Journal of Lie Theory
Volume **24** (2014) 439–455

On Some Subvarieties of the Cartesian Powers of a Semisimple Lie Algebra Related to a Parabolic Subalgebra

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Communicated by J. Faraut

Abstract. In this paper, we describe some desingularizations of some subvarieties of the cartesian powers of a semisimple Lie algebra of finite dimension.
Mathematics Subject Classification 2000: 14A10, 14L17, 22E20, 22E46.
Key Words and Phrases: Lie algebra, parabolic subalgebra, desingularization, normalization.

1. Introduction

The basic field $\Bbbk$ is an algebraic closed field of characteristic zero. Let $\mathfrak{g}$ be a semisimple Lie algebra of finite dimension and let $\mathbf{G}$ be its adjoint group. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$ and let $N_{\mathbf{G}}(\mathfrak{h})$ be the normalizer of $\mathfrak{h}$ in $\mathbf{G}$. Let denote by $\mathfrak{p} \neq \mathfrak{g}$ a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, $\mathfrak{l}$ its reductive factor containing $\mathfrak{h}$, $\mathfrak{p}_{\mathrm{u}}$ its nilpotent radical and $\mathbf{P}$ its normalizer in $\mathbf{G}$. Let set:

$$\mathcal{P}^{(k)} := \left\{(x_1, \ldots, x_k) \in \mathfrak{g}^k \mid \exists g \in \mathbf{G} \text{ such that } (g(x_1), \ldots, g(x_k)) \in \mathfrak{p}^k\right\},$$
$$\mathcal{P}_{\mathrm{u}}^{(k)} := \left\{(x_1, \ldots, x_k) \in \mathfrak{g}^k \mid \exists g \in \mathbf{G} \text{ such that } (g(x_1), \ldots, g(x_k)) \in \mathfrak{p}_{\mathrm{u}}^k\right\}.$$

For X a $\mathbf{G}$-variety, we denote by $\mathbf{G} \times_{\mathbf{P}} X$ the quotient of $\mathbf{G} \times X$ under the right action of $\mathbf{P}$ given by $(g, x).p := (gp, p^{-1}.x)$. The variety $\mathbf{G} \times_{\mathbf{P}} X$ is a vector bundle over $\mathbf{G}/\mathbf{P}$. The main result of this note is the following theorem:

Theorem 1. *(i) For $k \geqslant 2$, $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ is a desingularization of $\mathcal{P}^{(k)}$. Moreover, $\mathcal{P}^{(k)}$ is not normal.*

(ii) For $k \geqslant 2$, the canonical morphism $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\mathrm{u}}^k \to \mathcal{P}_{\mathrm{u}}^{(k)}$ is projective and factorizes through a desingularization of an affine variety $X^{(k)}$.

For $\mathfrak{p} = \mathfrak{b}$, this theorem is given in [2], a joint work with J-Y. Charbonnel. Moreover, in this case, $\mathcal{P}_{\mathrm{u}}^{(k)}$ is normal and has rational singularities. Since this

ISSN 0949–5932 / \$2.50

result is important, so it is for its generalization. In fact, this note is the first step to a generalization of some results of [2]. The goal of this paper should be finding an explicit normalization of the varieties $\mathcal{P}^{(k)}$ and $\mathcal{P}_{\mathrm{u}}^{(k)}$ and showing that these varieties have rational singularities when $\mathfrak{p}$ is a parabolic subalgebra. Unfortunately, we are not able to prove these results for the moment. So, the purpose of this note is to describe the good candidates $\chi^{(k)}$ and $X^{(k)}$ for the normalizations of $\mathcal{P}^{(k)}$ and $\mathcal{P}_{\mathrm{u}}^{(k)}$.

Let $S(\mathfrak{g})$ and $S(\mathfrak{h})$ be the symmetric algebras of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, let $S(\mathfrak{g})^{\mathbf{G}}$ be the subalgebra of $\mathbf{G}$-invariant elements of $S(\mathfrak{g})$ and let $S(\mathfrak{h})^{W}$ be the subalgebra of W-invariant elements of $S(\mathfrak{h})$, where W is the Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The subalgebras $S(\mathfrak{g})^{\mathbf{G}}$ and $S(\mathfrak{h})^{W}$ are identified by Chevalley Restriction Theorem. In order to find a normalization of $\mathcal{P}^{(k)}$, we use, in section 3, a normal variety χ_0 introduced in [2, §3], which is the closed subvariety of $\mathfrak{g}\times\mathfrak{h}$ such that its algebra of regular functions $\Bbbk[\chi_0]$ equals $S(\mathfrak{g})\otimes_{S(\mathfrak{h})^{W}} S(\mathfrak{h})$, i.e.

$$\chi_0=\left\{(x,h)\in\mathfrak{g}\times\mathfrak{h}|p(x)=p(h),\forall p\in S(\mathfrak{h})^{W}\right\},$$

and we introduce the normal variety $\chi := \chi_0//W_{\mathfrak{l}}$, where $W_{\mathfrak{l}}$ is the Weyl group of the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ with respect to $\mathfrak{h}$. Let

$$\begin{gathered}\sigma_k':\mathbf{G}\times\mathfrak{p}^k\to\chi^k\\ (g,x_1,\ldots,x_k)\mapsto\left(g(x_1),\ldots,g(x_k),\overline{x_1},\ldots,\overline{x_k}\right),\end{gathered}$$

where $\bar{x}_i$ is the image of $\widetilde{x_i}$ in $\mathfrak{l}//\mathbf{L}$ (see notations). Denote by σ_k the morphism from $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ to χ^k defined through the quotient by σ_k'. Let $\chi^{(k)}$ be the image of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ by σ_k. We prove the following theorem:

Theorem 2. *(i) The variety $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ is a desingularization of a subvariety $\chi^{(k)}$ of χ^k.*

(ii) The variety $\chi^{(k)}$ is an irreducible component of the inverse image of $\mathcal{P}^{(k)}$ in χ^k by the canonical projection $\mathrm{pr}_{1,k}$ from χ^k to onto $\mathfrak{g}^k$.

In section 4, to find a normalization of $\mathcal{P}_{\mathrm{u}}^{(k)}$, we introduce the variety $X := Spec\mathcal{A}$ and $X^{(k)}$ a subvariety of X^k, where $\mathcal{A}$ is the integral closure of the algebra of regular functions $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$ of $\mathbf{G}(\mathfrak{p}_{\mathrm{u}})$ in the field of rational functions $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ and we prove the following theorem:

Theorem 3. *(i) The variety $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^k$ is a desingularization of $X^{(k)}$.*

(ii) There exists a pure morphism in the sense of [7, Appendix to §7] from $\left(X^{(k)}\right)_{\mathrm{n}}$ the normalisation of $X^{(k)}$ to $\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}$ the normalisation of $\mathcal{P}_{\mathrm{u}}^{(k)}$.

2. Notations

We consider the diagonal action of $\mathbf{G}$ on $\mathfrak{g}^k$. Let $\mathcal{R}$ be the root system of $\mathfrak{h}$ in $\mathfrak{g}$ and let $\mathcal{R}_+$ be the positive root system of $\mathcal{R}$ defined by $\mathfrak{b}$. The Weyl group of $\mathcal{R}$

is denoted by W and the basis of $\mathcal{R}_+$ is denoted by Π. Set:

$$\mathcal{R}_{\mathfrak{l}} := \{\alpha \in \mathcal{R} \mid \mathfrak{g}^\alpha \subset \mathfrak{l}\}$$
$$\mathfrak{p}_- := \mathfrak{l} \oplus \bigoplus_{\alpha \in \mathcal{R}_+ \setminus \mathcal{R}_{\mathfrak{l}}} \mathfrak{g}^{-\alpha}.$$

Let $W_{\mathfrak{l}}$ be the Weyl group of $\mathcal{R}_{\mathfrak{l}}$ and let $\mathbf{P}_-$ be the normalizer of $\mathfrak{p}_-$ in $\mathbf{G}$. Denote by $\mathbf{L}$ the identity component of the normalizer of $\mathfrak{l}$ in $\mathbf{G}$ and denote by $\mathbf{P}_{\mathrm{u}}$ and $\mathbf{P}_{-,\mathrm{u}}$ the unipotent radicals of $\mathbf{P}$ and $\mathbf{P}_-$ respectively. Let Δ be the Richardson orbit of $\mathfrak{p}$ and let $\mathfrak{p}_{\mathrm{u},\Delta} := \Delta \cap \mathfrak{p}_{\mathrm{u}}$. We use also the following notations:

- All topological terms refer to the Zariski topology. For Y an open subset of the algebraic variety X, Y is called a big open subset if the codimension of $X \setminus Y$ in X is at least 2. Let $\mathcal{O}_X$ be the structural sheaf of X, let $\Bbbk[X]$ be the algebra of regular functions on X, and let $\Bbbk(X)$ be the field of rational functions on X when X is irreducible.

- If $\mathbf{G} \times A \longrightarrow A$ is an action of $\mathbf{G}$ on the algebra A, denote by $A^{\mathbf{G}}$ the subalgebra of the $\mathbf{G}$-invariant elements of A.

- We identify the algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ by the Killing form $< .,. >$ of $\mathfrak{g}$, by this way, $\mathfrak{l}$ identifies with its dual $\mathfrak{l}^*$.

- $\mathrm{b}_{\mathfrak{g}}$ the dimension of the Borel subalgebra of $\mathfrak{g}$ and $\mathrm{rk}\mathfrak{g}$ the rank of $\mathfrak{g}$.

- $p_1, \ldots, p_{\mathrm{rk}\mathfrak{g}}$ homogeneous invariant functions generate $S(\mathfrak{g})^G$ of degrees $d_1, \ldots, d_{\mathrm{rk}\mathfrak{g}}$.

- For $i = 1, \ldots, \mathrm{rk}\mathfrak{g}$, the 2-order polarizations of p_i of bidegree $(d_i - n, n)$, denoted by $p_i^{(n)}$, are the unique elements in $(S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}))^{\mathbf{G}}$ satisfying the following relation

$$p_i(ax + by) = \sum_{n=0}^{d_i} a^{d_i - n} b^n p_i^{(n)}(x, y),$$

  for all $a, b \in \mathbb{C}$ and for all $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

- For $i \in \{1, \ldots, \mathrm{rk}\mathfrak{g}\}$, ε_i is the element of $S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ defined by

$$< \varepsilon_i(x), v >= p_i'(x)(v), \ \forall x, v \in \mathfrak{g},$$

  where $p_i'(x)$ is the differential of p_i at x for all $i \in \{1, \ldots \mathrm{rk}\mathfrak{g}\}$.

- For $i \in \{1, \ldots, \mathrm{rk}\mathfrak{g}\}$ and for $m \in \{0, \ldots, d_i - 1\}$, the 2-polarizations of ε_i of bidegree $(d_i - m - 1, m)$ denoted by $\varepsilon_i^{(m)}$ are the unique elements in $S(\mathfrak{g}) \otimes_{\mathbb{C}} S(\mathfrak{g}) \otimes_{\mathbb{C}} \mathfrak{g}$ satisfying the following relation

$$\varepsilon_i(ax + by) = \sum_{n=0}^{d_i - 1} a^{d_i - n - 1} b^n \varepsilon_i^{(n)}(x, y),$$

  for all $a, b \in \mathbb{C}$ and $(x, y) \in \mathfrak{g} \times \mathfrak{g}$.

- $\mathfrak{g}_{\mathrm{reg}}$ is the set of regular elements of $\mathfrak{g}$.
- For a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$, $\mathfrak{a}_{\mathrm{reg}} := \mathfrak{a} \cap \mathfrak{g}_{\mathrm{reg}}$.
- For $(x,y) \in \mathfrak{g} \times \mathfrak{g}$, $V_{x,y}$ is the space generated by the set
$$\left\{\varepsilon_i^{(m)}(x,y), i \in \{1,\dots \mathrm{rk}\mathfrak{g}\}, m \in \{0,\dots,d_i-1\}\right\}.$$
- More generally, for a reductive algebra $\mathfrak{l}$ and for $(x,y) \in \mathfrak{l} \times \mathfrak{l}$, $V_{x,y}^{\mathfrak{l}}$ is defined analogously.
- For $(x_1,\dots,x_k) \in \mathfrak{g}^k$, $P_{x_1,\dots,x_k}$ is the space of $\mathfrak{g}$ generated by $x_1,\dots,x_k$.
- $\Omega_{\mathfrak{g}} := \{(x,y) \in \mathfrak{g} \times \mathfrak{g} \mid P_{x,y} \backslash \{0\} \subset \mathfrak{g}_{\mathrm{reg}} \text{ and } \dim P_{x,y} = 2\}$.
- For x in $\mathfrak{g}$, $\mathbf{G}^x$ is the centralizer of x in $\mathbf{G}$.
- For a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ and for x in $\mathfrak{a}$, $\mathfrak{a}^x$ is the centralizer of x in $\mathfrak{a}$.
- ϖ is the canonical morphism from $\mathfrak{p}$ to $\mathfrak{l}$.
- For x in $\mathfrak{p}$, $\widetilde{x}$ is the image of x by ϖ.
- (e,h,f) is a principal standard $\mathfrak{sl}_2$-triplet.

3. On the variety $\mathcal{P}^{(k)}$

3.1. On parabolic subalgebras.

Lemma 4. *Let $\mathfrak{a}$ be an algebraic subalgebra of $\mathfrak{g}$.*

(i) Let suppose that $\mathfrak{a}$ contains $\mathfrak{g}^x$ for all x in a nonempty open subset of $\mathfrak{a}$ and let suppose that $\mathfrak{a}_{\mathrm{reg}}$ is not empty. Then $V_{x,y}$ is contained in $\mathfrak{a}$ for all (x,y) in $\mathfrak{a} \times \mathfrak{a}$.

(ii) Let suppose that $\mathfrak{a}$ contains a Cartan subalgebra of $\mathfrak{g}$. Then $V_{x,y}$ is contained in $\mathfrak{a}$ for all (x,y) in $\mathfrak{a} \times \mathfrak{a}$.

Proof. (i) By hypothesis, for all x in a nonempty open subset of $\mathfrak{a}$, x is a regular element and $\mathfrak{g}^x$ is contained in $\mathfrak{a}$. So, $\varepsilon_1(x),\dots,\varepsilon_{\mathrm{rk}\mathfrak{g}}(x)$ belong to $\mathfrak{a}$ for all x in a nonempty open subset of $\mathfrak{a}$ and hence for all x in $\mathfrak{a}$ by continuity. As a result, for all (x,y) in $\mathfrak{a} \times \mathfrak{a}$, $\left\{\varepsilon_i^{(m)}(x,y), i \in \{1,\dots,\mathrm{rk}\mathfrak{g}\}, m \in \{0,\dots,d_i-1\}\right\}$ is contained in $\mathfrak{a}$, whence the assertion.

(ii) Let $\mathfrak{c}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{a}$. Since $\mathfrak{a}$ is an algebraic subalgebra of $\mathfrak{g}$, every semisimple element of $\mathfrak{a}$ is conjugate under the adjoint group of $\mathfrak{a}$ to an element of $\mathfrak{c}$. Hence for every regular semisimple element x of $\mathfrak{g}$, belonging to $\mathfrak{a}$, $\mathfrak{g}^x$ is contained in $\mathfrak{a}$. As a result, the assertion is a consequence of (i) since the subset of regular semisimple elements of $\mathfrak{g}$, belonging to $\mathfrak{a}$, is a nonempty open subset of $\mathfrak{a}$. ■

Corollary 5. *For all (x,y) in $\mathfrak{p} \times \mathfrak{p}$, $V_{x,y}$ is contained in $\mathfrak{p}$. In particular, for all (x,y) in a nonempty open subset of $\mathfrak{b} \times \mathfrak{b}$, $V_{x,y} = \mathfrak{b}$.*

Proof. Since $\mathfrak{h}$ is contained in $\mathfrak{p}$, for all (x,y) in $\mathfrak{p}\times\mathfrak{p}$, $V_{x,y}$ is contained in $\mathfrak{p}$ by Lemma 4, (ii). According to [1, Corollary 2],

$$(x,y)\in\Omega_{\mathfrak{g}} \Leftrightarrow \dim V_{x,y} = \mathrm{b}_{\mathfrak{g}}.$$

Since (h,e) belongs to $\Omega_{\mathfrak{g}}$, $\Omega_{\mathfrak{g}}\cap\mathfrak{b}\times\mathfrak{b}$ is a nonempty open subset, whence the corollary. ∎

Lemma 6. *Let $\mathfrak{l}'_{\mathrm{reg}}$ be the set of regular elements of $\mathfrak{l}$. Set:*

$$\begin{aligned} R_{\mathfrak{p}} &:= \left\{x\in\mathfrak{p}_{\mathrm{reg}} | \widetilde{x}\in\mathfrak{l}'_{\mathrm{reg}}\right\} \\ R'_{\mathfrak{p}} &:= \left\{x\in R_{\mathfrak{p}} \mid \mathfrak{g}^{x}\cap\mathfrak{p}_{\mathrm{u}}=\{0\}\right\}. \end{aligned}$$

(i) For all x in $R_{\mathfrak{p}}$, $\varpi(\mathfrak{g}^{x})=\mathfrak{l}^{\widetilde{x}}$ if and only if $x\in R'_{\mathfrak{p}}$.
(ii) The subset $R'_{\mathfrak{p}}$ is open in $\mathfrak{p}$.
(iii) For all (x,y) in $\mathfrak{p}\times\mathfrak{p}$, $V_{x,y}$ is contained in $V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}}+\mathfrak{p}_{\mathrm{u}}$.
(iv) For all (x,y) in $R'_{\mathfrak{p}}\times\mathfrak{p}$, $\varpi(V_{x,y})=V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}}$.

Proof. (i) Let x be in $R_{\mathfrak{p}}$. By Lemma 4 (i), $\mathfrak{g}^{x}$ is contained in $\mathfrak{p}$. Since ϖ is a morphism of Lie algebra, $\varpi(\mathfrak{g}^{x})$ is contained in $\mathfrak{l}^{\widetilde{x}}$. Furthermore, $\dim\varpi(\mathfrak{g}^{x})=\mathrm{rk}\mathfrak{g}$ if and only if $\mathfrak{g}^{x}\cap\mathfrak{p}_{\mathrm{u}}=\{0\}$, whence the assertion since $\mathfrak{l}$ has rank $\mathrm{rk}\mathfrak{g}$.

(ii) For x regular semisimple in $\mathfrak{p}$, $\mathfrak{g}^{x}$ is a Cartan subalgebra contained in $\mathfrak{p}$. As a result, $\mathfrak{g}^{x}\cap\mathfrak{p}_{\mathrm{u}}=\{0\}$. So by (i), $R_{\mathfrak{p}}$ and $R'_{\mathfrak{p}}$ are not empty since for such x, $\widetilde{x}$ is a regular semisimple element of $\mathfrak{l}$. The subset $R_{\mathfrak{p}}$ of $\mathfrak{p}$ is open since the subsets of regular elements of $\mathfrak{g}$ and $\mathfrak{l}$ are open in $\mathfrak{g}$ and $\mathfrak{l}$ respectively. The map $x\mapsto\mathfrak{g}^{x}$ from $R_{\mathfrak{p}}$ to the Grassmanian $\mathrm{Gr}_{\mathrm{rk}\mathfrak{g}}(\mathfrak{g})$ is regular. So $R'_{\mathfrak{p}}$ is an open subset of $\mathfrak{p}$.

(iii) Let $L_{\mathfrak{l}}$ be the submodule of elements φ of $\mathrm{S}(\mathfrak{l})\otimes_{\Bbbk}\mathfrak{l}$ such that $[\varphi(x),x]=0$, for all x in $\mathfrak{l}$. Then $L_{\mathfrak{l}}$ is a free module of rank $\mathrm{rk}\mathfrak{g}$ according to [6, Theorem 9]. Denote by $\varphi_{1},\ldots,\varphi_{\mathrm{rk}\mathfrak{g}}$ a basis of $L_{\mathfrak{l}}$ and denote by $R_{\mathfrak{p},\mathfrak{l}}$ the subset of elements x of $\mathfrak{p}$ such that $\widetilde{x}$ is in $\mathfrak{l}'_{\mathrm{reg}}$. According to [10, Theorem 4.12], $\mathfrak{l}'_{\mathrm{reg}}$ is a big open subset of $\mathfrak{l}$. So $R_{\mathfrak{p},\mathfrak{l}}$ is a big open subset of $\mathfrak{p}$. For x in $R_{\mathfrak{p},\mathfrak{l}}$ and for $i=1,\ldots,\mathrm{rk}\mathfrak{g}$, $\varpi\circ\varepsilon_{i}(x)$ belongs to $\mathfrak{l}^{\widetilde{x}}$. So there exists a unique element $(a_{i,1}(x),\ldots,a_{i,\mathrm{rk}\mathfrak{g}}(x))$ of $\Bbbk^{\mathrm{rk}\mathfrak{g}}$ such that

$$\varpi\circ\varepsilon_{i}(x)=a_{i,1}(x)\varphi_{1}(\widetilde{x})+\cdots+a_{i,\mathrm{rk}\mathfrak{g}}(x)\varphi_{\mathrm{rk}\mathfrak{g}}(\widetilde{x}).$$

The functions $a_{i,1},\ldots,a_{i,\mathrm{rk}\mathfrak{g}}$ so defined on $R_{\mathfrak{p},\mathfrak{l}}$ have regular extensions to $\mathfrak{p}$ since $\mathfrak{p}$ is normal and since $R_{\mathfrak{p},\mathfrak{l}}$ is a big open subset of $\mathfrak{p}$. As a result, for all (x,y) in $\mathfrak{p}\times\mathfrak{p}$ and for all (a,b) in $\Bbbk^{2}$, $\varpi\circ\varepsilon_{i}(ax+by)$ is a linear combination of the elements $\varphi_{1}(ax+by),\ldots,\varphi_{\mathrm{rk}\mathfrak{g}}(ax+by)$. Hence, by [1], $\varpi(V_{x,y})$ is contained in $V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}}$ for all (x,y) in $\mathfrak{p}\times\mathfrak{p}$, whence the assertion.

(iv) Let (x,y) be in $R'_{\mathfrak{p}}\times\mathfrak{p}$. For all z in a nonempty open subset of $P_{x,y}$, z belongs to $R'_{\mathfrak{p}}$ since x belongs to $R'_{\mathfrak{p}}$. So by (i), $\mathfrak{l}^{\widetilde{z}}$ is contained in $\varpi(V_{x,y})$ for all z in a nonempty open subset of $P_{x,y}$. As a result, $V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}}$ is contained in $\varpi(V_{x,y})$, whence the assertion by (iii). ∎

Corollary 7. *For all (x,y) in $\Omega_{\mathfrak{g}}\cap\mathfrak{p}\times\mathfrak{p}$, $V_{x,y}=V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}}+\mathfrak{p}_{\mathrm{u}}$.*

Proof. Let $(x,y) \in \Omega_{\mathfrak{g}} \cap \mathfrak{p} \times \mathfrak{p}$. Since $(x,y) \in \mathfrak{p} \times \mathfrak{p}$,

$$\dim V_{x,y} \leqslant \dim V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}} + \dim \mathfrak{p}_{\mathrm{u}},$$

by Lemma 6 (iii). Moreover,

$$\dim V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}} + \dim \mathfrak{p}_{\mathrm{u}} \leqslant \mathrm{b}_{\mathfrak{l}} + \dim \mathfrak{p}_{\mathrm{u}} = \mathrm{b}_{\mathfrak{g}}.$$

Otherwise, $\dim V_{x,y} = \mathrm{b}_{\mathfrak{g}}$, since $(x,y) \in \Omega_{\mathfrak{g}}$. Hence $V_{x,y} = V^{\mathfrak{l}}_{\widetilde{x},\widetilde{y}} + \mathfrak{p}_{\mathrm{u}}$. ∎

Set:

$$\mathcal{R}'_+ := \{\alpha \in \mathcal{R}_+ \mid \mathfrak{g}^{\alpha} \subset \mathfrak{p}_{\mathrm{u}}\}.$$

Let $\beta_1, \ldots, \beta_{\mathrm{rk}\mathfrak{g}}$ be in Π, let s_i be the reflexion associated to β_i for all $i \in \{1, \ldots, \mathrm{rk}\mathfrak{g}\}$ and let I be the set of $i \in \{1, \ldots, \mathrm{rk}\mathfrak{g}\}$ such that $\beta_i \in \mathcal{R}'_+$.

Lemma 8. *Let w be in W.*

(i) If $w\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+$, then $w \in W_{\mathfrak{l}}$.

(ii) If $w\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+ \cup \mathcal{R}_{\mathfrak{l}}$, then $w \in W_{\mathfrak{l}}$.

Proof. (i) Let $s_{i_1} \ldots s_{i_p}$ be a reduced decomposition of w. We proceed by induction on the length of w denoted by $l(w)$. For $l(w) = 1$, $w = s_j$ for some $j \in \{1, \ldots, \mathrm{rk}\mathfrak{g}\}$. If $j \in I$, then $s_j(\beta_j) = -\beta_j$ but $w\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+$. Hence $j \notin I$ and $w \in W_{\mathfrak{l}}$. Suppose the property true for $l(w) \leqslant p-1$. By [9, Lemma 18.8.3 (ii)],

$$w\left(\beta_{i_p}\right) \in -\mathcal{R}_+.$$

Then $i_p \notin I$. So $s_{i_p}\left(\mathcal{R}'_+\right) = \mathcal{R}'_+$ and

$$s_{i_1} \ldots s_{i_{p-1}}\left(\mathcal{R}'_+\right) = w\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+.$$

Then, by induction hypothesis,

$$s_{i_1} \ldots s_{i_{p-1}} \in W_{\mathfrak{l}}.$$

Hence $w \in W_{\mathfrak{l}}$ since $s_{i_p} \in W_{\mathfrak{l}}$, whence the assertion.

(ii) Let g_w be in $N_{\mathbf{G}}(\mathfrak{h})$ a representative of w. Since $g_w(\mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{h})$ is contained in $\mathfrak{p}$, it is contained in a Borel subalgebra $\mathfrak{b}'$ contained in $\mathfrak{p}$, containing $\mathfrak{h}$. Then there exists $w' \in W_{\mathfrak{l}}$, such that $g_{w'}(\mathfrak{b}') = \mathfrak{b}$, for some representative $g_{w'}$ of w' in $N_{\mathbf{G}}(\mathfrak{h})$. So $g_{w'}g_w(\mathfrak{p}_{\mathrm{u}} \oplus \mathfrak{h})$ is contained in $\mathfrak{b}$ and $w'w\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+$. Hence, by (i), $w'w \in W_{\mathfrak{l}}$, whence $w \in W_{\mathfrak{l}}$ since $w' \in W_{\mathfrak{l}}$. ∎

Proposition 9. *Let $\mathfrak{p}$ and $\mathfrak{p}'$ be two parabolic subalgebras of $\mathfrak{g}$ such that $\mathfrak{p}_{\mathrm{u}}$ is contained in $\mathfrak{p}'$. If $\mathfrak{p}$ and $\mathfrak{p}'$ are conjugate under $\mathbf{G}$, then $\mathfrak{p} = \mathfrak{p}'$.*

Proof. We suppose that $\mathfrak{p}' = g(\mathfrak{p})$, for some g in $\mathbf{G}$. By Bruhat decomposition there exist $u, b \in \mathbf{B}$ and $w \in W$ such that $g = ug_wb$, where g_w is a representative of w in $N_G(\mathfrak{h})$. Since $g(\mathfrak{p})$ contains $\mathfrak{p}_{\mathrm{u}}$ and since $u^{-1}(\mathfrak{p}_{\mathrm{u}}) = \mathfrak{p}_{\mathrm{u}}$, $ug_w(\mathfrak{p})$ and $g_w(\mathfrak{p})$ contains $\mathfrak{p}_{\mathrm{u}}$. Hence $\mathfrak{p}$ contains $g_w^{-1}(\mathfrak{p}_{\mathrm{u}})$. Then $w^{-1}\left(\mathcal{R}'_+\right) \subseteq \mathcal{R}_+ \cup \mathcal{R}_{\mathfrak{l}}$. Hence, by Lemma 8 (ii), $w^{-1} \in W_{\mathfrak{l}}$, whence $g_w \in \mathbf{L}$ and $g \in \mathbf{P}$. ∎

3.2. A desingularization of $\mathcal{P}^{(k)}$.

We consider the action of $\mathbf{P}$ on $\mathbf{G}\times\mathfrak{p}^k$ given by

$$p.(g,x_1,\dots,x_k)=\left(gp^{-1},p\left(x_1\right),\dots,p\left(x_k\right)\right).$$

Let μ'_k be the morphism from $\mathbf{G}\times\mathfrak{p}^k$ to $\mathfrak{g}^k$ defined by

$$\mu'_k(g,x_1,\dots,x_k)=\left(g\left(x_1\right),\dots,g\left(x_k\right)\right)$$

and let μ_k be the morphism from $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ to $\mathfrak{g}^k$ defined through the quotient by μ'_k. Let

$$\mathcal{P}^{(k)}:=\left\{(x_1,\dots,x_k)\in\mathfrak{g}^k\mid\exists g\in G\text{ such that }\left(g\left(x_1\right),\dots,g\left(x_k\right)\right)\in\mathfrak{p}^k\right\}.$$

Then $\mathcal{P}^{(k)}$ is the image of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ by μ_k.

The following Lemma is well known:

Lemma 10. *Let P and Q be parabolic subgroups of $\mathbf{G}$ such that P is contained in Q. Let X be a Q-variety and let Y be a closed subset of X, invariant under P. Then $Q.Y$ is a closed subset of X. Moreover, the canonical map from $Q\times_P Y$ to $Q.Y$ is a projective morphism.*

Proof. The proof is given in [2, Lemma 1.4]. ■

Proposition 11. *Let $k\geqslant 2$, the variety $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ is a desingularization of $\mathcal{P}^{(k)}$ and μ_k is the desingularization morphism. The subvariety $\mathcal{P}^{(k)}$ of $\mathfrak{g}^k$ is closed, but it is not normal.*

Proof. According to Lemma 10, μ_k is a projective morphism and $\mathcal{P}^{(k)}$ is closed in $\mathfrak{g}^k$ since $\mathcal{P}^{(k)}$ is the image of $\mathbf{G}\times\mathfrak{p}^k$ by μ'_k.

Since $\Omega_{\mathfrak{g}}$ is an open subset of $\mathfrak{g}\times\mathfrak{g}$, $\Omega_{\mathfrak{g}}\times\mathfrak{g}^{k-2}\cap\mathcal{P}^{(k)}$ is an open subset of $\mathcal{P}^{(k)}$ and it is nonempty since $(e,h)\in\Omega_{\mathfrak{g}}\cap\mathfrak{p}^2$. Let $(x_1,\dots,x_k)$ be in $\Omega_{\mathfrak{g}}\times\mathfrak{g}^{k-2}\cap\mathfrak{p}^k$. Suppose that $(x_1,\dots,x_k)\in\left(g\left(\mathfrak{p}\right)\right)^k$ for some $g\in G$. By Corollary 5, V_{x_1,x_2} is contained in $\mathfrak{p}$ and $g\left(\mathfrak{p}\right)$ and by Corollary 7, V_{x_1,x_2} contains $\mathfrak{p}_{\mathrm{u}}$. Hence $\mathfrak{p}_{\mathrm{u}}$ is contained in $g\left(\mathfrak{p}\right)$. Then, by Proposition 9, $g\left(\mathfrak{p}\right)=\mathfrak{p}$. Hence, for all $(x_1,\dots,x_k)\in\Omega_{\mathfrak{g}}\times\mathfrak{g}^{k-2}\cap\mathcal{P}^{(k)}$, $\mid\mu_k^{-1}\left(x_1,\dots,x_k\right)\mid=1$, Whence μ_k is a birational morphism. Since $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}^k$ is a smooth variety as a vector bundle over the smooth variety $\mathbf{G}/\mathbf{P}$, it is a desingularization of $\mathcal{P}^{(k)}$ of morphism μ_k.

Let x be in $\mathfrak{h}_{\mathrm{reg}}$ and let g be in $\mathbf{G}$. Then

$$g^{-1}\left(x\right)\in\mathfrak{p}\Leftrightarrow g^{-1}\left(\mathfrak{h}\right)\subset\mathfrak{p}\Leftrightarrow g\in N_{\mathbf{G}}\left(\mathfrak{h}\right)\mathbf{P}.$$

According to Lemma 8, (ii), $N_G\left(\mathfrak{h}\right)\mathbf{P}/\mathbf{P}=W/W_{\mathfrak{l}}$ so that $\mid\mu_k^{-1}\left(x,0,\dots,0\right)\mid>1$. Since μ_k is proper and birational, if $\mathcal{P}^{(k)}$ would be normal, then by Zariski's main Theorem [8, §III.9], a finite fiber of μ_k would have cardinality 1. So $\mathcal{P}^{(k)}$ is not normal. ■

Let $\mathfrak{l}//\mathbf{L}$ be the categorical quotient of $\mathfrak{l}$ by $\mathbf{L}$. Let χ_0 be the closed subvariety of $\mathfrak{g}\times\mathfrak{h}$ such that $\Bbbk\left[\chi_0\right]=S\left(\mathfrak{g}\right)\otimes_{S(\mathfrak{h})^W}S\left(\mathfrak{h}\right)$ and let χ be equal to $\chi_0//W_{\mathfrak{l}}$. Then χ is a subset of $\mathfrak{g}\times\mathfrak{l}//\mathbf{L}$. The group $\mathbf{G}$ acts on χ_0 and χ on the first factor and W acts on χ_0 on the second factor. For $x\in\mathfrak{p}$, let $\bar{x}$ be the image of $\widetilde{x}$ in $\mathfrak{l}//\mathbf{L}$.

Lemma 12. *(i) Let x and x' be in $\mathfrak{p}_{\text{reg}}$. If $(x', \bar{x'})$ is in $\mathbf{G}.(x, \bar{x})$, then x' is in $\mathbf{P}(x)$.*

(ii) For x in $\mathfrak{p}_{\text{reg}}$, $\mathbf{G}^x$ is contained in $\mathbf{P}$.

(iii) The map

$$\begin{array}{rccc} \theta : & \mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\text{reg}} & \longrightarrow & \chi \\ & (g, x) & \longmapsto & (g(x), \bar{x}) \end{array}$$

is an isomorphism onto an open subset of χ.

Proof. (i) Let x and x' be in $\mathfrak{p}_{\text{reg}}$ such that $(x', \bar{x'}) \in \mathbf{G}.(x, \bar{x})$ and let $x = x_s + x_n$ and $x' = x'_s + x'_n$ be the Jordan decomposition of x and x' respectively. Since $\bar{x} = \bar{x'}$, there exists l in $\mathbf{L}$ such that $\widetilde{x}_s = l\left(\widetilde{x'}_s\right)$, where $\widetilde{x}_s$ and $\widetilde{x'}_s$ are the semisimple components of $\widetilde{x}$ and $\widetilde{x'}$ respectively. Then we can suppose that $\widetilde{x}_s = \widetilde{x'}_s$. The semisimple components x_s and x'_s of x and x' are conjugate under $\mathbf{P}$ since they are conjugate to $\widetilde{x}_s$ under $\mathbf{P}$. Then there exist p and p' in $\mathbf{P}$ such that $p(x_s) = p'(x'_s) \in \mathfrak{h}$. Since $p(x_s)$ is a semisimple element, $\mathfrak{g}^{p(x_s)}$ is a reductive algebra. Then $\mathfrak{p} \cap \mathfrak{g}^{p(x_s)}$ is a parabolic subalgebra of $\mathfrak{g}^{p(x_s)}$, since it contains the Borel subalgebra $\mathfrak{b} \cap \mathfrak{g}^{p(x_s)}$ of $\mathfrak{g}^{p(x_s)}$. Then $p(x_n)$ and $p'(x'_n)$ are regular nilpotent elements in $\mathfrak{p} \cap \mathfrak{g}^{p(x_s)}$, hence they are conjugate under the parabolic subgroup $\mathbf{Q}$ of $\mathbf{G}^{p(x_s)}$ of Lie algebra $\mathrm{ad}\left(\mathfrak{p} \cap \mathfrak{g}^{p(x_s)}\right)$. Hence $p(x)$ and $p'(x')$ are conjugate under $\mathbf{P}$.

(ii) By [6, Proposition 14], $\mathbf{G}^x$ is connected since x is regular . Moreover $\mathfrak{g}^x$ is contained in $\mathfrak{p}$ since x is in $\mathfrak{p}$. Hence $\mathbf{G}^x$ is contained in $\mathbf{P}$.

(iii) Let (g, x) and (g', x') be in $\mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\text{reg}}$ such that $\theta(g, x) = \theta(g', x')$. By (i),

$$\exists\, p \in \mathbf{P} \text{ such that } x' = p(x),$$

then

$$g^{-1} g' p \in \mathbf{G}^x.$$

Then, by (ii),

$$g^{-1} g' \in \mathbf{P}_{-,\mathrm{u}} \cap \mathbf{P} = \{1_{\mathfrak{g}}\}.$$

Whence $(g, x) = (g', x')$ and θ is injective. Since χ is the categorical quotient of the variety χ_0 by the finite subset $W_{\mathfrak{l}}$ and since χ_0 is normal, by [2, Lemma 3.1], χ is a normal variety. Then, by Zariski's Main Theorem [8, §III.9], θ is an isomorphism from $\mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\text{reg}}$ onto an open subset of χ. ∎

Let σ'_1 be the morphism from $\mathbf{G} \times \mathfrak{p}$ to χ defined by $\sigma'_1(g, x) = (g(x), \bar{x})$ and let σ_1 be the morphism from $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}$ to χ defined by σ'_1 through the quotient.

Corollary 13. *The morphism σ_1 is birational.*

Proof. Let $(x, \bar{x})$ be in χ such that x in $\mathfrak{p}_{\text{reg}}$ and let (g_1, x_1) be in $\mathbf{G} \times \mathfrak{p}$ such that $\sigma'_1(g_1, x_1) = (x, \bar{x})$. Then $(g_1(x_1), \overline{x_1}) = (x, \bar{x})$ and $(x_1, \overline{x_1}) \in \mathbf{G}.(x, \bar{x})$. Hence, by Lemma 12 (i), there exists $p \in \mathbf{P}$ such that $x_1 = p(x)$. So $g_1 p(x) = x$ and $g_1 p \in \mathbf{G}^x$. By Lemma 12 (ii), $g_1 \in \mathbf{P}$ since $x \in \mathfrak{p}_{\text{reg}}$. Hence the restriction of σ_1 to $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\text{reg}}$ is injective, whence σ_1 is birational since it is surjective. ∎

Recall that

$$\sigma'_k : \mathbf{G} \times \mathfrak{p}^k \to \chi^k$$
$$(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k}),$$

where $\bar{x}_i$ is the image of $\widetilde{x_i}$ in $\mathfrak{l}//\mathbf{L}$. Recall that σ_k is the morphism from $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ to χ^k defined through the quotient by σ'_k and $\chi^{(k)}$ is the image of $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ by σ_k and let $\mathrm{pr}_{1,k}$ be the canonical projection from χ^k onto $\mathfrak{g}^k$. Then we have the following commutative diagram

$$\begin{array}{ccc} G \times_{\mathbf{P}} \mathfrak{p}^k & \xrightarrow{\sigma_k} & \chi^{(k)} \\ & \searrow^{\mu_k} & \downarrow \mathrm{pr}_{1,k} \\ & & \mathcal{P}^{(k)}. \end{array}$$

Proposition 14. *(i) The variety $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ is a desingularization of $\chi^{(k)}$ of morphism σ_k.*

(ii) For $x = (x_1, \ldots, x_k, y_1, \ldots, y_k) \in \chi^{(k)}$ such that $x_i \in \mathfrak{g}_{\mathrm{reg}}$ for some i, the fiber of σ_k at x has one element.

Proof. (i) For $k = 1$, σ_1 is birational by Corallary 13 and it is projective by Lemma 10. For $k \leqslant 2$, since $\mu_k = \mathrm{pr}_{1,k} \circ \sigma_k$ and since μ_k is a projective and birational morphism, σ_k is projective and birational. Then σ_k is a desingularization of $\chi^{(k)}$ since $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ is a smooth variety.

(ii) Let x be in $\chi^{(k)}$ such that the first component of $\mathrm{pr}_{1,k}(x)$ is in $\mathfrak{g}_{\mathrm{reg}}$. Let $\overline{(g, x_1, \ldots, x_k)}$ and $\overline{(g', x'_1, \ldots, x'_k)}$ be in $\sigma_k^{-1}(x)$ and let $(g, x_1, \ldots, x_k)$ and $(g', x'_1, \ldots, x'_k)$ be in $\mathbf{G} \times \mathfrak{p}^k$ representatives of $\overline{(g, x_1, \ldots, x_k)}$ and $\overline{(g', x'_1, \ldots, x'_k)}$ respectively. So

$$(g(x_1), \overline{x_1}) = \left(g'(x'_1), \overline{x'_1}\right).$$

Hence, by Lemma 12, $g^{-1}g' \in \mathbf{P}$ and $g(\mathfrak{p}) = g'(\mathfrak{p})$, whence the assertion since σ_k is $\mathfrak{S}_k$-equivariant. ■

As a generalization of [2, Lemma 3.8(iii)], we have the following corollary:

Corollary 15. *The variety $\chi^{(k)}$ is an irreducible component of the inverse image of $\mathcal{P}^{(k)}$ in χ^k by $\mathrm{pr}_{1,k}$.*

Proof. Since $\chi_0 \to \mathfrak{g}$ is finite, $\chi \to \mathfrak{g}$ and $\mathrm{pr}_{1,k}$ are finite morphism. So $\mathrm{pr}_{1,k}^{-1}\left(\mathcal{P}^{(k)}\right)$, $\chi^{(k)}$ and $\mathcal{P}^{(k)}$ have the same dimension, whence the corollary since $\chi^{(k)}$ is irreducible as an image of an irreducible variety. ■

In this subsection, most of the results are generalization of results of [2] when $\mathfrak{p} \supsetneq \mathfrak{b}$.

Proposition 16. *Let*

$$W_k := \left\{(x_1, \ldots, x_k, \overline{y_1}, \ldots, \overline{y_k}) \in \chi^{(k)} \mid \exists i \in \{1, \ldots, n\} \text{ such that } x_i \in \mathfrak{g}_{\mathrm{reg}}\right\}.$$

(i) The subset W_k is a smooth open subset of $\chi^{(k)}$.

(ii) The codimensions of $\chi^{(k)} \setminus W_k$ in $\chi^{(k)}$ and of $\mathbf{G} \times_{\mathbf{P}} \left(\mathfrak{p}^k \setminus \mathrm{pr}_{1,k}(W_k)\right)$ in $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ are at least $2k$.

(iii) The restriction of σ_k to $\sigma_k^{-1}(W_k)$ is an isomorphism onto W_k.

Proof. (i) We recall that θ is the map

$$\begin{array}{ccc} \mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\mathrm{reg}} & \longrightarrow & \chi \\ (g, x) & \longmapsto & (g(x), \bar{x}) . \end{array}$$

Let W_k' be the inverse image of $\theta(\mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\mathrm{reg}})$ by the projection

$$\begin{array}{ccc} \chi^{(k)} & \longrightarrow & \chi \\ (x_1, \ldots, x_k, \overline{y_1}, \ldots, \overline{y_k}) & \longmapsto & (x_1, \overline{y_1}) . \end{array}$$

By Lemma 12 (iii), the image of θ is an open subset of χ. Then W_k' is an open subset of $\chi^{(k)}$.
Let $(x_1, \ldots, x_k)$ be in $\mathfrak{p}^k$ and let g be in $\mathbf{G}$ such that $(g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k})$ is in W_k'. Then x_1 is in $\mathfrak{p}_{\mathrm{reg}}$ and for some g' in $\mathbf{P}_{-,\mathrm{u}}$ and x_1' in $\mathfrak{p}_{\mathrm{reg}}$

$$g.(x_1, \overline{x_1}) = g'.\left(x_1', \overline{x_1'}\right).$$

By Lemma 12 (i), there exists $p \in \mathbf{P}$ such that $x_1' = p(x_1)$. So $g^{-1}g'p$ is in $\mathbf{G}^{x_1}$. Hence $g \in \mathbf{P}_{-,\mathrm{u}}\mathbf{P}$, since $\mathbf{G}^{x_1} \subset \mathbf{P}$ by Lemma 12 (ii). As a result, the map

$$\begin{array}{ccc} \mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\mathrm{reg}} \times \mathfrak{p}^{k-1} & \longrightarrow & W_k' \\ (g, x_1, \ldots, x_k) & \longmapsto & (g(x_1), \ldots, g(x_k), \overline{x_1}, \ldots, \overline{x_k}) \end{array}$$

is an isomorphism whose inverse is

$$\begin{array}{ccc} W_k' & \longrightarrow & \mathbf{P}_{-,\mathrm{u}} \times \mathfrak{p}_{\mathrm{reg}} \times \mathfrak{p}^{k-1} \\ (x_1, \ldots, x_k, \overline{y_1}, \ldots, \overline{y_k}) & \longmapsto & (q, z, q^{-1}(x_2), \ldots, q^{-1}(x_k)) , \end{array}$$

with q and z are the first and the second components of $\theta^{-1}(x_1, \overline{y_1})$. As a result, W_k' and $G.W_k'$ are smooth open subsets of $\chi^{(k)}$, whence the assertion since $\chi^{(k)}$ is $\mathfrak{S}_k$-invariant.

(ii) By definition, $\mathbf{G} \times_{\mathbf{P}} \left(\mathfrak{p}^k \setminus \mathrm{pr}_{1,k}(W_k)\right)$ is equal to $\mathbf{G} \times_{\mathbf{P}} (\mathfrak{p} \setminus \mathfrak{p}_{\mathrm{reg}})^k$ and $\chi^{(k)} \setminus W_k$ is equal to the image of $\mathbf{G} \times_{\mathbf{P}} (\mathfrak{p} \setminus \mathfrak{p}_{\mathrm{reg}})^k$ by σ_k. Since $\mathfrak{b} \setminus \mathfrak{b}_{\mathrm{reg}}$ has codimension 2 in $\mathfrak{b}$, $\mathfrak{p} \setminus \mathfrak{p}_{\mathrm{reg}}$ has codimension at least 2 in $\mathfrak{p}$. Then

$$\dim \chi^{(k)} \setminus W_k \leqslant \dim \mathbf{G} \times_{\mathbf{P}} (\mathfrak{p} \setminus \mathfrak{p}_{\mathrm{reg}})^k \leqslant k \dim \mathfrak{p} + \dim \mathfrak{p}_{\mathrm{u}} - 2k$$
$$\dim \mathbf{G} \times_{\mathbf{P}} \left(\mathfrak{p}^k \setminus \mathrm{pr}_{1,k}(W_k)\right) \leqslant \dim \mathbf{G} \times_{\mathbf{P}} (\mathfrak{p} \setminus \mathfrak{p}_{\mathrm{reg}})^k \leqslant k \dim \mathfrak{p} + \dim \mathfrak{p}_{\mathrm{u}} - 2k.$$

(iii) By Proposition 14 (ii), the restriction of σ_k to $\sigma_k^{-1}(W_k)$ is bijective. Whence the assertion by Zariski Main Theorem [8, §III.9] since W_k is a smooth open subset of $\chi^{(k)}$ by (i). ■

4. On the variety $\mathcal{P}_{\mathrm{u}}^{(k)}$

Recall that μ_k is the morphism from $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}^k$ to $\mathfrak{g}$ defined by the map

$$(g, x_1, \ldots, x_k) \mapsto (g(x_1), \ldots, g(x_k))$$

from $\mathbf{G} \times \mathfrak{p}^k$ to $\mathfrak{g}^k$ through the quotient. Let $\mu_{k,\mathrm{u}}$ be the restriction of μ_k to $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\mathrm{u}}^k$ and let

$$\mathcal{P}_{\mathrm{u}}^{(k)} := \left\{(x_1, \ldots, x_k) \in \mathfrak{g}^k \mid \exists g \in G \text{ such that } (g(x_1), \ldots, g(x_k)) \in \mathfrak{p}_{\mathrm{u}}^k\right\}.$$

Let $\Bbbk(\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\mathrm{u}})$ and $\Bbbk(\mathbf{G}(\mathfrak{p}_{\mathrm{u}}))$ be the fields of the rational functions of $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\mathrm{u}}$ and $\mathbf{G}(\mathfrak{p}_{\mathrm{u}})$ respectively.

Lemma 17. *The morphism $\mu_{1,\mathrm{u}}$ is a projective morphism. Moreover,* $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ *is an algebraic extension of finite degree of* $\Bbbk(\mathbf{G}(\mathfrak{p}_{\mathrm{u}}))$.

Proof. According to Lemma 10, $\mu_{1,\mathrm{u}}$ is a projective morphism. Since $\mathbf{G}(\mathfrak{p}_{\mathrm{u}})$ is the image of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ by $\mu_{1,\mathrm{u}}$, $\mu_{1,\mathrm{u}}$ induces an embedding from $\Bbbk(\mathbf{G}(\mathfrak{p}_{\mathrm{u}}))$ into $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$. Hence $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ is an algebraic extension of $\Bbbk(\mathbf{G}(\mathfrak{p}_{\mathrm{u}}))$ of finite degree since $\dim\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}=\dim\mathbf{G}(\mathfrak{p}_{\mathrm{u}})$. ■

Let $\mathcal{A}$ be the integral closure of $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$ in $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ and let $X = Spec\mathcal{A}$. Then $\Bbbk[X]$ is invariant under the action of $\mathbf{G}$ in $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ since so it is for $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$, whence an action of $\mathbf{G}$ on X. Denote by α the $\mathbf{G}$-equivariant morphism from X to $\mathbf{G}(\mathfrak{p}_{\mathrm{u}})$ such that its comorphism is the canonical injection from $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$ into $\mathcal{A}$. The variety X is the normalization of $G(\mathfrak{p}_{\mathrm{u}})$ in $\Bbbk(G\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ ([9, §17.2]).

Proposition 18. *There exists a unique $\mathbf{G}$-equivariant morphism $\sigma_{1,\mathrm{u}}$ from $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ to X such that $\mu_{1,\mathrm{u}}=\alpha\circ\sigma_{1,\mathrm{u}}$ Moreover, $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ is a desingularization of X of morphism $\sigma_{1,\mathrm{u}}$.*

Proof. The variety $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ is a smooth variety since it is a vector bundle over the smooth variety $\mathbf{G}/\mathbf{P}$. Identify $\Bbbk(\mathbf{G}(\mathfrak{p}_{\mathrm{u}}))$ with a subfield of $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ by the comorphism of $\mu_{1,\mathrm{u}}$. Let U be an affine open subset of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$. Then $\Bbbk[U]$ contains $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$ and the comorphism of the restriction map of $\mu_{1,\mathrm{u}}$ to U is the canonical injection of $\Bbbk[\mathbf{G}(\mathfrak{p}_{\mathrm{u}})]$ in $\Bbbk[U]$. Since U is a smooth variety, it is normal and $\Bbbk[U]$ is integrally closed in $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$. Hence $\Bbbk[U]$ contains $\mathcal{A}$. Then there exists a morphism $\sigma_{1,\mathrm{u}}^{U}$ from U to X such that its comorphism is the canonical injection of $\mathcal{A}$ in $\Bbbk[U]$. Then the morphisms $\sigma_{1,\mathrm{u}}^{U}$, where U is an affine open subset of $G\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ glue together as in [5, step 3 of proof of Ch. II, Theorem 3.3] to give the morphism $\sigma_{1,\mathrm{u}}$. The morphism $\sigma_{1,\mathrm{u}}$ is unique since its comorphism is the canonical injection from $\mathcal{A}$ to $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$. Since $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ is the fraction field of $\mathcal{A}$, $\sigma_{1,\mathrm{u}}$ is birational. Moreover, since $\mu_{1,\mathrm{u}}$ is a projective morphism, $\sigma_{1,\mathrm{u}}$ is too, whence the proposition. ■

Denote by κ the quotient map from $\mathbf{G}\times\mathfrak{p}_{\mathrm{u}}$ to $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ and denote by ι the morphism from $\{1\}\times\mathfrak{p}_{\mathrm{u}}$ to X defined by $\iota(1,x)=\sigma_{1,\mathrm{u}}\circ\kappa(1,x)$. Recall that Δ is the Richardson orbit of $\mathfrak{p}$ and recall that $\mathfrak{p}_{\mathrm{u},\Delta}:=\Delta\cap\mathfrak{p}_{\mathrm{u}}$. Let X' be the image of $\{1\}\times\mathfrak{p}_{\mathrm{u},\Delta}$ by ι and let δ be the morphism from $\mathbf{P}_{-,\mathrm{u}}\times X'$ to X defined by $\delta(p,x)=p.x$.

Proposition 19. *(i) For $x\in\mathfrak{p}_{\mathrm{u},\Delta}$, the stabilizer of $\iota(1,x)$ in $\mathbf{G}$ equals $\mathbf{P}\cap\mathbf{G}^{x}$.*

(ii) The morphism δ is an isomorphism onto an open subset of X.

(iii) Let g be in $\mathbf{G}$ and let x be in X'. If $g(x)$ is in X', then g is in $\mathbf{P}$.

Proof. (i) Let x be in $\mathfrak{p}_{\mathrm{u},\Delta}$. Then, by [3, Theorem 7.1.1], $\mathbf{G}^{x}$ and $\mathbf{P}\cap\mathbf{G}^{x}$ have the same identity component. Then $\mathbf{P}\cap\mathbf{G}^{x}$ is a subgroup of finite index of $\mathbf{G}^{\iota(1,x)}$ and $\Bbbk(\mathbf{G}/\mathbf{P}\cap\mathbf{G}^{x})$ is an algebraic extension of $\Bbbk\left(\mathbf{G}/\mathbf{G}^{\iota(1,x)}\right)$ of degree the index of $\mathbf{P}\cap\mathbf{G}^{x}$ in $\mathbf{G}^{\iota(1,x)}$. Moreover, $\Bbbk\left(\mathbf{G}/\mathbf{G}^{\iota(1,x)}\right)$ is an algebraic extension of finite degree of a transcendental extension of $\Bbbk$. Since $\sigma_{1,\mathrm{u}}$ is a birational morphism and since $G.\iota(1,x)$ is an open subset of X, $\Bbbk\left(\mathbf{G}/\mathbf{G}^{\iota(1,x)}\right)$ and $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ are isomorphic. Since $\mathbf{G}\times_{\mathbf{P}}\mathbf{P}(x)$ is an open subset of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$, $\Bbbk(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}})$ and $\Bbbk(\mathbf{G}/\mathbf{P}\cap\mathbf{G}^{x})$

are isomorphic. So that $\Bbbk\left(\mathbf{G}/\mathbf{G}^{\iota(1,x)}\right)$ and $\Bbbk\left(\mathbf{G}/\mathbf{P}\cap\mathbf{G}^{x}\right)$ are isomorphic. Hence $\mathbf{G}^{\iota(1,x)}=\mathbf{P}\cap\mathbf{G}^{x}$.

(ii) Let (p_1,x_1) and (p_2,x_2) be in $\mathbf{P}_{-,\mathrm{u}}\times X'$ such that $p_1.x_1=p_2.x_2$ and let x_1' and x_2' be in $\mathfrak{p}_{\mathrm{u},\Delta}$ such that $\iota\left(1,x_1'\right)=x_1$ and $\iota\left(1,x_2'\right)=x_2$. Let q be in $\mathbf{P}$ such that $x_2'=q\left(x_1'\right)$. Then $x_2=q.x_1$. So

$$\begin{array}{rcl} p_1.x_1 & = & p_2.x_2=p_2q.x_1 \\ \Rightarrow\ p_1^{-1}p_2q & \in & \mathbf{G}^{x_1}=\mathbf{G}^{\iota\left(1,x_1'\right)}\ =\mathbf{P}\cap\mathbf{G}^{x_1'}, \end{array}$$

by (i). So

$$\begin{array}{rcl} p_1^{-1}p_2 & \in & \mathbf{P}\cap\mathbf{P}_{-,\mathrm{u}}\ =\ \{1_{\mathfrak{g}}\} \\ \Rightarrow\ p_1=p_2 & \Rightarrow & x_1=x_2. \end{array}$$

Hence δ is an injective morphism. Moreover, δ is dominant since X and $\mathbf{P}_{-,\mathrm{u}}\times X'$ have the same dimension. Since X is a normal variety by definition, by Zariski's Main Theorem [8, §III.9], δ is an isomorphism onto an open subset of X.

(iii) Let $g\in G$ and let $x\in X'$ such that $g(x)\in X'$. Then there exist

$$x_1,\, y_1\in\mathfrak{p}_{\mathrm{u},\Delta}\text{ and }p\in\mathbf{P}$$

Such that

$$x=\iota\left(1,x_1\right),\quad g.x=\iota\left(1,y_1\right)\quad\text{and}\quad y_1=p\left(x_1\right).$$

So

$$g.\iota\left(1,x_1\right)=g.x=\iota\left(1,y_1\right)=p.\iota\left(1,x_1\right)$$

and then $p^{-1}g\in\mathbf{G}^{\iota(1,x_1)}$. Hence, by (i), $g\in\mathbf{P}$. ■

Remark 1. In several interesting cases, $\mathbf{G}^{x}$ is connected. When so it is, $\mathbf{G}^{x}$ is contained in $\mathbf{P}$ so that $X'=\mathfrak{p}_{\mathrm{u},\Delta}$ and $X=\mathbf{G}\left(\mathfrak{p}_{\mathrm{u}}\right)$.

Let ϑ be the morphism from $\mathbf{G}^{k}\times_{\mathbf{P}^{k}}\mathfrak{p}_{\mathrm{u}}^{k}$ to X^{k} defined by

$$\vartheta\overline{\left(x_1,\ldots,x_k\right)}=\left(\sigma_{1,\mathrm{u}}\left(x_1\right),\ldots,\sigma_{1,\mathrm{u}}\left(x_k\right)\right),$$

let $\sigma_{k,\mathrm{u}}$ be the restriction of ϑ to $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}$ and let $X^{(k)}$ be the image of $\sigma_{k,\mathrm{u}}$. Then we have the following commutative diagram

$$\begin{array}{ccc} G\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k} & \xrightarrow{\sigma_{k,\mathrm{u}}} & X^{(k)} \\ & \underset{\mu_{k,\mathrm{u}}}{\searrow} & \downarrow{\scriptstyle\varphi} \\ & & \mathcal{P}_{\mathrm{u}}^{(k)}, \end{array}$$

where φ is the morphism induced by the canonical projection α from X to $\mathbf{G}\left(\mathfrak{p}_{\mathrm{u}}\right)$ such that $\mu_{k,\mathrm{u}}=\varphi\circ\sigma_{k,\mathrm{u}}$.

Proposition 20. *Let*

$$V_k=\left\{\left(x_1,\ldots x_k\right)\in X^{(k)}\mid\exists\ i\in\{1,\ldots,k\}\ \textit{such that}\ \alpha\left(x_i\right)\in\mathfrak{p}_{\mathrm{u},\Delta}\right\}.$$

(i) The subset V_k is a smooth open subset of $X^{(k)}$.

(ii) The morphism φ from $X^{(k)}$ to $\mathcal{P}_{\mathrm{u}}^{(k)}$ is finite and surjective.

(iii) The codimension of $X^{(k)}\backslash V_k$ in $X^{(k)}$ and the codimension of

$$\mathbf{G}\times_{\mathbf{P}}\left(\mathfrak{p}_{\mathrm{u}}^{k}\setminus\varphi\left(V_k\right)\right) \text{ in } \mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}$$

are at least k.

(iv) The restriction of $\sigma_{k,\mathrm{u}}$ to $\sigma_{k,\mathrm{u}}^{-1}\left(V_k\right)$ is an isomorphism onto V_k.

Proof. (i) Set $V_k' := \mathbf{P}_{-,\mathrm{u}}\left(X'\right)\times X^{k-1}\cap X^{(k)}$ and

$$V' := \left\{\left(x_1,\ldots,x_k\right)\in\left(\iota\left(\{1\}\times\mathfrak{p}_{\mathrm{u}}\right)\right)^k \mid x_1\in\iota\left(\{1\}\times\mathfrak{p}_{\mathrm{u},\Delta}\right)\right\}.$$

By Proposition 19 (ii), V_k' is an open subset of $X^{(k)}$. Let $g\in G$ and $x_1\in X'$ such that $g.x_1\in\mathbf{P}_{-,\mathrm{u}}\left(X'\right)$. Then there exist

$$q\in\mathbf{P}_{-,\mathrm{u}},\quad y_1\in X',\quad x_1',y_1'\in\mathfrak{p}_{\mathrm{u},\Delta}\quad\text{and}\quad p\in\mathbf{P}$$

such that

$$g\left(x_1\right)=q\left(y_1\right),\quad x_1=\iota\left(1,x_1'\right),\quad y_1=\iota\left(1,y_1'\right)\quad\text{and}\quad y_1'=p\left(x_1'\right).$$

So

$$g.\iota\left(1,x_1'\right)=g\left(x_1\right)=q.\iota\left(1,y_1'\right)=qp.\iota\left(1,x_1'\right)$$

and $p^{-1}q^{-1}g\in\mathbf{G}^{\iota\left(1,x_1'\right)}$. Hence, by Proposition 19 (i), $g\in\mathbf{P}_{-,\mathrm{u}}\mathbf{P}$. As a result

$$\begin{array}{ccc}\mathbf{P}_{-,\mathrm{u}}\times V' & \longrightarrow & V_k' \\ \left(g,x_1,\ldots,x_k\right) & \longmapsto & \left(g.x_1,\ldots,g.x_k\right)\end{array}$$

is an isomorphism whose inverse is given by

$$\begin{array}{ccc}V_k' & \longrightarrow & \mathbf{P}_{-,\mathrm{u}}\times V' \\ \left(x_1,\ldots,x_k\right) & \longmapsto & \left(g_1,y_1,g_1^{-1}.x_2,\ldots,g_1^{-1}.x_k\right),\end{array}$$

where $\left(g_1,y_1\right)=\delta^{-1}\left(x_1\right)$. Let ι_k be the morphism from $\{1\}\times\mathfrak{p}_{\mathrm{u}}^k$ to X^k defined by

$$\iota_k\left(1,x_1,\ldots,x_k\right)=\left(\iota\left(1,x_1\right),\ldots,\iota\left(1,x_k\right)\right)$$

and let Z_k be the image of ι_k. Then we have the following commutative diagram

$$\begin{array}{ccc}\{1\}\times\mathfrak{p}_{\mathrm{u}}^k & \xrightarrow{\iota_k} & Z_k \\ & \searrow & \downarrow{\scriptstyle \varphi_{|Z_k}} \\ & & \mathfrak{p}_{\mathrm{u}}^k,\end{array}$$

where $\varphi_{|Z_k}$ is the restriction of φ to Z_k. Hence $\Bbbk[\{1\}\times\mathfrak{p}_{\mathrm{u}}^k]\supseteq\Bbbk[Z_k]\supseteq\Bbbk[\mathfrak{p}_{\mathrm{u}}^k]$. Since $\Bbbk[\{1\}\times\mathfrak{p}_{\mathrm{u}}^k]$ and $\Bbbk[\mathfrak{p}_{\mathrm{u}}^k]$ are equal, ι_k is an isomorphism onto its image. So it is a closed immersion. Then V' is smooth since $V'=\iota_k\left(\{1\}\times\mathfrak{p}_{\mathrm{u},\Delta}\times\mathfrak{p}_{\mathrm{u}}^k\right)$ and since $\mathfrak{p}_{\mathrm{u},\Delta}$ is smooth as a $\mathbf{P}$-orbit. Hence V_k' and $\mathbf{G}.V_k'$ are smooth open subsets of $X^{(k)}$, whence the assertion since $X^{(k)}$ is $\mathfrak{S}_k$-invariant.

(ii) Since α is a finite morphism, φ is too and since $\mu_{k,\mathrm{u}}$ is surjective, φ is too.

(iii) By definition of V_k, $X^{(k)} \backslash V_k$ is equal to the image of $\mathbf{G} \times_{\mathbf{P}} (\mathfrak{p}_{\rm u} \setminus \mathfrak{p}_{{\rm u},\Delta})^k$ by $\sigma_{k,{\rm u}}$ and $\mathbf{G} \times_{\mathbf{P}} \left(\mathfrak{p}_{\rm u}^k \setminus \varphi(V_k)\right)$ is equal to $\mathbf{G} \times_{\mathbf{P}} (\mathfrak{p}_{\rm u} \setminus \mathfrak{p}_{{\rm u},\Delta})^k$. Hence

$$\dim X^{(k)} \setminus V_k \leqslant \dim \mathbf{G} \times_{\mathbf{P}} (\mathfrak{p}_{\rm u} \setminus \mathfrak{p}_{{\rm u},\Delta})^k \leqslant (k+1)\dim \mathfrak{p}_{\rm u} - k$$
$$\dim \mathbf{G} \times_{\mathbf{P}} (\mathfrak{p}^k \setminus \varphi(V_k)) \leqslant \dim \mathbf{G} \times_{\mathbf{P}} (\mathfrak{p}_{\rm u} \setminus \mathfrak{p}_{{\rm u},\Delta})^k \leqslant (k+1)\dim \mathfrak{p}_{\rm u} - k,$$

whence the assertion since $\dim X^{(k)} = \dim \mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\rm u}^k = (k+1)\dim \mathfrak{p}_{\rm u}$.

(iv) Let $(1, x_1, \ldots, x_k)$ and $(g, y_1, \ldots, y_k)$ be in $\mathbf{G} \times \mathfrak{p}_{\rm u}^k$ such that x_1 and y_1 are in $\mathfrak{p}_{{\rm u},\Delta}$ and

$$\sigma_{k,{\rm u}}\overline{(1, x_1, \ldots, x_k)} = \sigma_{k,{\rm u}}\overline{(g, y_1, \ldots, y_k)} \in V_k,$$

where $\overline{(1, x_1, \ldots, x_k)}$ and $\overline{(g, y_1, \ldots, y_k)}$ are the images of $(1, x_1, \ldots, x_k)$ and $(g, y_1, \ldots, y_k)$ in $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\rm u}^k$ by the quotient map. Then there exists $p \in \mathbf{P}$ such that $y_1 = p(x_1)$. So

$$\iota(1, x_1) = \iota(g, y_1) = g.\iota(1, y_1) = gp.\iota(1, x_1),$$

then

$$\iota(1, x_1) = gp.\iota(1, x_1) \in X'.$$

So, by Proposition 19 (i), $g \in \mathbf{P}$ and $\overline{(1, x_1, \ldots, x_k)} = \overline{(g, y_1, \ldots, y_k)}$. Hence the restriction of $\sigma_{k,{\rm u}}$ to $\sigma_{k,{\rm u}}^{-1}(V_k)$ is injective since V_k is $\mathfrak{S}_k$-invariant, whence the assertion by Zariski Main Theorem [8, §III.9] since V_k is smooth open subset of $X^{(k)}$ by (i). ■

Proposition 21. *The variety $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\rm u}^k$ is a desingularization of $X^{(k)}$ of morphism $\sigma_{k,{\rm u}}$.*

Proof. Since $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\rm u}^k$ is a vector bundle on the smooth variety $\mathbf{G}/\mathbf{P}$, it is a smooth variety. According to Lemma 10, $\mu_{k,{\rm u}}$ is a projective morphism then $\sigma_{k,{\rm u}}$ is too since $\mu_{k,{\rm u}} = \varphi \circ \sigma_{k,{\rm u}}$. Hence, by Proposition 20 (iv), $\mathbf{G} \times_{\mathbf{P}} \mathfrak{p}_{\rm u}^k$ is a desingularization of $X^{(k)}$. ■

Let x be in $\mathfrak{p}_{{\rm u},\Delta}$ and let $(\mathbf{G}^x)_0$ be the neutral component of $\mathbf{G}^x$. The canonical map $\mathbf{G}/(\mathbf{G}^x)_0 \to \mathbf{G}/\mathbf{G}^x$ induces an embedding of $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})]$ into $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)$. Let $\mathcal{C}$ be the integral closure of $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})]$ in $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)$.

Proposition 22. *(i) The field $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)$ is a Galois extension of the field $\Bbbk(\mathbf{G}/\mathbf{G}^x)$ of Galois group $\Gamma = \mathbf{G}^x/(\mathbf{G}^x)_0$.*

(ii) The subalgebra $\mathcal{C}$ is invariant under the action of Γ and $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})_{\rm n}]$ is the set of fixed points by Γ in $\mathcal{C}$, where $\mathbf{G}(\mathfrak{p}_{\rm u})_{\rm n}$ is the normalization of $\mathbf{G}(\mathfrak{p}_{\rm u})$.

Proof. (i) Since $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)^\Gamma = \Bbbk(\mathbf{G}/\mathbf{G}^x)$, $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)$ is a Galois extension of the Galois groupe Γ.

(ii) Since $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})]$ is invariant under the action of Γ, $\mathcal{C}$ is too. Moreover, $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})_{\rm n}]$ is contained in $\Bbbk(\mathbf{G}(\mathfrak{p}_{\rm u})) = \Bbbk(\mathbf{G}/\mathbf{G}^x)$ the set of fixed points of Γ in $\Bbbk(\mathbf{G}/(\mathbf{G}^x)_0)$. Let a be in $\mathcal{C}$ such that a is a fixed point by Γ. Then a is in $\Bbbk(\mathbf{G}/\mathbf{G}^x)$. Hence a is in $\Bbbk[\mathbf{G}(\mathfrak{p}_{\rm u})_{\rm n}]$ since it is in $\mathcal{C}$, whence the assertion. ■

Proposition 23. *(i) The field* $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}\right)$ *is the field of rational fractions* $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}\right)(\pi_1,\ldots,\pi_m)$ *over* $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}\right)$*, where* $m=\dim\mathfrak{p}_{\mathrm{u}}^{k-1}$*. Moreover, it is a subfield of* $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)(\pi_1,\ldots,\pi_m)$*.*

(ii) The field $\Bbbk\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)$ *is the set of fixed points of* Γ *in* $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)(\pi_1,\ldots,\pi_m)$ *under the trivial extension of its action on* $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)$*.*

Proof. (i) Since $\mathbf{P}_{-,\mathrm{u}}$ is isomorphic to an open subset of $\mathbf{G}/\mathbf{P}$ and since $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}$ is a vector bundle over $\mathbf{G}/\mathbf{P}$, $\mathbf{P}_{-,\mathrm{u}}\times\mathfrak{p}_{\mathrm{u}}^{k}$ is isomorphic to an open subset of $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}$. Moreover, $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}$ is a vector bundle over $\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}$ whose fibers are isomorphic to $\mathfrak{p}_{\mathrm{u}}^{k-1}$. Hence $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}^{k}\right)$ equals $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}\right)(\pi_1,\ldots,\pi_m)$, where $m=\dim\mathfrak{p}_{\mathrm{u}}^{k-1}$. So it is a subfield of $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)(\pi_1,\ldots,\pi_m)$ since $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)$ is an extension of $\Bbbk\left(\mathbf{G}\times_{\mathbf{P}}\mathfrak{p}_{\mathrm{u}}\right)$.

(ii) Let ρ be the first projection from $\mathcal{P}_{\mathrm{u}}^{(k)}\cap\mathbf{G}\left(\mathfrak{p}_{\mathrm{u},\Delta}\right)\times\mathfrak{g}^{k-1}$ to $\mathbf{G}\left(\mathfrak{p}_{\mathrm{u},\Delta}\right)$, let x_1 be in $\mathbf{G}\left(\mathfrak{p}_{\mathrm{u},\Delta}\right)$ and let z be in $\rho^{-1}(x_1)$ such that it is a smooth point of $\mathcal{P}_{\mathrm{u}}^{(k)}$ and ρ is smooth at z. Then there exists a smooth affine open subset O containing z in $\mathcal{P}_{\mathrm{u}}^{(k)}$ such that ρ_O the restriction of ρ to O is a submersion onto an open subset of $\mathbf{G}\left(\mathfrak{p}_{\mathrm{u},\Delta}\right)$. Hence the fiber of ρ_O at $\rho(z)$ is an open subset of an affine space. So the restrictions of the affine coordinates $\pi_1,\ldots,\pi_m$ of this space to $\rho_O^{-1}(\rho(z))$ are the restrictions of regular functions on O which we denote again $\pi_1,\ldots,\pi_m$. Let ψ be the morphism defined by

$$\begin{array}{rccc}\psi: & O & \longrightarrow & \rho(O)\times\Bbbk^{m}\\ & y & \longmapsto & \left(\rho(y),\pi_1(y),\ldots,\pi_m(y)\right).\end{array}$$

Then ψ is a submersion at z. So, for all y in an open subset O' of O, containing z, the intersection of the kernels of the differential at y of the restrictions of $\pi_1,\ldots,\pi_m$ to $F_{\rho(y)}$ the fiber of ρ at $\rho(y)$ equals $\{0\}$ so that ψ is locally injective on $F_{\rho(y)}$. Let

$$\Lambda:=\left\{(x',x'')\in O'\times O'\mid\psi(x')=\psi(x'')\right\}$$

and let $\Lambda_{O'}$ be the diagonal of O'. Hence the dimension of $\Lambda\cap F_{\rho(y)}\times F_{\rho(y')}$ is zero, for (y,y') in Λ. Then the dimensions of Λ and of O are equal. Hence $\Lambda_{O'}$ is an irreducible component of Λ. Then there exists an open subset O'' of O such that the restriction of ψ to O'' is injective. Hence $\Bbbk\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)=\Bbbk\left(\mathbf{G}\left(\mathfrak{p}_{\mathrm{u}}\right)\right)(\pi_1,\ldots,\pi_m)$, whence the assertion since $\Bbbk\left(\mathbf{G}\left(\mathfrak{p}_{\mathrm{u}}\right)\right)=\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)^{\Gamma}$. ∎

Let $\mathcal{C}^{(k)}$ be the integral closure of $\Bbbk\left[\mathcal{P}_{\mathrm{u}}^{(k)}\right]$ in $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_0\right)[\pi_1,\ldots,\pi_m]$, let $\left(\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}},\nu_k\right)$ be the normalization of $\mathcal{P}_{\mathrm{u}}^{(k)}$ and let $\left(\left(X^{(k)}\right)_{\mathrm{n}},\eta_k\right)$ be the normalization of $X^{(k)}$.

Proposition 24. *(i) The algebra* $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ *is the set of fixed points of* Γ *in* $\mathcal{C}^{(k)}$*.*

(ii) The algebra $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ *is a direct factor of* $\mathcal{C}^{(k)}$ *as a* $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$*-module. (iii) There exists a pure morphism* β *in the sense of [7, Appendix to §7] from* $\left(X^{(k)}\right)_{\mathrm{n}}$ *to* $\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}$ *such that* $\varphi\circ\eta_k=\nu_k\circ\beta$*.*

Proof. (i) Since $\Bbbk\left[\mathcal{P}_{\mathrm{u}}^{(k)}\right]$ is invariant under the action of Γ, $\mathcal{C}^{(k)}$ is too. Moreover, $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ is contained in $\Bbbk\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)$ the set of fixed points of Γ in $\Bbbk\left(\mathbf{G}/\left(\mathbf{G}^{x}\right)_{0}\right)(\pi_1,\ldots,\pi_m)$. Let a be in $\mathcal{C}^{(k)}$ such that a is a fixed point by Γ. Then a is in $\Bbbk\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)$ and it satisfies an equation of integral dependence on $\Bbbk\left[\mathcal{P}_{\mathrm{u}}^{(k)}\right]$. Hence a is in $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$, whence the assertion.

(ii) Let Φ be the map

$$\begin{array}{rccl} \Phi: & \mathcal{C}^{(k)} & \longrightarrow & \Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right] \\ & c & \longmapsto & c^{\#} = \frac{1}{|\Gamma|}\sum\limits_{\gamma\in\Gamma}\gamma.c \end{array} .$$

Then Φ is a projection from $\mathcal{C}^{(k)}$ onto $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ by (i). Since $(bc)^{\#} = bc^{\#}$, for $b \in \Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ and $c \in \mathcal{C}^{(k)}$, $\mathcal{C}^{(k)}$ is the direct sum as $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$-module of $\Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right]$ and $\mathrm{M}_k := \mathrm{Ker}\Phi$.

(iii) Since $\left(X^{(k)}\right)_{\mathrm{n}}$ is normal and since $\varphi\circ\eta_k$ is surjective, there exists a morphism β from $\left(X^{(k)}\right)_{\mathrm{n}}$ to $\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}$ such that $\varphi\circ\eta_k = \nu_k\circ\beta$ ([5], ch. II, Ex. 3.8). By (ii),

$$\Bbbk\left[\left(X^{(k)}\right)_{\mathrm{n}}\right] = \Bbbk\left[\left(\mathcal{P}_{\mathrm{u}}^{(k)}\right)_{\mathrm{n}}\right] \oplus \left(\mathrm{M}_k \cap \Bbbk\left[\left(X^{(k)}\right)_{\mathrm{n}}\right]\right).$$

Hence β is a pure morphism in the sense of [7, Appendix to §7]. ∎

Acknowledgements. I would like to thank Jean-Yves Charbonnel for his advice and help and for his rigorous attention to this work. I also thank the referee for the careful reading of our paper, and for the useful pieces of advice given.

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Received November 27, 2012
and in final form October 17, 2013