} = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 0})^r : m_1 \geq m_2 \geq \dots \geq m_r \geq 0\}$, and denote the space of holomorphic polynomials on $V^{\mathbb{C}}$ by $\mathcal{P}(V^{\mathbb{C}})$. For $\mathbf{m} \in \mathbb{Z}_{++}^r$, we define $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) := \mathbb{C}\text{-span}\{\Delta_{\mathbf{m}} \circ l : l \in L^{\mathbb{C}}\}$. Then clearly $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ becomes a $L^{\mathbb{C}}$ -module. Moreover, we have

Theorem 2.4 (Hua–Kostant–Schmid, see [8, Theorem XI.2.4]).

$$\mathcal{P}(V^{\mathbb{C}}) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}).$$

These $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$'s are mutually inequivalent, and irreducible as $L^{\mathbb{C}}$ -modules.

Since Δ_l vanishes on $\overline{\mathcal{X}_{l-1}}$, all polynomials in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ vanish on $\overline{\mathcal{X}_{l-1}}$ if and only if $m_l \neq 0$.

We write $d_{\mathbf{m}} := \dim \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$, and $\Phi_{\mathbf{m}}(x) := \int_{K_L} \Delta_{\mathbf{m}}(kx) dk$. Then the K_L -fixed subspace in $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ is spanned by $\Phi_{\mathbf{m}}$ (see [8, Proposition XI.3.1]).

2.5. Inner products on $\mathcal{P}(V^{\mathbb{C}})$. For $f, g \in \mathcal{P}(V^{\mathbb{C}})$, we denote the *Fischer inner product* by $\langle f, g \rangle_F$:

$$\langle f, g \rangle_F := \frac{1}{\pi^n} \int_{V^{\mathbb{C}}} f(w) \overline{g(w)} e^{-(w|w)} dw = f \left(\frac{\partial}{\partial w} \right) \overline{g(w)} \Big|_{w=0}$$

(For the second equality see [8, Proposition XI.1.1]). Then the reproducing kernel of $\overline{\mathcal{P}(V^{\mathbb{C}})}^F$ (Hilbert completion of $\mathcal{P}(V^{\mathbb{C}})$) is given by $e^{(z|w)}$. We denote by $K^{\mathbf{m}}(z, w) = K_w^{\mathbf{m}}(z)$ the reproducing kernel of $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$ with respect to $\langle \cdot, \cdot \rangle_F$. Then clearly,

$$e^{(z|w)} = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} K^{\mathbf{m}}(z, w),$$

Also, by [8, Proposition XI.3.3, Proposition XI.4.1.(ii)], we have

$$K^{\mathbf{m}}(gz, w) = K^{\mathbf{m}}(z, g^*w) \quad \text{for any } g \in \text{Str}(V^{\mathbb{C}}),$$

$$K_e^{\mathbf{m}}(z) = \frac{1}{\|\Phi_{\mathbf{m}}\|_F^2} \Phi_{\mathbf{m}}(z) = \frac{d_{\mathbf{m}}}{\binom{n}{r}_{\mathbf{m}}} \Phi_{\mathbf{m}}(z)$$

and

$$K^{\mathbf{m}}(x, \bar{x}) = K^{\mathbf{m}}(x^2, e)$$

for $x \in V$, and therefore for any $x \in V^{\mathbb{C}}$ by analytic continuation.

Also, for $\lambda > \frac{2n}{r} - 1$, we denote the *weighted Bergman inner product* on D by $\langle \cdot, \cdot \rangle_{\lambda}$:

$$\langle f, g \rangle_{\lambda} := \frac{1}{\pi^n} \frac{\Gamma_{\Omega}(\lambda)}{\Gamma_{\Omega}(\lambda - \frac{n}{r})} \int_D f(w) \overline{g(w)} h(w, w)^{\lambda - \frac{2n}{r}} dw.$$

Then, these two inner products are related as follows:

Theorem 2.5 (Faraut–Korányi, see [8, Theorem XIII.2.7]). *If $f, g \in \mathcal{P}(V^{\mathbb{C}})$ are decomposed as $f = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} f_{\mathbf{m}}$, $g = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} g_{\mathbf{m}}$ ($f_{\mathbf{m}}, g_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$), then*

$$\langle f, g \rangle_{\lambda} = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}} \rangle_F. \tag{2.7}$$

Although the left hand side is only defined for $\lambda > \frac{2n}{r} - 1$, the right hand side extends meromorphically for $\lambda \in \mathbb{C}$. Therefore we can redefine $\langle \cdot, \cdot \rangle_\lambda$ with this formula for any $\lambda \in \mathbb{C}$ by restricting the domain. For $\lambda \in \mathbb{C}$ we set

$$\begin{aligned} \text{rank } \lambda &:= \max \{l \in \{0, 1, \dots, r\} : (\lambda)_{\mathbf{m}} \neq 0 \text{ for any } \mathbf{m} \in \mathbb{Z}_{++}^r \cap \{m_{l+1} = 0\}\} \\ &= \begin{cases} l & \text{if } \lambda \in (l\frac{d}{2} + \mathbb{Z}_{\leq 0}) \setminus \bigcup_{j=0}^{l-1} (j\frac{d}{2} + \mathbb{Z}_{\leq 0}) \quad (l = 0, 1, \dots, r-1), \\ r & \text{if } \lambda \notin \bigcup_{j=0}^{r-1} (j\frac{d}{2} + \mathbb{Z}_{\leq 0}). \end{cases} \end{aligned} \tag{2.8}$$

For example, if $d = 2$, *i.e.*, $V = \text{Herm}(r, \mathbb{C})$, then

$$\text{rank } \lambda = \begin{cases} 0 & (\lambda \in \mathbb{Z}_{\leq 0}), \\ l & (\lambda = l, l = 1, \dots, r-1), \\ r & (\lambda \notin r-1 + \mathbb{Z}_{\leq 0}). \end{cases}$$

Then $\langle \cdot, \cdot \rangle_\lambda$ defines a sesquilinear form on $\bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r, m_{\text{rank } \lambda + 1} = 0} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$. This form $\langle \cdot, \cdot \rangle_\lambda$ is positive definite if and only if

$$\lambda \in \mathcal{W} := \left\{0, \frac{d}{2}, \dots, (r-1)\frac{d}{2}\right\} \cup \left((r-1)\frac{d}{2}, \infty\right). \tag{2.9}$$

This set \mathcal{W} is called the *(Berezin-)Wallach set* (see [20] or [4]).

2.6. Invariant differential operators. For $\lambda \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$, we recall the differential operators $D^{(k)}$ from [8, Section XIV.2]:

$$D^{(k)}(\lambda) := \Delta(x)^{\frac{n}{r} - \lambda} \Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda - \frac{n}{r} + k}$$

where $\Delta\left(\frac{\partial}{\partial x}\right)$ is the differential operator characterized by $\Delta\left(\frac{\partial}{\partial x}\right) e^{(x|y)} = \Delta(y) e^{(x|y)}$. Then these operators commute with the $L^{\mathbb{C}}$ -action

$$(i.e., D^{(k)}(\lambda)(f \circ l) = (D^{(k)}(\lambda)f) \circ l \text{ for } f \in \mathcal{P}(V^{\mathbb{C}}) \text{ and } l \in L^{\mathbb{C}}).$$

Moreover:

Proposition 2.6. *We have*

$$D^{(k)}(\lambda)e^{(x|y)} = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} (-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}} (\lambda + \mathbf{m})_{k-\mathbf{m}} K^{\mathbf{m}}(x, y) e^{(x|y)},$$

and if $(\lambda)_{\mathbf{m}} \neq 0$ for any $\mathbf{m} \in \mathbb{Z}_{++}^r$, $|\mathbf{m}| \leq rk$,

$$D^{(k)}(\lambda)e^{(x|y)} = (\lambda)_{k1} F_1(-k, \lambda; -x, y) e^{(x|y)}$$

where

$${}_1F_1(-k, \lambda; -x, y) := \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} \frac{(-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x, y). \tag{2.10}$$

Proof. We follow the proof of [8, Proposition XIV.1.5]. For $x \in \Omega$ and

$$\lambda < -k + 1,$$

$$\begin{aligned} D^{(k)}(\lambda)e^{(x|e)} &= \Delta(x)^{\frac{n}{r}-\lambda} \Delta\left(\frac{\partial}{\partial x}\right)^k \Delta(x)^{\lambda-\frac{n}{r}+k} e^{(x|e)} \\ &= \Delta(x)^{\frac{n}{r}-\lambda} \Delta\left(\frac{\partial}{\partial x}\right)^k \frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)} \int_\Omega e^{(x|e-y)} \Delta(y)^{-\lambda+\frac{n}{r}-k} \Delta(y)^{-\frac{n}{r}} dy \\ &= \Delta(x)^{\frac{n}{r}-\lambda} \frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)} \int_\Omega e^{(x|e-y)} \Delta(e-y)^k \Delta(y)^{-\lambda-k} dy \\ &= \Delta(x)^{\frac{n}{r}-\lambda} \frac{1}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)} \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} d_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \int_\Omega e^{(x|e-y)} \Phi_{\mathbf{m}}(y) \Delta(y)^{-\lambda-k} dy \\ &= \Delta(x)^{\frac{n}{r}-\lambda} \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} d_{\mathbf{m}} \frac{(-k)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{\Gamma_\Omega\left(\mathbf{m}-\lambda+\frac{n}{r}-k\right)}{\Gamma_\Omega\left(-\lambda+\frac{n}{r}-k\right)} \Phi_{\mathbf{m}}(x^{-1}) \Delta(x)^{\lambda-\frac{n}{r}+k} e^{(x|e)} \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} \frac{d_{\mathbf{m}} (-k)_{\mathbf{m}} \left(-\lambda+\frac{n}{r}-k\right)_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{k-\mathbf{m}^*}(x) e^{(x|e)} \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} \frac{d_{k-\mathbf{m}^*} (-k)_{k-\mathbf{m}^*} \left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} \Phi_{\mathbf{m}}(x) e^{(x|e)}. \end{aligned}$$

Here we used [8, Lemma XI.2.3] at the 2nd and 5th equalities, and [8, Corollary XII.1.3] at the 4th equality. At the 6th equality we used $\Phi_{\mathbf{m}}(x^{-1})\Delta(x)^k = \Phi_{k-\mathbf{m}^*}(x)$, which follows from the linear isomorphism $\mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}}) \rightarrow \mathcal{P}_{k-\mathbf{m}^*}(V^{\mathbb{C}})$, $p \mapsto \Delta(x)^k p(x^{-1})$. Now, $d_{\mathbf{m}} = d_{k-\mathbf{m}^*}$ holds by this isomorphism, and by (2.6),

$$\begin{aligned} \frac{(-k)_{k-\mathbf{m}^*}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} &= \frac{(-1)^{|k-\mathbf{m}^*|} \left(\frac{n}{r}+\mathbf{m}\right)_{k-\mathbf{m}}}{\left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|k-\mathbf{m}^*|} \left(\frac{n}{r}\right)_k}{\left(\frac{n}{r}\right)_{\mathbf{m}} \left(\frac{n}{r}\right)_{k-\mathbf{m}^*}} = \frac{(-1)^{|k-\mathbf{m}^*|} (-k)_{\mathbf{m}}}{(-1)^{|\mathbf{m}|} \left(\frac{n}{r}\right)_{\mathbf{m}}}, \\ &\quad \left(-\lambda+\frac{n}{r}-k\right)_{k-\mathbf{m}^*} = (-1)^{|k-\mathbf{m}^*|} (\lambda+\mathbf{m})_{k-\mathbf{m}}. \end{aligned}$$

Therefore,

$$D^{(k)}(\lambda)e^{(x|e)} = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} (-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}} (\lambda+\mathbf{m})_{k-\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) e^{(x|e)}.$$

By the $L^{\mathbb{C}}$ -invariance of $D^{(k)}(\lambda)$, for $y \in \Omega$,

$$\begin{aligned} D^{(k)}(\lambda)e^{(x|y)} &= D^{(k)}(\lambda)e^{(P(y^{\frac{1}{2}})x|e)} \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} (-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}} (\lambda+\mathbf{m})_{k-\mathbf{m}} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \Phi_{\mathbf{m}}(P(y^{\frac{1}{2}})x) e^{(P(y^{\frac{1}{2}})x|e)} \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk} (-1)^{|\mathbf{m}|} (-k)_{\mathbf{m}} (\lambda+\mathbf{m})_{k-\mathbf{m}} K^{\mathbf{m}}(x, y) e^{(x|y)}. \end{aligned}$$

This holds for any $x, y \in V^{\mathbb{C}}$ and $\lambda \in \mathbb{C}$ by analytic continuation. The second equality follows from

$$(\lambda+\mathbf{m})_{k-\mathbf{m}} = \frac{(\lambda)_k}{(\lambda)_{\mathbf{m}}}. \quad \blacksquare$$

Using these differential operators, we can calculate $\langle f, g \rangle_\lambda$ for $\lambda \in \mathbb{C}$: for $\operatorname{Re} \lambda + k > \frac{2n}{r} - 1$ and $f, g \in \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r, m_{\operatorname{rank} \lambda + 1} = 0} \mathcal{P}_{\mathbf{m}}(V^{\mathbb{C}})$,

$$\langle f, g \rangle_\lambda = \begin{cases} \frac{c_{\lambda+k}}{(\lambda)_k} \int_D (D^{(k)}(\lambda)f)(w) \overline{g(w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw & (\operatorname{rank} \lambda = r) \\ \lim_{\mu \rightarrow \lambda} \frac{c_{\mu+k}}{(\mu)_k} \int_D (D^{(k)}(\mu)f)(w) \overline{g(w)} h(w, w)^{\mu+k-\frac{2n}{r}} dw & (\operatorname{rank} \lambda < r) \end{cases} \tag{2.11}$$

where $c_\lambda = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega(\lambda - \frac{n}{r})}$ (see [8, Proposition XIV.2.2, Proposition XIV.2.5]). We can prove easily that this equality holds not only for polynomials, but also for holomorphic functions $f, g \in \mathcal{O}(D)$ with $D^{(k)}(\lambda)f$ and g bounded on \overline{D} .

3. Proof for main theorem

For $\lambda \in \mathbb{C}$ with $\operatorname{rank} \lambda = r$, the I and J-Bessel functions are defined by

$$\begin{aligned} \mathcal{I}_\lambda(x) &:= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x), \\ \mathcal{J}_\lambda(x) &:= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) = \mathcal{I}_\lambda(-x). \end{aligned}$$

If $\operatorname{rank} \lambda < r$, then $(\lambda)_{\mathbf{m}} = 0$ for some \mathbf{m} , so we cannot define these functions on entire $V^{\mathbb{C}}$. However, if $x \in \overline{\mathcal{X}_l}$, $\Phi_{\mathbf{m}}(x) = 0$ for $m_{l+1} \neq 0$, and therefore for any $\lambda \in \mathbb{C}$ we can define I and J-Bessel functions for $x \in \overline{\mathcal{X}_{\operatorname{rank} \lambda}}$ (see (2.3) and (2.8)) by

$$\begin{aligned} \mathcal{I}_\lambda(x) &:= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, m_{\operatorname{rank} \lambda + 1} = 0} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{1}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x), \\ \mathcal{J}_\lambda(x) &:= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r, m_{\operatorname{rank} \lambda + 1} = 0} \frac{d_{\mathbf{m}}}{\left(\frac{n}{r}\right)_{\mathbf{m}}} \frac{(-1)^{|\mathbf{m}|}}{(\lambda)_{\mathbf{m}}} \Phi_{\mathbf{m}}(x) = \mathcal{I}_\lambda(-x). \end{aligned}$$

Now we are ready to state the main theorem.

Theorem 3.1. For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\operatorname{rank} \lambda}}$, take $k \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{Re} \lambda + k > \frac{2n}{r} - 1$. Then we have the integral expressions

$$\begin{aligned} \mathcal{I}_\lambda(x^2) &= c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -x, w) e^{2(x|\operatorname{Re} w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw, \\ \mathcal{J}_\lambda(x^2) &= c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -ix, w) e^{2i(x|\operatorname{Re} w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw. \end{aligned}$$

where

$$c_\lambda = \frac{1}{\pi^n} \frac{\Gamma_\Omega(\lambda)}{\Gamma_\Omega\left(\lambda - \frac{n}{r}\right)}, \quad {}_1F_1(-k, \lambda; x, w) = \sum_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r, |\mathbf{m}| \leq rk, \\ m_{\operatorname{rank} \lambda + 1} = 0}} \frac{(-k)_{\mathbf{m}}}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x, w).$$

When $\text{rank } \lambda = r$, the definition of ${}_1F_1$ clearly coincides with the one in (2.10).

Proof. We calculate $\langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)} \rangle_\lambda$ in two ways. By (2.7),

$$\begin{aligned} \langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)} \rangle_\lambda &= \left\langle \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} K_{\bar{x}}^{\mathbf{m}}, \sum_{\mathbf{n} \in \mathbb{Z}_{++}^r} K_x^{\mathbf{n}} \right\rangle_\lambda = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} \langle K_{\bar{x}}^{\mathbf{m}}, K_x^{\mathbf{m}} \rangle_F \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x, \bar{x}) = \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} K^{\mathbf{m}}(x^2, e) \\ &= \sum_{\mathbf{m} \in \mathbb{Z}_{++}^r} \frac{1}{(\lambda)_{\mathbf{m}}} \frac{d_{\mathbf{m}}}{\binom{r}{\mathbf{m}}} \Phi_{\mathbf{m}}(x^2) = \mathcal{I}(x^2). \end{aligned}$$

On the other hand, by (2.11) and Proposition 2.6,

$$\begin{aligned} \langle e^{(\cdot|\bar{x})}, e^{(\cdot|x)} \rangle_\lambda &= \lim_{\mu \rightarrow \lambda} \frac{c_{\mu+k}}{(\mu)_k} \int_D (D^{(k)}(\mu) e^{(w|\bar{x})}) \overline{e^{(w|x)}} h(w, w)^{\mu+k-\frac{2n}{r}} dw \\ &= \lim_{\mu \rightarrow \lambda} c_{\mu+k} \int_D {}_1F_1(-k, \mu; -x, w) e^{(w|\bar{x})} \overline{e^{(w|x)}} h(w, w)^{\mu+k-\frac{2n}{r}} dw \\ &= c_{\lambda+k} \int_D {}_1F_1(-k, \lambda; -x, w) e^{2(x|\text{Re } w)} h(w, w)^{\lambda+k-\frac{2n}{r}} dw. \end{aligned}$$

The formula for $\mathcal{J}_\lambda(x^2)$ follows by replacing x by ix . ■

From this theorem we can easily deduce the following corollary.

Corollary 3.2. For $\lambda \in \mathbb{C}$, $x \in \overline{\mathcal{X}_{\text{rank } \lambda}}$, if $\text{Re } \lambda + k > \frac{2n}{r} - 1$ for some $k \in \mathbb{Z}_{\geq 0}$, then there exists a positive constant $C_{\lambda,k} > 0$ such that

$$|\mathcal{I}_\lambda(x^2)| \leq C_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Re } x|_1}, \quad |\mathcal{J}_\lambda(x^2)| \leq C_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Im } x|_1}$$

where $|x|_1$ is the norm defined in Definition 2.1.

Proof. By Proposition 2.2, for $w \in D$, $x \in V^{\mathbb{C}}$,

$$|(\text{Re } x | \text{Re } w)| \leq |\text{Re } x|_1 |\text{Re } w|_\infty \leq |\text{Re } x|_1 \frac{|w|_\infty + |\bar{w}|_\infty}{2} \leq |\text{Re } x|_1.$$

Also, since ${}_1F_1(-k, \lambda; -x, w)$ is a polynomial of degree rk with respect to both x and w ,

$$|{}_1F_1(-k, \lambda; -x, w)| \leq C'_{\lambda,k} (1 + |x|_1^{rk}) (1 + |w|_\infty^{rk}) \leq 2C'_{\lambda,k} (1 + |x|_1^{rk}).$$

Therefore, by Theorem 3.1,

$$\begin{aligned} |\mathcal{I}_\lambda(x^2)| &\leq |c_{\lambda+k}| \int_D |{}_1F_1(-k, \lambda; -x, w)| e^{2(\text{Re } x | \text{Re } w)} h(w, w)^{\text{Re } \lambda + k - \frac{2n}{r}} dw \\ &\leq 2|c_{\lambda+k}| C'_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Re } x|_1} \int_D h(w, w)^{\text{Re } \lambda + k - \frac{2n}{r}} dw \\ &= C_{\lambda,k} (1 + |x|_1^{rk}) e^{2|\text{Re } x|_1}. \end{aligned}$$

The proof for $\mathcal{J}_\lambda(x^2)$ is similar. ■

Remark 3.3. In [17, Lemma 3.1] Möllers gave another estimate of $\mathcal{J}_\lambda(x)$:

$$|\mathcal{J}_\lambda(x^2)| \leq C(1 + |x|_2^2)^{\frac{r(2n-1)}{4}} e^{2r|x|_2} \quad \text{for any } \lambda \in \mathcal{W}, x \in \overline{\mathcal{X}_{\text{rank } \lambda}} \subset V^{\mathbb{C}}.$$

However, our estimate is sharper because our leading term is given by $e^{2|\text{Im } x|_1}$. Especially in our estimate $\mathcal{J}_\lambda(x)$ is uniformly bounded on V if $\text{Re } \lambda$ is sufficiently large. This difference comes from that of methods of proofs: in [17] the Taylor expansion was used, while in this paper we use the integral formula. However, in general Taylor series is not strong enough for L^∞ estimates. For example, the bound of cosine function is calculated as follows:

$$|\cos x| = \left| \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \right| \leq \sum_{m=0}^{\infty} \frac{1}{(2m)!} |x|^{2m} \leq \sum_{m=0}^{\infty} \frac{1}{m!} |x|^m = e^{|x|}.$$

However, it is well-known that cosine function is bounded uniformly on \mathbb{R} . So this bound is not sharp.

4. Applications

For $\lambda > \frac{n}{r} - 1$, $t \in \mathbb{C} \setminus \pi i\mathbb{Z}$, $\text{Re } t \geq 0$, we define a integral operator on Ω : for a measurable function $\varphi : \Omega \rightarrow \mathbb{C}$, we define

$$\tau_\lambda(t)\varphi(x) := \frac{1}{\Gamma_\Omega(\lambda)} \int_\Omega \varphi(y) \frac{e^{-\coth t(\text{tr } x + \text{tr } y)}}{\sinh^{r\lambda} t} \mathcal{I}_\lambda \left(\frac{1}{\sinh^2 t} P(x^{\frac{1}{2}})y \right) \Delta(y)^{\lambda - \frac{n}{r}} dy.$$

Since \mathcal{I}_λ is K -invariant, by [8, Lemma XIV.1.2] we can replace $P(x^{\frac{1}{2}})y$ by $P(y^{\frac{1}{2}})x$.

Remark 4.1. For $\lambda > \frac{2n}{r} - 1$, the Laplace transform

$$\mathcal{L}_\lambda : L^2(\Omega, \Delta(x)^{\lambda - \frac{n}{r}} dx) \longrightarrow L^2(V + \sqrt{-1}\Omega, \Delta(\text{Im } z)^{\lambda - \frac{2n}{r}} dz) \cap \mathcal{O}(V + \sqrt{-1}\Omega)$$

is defined by

$$\mathcal{L}_\lambda \varphi(z) := \frac{2^n}{\Gamma_\Omega(\lambda)} \int_\Omega e^{i(z|x)} \varphi(x) \Delta(2x)^{\lambda - \frac{n}{r}} dx.$$

Then we can prove by the similar method to [8, Theorem XV.4.1] that

$$\begin{aligned} \mathcal{L}_\lambda \tau_\lambda(t) \mathcal{L}_\lambda^{-1} F(z) &= \Delta(-\sin(it)z + \cos(it)e)^{-\lambda} \\ &\quad \times F((\cos(it)z + \sin(it)e)(-\sin(it)z + \cos(it)e)^{-1}). \end{aligned}$$

If t is purely imaginary, then this coincides with the restriction of the holomorphic discrete series representation of the simple Hermitian Lie group $\text{Bihol}(V + \sqrt{-1}\Omega)$, to the center of the maximal compact subgroup $\text{Stab}(ie)$. That is, τ_λ can be regarded as the natural complexification of the action of

$$Z(\text{Stab}(ie)) \subset \text{Bihol}(V + \sqrt{-1}\Omega).$$

Especially, $\tau_\lambda(s)\tau_\lambda(t) = \tau_\lambda(s+t)$ holds for $\lambda > \frac{2n}{r} - 1$.

Remark 4.2. Let E be an Euclidean vector space of dimension N with inner product $(\cdot|\cdot)_E$. Then the Hermite semigroup on $L^2(E)$ is given by

$$\tilde{\tau}(t)f(\xi) := \frac{1}{(2\pi \sinh t)^{\frac{N}{2}}} \int_E f(\eta) \exp\left(-\frac{1}{2} \coth t(|\xi|_E^2 + |\eta|_E^2) + \frac{1}{\sinh t}(\xi|\eta)_E\right) d\eta \tag{4.12}$$

for $f \in L^2(E)$, $t \in \mathbb{C} \setminus \pi i\mathbb{Z}$, $\text{Re } t \geq 0$ (see, e.g., [7, Section 5.2]). From now on we assume there exists a self-adjoint representation $\phi : V \rightarrow \text{End}(E)$. We also assume $N > r(r - 1)d$. Let $Q : E \rightarrow V$ be the quadratic map defined by

$$(\phi(x)\xi|\xi)_E = (x|Q(\xi))_V \quad \text{for any } x \in V, \xi \in E.$$

Let $\Sigma := Q^{-1}(e) \subset E$ be the Stiefel manifold. Then we have

$$\int_{\Sigma} e^{-i(\xi|\sigma)} d\sigma = \mathcal{J}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2}\right)\right) \tag{4.13}$$

(see [8, Proposition XVI.2.3]). We extend Q to $Q : E^{\mathbb{C}} \rightarrow V^{\mathbb{C}}$ bilinearly. Then since $\mathcal{J}_{\lambda}(x) = \mathcal{I}_{\lambda}(-x)$ we have

$$\int_{\Sigma} e^{(\xi|\sigma)} d\sigma = \mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{\xi}{2}\right)\right).$$

If $f \in L^2(E)$ is written as $f(\xi) = F\left(\frac{1}{2}Q(\xi)\right)$ with a function F on V , then (4.12) can be rewritten as

$$\begin{aligned} \tilde{\tau}(t)f(\xi) &= \frac{1}{(2\pi \sinh t)^{\frac{N}{2}}} \int_E F\left(\frac{1}{2}Q(\eta)\right) \exp\left(-\frac{1}{2} \coth t(|\xi|_E^2 + |\eta|_E^2) + \frac{1}{\sinh t}(\xi|\eta)_E\right) d\eta \\ &= \frac{1}{(\pi \sinh t)^{\frac{N}{2}}} \int_E F(Q(\eta)) \exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + |\eta|_E^2\right) + \frac{\sqrt{2}}{\sinh t}(\xi|\eta)_E\right) d\eta \\ &= \frac{1}{\Gamma_{\Omega}\left(\frac{N}{2r}\right) \sinh^{\frac{N}{2}} t} \int_{\Omega} \int_{\Sigma} F(Q(\phi(y^{\frac{1}{2}})\sigma)) \exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + |\phi(y^{\frac{1}{2}})\sigma|_E^2\right)\right) \\ &\quad \times \exp\left(\frac{\sqrt{2}}{\sinh t}(\xi|\phi(y^{\frac{1}{2}})\sigma)_E\right) \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} d\sigma dy \\ &= \frac{1}{\Gamma_{\Omega}\left(\frac{N}{2r}\right)} \int_{\Omega} \int_{\Sigma} F(y) \frac{\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + \text{tr } y\right)\right)}{\sinh^{\frac{N}{2}} t} \exp\left(\frac{\sqrt{2}}{\sinh t}(\phi(y^{\frac{1}{2}})\xi|\sigma)_E\right) \\ &\quad \times \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} d\sigma dy \\ &= \frac{1}{\Gamma_{\Omega}\left(\frac{N}{2r}\right)} \int_{\Omega} F(y) \frac{\exp\left(-\coth t\left(\frac{1}{2}|\xi|_E^2 + \text{tr } y\right)\right)}{\sinh^{\frac{N}{2}} t} \mathcal{I}_{\frac{N}{2r}}\left(Q\left(\frac{1}{\sqrt{2} \sinh t} \phi(y^{\frac{1}{2}})\xi\right)\right) \\ &\quad \times \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} dy \\ &= \frac{1}{\Gamma_{\Omega}\left(\frac{N}{2r}\right)} \int_{\Omega} F(y) \frac{\exp\left(-\coth t\left(\frac{1}{2} \text{tr } Q(\xi) + \text{tr } y\right)\right)}{\sinh^{\frac{N}{2}} t} \mathcal{I}_{\frac{N}{2r}}\left(\frac{1}{2 \sinh^2 t} P(y^{\frac{1}{2}})Q(\xi)\right) \\ &\quad \times \Delta(y)^{\frac{N}{2r} - \frac{n}{r}} dy \\ &= \tau_{\frac{N}{2r}}(t) F\left(\frac{1}{2}Q(\xi)\right) \end{aligned}$$

where we used [8, Proposition XVI.2.1] at the 3rd equality and [8, Lemma XVI.2.2. (ii)] at the 4th, 6th equalities. Therefore $\tau_{\frac{N}{2r}}(t)$ coincides with the action of the Hermite semigroup on radial functions on E .

Remark 4.3. For $x \in \overline{\mathcal{X}}_1$ (see (2.3)), $\mathcal{I}_\lambda(x) = \Gamma(\lambda)\tilde{I}_{\lambda-1}(2\sqrt{|x|}_2)$ holds (see [17, Example 3.3]), and by analytic continuation the distribution $\frac{1}{\Gamma_\Omega(\lambda)}\Delta(x)^{\lambda-\frac{n}{r}}\mathbf{1}_\Omega dx$ at $\lambda = \frac{d}{2}$ gives the semi-invariant measure on $\overline{\mathcal{X}}_1 \cap \overline{\Omega}$ (see [8, Proposition VII.2.3]). Therefore for $V = \mathbb{R}^{1,n-1}$ the action τ_λ at $\lambda = \frac{d}{2}$ coincides with the action of the holomorphic semigroup on the minimal representation of $O(p, 2)$ (see [14, Theorem B] or [15, Theorem 5.1.1]).

Remark 4.4. We set

$$H_\lambda\varphi(x) := i^{r\lambda}\tau_\lambda\left(\frac{\pi i}{2}\right)\varphi(x) = \frac{1}{\Gamma_\Omega(\lambda)}\int_\Omega\varphi(y)\mathcal{J}\left(P(x^{\frac{1}{2}})y\right)\Delta(y)^{\lambda-\frac{n}{r}}dy.$$

This is called the generalized Hankel transform ([8, Section XV.4]). Similar to Remark 4.2, this is regarded as a variant of the Fourier transform. Therefore it is expected that this Hankel transform has similar properties as the Fourier transform such as a Paley-Wiener type theorem, which determines the image of the compactly supported functions. This is done by, *e.g.*, [1], [16, Remark 5.4] for classical $V = \mathbb{R}$ case, but not for generalized case. In this paper we don't touch this topic in detail.

We set $K_\lambda(x, y; t) := e^{-\coth t(\operatorname{tr} x + \operatorname{tr} y)}\mathcal{I}_\lambda\left(\sinh^{-2}tP(x^{\frac{1}{2}})y\right)$, the kernel function of $\tau_\lambda(t)$. Then we can deduce from Theorem 3.2 that

Theorem 4.5. Take $k \in \mathbb{Z}_{\geq 0}$ such that $\lambda + k > \frac{2n}{r} - 1$. Then if $t = u + iv$, $u, v \in \mathbb{R}$, $u \geq 0$,

$$|K_\lambda(x, y; t)| \leq C_{\lambda,t}\left(1 + (\operatorname{tr} x \operatorname{tr} y)^{\frac{rk}{2}}\right)\exp\left(-\frac{\sinh u}{\cosh u + |\cos v|}(\operatorname{tr} x + \operatorname{tr} y)\right).$$

Especially, if $u = \operatorname{Re} t > 0$ then the integral defining $\tau_\lambda(t)$ converges if φ is of polynomial growth, and the resulting $\tau_\lambda(t)\varphi$ has exponential decay. Even if $u = \operatorname{Re} t = 0$, if $\lambda > \frac{2n}{r} - 1$ and $t \notin \pi i\mathbb{Z}$, the integral converges if $\varphi \in L^1(\Omega, \Delta(x)^{\lambda-\frac{n}{r}}dx)$, and the resulting $\tau_\lambda(t)\varphi$ is bounded. In order to prove this theorem, we prepare the following lemma.

Lemma 4.6. (1) For $x \in \Omega$ the directional derivative of $x \mapsto \sqrt{x}$ is

$$D_u\sqrt{x} = \frac{1}{2}L(\sqrt{x})^{-1}u.$$

(2) For $x, y \in V$ if $[L(x), L(y)] = 0$, then there exists a Jordan frame $\{c_1, \dots, c_r\}$ such that $x, y \in \mathbb{R}\text{-span}\{c_1, \dots, c_r\}$.

(3) For $x, y \in \Omega$, $\operatorname{tr}\sqrt{P(x^{\frac{1}{2}})y} \leq \sqrt{\operatorname{tr} x \operatorname{tr} y} \leq \frac{\operatorname{tr} x + \operatorname{tr} y}{2}$.

Proof. (1) $u = D_u x = D_u (\sqrt{x})^2 = 2\sqrt{x}D_u\sqrt{x} = 2L(\sqrt{x})D_u\sqrt{x}$ and then $D_u\sqrt{x} = \frac{1}{2}L(\sqrt{x})^{-1}u$ follows.

(2) See [8, Lemma X.2.2].

(3) The second inequality is clear. For the first inequality, we take $k_0 \in K$ such that $\operatorname{tr} \sqrt{P(x^{\frac{1}{2}})ky}$ ($k \in K_L$) attains its maximum at $k = k_0$. We put $k_0 y =: y_0$. Then for any $D \in \mathfrak{k}_t = \operatorname{Lie}(K_L)$,

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \operatorname{tr} \sqrt{P(x^{\frac{1}{2}})e^{tD}y_0} = \frac{1}{2} \operatorname{tr} \left(L \left(\sqrt{P(x^{\frac{1}{2}})y_0} \right)^{-1} P(x^{\frac{1}{2}})Dy_0 \right) \\ &= \frac{1}{2} \left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \Big| P(x^{\frac{1}{2}})Dy_0 \right) = \frac{1}{2} \left(P(x^{\frac{1}{2}}) \sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \Big| Dy_0 \right). \end{aligned}$$

We put $P(x^{\frac{1}{2}}) \sqrt{P(x^{\frac{1}{2}})y_0}^{-1} =: z$. If $D = [L(u), L(v)]$ ($u, v \in V$), then

$$\begin{aligned} 0 &= (z|[L(u), L(v)]y_0) = (z|u(vy_0)) - (z|v(uy_0)) = (zu|vy_0) - (zv|uy_0) \\ &= (y_0(zu)|v) - (v|(uy_0)z) = ([L(y_0), L(z)]u|v). \end{aligned}$$

Since $(\cdot|\cdot)$ is non-degenerate, $[L(y_0), L(z)] = 0$. Also,

$$\begin{aligned} P(z)y_0 &= P \left(P(x^{\frac{1}{2}}) \sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \right) y_0 \\ &= P(x^{\frac{1}{2}})P \left(\sqrt{P(x^{\frac{1}{2}})y_0}^{-1} \right) P(x^{\frac{1}{2}})y_0 = P(x^{\frac{1}{2}})e = x. \end{aligned}$$

So especially $[L(x), L(y_0)] = 0$. Let $x = \sum_{j=1}^r t_j c_j$, $y = \sum_{j=1}^r s_j d_j$ ($t_j, s_j > 0$, and $\{c_j\}_{j=1}^r, \{d_j\}_{j=1}^r$ are Jordan frames). Then,

$$\begin{aligned} \operatorname{tr} \sqrt{P(x^{\frac{1}{2}})y} &\leq \operatorname{tr} \sqrt{P(x^{\frac{1}{2}})y_0} = \operatorname{tr} \sqrt{P \left(\sum_{j=1}^r t_j^{\frac{1}{2}} c_j \right) \sum_{j=1}^r s_j c_j} \\ &= \sum_{j=1}^r \sqrt{t_j s_j} \leq \sqrt{\left(\sum_{j=1}^r t_j \right) \left(\sum_{j=1}^r s_j \right)} = \sqrt{\operatorname{tr} x \operatorname{tr} y} \end{aligned}$$

and the proof is completed. ■

Now we are ready to prove Theorem 4.5.

Proof. [Proof of Theorem 4.5] By Corollary 3.2,

$$\begin{aligned}
|K_\lambda(x, y; t)| &\leq C'_\lambda e^{-\operatorname{Re} \coth t(\operatorname{tr} x + \operatorname{tr} y)} \left(1 + \left| \frac{1}{\sinh t} \sqrt{P(x^{\frac{1}{2}})y} \right|_1^{rk} \right) e^{2 \left| \operatorname{Re} \frac{1}{\sinh t} \sqrt{P(x^{\frac{1}{2}})y} \right|_1} \\
&= C'_\lambda e^{-\operatorname{Re} \coth t(\operatorname{tr} x + \operatorname{tr} y)} \left(1 + \frac{1}{|\sinh t|^{rk}} \operatorname{tr} \left(\sqrt{P(x^{\frac{1}{2}})y} \right)^{rk} \right) e^{2 \left| \operatorname{Re} \frac{1}{\sinh t} \right| \operatorname{tr} \left(\sqrt{P(x^{\frac{1}{2}})y} \right)} \\
&\leq C_{\lambda, t} \exp \left(-\frac{\cosh u \sinh u}{\cosh^2 u - \cos^2 v} (\operatorname{tr} x + \operatorname{tr} y) \right) \left(1 + \sqrt{\operatorname{tr} x \operatorname{tr} y}^{rk} \right) \\
&\quad \times \exp \left(\frac{\sinh u |\cos v|}{\cosh^2 u - \cos^2 v} (\operatorname{tr} x + \operatorname{tr} y) \right) \\
&= C_{\lambda, t} \left(1 + (\operatorname{tr} x \operatorname{tr} y)^{\frac{rk}{2}} \right) \exp \left(-\frac{\sinh u}{\cosh u + |\cos v|} (\operatorname{tr} x + \operatorname{tr} y) \right)
\end{aligned}$$

and this completes the proof. ■

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