Journal of Lie Theory
Volume **24** (2014) 529–543

A New Formula for the Pfaffian-Type Segal-Sugawara Vector

Natasha Rozhkovskaya

Communicated by G. Olshanski

Abstract. A combinatorial formula for Pfaffian for the universal enveloping algebra $U(\hat{\mathfrak{o}}_{2n})$ of the affine Kac–Moody algebra $\hat{\mathfrak{o}}_{2n}$ is proved. It allows us easily to compute the image of the Segal-Sugawara vector under the Harish-Chandra homomorphism and to deduce formulas for classical Pfaffian of universal enveloping algebra $U(\mathfrak{o}_{2n})$ of the even orthogonal Lie algebra.
Mathematics Subject Classification 2000: 17B35, 17B67.
Key Words and Phrases: Pfaffian, affine orthogonal Lie algebra, Feigin–Frenkel center, Harish-Chandra homomorphism.

1. Introduction

Let $U(\mathfrak{g})$ be the universal enveloping algebra of a simple Lie algebra $\mathfrak{g}$. The Harish-Chandra isomorphism identifies the center $Z(\mathfrak{g})$ of the algebra $U(\mathfrak{g})$ with the algebra of polynomials over the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ that are invariant under a certain action of the corresponding Weyl group W. The elements of $Z(\mathfrak{g})$ act in finite-dimensional irreducible representations of $\mathfrak{g}$ by multiplication by scalars, and these eigenvalues can be found from the images of central elements under the Harish-Chandra isomorphism.

For the even orthogonal Lie algebra $\mathfrak{g} = \mathfrak{o}_{2n}$ the center is generated by n elements those images under the Harish-Chandra isomorphism are W-invariant polynomials over $\mathfrak{h}$ of degrees $2, 4, 6, \dots, 2n-2$ and n. There are several constructions that define these generators explicitly (see, e.g. [5, 8, 10, 12]). In particular, the generator of degree n can be realized as a non-commutative Pfaffian of a certain matrix with coefficients in $U(\mathfrak{o}_{2n})$. Namely, in the notations of Section 2, if F is the $2n \times 2n$ matrix those entries $\{F_{ij}\}$, $(i, j \in \{-n, \dots, -1, 1, \dots n\})$ are standard generators of $U(\mathfrak{o}_{2n})$, and J_{2n} is the matrix of the form that defines $\mathfrak{o}_{2n}$, the Pfaffian of $\tilde{F} = FJ_{2n}$ is the central element defined by the formula

$$\operatorname{Pf} \tilde{F} = \frac{1}{2^n n!} \sum_{\sigma \in S_{[-n;n]}} \operatorname{sgn}(\sigma)(FJ_{2n})_{\sigma(-n)\sigma(-n+1)} \dots (FJ_{2n})_{\sigma(n-1)\sigma(n)}. \tag{1}$$

ISSN 0949–5932 / \$2.50

In [4] a combinatorial formula for $\operatorname{Pf}\tilde{F}$ is proved, which (with some change of notations to match the notations of this note) can be formulated as follows:

$$\operatorname{Pf}\tilde{F} = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{I,J\subset\{-n,\dots,-1\} \\ |I|=|J|=2k}} \operatorname{sgn}(\bar{I},I)\operatorname{sgn}(\bar{J},J)\det\left(\tilde{F}_{\bar{I}\bar{J}} + \rho(|J|)\right)\operatorname{Pf}\tilde{F}_{-J\,J}\operatorname{Pf}\tilde{F}_{I\,-I}. \quad (2)$$

Here for a subset I of $\{-n,\dots,-1\}$, the set $\bar{I}$ is the complement of I in $\{-n,\dots,-1\}$; a submatrix $\tilde{F}_{IJ}$ is defined by $\tilde{F}_{IJ} = (\tilde{F}_{ij})_{i\in I, j\in J}$; $\rho(k)$ stands for the diagonal matrix $\operatorname{diag}(k-1,k-2,\dots,1,0)$; the expressions $\det(\tilde{F}_{\bar{I}\bar{J}}+\rho(|J|))$, $\operatorname{Pf}\tilde{F}_{-J\,J}$, $\operatorname{Pf}\tilde{F}_{I\,-I}$ are certain noncommutative analogues of the determinant and Pfaffian of a matrix; $\operatorname{sgn}(\bar{I},I)$, $\operatorname{sgn}(\bar{J},J)$ are the signs of certain permutations. (We refer the reader for full details to [4].) The advantage of this formula is that it immediately provides the eigenvalues of the central element $\operatorname{Pf}\tilde{F}$ on the highest weight modules. The purpose of this note is to prove a similar to (2) combinatorial formula for a Pfaffian of the matrix of generators for the universal enveloping algebra $U(\hat{\mathfrak{o}}_{2n})$ of the affine Kac–Moody algebra $\hat{\mathfrak{o}}_{2n}$ using the techniques of [9] and [4]. The main results of this paper are Theorems 3.1 and 4.3.

In general, for a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, the vacuum module $V_{-h^\vee}(\mathfrak{g})$ over the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level $-h^\vee$ is defined as the quotient of the universal enveloping algebra $U(\hat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K+h^\vee$, where $h^\vee$ denotes the dual Coxeter number for $\mathfrak{g}$. As a vertex algebra, $V_{-h^\vee}(\mathfrak{g})$ has a non-trivial center $\mathfrak{z}(\hat{\mathfrak{g}})$ defined by

$$\mathfrak{z}(\hat{\mathfrak{g}}) = \{S \in V_{-h^\vee}(\mathfrak{g}) \mid \mathfrak{g}[t]S = 0\}.$$

Its structure is described by the Feigin-Frenkel theorem (1992); see [1] for detailed exposition. In [2, 3, 6] complete sets of explicit generators of the commutative algebra $\mathfrak{z}(\hat{\mathfrak{g}})$ are constructed for classical Lie algebras. In the case of $\mathfrak{g} = \mathfrak{o}_{2n}$ the set of generators contains an element $\operatorname{Pf}\tilde{F}[-1]$ defined below by (3), and this element is the main focus of this note.

Our goal is to prove a combinatorial formula (Theorem 3.1 and Theorem 4.3) for $\operatorname{Pf}\tilde{F}[-1]$ in the spirit of formulas for Pfaffians proved in [4]. The immediate application of this result is that it allows to compute easily the image of the Pfaffian $\operatorname{Pf}\tilde{F}[-1]$ under the Harish-Chandra homomorphism (Section 5). The algebra $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ is a commutative subalgebra of $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}}$ – the algebra of invariants under the action of the Cartan subalgebra $\mathfrak{h}$. A restriction of a Harish-Chandra homomorphism $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}} \to U(t^{-1}\mathfrak{h}[t^{-1}])$ to the subalgebra $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ yields an isomorphism of $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ to the classical $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{o}_{2n})$ (see [1, 7]). The Harish-Chandra images of generators of $\mathfrak{z}(\hat{\mathfrak{g}})$ for classical Lie algebras $\mathfrak{g}$ were computed in [2, 7]. Below Corollary 5.1 deduces the Harish-Chandra image of $\operatorname{Pf}\tilde{F}[-1]$ from the Theorem 3.1 (or Theorem 4.3) of the paper.

Moreover, we illustrate in Section 6 that the formulas of [4] for Pfaffians in classical case can be deduced from the formulas for affine Lie algebra $\hat{\mathfrak{o}}_{2n}$ proved in this note.

2. Application of exterior calculus

Let J_{2n} be a $2n \times 2n$ symmetric matrix with ones on the anti-diagonal and zeros elsewhere. Consider the corresponding realization of $\mathfrak{o}_{2n}$ as a Lie algebra of $2n\times 2n$ complex-valued matrices X that satisfy $X^T J_{2n} + J_{2n}X = 0$. We will use the notation $[-n;n] = \{-n,\dots,-1,1,\dots,n\}$ (the interval of integers from $-n$ to n with the element 0 being omitted). Let $\{E_{i,j}\}_{i,j\in[-n;n]}$ be the matrices that have one in the ith row and jth column and zeros elsewhere. Set for $i,j \in [-n;n]$

$$F_{i,j} = E_{i,j} - E_{-j,-i}, \quad (F_{i,j} = -F_{-j,-i}).$$

The Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{o}_{2n}$ is generated by elements F_{ii} $(i = -n,\dots,-1)$. Using the bilinear form

$$\langle X, Y\rangle = \frac{1}{2}\operatorname{tr}(XY) \quad \text{for } X, Y \in \mathfrak{o}_{2n},$$

one defines the extended affine Kac–Moody algebra $\hat{\mathfrak{o}}_{2n}\oplus\mathbb{C}\tau = \mathfrak{o}_{2n}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}\tau$ and set $X[r] = Xt^r$, $r \in \mathbb{Z}$. Then the commutation relations in $\hat{\mathfrak{o}}_{2n}$ are

$$\begin{aligned}[X[r], Y[s]] &= [X,Y][r+s] + r\delta_{r,-s}\langle X,Y\rangle K,\\ [\tau, X[r]] &= -rX[r-1], \quad [\hat{\mathfrak{o}}_{2n}, K] = 0.\end{aligned}$$

In particular, for $i,j,k,l \in [-n;n]$ one gets

$$\begin{aligned}[F_{i,j}[r], F_{k,l}[s]] = {}& \delta_{j,k}F_{i,l}[r+s]+\delta_{i,l}F_{-j,-k}[r+s]-\delta_{j,-l}F_{i,-k}[r+s]-\delta_{i,-k}F_{-j,l}[r+s]\\ &+ r\delta_{r,-s}(\delta_{j,k}\delta_{i,l} - \delta_{i,-k}\delta_{j,-l})K.\end{aligned}$$

$U(\hat{\mathfrak{o}}_{2n})$ stands for the universal enveloping algebra of $\hat{\mathfrak{o}}_{2n}$. Define $F[r]$ as $U(\hat{\mathfrak{o}}_{2n})$-valued matrix of size $2n\times 2n$ with entries $(F[r])_{i,j} = F_{i,j}[r]$ for $i,j \in [-n;n]$. Then the matrix $\tilde{F}[-1] = F[-1]J_{2n}$ is skew-symmetric and, similarly to the classical case, one defines the Pfaffian of $\tilde{F}[-1]$ as

$$\operatorname{Pf}\tilde{F}[-1] = \frac{1}{2^n n!}\sum_{\sigma\in S_{[-n;n]}} \operatorname{sgn}(\sigma)\tilde{F}_{\sigma(-n),\sigma(-n+1)}[-1]\dots\tilde{F}_{\sigma(n-1),\sigma(n)}[-1], \tag{3}$$

where the sum is over all permutations of the elements of the set $[-n;n]$. By Lemma 4.3 in [6], the Pfaffian of $\tilde{F}[-1]$ is an element of Feigin-Frenkel center $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$. Our goal is to prove a combinatorial formula for $\operatorname{Pf}\tilde{F}[-1]$ that, in particular, allows easily to compute the image of this Segal-Sugawara vector under the Harish-Chandra homomorphism.

As in [9], it is convenient for us to express Pfaffian-like elements as coefficients of powers of certain two-forms. Consider the algebra Λ generated by elements $\{e_i\}_{i\in[-n;n]}$ with the defining relations $e_ie_j = -e_je_i$ for all i and j. Define the $U(\hat{\mathfrak{o}}_{2n})$-valued two-form

$$\Omega[s] = \sum_{i,j\in[-n;n]} e_ie_{-j}F_{i,j}[s] \in \Lambda\otimes U(\hat{\mathfrak{o}}_{2n}).$$

Lemma 2.1. $(\Omega[-1])^n = e_{-n}e_{-n+1}\dots e_{-1}e_1\dots e_n 2^n n!\,\mathrm{Pf}\ \tilde{F}[-1]$.

Proof. (Cf. Proposition 1.1 in [9] and Lemma 4.1 in [4]):

$$\begin{aligned}
\Omega[-1]^n &= \sum_{i_1,j_1,\dots,i_n,j_n\in[-n;n]} e_{i_1}e_{-j_1}\dots e_{i_n}e_{-j_n}F_{i_1,j_1}[-1]\dots F_{i_n,j_n}[-1]\\
&= \sum_{i_1,j_1,\dots,i_n,j_n\in[-n;n]} e_{i_1}e_{j_1}\dots e_{i_n}e_{j_n}\tilde{F}_{i_1,j_1}[-1]\dots \tilde{F}_{i_n,j_n}[-1]\\
&= e_{-n}e_{-n+1}\dots e_{n-1}e_n \sum_{\sigma\in S_{[-n;n]}} \mathrm{sgn}(\sigma)\tilde{F}_{\sigma(-n),\sigma(-n+1)}[-1]\dots \tilde{F}_{\sigma(n-1),\sigma(n)}[-1]\\
&= e_{-n}e_{-n+1}\dots e_{n-1}e_n\, 2^n n!\,\mathrm{Pf}\ \tilde{F}[-1]. \qquad \blacksquare
\end{aligned}$$

We also introduce the following forms:

$$\begin{aligned}
\Theta[r] &= \sum_{i=-n}^{-1}\sum_{j=1}^{n} e_ie_{-j}F_{i,j}[r], \quad \tilde{\Theta}[r] = \sum_{i=1}^{n}\sum_{j=-n}^{-1} e_ie_{-j}F_{i,j}[r],\\
\Xi[r] &= \sum_{i,j=-n}^{-1} e_ie_{-j}F_{i,j}[r], \quad \Psi = \sum_{j=1}^{n} e_je_{-j},\\
\xi[-1] &= \Xi[-1] + 2\Psi\tau, \quad \tilde{\xi}[-1] = \Xi[-1] - 2\Psi\tau,\\
X[-1] &= \Theta[-1] + \xi[-1], \quad Y[-1] = \tilde{\Theta}[-1] + \tilde{\xi}[-1].
\end{aligned}$$

The following properties of the forms hold:

Lemma 2.2. *We have the following equations:*

$$\Psi = \sum_{j=-n}^{-1} e_{-j}e_j = -\sum_{j=-n}^{-1} e_je_{-j} = -\sum_{j=1}^{n} e_{-j}e_j.$$

$$[\Xi[r],\Xi[s]] = -r\delta_{r,-s}\Psi^2K, \tag{4}$$

$$[\Theta[r],\Theta[s]] = [\tilde{\Theta}[r],\tilde{\Theta}[s]] = 0, \tag{5}$$

$$[\Theta[r],\tilde{\Theta}[s]] = -4\Psi\Xi[r+s] + 2r\delta_{r,-s}\Psi^2K, \tag{6}$$

$$[\Xi[r],\Theta[s]] = 2\Psi\Theta[r+s], \quad [\Xi[r],\tilde{\Theta}[s]] = -2\Psi\tilde{\Theta}[r+s], \tag{7}$$

$$[\Xi[-1]+2a\Psi\tau,\Theta[s]] = 2\Psi(1-as)\Theta[s-1], \tag{8}$$

$$[\Xi[-1]+2b\Psi\tau,\tilde{\Theta}[s]] = 2\Psi(-1-bs)\tilde{\Theta}[s-1], \tag{9}$$

$$[X[-1],Y[-1]] = 0. \tag{10}$$

Proof. All of these relations can be checked directly. We illustrate this with

calculations of (6). With $i,l \in \{-n,\dots,-1\}$ and $j,k \in \{1,\dots,n\}$,

$$\begin{aligned}
[\Theta[r],\tilde{\Theta}[s]] &= \sum_{ijkl} e_i e_{-j} e_k e_{-l} [F_{i,j}[r], F_{k,l}[s]] = \sum_{ijkl} e_i e_{-j} e_k e_{-l} \\
&\times \big(\delta_{j,k} F_{i,l}[r+s] + \delta_{i,l} F_{-j,-k}[r+s] - \delta_{j,-l} F_{i,-k}[r+s] - \delta_{i,-k} F_{-j,l}[r+s] \\
&+ r\delta_{r,-s}(\delta_{j,k}\delta_{i,l} - \delta_{i,-k}\delta_{j,-l})K\big) \\
&= -\Psi\big(\sum_{i,l} e_i e_{-l} F_{i,l}[r+s] + \sum_{j,k} e_{-j} e_k F_{-j,-k}[r+s] \\
&+ \sum_{i,k} e_i e_k F_{i,-k}[r+s] + \sum_{j,l} e_{-j} e_{-l} F_{-j,l}[r+s]\big) + 2r\delta_{r,-s}\Psi^2 K \\
&= -4\Psi\Xi[r+s] + 2r\delta_{r,-s}\Psi^2 K. \qquad \blacksquare
\end{aligned}$$

Observe that $\Omega[-1] = \tilde{\Theta}[-1] + 2\Xi[-1] + \tilde{\Theta}[-1] = X[-1] + Y[-1]$, and since $X[-1]$ and $Y[-1]$ commute, we can write

$$\Omega[-1]^n = (X[-1] + Y[-1])^n = \sum_{m+k=n} \binom{n}{m} Y[-1]^m X[-1]^k.$$

From now on we will use the following notation: for a non-negative integer a and a partition $\alpha \vdash a$, $\alpha = (a_1 \geq \dots \geq a_l \geq 0)$, denote as $\tilde{\Theta}[-\alpha] = \tilde{\Theta}[-a_1]\dots\tilde{\Theta}[-a_l]$, and as $\Theta[-\alpha] = \Theta[-a_1]\dots\Theta[-a_l]$. We also set $\tilde{\Theta}[-(0)] = \Theta[-(0)] = 1$.

Proposition 2.3.

$$\begin{aligned}
Y[-1]^m &= \sum_{a=0}^{m} \sum_{\alpha\vdash a} \frac{m!}{(m-a)!m_1!m_2!\dots} (-2\Psi)^{a-l(\alpha)} \tilde{\Theta}[-\alpha]\,(\tilde{\xi}[-1])^{m-a}, \\
X[-1]^m &= \sum_{a=0}^{m} \sum_{\alpha\vdash a} \frac{m!}{(m-a)!m_1!m_2!\dots} (-2\Psi)^{a-l(\alpha)} (\xi[-1])^{m-a} \Theta[-\alpha],
\end{aligned}$$

where the sum is taken over all partitions $\alpha = (1^{m_1}, 2^{m_2}\dots) \vdash a$, and $l(\alpha)$ is the length of α: $l(\alpha) = m_1 + m_2 + \dots$.

Proof. We prove the statement in two steps. First, we show that $Y[-1]^m$ and $X[-1]^m$ can be expressed as linear combinations of ordered terms labeled by *compositions* of integer numbers a, $1 \leq a \leq m$.

Lemma 2.4.

$$\begin{aligned}
Y[-1]^m &= \sum_{a=1}^{m} \sum_{\bar{a}} C(\bar{a},m)(-2\Psi)^{a-l} \tilde{\Theta}[-a_1]\tilde{\Theta}[-a_2]\dots\tilde{\Theta}[-a_l]\,(\tilde{\xi}[-1])^{m-a}, \\
X[-1]^m &= \sum_{a=1}^{m} \sum_{\bar{a}} C(\bar{a},m)(-2\Psi)^{a-l} (\xi[-1])^{m-a} \Theta[-a_1]\ \Theta[-a_2]\dots\Theta[-a_l],
\end{aligned}$$

where the sum is taken over all ordered l-tuples $\bar{a} = (a_1, \dots, a_l)$, with $0 \le l \le m$, $a_i \ge 1$ and $\sum_{i=1}^{l} a_i = a \le m$. Here

$$C(\bar{a}, m) = \frac{m!}{(m-a)!} \prod_{s=1}^{l} \frac{a_s}{(a_1 + \cdots + a_s)}.$$

Proof. $Y[-1]^m = (\tilde{\Theta}[-1] + \tilde{\xi}[-1])^m$ is the sum of monomials of the form

$$\tilde{\xi}[-1]^{p_1} \tilde{\Theta}[-1] \tilde{\xi}[-1]^{p_2} \tilde{\Theta}[-1] \dots \tilde{\xi}[-1]^{p_l} \tilde{\Theta}[-1] \tilde{\xi}[-1]^{p_{l+1}}, \tag{11}$$

where $p_i \ge 0$ and $p_1 + \cdots + p_{l+1} = m - l$. From (9),

$$\tilde{\xi}[-1]^p \tilde{\Theta}[-1] = \sum_{s=1}^{p+1} \binom{p}{s-1} (-2\Psi)^{s-1} s! \tilde{\Theta}[-s] \tilde{\xi}[-1]^{p-s+1}, \tag{12}$$

and we obtain that the terms of the monomials (11) can be permuted to obtain that $Y[-1]^m$ as a linear combination

$$Y[-1]^m = \sum_{\bar{a}} c(\bar{a}, m) (-2\Psi)^{a-l} a_1! a_2! \dots a_l! \tilde{\Theta}[-a_1] \tilde{\Theta}[-a_2] \dots \tilde{\Theta}[-a_l] \, (\tilde{\xi}[-1])^{m-a} \tag{13}$$

with certain coefficients $c(\bar{a}, m)$ and the sum taken over all ordered l-tuples $\bar{a} = (a_1, \dots, a_l)$ that satisfy $a_i \ge 1$ $(i = 1, \dots, l)$, $a = \sum_{i=1}^{l} a_i$. Our goal is to compute $c(\bar{a}, m)$. Fix the ordered l-tuple $(a_1, \dots, a_l)$ and consider the corresponding term in the sum (13). We use the following notations:

$$\begin{aligned}
&A_1 = a_1 - 1, \quad A_2 = (a_1 - 1) + (a_2 - 1), \quad \dots, \quad A_l = (a_1 - 1) + \cdots + (a_l - 1), \\
&P_1 = p_1 + \cdots + p_{l+1} = m - l, \quad P_2 = p_2 + \cdots + p_{l+1}, \quad \dots, \\
&P_{l+1} = p_{l+1}, \quad P_k = 0 \quad (k > l + 1).
\end{aligned}$$

Then from (12), $c(\bar{a}, m) =$

$$\begin{aligned}
&\sum_{p_i \ge 0,\, p_1 + \cdots + p_{l+1} = m-l} \binom{p_1}{a_1 - 1} \binom{p_1 + p_2 - A_1}{a_2 - 1} \dots \binom{\sum_{i=1}^{l} p_i - A_{l-1}}{a_l - 1} \\
&= \sum_{\substack{p_i \ge 0, \\ p_1 + \cdots + p_{l+1} = m-l}} \binom{m - l - P_2}{a_1 - 1} \binom{m - l - P_3 - A_1}{a_2 - 1} \dots \binom{m - l - P_{l+1} - A_{l-1}}{a_l - 1} \\
&= \sum_{p_{l+1}=0}^{m-l} \binom{m - l - P_{l+1} - A_{l-1}}{a_l - 1} \dots \sum_{p_3=0}^{m-l-P_4} \binom{m - l - P_3 - A_1}{a_2 - 1} \sum_{p_2=0}^{m-l-P_3} \binom{m - l - P_2}{a_1 - 1}.
\end{aligned}$$

We compute the sums of this expression starting from the last one. Recall that

$$\sum_{j=0}^{r} \binom{j}{k} = \binom{r+1}{k+1} \quad \text{and} \quad \binom{j-b}{a} \binom{j}{b} = \binom{j}{a+b} \binom{a+b}{a}.$$

We get

$$\sum_{p_2=0}^{m-l-P_3} \binom{m-l-P_2}{a_1-1} = \binom{m-l+1-P_3}{a_1}$$

and

$$\binom{m-l-P_3-A_1}{a_2-1}\binom{m-l+1-P_3}{a_1} = \binom{m-l+1-P_3}{a_1+a_2-1}\binom{a_1+a_2-1}{a_2-1}.$$

Similarly,

$$\sum_{p_s=0}^{m-l-P_{s+1}} \binom{m-l+s-2-P_s}{a_1+\dots+a_{s-1}-1} = \binom{m-l+s-1-P_{s+1}}{a_1+\dots+a_{s-1}}$$

and

$$\begin{aligned}&\binom{m-l-P_{s+1}-A_{s-1}}{a_s-1}\binom{m-l+s-1-P_{s+1}}{a_1+\dots+a_{s-1}}\\ &= \binom{m-l+s-1-P_{s+1}}{a_1+\dots+a_s-1}\binom{a_1+\dots+a_s-1}{a_s-1}.\end{aligned}$$

By induction,

$$\begin{aligned}c(\bar{a},m) =& \binom{m}{a_1+\dots+a_l}\binom{a_1+\dots+a_l-1}{a_l-1}\dots\binom{a_1+a_2-1}{a_2-1}\\ =& \frac{m!}{(m-a)!}\prod_{s=1}^{l}\frac{1}{(a_1+\dots+a_s)(a_s-1)!}.\end{aligned}$$

Set $C(\bar{a},m) = a_1!a_2!\dots a_l!\,c(\bar{a},m)$, and the statement of Lemma 2.4 follows from (13).

For $X[-1]^m$ the same arguments apply since

$$\Theta[-1]\xi[-1]^p = \sum_{s=1}^{p+1}\binom{p}{s-1}(-2\Psi)^{s-1}s!\xi[-1]^{p-s+1}\Theta[-s]. \qquad \blacksquare$$

Observe now that $\tilde{\Theta}[-a]$ and $\tilde{\Theta}[-b]$ commute for $a,b\geq 1$. Thus the factors of any monomial $\tilde{\Theta}[-a_1]\tilde{\Theta}[-a_2]\dots\tilde{\Theta}[-a_l]$ can be rearranged into a monomial $\tilde{\Theta}[-a_{i_1}]\tilde{\Theta}[-a_{i_2}]\dots\tilde{\Theta}[-a_{i_l}]$ so that $(a_{i_1}\geq\dots\geq a_{i_l}\geq 1)$ is a *partition* of a, and

$$Y[-1]^m = \sum_{a=1}^{m}\sum_{\alpha\vdash a}\frac{m!}{(m-a)!}C(\alpha)\prod_{s=1}^{l}a_s\,(-2\Psi)^{a-l}\tilde{\Theta}[-\alpha]\,(\tilde{\xi}[-1])^{m-a},$$

where the sum is taken over all partitions $\alpha=(a_1\geq a_2\geq\dots\geq a_l\geq 1)$ of a, and

$$C(\alpha) = \sum\frac{1}{(a_{i_1})(a_{i_1}+a_{i_2})\dots(k_1a_1\dots+k_la_l)},$$

– the sum is taken over all distinct permutations $(a_{i_1},\dots,a_{i_l})$ of the parts of α.

The following lemma allows to compute the coefficients of that linear combination.

Lemma 2.5. *Let $\{a_1, a_2, \dots, a_r)$ be a set of distinct positive numbers and let $\alpha = (a_1, \dots a_1, a_2, \dots, a_2, \dots a_r, \dots, a_r)$ be an ordered l-tuple, where the part a_i appears with multiplicity k_i (therefore, $l = k_1 + \cdots + k_r$). Then*

$$\sum \frac{1}{(a_{i_1})(a_{i_1} + a_{i_2}) \dots (k_1 a_1 + \cdots + k_r a_r)} = \frac{1}{k_1! a_1^{k_1} \dots k_r! a_r^{k_r}}, \tag{14}$$

where the sum on the left-hand side is taken over distinct permutations $(a_{i_1}, \dots, a_{i_l})$ of $\alpha = (a_1, \dots a_1, a_2, \dots, a_2, \dots a_r, \dots, a_r)$.

Proof. Denote the sum on the left-hand side of (14) as $C(\alpha)$. Assume first that $k_i = 1$ $(i = 1, \dots, r)$. Then for $r = 2$,

$$C((a_1, a_2)) = \frac{1}{a_1(a_1 + a_2)} + \frac{1}{a_2(a_1 + a_2)} = \frac{1}{a_1 a_2}. \tag{15}$$

By induction for general α with all distinct parts we show that

$$C(\alpha) = \sum_{\sigma \in S_r} \prod_{s=1}^{r} \frac{1}{(a_{\sigma(1)} + \cdots + a_{\sigma(s)})} \tag{16}$$

$$= \sum_{k=1}^{r} \sum_{\sigma \in S_r, \sigma(r)=k} \frac{1}{(a_{\sigma(1)} + \cdots + a_{\sigma(r-1)} + a_k)} \prod_{s=1}^{r-1} \frac{1}{(a_{\sigma(1)} + \cdots + a_{\sigma(s-1)})} \tag{17}$$

$$= \sum_{k=1}^{r} \sum_{\sigma \in S_r, \sigma(r)=k} \frac{1}{(a_{\sigma(1)} + \cdots + a_{\sigma(r-1)} + a_k)} \frac{1}{a_1 \dots \hat{a}_k \dots a_r} = \frac{1}{a_1 \dots a_r}. \tag{18}$$

($\hat{a}_k$ means that the factor a_k is omitted). Now suppose that not all of $k_i = 1$ in α. For simplicity of the argument, let us assume that $k_1 \neq 1$, and the rest of $k_i = 1$. Then consider $\alpha_\varepsilon = (a_1 + \varepsilon_1, a_1 + \varepsilon_2, \dots, a_1 + \varepsilon_{k_1}, a_2, \dots, a_r)$ with such values of ε_i that all the terms of α_ε are distinct. $C(\alpha_\epsilon)$ is a rational function of $(\varepsilon_1, \dots \varepsilon_{k_1})$. When all $\varepsilon_i \to 0$,

$$\frac{1}{(a_1 + \varepsilon_1) \dots (a_{k_1} + \varepsilon_{k_1}) a_{k_1+1} \dots a_r} \to \frac{1}{(a_1)^{k_1} a_{k_1+1} \dots a_r}.$$

On the other hand,

$$C(\alpha_\epsilon) \to \sum_{\sigma \in S_{r-k_1+1}} \prod_{s=1}^{r} \frac{k_1!}{(a_{\sigma(1)} + \cdots + a_{\sigma(s)})},$$

where the sum is taken over all permutations of the set $\{1, 2, \dots, r\}$. By generalizing this argument one can show that if α has distinct parts $\{a_1, \dots, a_r\}$ with the corresponding multiplicities $\{k_1, \dots, k_r\}$, the value of $C(\alpha)$ is given by the formula $C(\alpha) = 1/(k_1! a_1^{k_1} \dots k_r! a_r^{k_r})$. ■

Applying Lemma 2.5 and rewriting the partition α in the form $\alpha = (1^{m_1}, 2^{m_2}, \dots)$, we get the statement of Proposition 2.3. ■

3. Formula for $\Omega[-1]^n$

We use notation ad_τ for the operator $f \mapsto [\tau, f]$ acting on $U(\hat{\mathfrak{o}}_{2n})$.

Theorem 3.1. *For the Lie algebra $\mathfrak{o}_{2n}$,*

$$\Omega[-1]^n = \sum_{l=0}^{\lfloor n/2 \rfloor} \sum_{\substack{a,b,\\ 2l \le a+b \le n}} \frac{2^{n-2l} n!}{(n-a-b)!} \left(\sum_{\alpha \vdash a, l(\alpha)=l} \frac{1}{m_1! m_2! \dots} \tilde{\Theta}[-\alpha] \right)$$
$$\times (-\Psi)^{a+b-2l} \left((\Xi[-1] - \Psi ad_\tau)^{n-a-b} \cdot 1 \right) \left(\sum_{\beta \vdash b, l(\beta)=l} \frac{1}{m_1'! m_2'! \dots} \Theta[-\beta] \right),$$

where we use the notations $\alpha = (1^{m_1}, 2^{m_2}, \dots) \vdash a$ and $\beta = (1^{m_1'}, 2^{m_2'}, \dots) \vdash b$.

Example 3.2. For $n = 2$ the partitions that contribute to the sum of the formula are $\alpha = \beta = \emptyset$ and $\alpha = \beta = (1)$. Accordingly, in $\mathfrak{o}_4$ we get

$$\Omega[-1]^2 = 4(\Xi[-1] - \Psi \mathrm{ad}_\tau)^2 \cdot 1 + 2\tilde{\Theta}[-1]\Theta[-1].$$

For $n = 3$ the partitions that contribute to the sum of the formula are $\alpha = \beta = \emptyset$; $\alpha = \beta = (1)$; $\alpha = (2), \beta = (1)$ or $\alpha = (1), \beta = (2)$. Accordingly, in $\mathfrak{o}_6$ we get

$$\Omega[-1]^3 = 8(\Xi[-1] - \Psi \mathrm{ad}_\tau)^3 \cdot 1 + 12\tilde{\Theta}[-1] \left((\Xi[-1] - \Psi \mathrm{ad}_\tau) \cdot 1 \right) \Theta[-1]$$
$$- 12\Psi(\tilde{\Theta}[-2]\Theta[-1] + \tilde{\Theta}[-1]\Theta[-2]).$$

Proof.

$$\Omega[-1]^n = \sum_{m=0}^{n} \binom{n}{m} Y[-1]^m X[-1]^{n-m}$$
$$= \sum_{m=0}^{n} \sum_{a=0}^{m} \sum_{b=0}^{n-m} \sum_{\alpha \vdash a \beta \vdash b} \binom{n}{m} \frac{m!\,(n-m)!}{(m-a)!(n-m-b)! m_1! m_2! \dots m_1'! m_2'! \dots}$$
$$(-2\Psi)^{b+a-l(\alpha)-l(\beta)} \tilde{\Theta}[-\alpha] (\tilde{\xi}[-1])^{m-a} (\xi[-1])^{n-m-b} \Theta[-\beta]$$
$$= \sum_{0 \le a+b \le n} \sum_{\alpha,\beta} \frac{n!}{(n-a-b)! m_1! m_2! \dots m_1'! m_2'! \dots} (-2\Psi)^{a+b-l(\alpha)-l(\beta)}$$
$$\times \tilde{\Theta}[-\alpha] \left(\sum_{m=a}^{n-b} \binom{n-a-b}{m-a} (\tilde{\xi}[-1])^{m-a} (\xi[-1])^{n-m-b} \right) \Theta[-\beta].$$

Denote as

$$P_r = \sum_{k=0}^{r} \binom{r}{k} (\tilde{\xi}[-1])^k (\xi[-1])^{r-k}, \quad P_0 = 1.$$

One has:

$$P_r = \tilde{\xi}[-1] P_{r-1} + P_{r-1} \xi[-1] = (\Xi[-1] - 2\Psi\tau) P_{r-1} + P_{r-1} (\Xi[-1] + 2\Psi\tau)$$
$$= \Xi[-1] P_{r-1} + P_{r-1} \Xi[-1] - 2\Psi[\tau, P_{r-1}].$$

If P_{r-1} is a sum of monomials in Ψ and $\Xi[s]$ with some negative values of s, then $[\tau, P_{r-1}]$ is also a sum of monomials in Ψ and $\Xi[s]$ with some negative values of s. This, together with the fact that $P_1 = 2\Xi[-1]$, implies that all P_r are linear combinations of monomials in Ψ and $\Xi[s]$ $(s < 0)$, each of them commutes with $\Xi[-1]$ and we can write

$$P_r = 2(\Xi[-1]P_{r-1} - \Psi[\tau, P_{r-1}]) = 2(\Xi[-1] - \Psi \mathrm{ad}_\tau)\, P_{r-1} = 2^r(\Xi[-1] - \Psi \mathrm{ad}_\tau)^r \cdot 1.$$

Thus

$$\sum_{m=a}^{n-b} \binom{n-a-b}{m-a} (\tilde{\xi}[-1])^{m-a} (\xi[-1])^{n-m-b} = 2^{n-a-b}(\Xi[-1] - \Psi \mathrm{ad}_\tau)^{n-a-b} \cdot 1.$$

Also notice that since $\Omega[-1]^n$ is a form of full degree, the number of parts α and β in non-zero terms is the same: $l(\alpha) = l(\beta) = l$. This completes the proof of the theorem. ■

4. Explicit formula for Pfaffian Pf $\tilde{F}[-1]$

Pfaffian $\mathrm{Pf}\,\tilde{F}[-1]$ is given by the coefficient of the monomial $e_{-n}e_{-n+1}\dots e_{n-1}e_n$ in the form $\Omega[-1]^n$. One can use Lemma 4.1 below to interpret through Pfaffian- and determinant-like elements the expressions $\tilde{\Theta}[-\alpha]$, $(\Xi[-1] - \Psi \mathrm{ad}_\tau)^r$, $\Theta[-\beta]$ in the statement of Theorem 3.1.

For subsets I, J of $[-n; n]$ such that $|I| = |J| = l$ we denote as $\Phi[-s]_{IJ}$ an $l \times l$-submatrix of the matrix $\Phi[-s] = F[-s] + \mathrm{ad}_\tau \cdot Id$

$$\Phi_{IJ}[-s] = (F_{i,j}[-s] + \mathrm{ad}_\tau \delta_{ij})_{i \in I, j \in J}.$$

and set

$$\det \Phi_{IJ}[-1] = \sum_{\sigma \in S_l} \mathrm{sgn}\,(\sigma) \Phi_{i_{\sigma(1)}, j_1} \dots \Phi_{i_{\sigma(l)}, j_l}.$$

Let $I = (i_1 < \dots < i_k)$ be a string of elements of a subset of $\{-n, \dots, -1\}$ written in increasing order. Denote $-I = (-i_1 > \dots > -i_k)$. Define

$$\mathrm{Pf}\,\Phi_{I,-I}[-\beta] = \frac{1}{2^l l!} \sum_{\sigma \in S_{2l}} \mathrm{sgn}\,(\sigma) F_{i_{\sigma(1)}, -i_{\sigma(2)}}[-\beta_1] \dots F_{i_{\sigma(2l-1)}, -i_{\sigma(2l)}}[-\beta_l], \tag{19}$$

$$\mathrm{Pf}\,\Phi_{-I,I}[-\alpha] = \frac{1}{2^l l!} \sum_{\sigma \in S_{2l}} \mathrm{sgn}\,(\sigma) F_{-i_{\sigma(1)}, i_{\sigma(2)}}[-\alpha_1] \dots F_{-i_{\sigma(2l-1)}, i_{\sigma(2l)}}[-\alpha_l]. \tag{20}$$

Lemma 4.1. *For partitions α, β and for $r \in \mathbb{Z}_+$,*

$$\tilde{\Theta}[-\alpha] = 2^l l! \sum_{-n \le i_1 < \dots < i_{2l} \le -1} e_{-i_1} \dots e_{-i_{2l}}\ \mathrm{Pf}\ \Phi_{-I,I}[-\alpha],$$

$$\Theta[-\beta] = 2^l l! \sum_{-n \le i_1 < \dots < i_{2l} \le -1} e_{i_1} \dots e_{i_{2l}}\ \mathrm{Pf}\ \Phi_{I,-I}[-\beta],$$

$$(\Xi[-1] - \Psi ad_\tau)^r \cdot 1 = r! \sum_{\substack{-n \le i_1 < \dots < i_r \le -1 \\ -n \le j_1 < \dots < j_r \le -1}} e_{i_1} \dots e_{i_r} e_{-j_r} \dots e_{-j_1}\, (det\, \Phi_{I,J}[-1]) \cdot 1.$$

Proof. The first and the second equalities follow from the definitions (19-20), and the proof of the third equality repeats word-to-word the proof of Proposition 4.5 in [4] if we observe that $\Xi[-1] - \Psi\mathrm{ad}_\tau = \sum_{j=-n}^{-1} \eta_j e_{-j}$, where $\eta_j = \sum_{i=-n}^{-1} e_i(F_{i,j}[-1] + \mathrm{ad}_\tau \delta_{ij})$ and satisfy $\eta_j\eta_k + \eta_k\eta_j = 0$. ■

Example 4.2. For $n = 2$ one gets $4(\Xi[-1]-\Psi\mathrm{ad}_\tau)^2 = 8e_{-2}e_{-1}e_1e_2 \det(\Phi[-1])\cdot 1$ and $\Theta[-1] = 2e_{-2}e_{-1}F_{-2,1}[-1]$, $\tilde{\Theta}[-1] = 2e_1e_2F_{1,-2}[-1]$. Therefore,

$$\begin{aligned}\mathrm{Pf}\,\tilde{F}[-1] &= \det(\Phi[-1])\cdot 1 - F_{1,-2}[-1]F_{-2,1}[-1]\\ =&F_{-2,-2}[-1]F_{-1,-1}[-1]-F_{-1,-2}[-1]F_{-2,-1}[-1]+F_{1,-2}[-1]F_{-2,1}[-1]+F_{-1,-1}[-2].\end{aligned}$$

This example illustrates that Theorem 3.1 can be used to compute the Pfaffian $\mathrm{Pf}\,\tilde{F}[-1]$: we find by that theorem the value of $\Omega[-1]^n$, and then by Lemma 2.1, the coefficient of the monomial $e_{-n}e_{-n+1}\dots e_{n-1}e_n$ in $\Omega[-1]^n$ is $2^n n!\,\mathrm{Pf}\,\tilde{F}[-1]$. For the sake of completeness we write explicitly the general formula for that coefficient (c.f. Theorem 4.7 in [4]), though it involves more technical notations. The statements of Theorems 3.1 above and 4.3 below are equivalent.

Consider subsets J, I_1, I_2, J_1, J_2 of $\{-n,\dots,-1\}$ those elements are written in increasing order and such that satisfy the properties:

$$I_1 \sqcup J_1 \sqcup J = J \sqcup J_2 \sqcup I_2 = \{-n,\dots,-1\}, \tag{21}$$

$$|I_1| = |I_2| = 2l, \quad |J_1| = |J_2| = n-a-b. \tag{22}$$

Write the elements of strings $[J, J_2, I_2]$ exactly in that order. We denote by $\mathrm{sgn}\,(J, J_2, I_2)$ the sign of the permutation of the elements of this string that puts them in the increasing order $(-n,\dots,-1)$. Similarly, $\mathrm{sgn}\,(-I_1,-J_1,-J)$ is the sign of the permutation of the elements of this string that puts them in the increasing order $(1,\dots,n)$.

Theorem 4.3.

$$\begin{aligned}\mathrm{Pf}\,\tilde{F}[-1] = \sum_{l=0}^{\lfloor n/2\rfloor} \sum_{\substack{a,b\\0\le a+b\le n}} \sum_{I_1,J_1,I_2,J_2,J} &sgn\,(J,J_2,I_2)sgn\,(-I_1,-J_1,-J)\\ &\times\left(\sum_{\alpha\vdash a, l(\alpha)=l} \frac{l!}{m_1!m_2!\dots}\,\mathrm{Pf}\;\Phi_{-I_1+I_1}[-\alpha]\right)\\ &\times((a+b-2l)!(det\,\Phi_{J_2,J_1}[-1]\cdot 1))\left(\sum_{\beta\vdash b, l(\beta)=l}\frac{l!}{m_1'!m_2'!\dots}\Phi_{+I_2-I_2}[-\beta]\right),\end{aligned} \tag{23}$$

the third sum is taken over subsets J, I_1, I_2, J_1, J_2 of $\{-n,\dots,-1\}$ that satisfy the properties (21, 22).

Proof. Using for a subset $I = (i_1 < \dots < i_k)$ of $\{-n,\dots,-1\}$ the notations $e_I = e_{i_1}\dots e_{i_k}$, $e_{-I} = e_{-i_1}\dots e_{-i_k}$, $e_{-\tilde{I}} = e_{-i_k}\dots e_{-i_1}$, combine Theorem 3.1 and

Lemma 4.1 to obtain

$$\Omega[-1]^n = 2^n n! \sum_{l=0}^{\lfloor n/2 \rfloor} \sum_{0\le a+b\le n} \sum_{I_1,J_1,I_2,J_2} (-\Psi)^{a+b-2l} e_{-I_1} e_{J_2} e_{-\tilde{J}_1} e_{I_2}$$
$$\times \sum_{\alpha\vdash a, l(\alpha)=l} \frac{l!}{m_1! m_2! \dots} \operatorname{Pf} \Phi_{-I_1 I_1}[-\alpha] (\det \Phi_{J_2 J_1}[-1] \cdot 1)$$
$$\times \sum_{\beta\vdash b, l(\beta)=l} \frac{l!}{m_1'! m_2'! \dots} \operatorname{Pf} \Phi_{I_2 - I_2}[-\beta]$$

Note that

$$(-\Psi)^r = (r)! \sum_{J\subset\{-n\cdots-1\},\, |J|=r} (-1)^{\frac{r(r-1)}{2}} e_J e_{-J},$$

and that

$$(-1)^{\frac{(a+b-2l)(a+b-2l-1)}{2}} e_J e_{-J} e_{-I_1} e_{J_2} e_{-\tilde{J}_1} e_{I_2} = e_J e_{J_2} e_{I_2} e_{-I_1} e_{-J_1} e_{-J}.$$

By comparing the order of the terms of this monomial with the monomial $e_{-n} e_{-n+1} \dots e_{-1} e_1 \dots e_{n-1} e_n$ we get formula (23). ■

5. Harish-Chandra image of the Pfaffian

The Feigin-Frenkel center $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ is a commutative subalgebra of $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}}$. Consider a left ideal I of $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])$ that is generated by the elements $F_{ij}[r]$ for $-n \le i < j \le n$ and $r < 0$. The quotient of $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}}$ by the two-sided ideal $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}} \cap I$ is isomorphic to the commutative algebra that is freely generated by the images of the elements $F_{ii}[r]$ $(i = -n, \dots, -1,\ r < 0)$. Denote these images as $\mu_i[r] = F_{ii}[r]$.We get an analogue of classical Harish-Chandra homomorphism

$$\chi : U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}} \to U(t^{-1}\mathfrak{h}[t^{-1}]),$$

and $U(t^{-1}\mathfrak{h}[t^{-1}])^{\mathfrak{h}}$ can be viewed as a polynomial algebra in variables $\{\mu_i[r], i = -n, \dots, -1, r < 0\}$. The restriction of χ to $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ yields an isomorphism $\mathfrak{z}(\hat{\mathfrak{o}}_{2n}) \to \mathcal{W}(\mathfrak{o}_{2n})$, where $\mathcal{W}(\mathfrak{o}_{2n})$ is the classical $\mathcal{W}$-algebra of $\mathfrak{o}_{2n}$. It is a subalgebra of $U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])^{\mathfrak{h}}$ annihilated by screening operators ([1], Chapter 7). The elements of $\mathcal{W}(\mathfrak{o}_{2n})$ are polynomials in $\mu_i[r]$ $(r < 0,\ i = -n, \dots, -1)$. We give a new direct proof of the following corollary established in [7].

Corollary 5.1.

$$\chi(\operatorname{Pf} \tilde{F}[-1]) = (\mu_{-n}[-1] + ad_\tau) \dots (\mu_{-1}[-1] + ad_\tau) \cdot 1.$$

Proof. Only the terms with $a = b = 0$ will survive under Harish-Chandra homomorphism in the formula of Theorem 3.1 for $\Omega[-1]^n$: $\chi(\Omega[-1]^n)$

$$= \chi(2^n (\Xi[-1] - T\mathrm{ad}_\tau)^n \cdot 1) = 2^n n!\, e_1 \dots e_n e_{-1} \dots e_{-n}\, \chi(\det(F[-1] + \mathrm{ad}_\tau) \cdot 1)$$
$$= 2^n n!\, e_1 \dots e_n e_{-1} \dots e_{-n}\, (\mu_{-n}[-1] + \mathrm{ad}_\tau) \dots (\mu_{-1}[-1] + \mathrm{ad}_\tau) \cdot 1$$ ■

6. Deducing the formula for Pfaffian for $U(\mathfrak{o}_{2n})$

In this section we would like to illustrate that the application of evaluation homomorphism to the statement of Theorem 3.1 implies the formulas for Pfaffian of the universal enveloping algebra $U(\mathfrak{o}_{2n})$ in [4] (namely, Proposition 4.4. there).

Let z be another variable and consider a homomorphism of algebras $\varphi\colon U(t^{-1}\mathfrak{o}_{2n}[t^{-1}])\to U(\mathfrak{o}_{2n})\otimes z^{-1}\mathbb{C}[z^{-1}]$, defined by $\varphi(F_{ij}[r]) = F_{ij}\otimes z^r$, $\varphi(K)=0$. Then the action of ad_τ is identified with the operator $-\partial_z$ by the rule $\varphi([\tau,f]) = -\partial_z(\varphi(f))$. Let us apply φ to the formula of the Theorem 3.1.

We have:

$$\varphi(\Omega[-1]^n) = \Omega^n z^{-n}, \quad \text{where} \quad \Omega = \sum_{i,j\in[-n;n]} e_i e_{-j} F_{i,j} \in \Lambda\otimes U(\mathfrak{o}_{2n}).$$

Note that for a partition $\alpha\vdash a$ of length $l(\alpha)=l$, the evaluation map gives $\varphi(\tilde\Theta[-\alpha]) = \tilde\Theta^l z^{-a}$ and $\varphi(\Theta[-\alpha]) = \Theta^l z^{-a}$, where $\tilde\Theta = \sum_{i=1}^{n}\sum_{j=-n}^{-1} e_i e_{-j} F_{i,j}$, $\Theta = \sum_{i=-n}^{1}\sum_{j=1}^{n} e_i e_{-j} F_{i,j}$. Also

$$\varphi((\Xi[-1] - \Psi\,\mathrm{ad}_\tau)^k\cdot 1) = ((\Xi z^{-1} + \Psi\,\partial_z)^k\cdot 1),$$

where $\Xi = \sum_{i,j=-n}^{-1} e_i e_{-j} F_{i,j}$. Using that in the notations $\alpha = (1^{m_1}, 2^{m_2}, \dots)$,

$$\sum_{\alpha\vdash a, l(\alpha)=l} \frac{l!}{m_1! m_2! \dots} = \binom{a-1}{l-1} \tag{24}$$

(see e.g. Corollary 6 in [11]), and that

$$\sum_{a+b=p} \binom{a-1}{l-1}\binom{b-1}{l-1} = \binom{p-1}{2l-1}, \tag{25}$$

we obtain from Theorem 3.1,

$$\Omega^n z^{-n} = \sum_{l=0}^{\lfloor n/2\rfloor} \frac{2^{n-2l} n!}{l! l!}\tilde\Theta^l \quad \Big(\sum_{2l\le p\le n} \frac{1}{(n-p)!}\binom{p-1}{2l-1} \tag{26}$$

$$\times\, (-\Psi)^{p-2l}\,(\Xi z^{-1} + \Psi\partial_z)^{n-p}\cdot 1\Big)\Theta^l z^{-p}. \tag{27}$$

Let us multiply both parts of (26) by z^{-n}. Observe that

$$((\Xi z^{-1} + \Psi\partial_z)^{n-p}\cdot 1) z^{n-p} = (\Xi)(\Xi-\Psi)(\Xi-2\Psi)\dots(\Xi-(n-p+1)\Psi),$$

and

$$\frac{1}{(n-p)!}\binom{p-1}{2l-1} = \frac{1}{(n-2l)!}\binom{n-2l}{n-p}\frac{(p-1)!}{(2l-1)!}.$$

Therefore, we can rewrite the central part expression of (26) as

$$\sum_{2l\le p\le n} \frac{1}{(n-p)!}\binom{p-1}{2l-1}(-\Psi)^{p-2l}\,((\Xi z^{-1} + \Psi\partial_z)^{n-p}\cdot 1)\, z^{n-p}$$

$$= \frac{1}{(n-2l)!}\sum_{p=0}^{n}\binom{n-2l}{n-p} y^{(p-2l)} x^{(n-p)} \tag{28}$$

with

$$y = -2l\Psi, \quad y^{(p-2l)} = (-2l\Psi)\dots(-(p-1)\Psi),$$
$$x = \Xi, \quad x^{(n-p)} = \Xi(\Xi-\Psi)\dots(\Xi-(n-p+1)\Psi).$$

Since Ξ and Ψ commute, (28) can be rewritten using binomial theorem for falling factorial powers as $\frac{1}{(n-2l)!}(x+y)^{(n-2l)} =$

$$\frac{1}{(n-2l)!}(\Xi-2l\Psi)^{(n-2l)} = \frac{1}{(n-2l)!}(\Xi-2l\Psi)\dots(\Xi-(n+1)\Psi).$$

Hence, from (26),

$$\Omega^n = \sum_{l=0}^{\lfloor n/2\rfloor} \frac{2^{n-2l}n!}{l!l!}\tilde{\Theta}^l \frac{1}{(n-2l)!}(\Xi-2l\Psi)\dots(\Xi-(-n+1)\Psi)\Theta^l. \tag{29}$$

Finally, using the commotion relation

$$\tilde{\Theta}^l(\Xi-2l\Psi)\dots(\Xi+(n-1)\Psi) = \Xi(\Xi-\Psi)\dots(\Xi-(n-2l-1)\Psi)\tilde{\Theta}^l,$$

we get

$$\Omega^n = \sum_{l=0}^{\lfloor n/2\rfloor} \frac{2^{n-2l}n!}{(n-2l)!l!l!}\Xi(\Xi-\Psi)\dots(\Xi-(n-2l-1)\Psi)\ \tilde{\Theta}^l\Theta^l, \tag{30}$$

which is the statement of Proposition 4.4. in [4].

Remark 6.1. (a) As it is discussed in the proof of Theorem 4.17 of [4], necessarily $p = q$ in Proposition 4.4 of [4].
(b) Our notation Ψ stands for $-\tau$ in [4].
(c) As it was mentioned before, Theorem 3.1 and Theorem 4.3 of this note are equivalent statements. Similarly, the results of [4] cited here in (30) and (2) are equivalent. We chose to illustrate how (30) is deduced from Theorem 3.1 rather then (2) from Theorem 4.3 in order to avoid technically involved notations of the other two statements.

Acknowledgement. The author is very grateful to A. I. Molev for brining this problem to her attention and for valuable discussions. She also would like to thank the Max Planck Institute for Mathematics for its warm hospitality.

References

[1] Frenkel, E., „Langlands correspondence for loop groups“, Cambridge Studies in Advanced Mathematics **103**, Cambridge University Press, Cambridge, 2007.

[2] Chervov, A. V., and A. I. Molev, *On higher order Sugawara operators*, Int. Math. Res. Not., (2009), 1612–1635.

[3] Chervov, A. V., and D. Talalaev, *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, arXiv: hep-th/0604128.

[4] Hashimoto, T., *A central element in the universal enveloping algebra of type D_n via minor summation formula of Pfaffians*, J. Lie Theory **18** (2008), 581–594.

[5] Howe, R., and T. Umeda, *The Capelli identity, the double commatant theorem, and multiplicity-free actions*, Math. Ann. **290** (1991), 565–619.

[6] Molev, A. I., *Feigin-Frenkel center in types B, C and D*, Invent. math. **191** (2013), 1–34.

[7] Molev, A. I., and E. E. Mukhin, *Yangian characters and classical $\mathcal{W}$-algebras*, arXiv:1212-4032.

[8] Molev, A. I., and M. Nazarov, *Capelli identities for classical Lie algebras*, Math. Ann. **313** (1999), 315–357.

[9] Itoh, M., and T. Umeda, *On central elements in the universal enveloping algebras of the orthogonal Lie algebra*, Composition Math. **127** (2001), 333–359.

[10] Perelomov, A. M., and V. S. Popov, *Casimir operators for the orthogonal and symplectic groups*, Soviet J. Nuclear Phys. **3** (1966), 819–824.

[11] Wenchang, C., *A partition identity on binomial coefficients and its applications*, Graphs and Combinatorics **5** (1989), 197–200.

[12] Wachi, A., *Central elements in the universal enveloping algebras for the split realization of the orthogonal Lie algebras*, Lett. in Math. Phys. **77** (2006), 155–168.

Natasha Rozhkovskaya
Department of Mathematics
Kansas State University
138 Cardwell Hall
Manhattan, KS, 66502
rozhkovs@math.ksu.edu

Received September 9, 2013
and in final form October 24, 2013