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Regularity in Milnor's Sense for Ascending Unions of Banach-Lie Groups

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Abstract. In this paper we give a criterion for the union of an ascending sequence of Banach-Lie groups to be a *regular* Lie group in Milnor's sense. It was shown in earlier work that this union carries a Lie group structure. We use the differential calculus by Michal-Bastiani (Keller's C_c^∞-calculus).

In a future article we will show, using similar techniques, that groups of germs of local diffeomorphisms are regular, thus solving an open problem posed by K.-H. Neeb.
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1. Introduction and statement of the results

Let G be a group which can be written as an ascending union $G = \bigcup_n G_n$ of subgroups $(G_n)_{n\in\mathbb{N}}$. It is a natural question in infinite dimensional Lie theory whether the group G carries a regular Lie group structure provided that the groups G_n do so. With *Lie group structure*, we mean a smooth manifold structure modeled on a locally convex topological vector space such that the group operations are smooth mappings (see Section 2 for details). Such a Lie group is called *regular* if every smooth curve in the Lie algebra can be uniquely integrated to a curve with values in the Lie group (see Definition 2.4 for details).

This notion of regularity was was introduced by Milnor in [14] to circumvent the lack of standard existence and uniqueness arguments of solutions for ordinary differential equations beyond Banach spaces. Under the assumption of regularity many concepts from finite dimensional Lie groups (and Banach Lie groups) can be transferred to the locally convex setting: For example, in the class of regular Lie groups, a simply connected group is uniquely determined by its Lie algebra (see e.g. [14, Theorem 8.1]). For non-regular Lie groups this no longer holds (see [8, Theorem B] for an instructive counter-example). Note that major classes of Lie

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groups are known to be regular (e.g. Banach-Lie groups (see [17]), unit groups of certain topological algebras (see [13]), diffeomorphism groups of compact manifolds (see [19]) or compact orbifolds (see [20]))

In [10], Glöckner showed that if each G_n is a finite dimensional Lie group, the union $G := \bigcup_n G_n$ carries a Lie group structure and that this so-obtained Lie group is regular. It is shown in [5] via different methods that the union of infinite dimensional Banach Lie groups carries a canonical infinite dimensional Lie group structure if some mild hypotheses are satisfied. The question whether this union happens to be a regular Lie group remained unanswered.

In this article, we show that regularity of the Lie group G is connected to an (a priori different) concept from the theory of locally convex direct limits, also called *regularity*: A locally convex direct limit $E = \bigcup_n E_n$ (or equivalently the sequence $(E_n)_{n\in\mathbb{N}}$) is called *regular* (or *boundedly regular*) if each bounded subset of E is also a bounded subset of one of the spaces E_n. See Definition 4.1 for details. With these notions we can now give the main result:

Theorem A (Regularity of countable unions of Banach-Lie groups). *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We are given an increasing sequence*

$$G_1 \subseteq G_2 \subseteq \cdots$$

of $\mathbb{K}$-analytic Banach-Lie groups, such that the inclusion maps $j_n \colon G_n \longrightarrow G_{n+1}$ are analytic group homomorphisms. We fix a norm $\|\cdot\|_{\mathfrak{g}_n}$ on the Banach-Lie algebra $\mathfrak{g}_n := \mathbf{L}(G_n)$, defining its topology. We assume that the locally convex direct limit $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ is Hausdorff and that all inclusion maps and all Lie brackets have operator norm at most 1. Also, we assume that the map $\exp_G := \bigcup_n \exp_{G_n} \colon \mathfrak{g} \longrightarrow \bigcup_n G_n$ is injective on some 0-neighborhood in $\mathfrak{g}$. Then

(a) *There exists a unique locally convex Lie group structure on the group $G := \bigcup_n G_n$, such that $\exp_G$ becomes a local diffeomorphism around 0.*

(b) *The Lie group G is regular (in Milnor's sense) if and only if the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is (boundedly) regular.*

Part (a) of this theorem is the construction from the author's article [5]. Part (b) is new.

It is not known to the author if the requirement that the exponential map be locally injective is necessary to obtain a Lie group structure on G. However, the construction via the exponential map is rather classical and was also used by Natarajan et al. in [16] to obtain the Lie group structure on the union of finite dimensional groups before Glöckner found a construction in [10] that did not use the exponential map. Unfortunately, these arguments are not applicable if each G_n is infinite dimensional, since the direct limit in the category of locally convex spaces does no longer coincide with the direct limit in the category of topological spaces which was used in [10].

Theorem A implies the regularity of many interesting examples, like the group of germs of Lie-group-valued mappings discussed by Helge Glöckner in [11, Section 10] and the groups associated to Dirichlet-series and Hölder-continuous functions, introduced in [5] and [4], respectively. The author is convinced that this result can also be used to construct a regular Lie group associated to Sobolev spaces at the critical exponent but so far the necessary details have not yet been worked out.

It is an important open question (see e.g. [14]), whether regularity of an infinite dimensional Lie group is equivalent to Mackey-completeness of its Lie algebra. Theorem A answers this question to the affirmative for this new class of Lie groups. We can even strengthen Theorem A under the assumption of a stronger regularity property for the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$, generalizing the duality between regularity of direct limits on the one hand and regularity of Lie groups on the other: We show that *compact regularity* of the direct limit (see Definition 4.1) implies *strong C^0-regularity* of the corresponding Lie group (see Definition 2.4) which is stronger than regularity in Milnor's sense (see Theorem 4.6).

The structure of the present article is as follows: In Section 2, we recall the basic notions of infinite dimensional Lie groups, (strong C^k-)regularity of Lie groups and introduce the concept of local Lie groups. In Section 3, we concentrate on local Banach Lie groups and obtain concrete estimates for their (strong C^0-) regularity. These constants introduced in Section 3, are used in Section 4 to show regularity of the union of local Banach Lie groups. Since regularity is a local concept, we conclude that unions of global Banach Lie groups are regular, as well.

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2. Lie groups and regularity

In this article, we use the notion of $C^k_{\mathbb{K}}$-maps in the sense of Michal-Bastiani, which can be found in [9], [12], [14] and in [17].

When $\gamma\colon U \subseteq X \longrightarrow Z$ is a $C^k_{\mathbb{K}}$-map, we will denote the differential by $d\gamma\colon U \times X \longrightarrow Z$. In the case that X and Z are normed spaces, we also use the notation $\gamma'\colon U \longrightarrow \mathcal{L}(X,Z)$. Recall that a $C^\infty_{\mathbb{K}}$-map between normed spaces in the sense of Michal-Bastiani is also C^∞ in the well-known notion of Fréchet differentiability. (see e.g. [22, Lemmas A.4.1 and A.4.3])

For complex locally convex spaces, a $C^\infty_{\mathbb{C}}$-map is also called *complex analytic* and is also denoted by $C^\omega_{\mathbb{C}}$ (Often, complex analyticity is defined via power series (see e.g. [17, Definition I.2.1 (b)]), but these definitions are equivalent in locally convex spaces (see [1, Propositions 7.4 and 7.7]).

A mapping $f\colon U \longrightarrow F$ from an open subset $U \subseteq E$ of a real locally convex space E with values in F is called *real analytic* or $C^\omega_{\mathbb{R}}$ if it extends to a complex analytic $F_{\mathbb{C}}$-valued map on an open neighborhood of U in the complexification $E_{\mathbb{C}}$.

Once we have the notion of $C^k_{\mathbb{K}}$-maps, the definitions of manifolds and Lie groups are analogous to the finite dimensional case: A *$C^k_{\mathbb{K}}$-manifold modeled on E* is a Hausdorff space M together with a maximal set *(atlas)* of homeomorphisms (*charts*) $\phi\colon U_\phi \longrightarrow V_\phi$ where $U_\phi \subseteq M$ and $V_\phi \subseteq E$ are open subsets, such that the transition maps are $C^k_{\mathbb{K}}$ and $M = \bigcup_\phi U_\phi$. Continuous mappings between $C^k_{\mathbb{K}}$-manifolds are called $C^k_{\mathbb{K}}$ if they are $C^k_{\mathbb{K}}$ after composition with suitable charts. The definitions of tangent spaces, tangent bundles, vector fields and similar concepts are similar to the finite dimensional case and can be found in [17, Definitions I.3.1, I.3.3 and I.3.6].

A $C^k_{\mathbb{K}}$-Lie group G is a group which is at the same time a $C^k_{\mathbb{K}}$-manifold such that the group operations are $C^k_{\mathbb{K}}$. The *Lie algebra of a Lie group* G, denoted by $\mathbf{L}(G)$, is the tangent space T_1G at the identity element, together with the Lie bracket, obtained from the Lie algebra of left-invariant vector fields on the group G. The Lie algebra $\mathbf{L}(G)$ is a locally convex Lie algebra. We refer to [17, Definitions II.1.5 and II.1.7] for details.

Although we are mostly interested in (global) Lie groups, we will use so called *local Lie groups* as a tool to show regularity of global ones:

Definition 2.1 (Local Lie group)**.** Let G be a smooth manifold, $D \subseteq G \times G$ an open subset, $1 \in G$, and let $m_G\colon D \longrightarrow G : (x,y) \mapsto x*y$, $\eta_G\colon G \longrightarrow G : x \mapsto x^{-1}$ be smooth maps. We call $(G, D, m_G, 1_G, \eta_G)$ a *local Lie group* if

(Loc1) Assume that $(x,y),(y,z) \in D$. If $(x*y,z)$ or $(x,y*z) \in D$, then both are contained in D and $(x*y)*z = x*(y*z)$.

(Loc2) For each $x \in G$ we have $(x,1_G),(1_G,x) \in D$ and $x*1_G = 1_G*x = x$.

(Loc3) For each $x \in G$ we have $(x,x^{-1}),(x^{-1},x) \in D$ and $x*x^{-1} = x^{-1}*x = 1$.

(Loc4) If $(x,y) \in D$, then $(y^{-1},x^{-1}) \in D$.

Remark 2.2. Every symmetric open identity neighborhood of a Lie group can be turned into a local Lie group. Conversely, not every local Lie group can be enlarged to a Lie group. While this holds for finite dimensional local Lie groups, it already fails for Banach-Lie groups (See [21] for a counter-example).

Like with global Lie groups, we can attach to every local Lie group G a Lie algebra $\mathbf{L}(G) := T_{1_G}G$ (See [17, Definition II.1.10].)

Remark 2.3. In general, it is a difficult question whether a locally convex Lie algebra is the Lie algebra of a local Lie group (see Section VI of [17] for a treatment of integrability of locally convex Lie algebras). However, for Banach-Lie algebras, this is possible (See Proposition 3.2).

Definition 2.4 (Regular (local and global) Lie groups)**.**

(a) Let $\gamma\colon [0,1] \longrightarrow G$ be a C^1-curve in a (global or local) Lie group G with $\gamma(0) = 1_G$. Then the *left logarithmic derivative* of γ is defined as:

$$\delta\gamma\colon [0,1] \longrightarrow \mathbf{L}(G) : t \mapsto (\gamma(t))^{-1} \cdot \gamma'(t).$$

The multiplication is to be understood as the canonical multiplication on the tangent bundle (local) group TG.

Conversely, if a C^0-curve $\eta\colon [0,1] \longrightarrow \mathbf{L}(G)$ is given, then there is at most one C^1-curve γ with $\delta\gamma = \eta$ and $\gamma(0) = 1_G$. This unique η is called the *left evolution* of γ and denoted by $\eta = \mathrm{Evol}(\gamma)$.

(b) Let G be a (global) Lie group and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then G is called *strongly C^k-regular* if every $\eta \in C^k([0,1], \mathbf{L}(G))$ has a left evolution and if

$$\mathrm{evol}\colon C^k([0,1], \mathbf{L}(G)) \longrightarrow C^k([0,1], G) : \eta \mapsto \mathrm{Evol}(\eta)(1)$$

is smooth.

(c) Let G be a local Lie group and let $k \in \mathbb{N}_0 \cup \{\infty\}$. Then G is called *strongly C^k-regular* as a local Lie group if there is an open 0-neighborhood $\Omega \subseteq C^k([0,1], \mathbf{L}(G))$ such that every $\eta \in \Omega$ has a left evolution and if

$$\mathrm{evol}|_\Omega\colon \Omega \longrightarrow C^k([0,1], G) : \eta \mapsto \mathrm{Evol}(\eta)(1)$$

is smooth.

Strongly C^∞-regular (global or local) Lie groups are also just called *regular*. This is the regularity concept originally introduced by Milnor in [14].

Remark 2.5. (i) If $k \leq \ell$, then strong C^k-regularity implies strong C^ℓ-regularity.

(ii) Sometimes *strong C^k-regularity* is just called *C^k-regularity*.

For further details concerning logarithmic derivatives and regularity, we refer to [14] and [17].

Regularity of local Lie groups is connected with regularity of global Lie groups via the following fact which can be found in [18, Lemma 9.5]:

Lemma 2.6. *Let G be a Lie group. Then it is (strongly C^k-)regular as Lie group if and only if it is (strongly C^k-)regular as a local Lie group.*

3. Regularity of local Banach Lie groups

In this section, we will establish a quantitative version of the well-known result that local Banach Lie groups are strongly C^0-regular. As a main tool, we use the *Baker-Campbell-Hausdorff-Series* (or *BCH*-series) (see [3, Definition 1, Ch. II, §6] or [17, Definition IV.1.3] for a formal definition). The question whether the *BCH*-series converges for a given pair (x, y) is in general not easy to answer, but on Banach Lie algebras, we have the following fact, which can be found in [3, Propositions 1 and 2, Ch. II, §7]:

Lemma 3.1 (*BCH*-series in Banach-Lie algebras)**.** *Let $(\mathfrak{g}, \|\cdot\|)$ be a Banach-Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with compatible norm, i.e. $\|[x,y]\| \leq \|x\|\,\|y\|$.*

(a) *The* BCH-*series converges on the set*

$$\Omega := \{(x,y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \log 2\}$$

to a $C^\omega_\mathbb{K}$ function $*\colon \Omega \longrightarrow \mathfrak{g}$.

(b) *If* x, y, z *are elements in* $\mathfrak{g}$ *with* $\|x\| + \|y\| + \|z\| < \log \frac{3}{2}$, *then*

$$(x, y) \in \Omega, (y, z) \in \Omega, (x * y, z) \in \Omega, (x, y * z) \in \Omega \text{ and } x * (y * z) = (x * y) * z.$$

This Lemma enables us to turn each Banach Lie algebra (with compatible norm) into a local Lie group in the sense of Definition 2.1:

Proposition 3.2. *Let* $R_0 := \frac{1}{3} \log \frac{3}{2}$ *be fixed and define* $B := \mathrm{B}^{\mathfrak{g}}_{R_0}(0)$. *By construction,* $B \times B$ *lies in the set* Ω, *defined in Lemma 3.1 and therefore two elements in* B *can always be* BCH-*multiplied. This enables us to define the set* $D := \{(x, y) \in B \times B : x * y \in B\} \subseteq B \times B$ *and the map* $m_B \colon D \longrightarrow B : (x, y) \mapsto x * y$. *Together with inversion map* $\eta_B \colon B \longrightarrow B : x \mapsto -x$ *we obtain a local Lie group* $(B, D, m_B, 0_{\mathfrak{g}}, \eta_B)$.

Proof. The four properties listed in Definition 2.1 follow easily from Lemma 3.1. ∎

We will state the following useful lemma for estimates for operator norms of homogeneous polynomials which we will apply later to the polynomials of the *BCH*-series:

Lemma 3.3. *Let* X *and* Y *be normed vector spaces over* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. *Let* $k \in \mathbb{N}_0$, *let* $f \colon X^k \longrightarrow Y$ *be a continuous symmetric* k-*linear map and* $p \colon X \longrightarrow Y : x \mapsto f(x, \ldots, x)$ *be the corresponding* k-*homogeneous polynomial. Then we have the following estimates:*

(a) $\|p\|_{\mathrm{op}} \leq \|f\|_{\mathrm{op}}$.

(b) $\|f\|_{\mathrm{op}} \leq \frac{(2k)^k}{k!} \|p\|_{\mathrm{op}} \leq (2e)^k \|p\|_{\mathrm{op}}$.

(c) $\|p'\|_{\mathrm{op}} \leq k \|f\|_{\mathrm{op}} \leq k(2e)^k \|p\|_{\mathrm{op}}$.

(d) $\|p''\|_{\mathrm{op}} \leq k(k-1)(2e)^{2k-1} \|p\|_{\mathrm{op}}$.

Proof. Part (a) is obviuos. Part (b) is a direct consequence of the polarization formula (See for example [2, Theorem A]) and the well-known formula $k^k/k! \leq e^k$. Part (c) follows from (b), together with the fact that the Fréchet-derivative of a k-homogeneous polynomial is a $(k-1)$-homogeneous polynomial. Part (d) follows by applying part (c) twice. ∎

Lemma 3.4. *Let* $\sum_{n=1}^{\infty} \eta_n X^n$ *denote the power series expansion of the function*

$$-\log(2 - \exp(2X))$$

around zero. Then the power series

$$\sum_{n \in \mathbb{N}} n(2e)^n \eta_n X^n \text{ and } \sum_{n \in \mathbb{N}} n(n-1)(2e)^{2n-1} \cdot \eta_n X^n$$

have positive radii of convergence.

Let $\mathfrak{g}$ *be a Banach-Lie algebra over* $\mathbb{K}$ *with compatible norm and* BCH-*series*

$$\mu(x,y) = x * y = \sum_{n\in\mathbb{N}} p_n(x,y)$$

with continuous homogeneous polynomials $p_n\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathfrak{g}$ *and Fréchet derivatives* $p_n'\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$ *and* $p_n''\colon \mathfrak{g}\times\mathfrak{g} \longrightarrow \mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathcal{L}(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})) \cong \mathrm{Lin}_c^2(\mathfrak{g}\times\mathfrak{g},\mathfrak{g})$. *On the space* $\mathfrak{g}\times\mathfrak{g}$, *we the use maximum norm. Then we have the following estimates:*

(a) $\|p_n\|_{\mathrm{op}} \leq \eta_n$.

(b) $\|p_n'\|_{\mathrm{op}} \leq n(2e)^n \cdot \eta_n$.

(c) $\|p_n''\|_{\mathrm{op}} \leq n(n-1)(2e)^{2n-1} \cdot \eta_n$.

Proof. The convergence of the two series is obvious. The proof of part (a) can be found in [3, Lemma 1, Ch. II, §7]. Part (b) and (c) follow from (a) using Lemma 3.3 ■

Lemma 3.5. *We fix a real number* $s_0 \in]0, \frac{1}{3}\log\frac{3}{2}[$ *such that*

$$\sum_{n\geq 2} n(2e)^n \eta_n \cdot (4s_0)^{n-1} \leq \frac{1}{4}$$

and that

$$\sum_{n\geq 2} n(n-1)(2e)^{2n-1}\eta_n \cdot (4s_0)^{n-2} \leq \frac{1}{8},$$

using the converging power series introduced in Lemma 3.4.

Let $\mathfrak{g}$ *be a Banach-Lie algebra over* $\mathbb{K}$ *with compatible norm, addition map* $\alpha_{\mathfrak{g}}(x,y) = x+y$ *and* BCH-*multiplication* $\mu(x,y) = x*y$. *Then for all* $a,b \in \mathrm{B}^{\mathfrak{g}}_{4s_0}(0)$, *we have the estimates:*

(a) $\|\mu'(a,b) - \alpha_{\mathfrak{g}}\|_{\mathrm{op}} \leq \frac{1}{2}$

(b) $\|\mu''(a,b)\|_{\mathrm{op}} \leq \frac{1}{8}$.

Proof. First of all, a number s_0 with the desired properties exists since the power series introduced in Lemma 3.4 have a positive radius of convergence.

(a) We write the *BCH*-multiplication as in Lemma 3.4:

$$\mu(a,b) = a * b = \sum_{n\in\mathbb{N}} p_n(a,b)$$

It is known that $p_1 = \alpha_{\mathfrak{g}}$. Now, we take the Fréchet derivative on both sides:

$$\mu'(a,b) = \sum_{n\in\mathbb{N}} p_n'(a,b).$$

Since p_1 is linear, we have $p_1'(a,b) = p_1 = \alpha_{\mathfrak{g}}$. Hence, we can estimate:

$$\begin{aligned}\|\mu'(a,b) - \alpha_{\mathfrak{g}}\|_{\mathrm{op}} &= \Big\|\sum_{n\in\mathbb{N}} p_n'(a,b) - p_1'(a,b)\Big\|_{\mathrm{op}} = \Big\|\sum_{n\geq 2} p_n'(a,b)\Big\|_{\mathrm{op}} \\ &\leq \sum_{n\geq 2} \|p_n'(a,b)\|_{\mathrm{op}} \leq \sum_{n\geq 2} \|p_n'\|_{\mathrm{op}} \|(a,b)\|^{n-1} \leq \sum_{n\geq 2} n(2e)^n \eta_n (4s_0)^{n-1} \leq \frac{1}{4}.\end{aligned}$$

(b) Taking once again the Fréchet derivative of μ', we obtain: $\mu''(a,b) = \sum_{n\in\mathbb{N}} p_n''(a,b)$ with $p_1'' = 0$. Now, we estimate:

$$\begin{aligned}\|\mu''(a,b)\|_{\mathrm{op}} &= \Big\|\sum_{n\geq 2} p_n''(a,b)\Big\|_{\mathrm{op}} \leq \sum_{n\geq 2} \|p_n''(a,b)\|_{\mathrm{op}} \leq \sum_{n\geq 2} \|p_n''\|_{\mathrm{op}} \|(a,b)\|^{n-2} \\ &\leq \sum_{n\geq 2} \left(n(2e)^n\right)^2 \eta_n (4s_0)^{n-2} \leq \frac{1}{8}.\end{aligned}$$ ■

Now, we are ready to prove the quantitative version of the regularity of local Banach-Lie groups, using the constants we introduced in the preceedings lemmas.

Theorem 3.6 (Quantitative strong C^0-regularity of local Banach-Lie groups). *Let $\left(\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0), D, *, 0_{\mathfrak{g}}, -\mathrm{id}\right)$ be the local Banach-Lie group, corresponding to a Banach-Lie algebra $(\mathfrak{g}, \|\cdot\|)$ as constructed in Proposition 3.2 and let $s_0 > 0$ be the number taken from Lemma 3.5 and consider the set*

$$V := \left\{\gamma \in C^1_*([0,1],\mathfrak{g}) : \|\gamma'\|_\infty < 4s_0\right\}$$

which is open in the Banach space

$$C^1_*([0,1],\mathfrak{g}) := \left\{\gamma \in C^1([0,1],\mathfrak{g}) : \gamma(0) = 0\right\}.$$

Then the left logarithmic derivative $\delta|_V\colon V \longrightarrow C([0,1],\mathfrak{g}) : \gamma \mapsto \delta\gamma$ is a diffeomorphism onto its open image $\delta(V)$ which contains $\mathrm{B}^{C([0,1],\mathfrak{g})}_{s_0}(0)$. In particular, the local Lie group $\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0)$ is strongly C^0-regular.

Proof. We set $R_0 := \frac{1}{3}\log\frac{3}{2}$ and $C := \log 2$. First, we will now show that

$$\delta\colon C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_{R_0}(0)\right) \longrightarrow C([0,1],\mathfrak{g})$$

is a smooth map. We fix the notation

$$\mu\colon \mathrm{B}^{\mathfrak{g}}_{R_0}(0) \times \mathrm{B}^{\mathfrak{g}}_{R_0}(0) \longrightarrow \mathrm{B}^{\mathfrak{g}}_{C}(0) : (x,y) \mapsto x * y.$$

It is an easy calculation to see that the left logarithmic derivative of a C^1-curve in $\mathrm{B}^{\mathfrak{g}}_R(0)$ can be written as:

$$\delta\gamma(t) = d\lambda_{-\gamma(t)}(\gamma(t), \gamma'(t)) = d\mu\Big((-\gamma(t), \gamma(t)), (0, \gamma'(t))\Big)$$

Since the BCH-multiplication is smooth, the map

$$d\mu\colon (\mathrm{B}^{\mathfrak{g}}_{R_0}(0) \times \mathrm{B}^{\mathfrak{g}}_{R_0}(0)) \times (\mathfrak{g} \times \mathfrak{g}) \longrightarrow \mathfrak{g}$$

is also smooth. Now, the map $\delta\colon C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_{R_0}(0)\right) \longrightarrow C([0,1],\mathfrak{g})$ can be written as $\delta = g \circ \Phi$, with

$$\begin{aligned} \Phi : C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_{R_0}(0)\right) &\longrightarrow C\left([0,1], (\mathrm{B}^{\mathfrak{g}}_{R_0}(0) \times \mathrm{B}^{\mathfrak{g}}_{R_0}(0)) \times (\mathfrak{g} \times \mathfrak{g})\right) \\ \gamma &\longmapsto \Big((-\gamma,\gamma),(0,\gamma')\Big) \end{aligned}$$

and

$$g\colon C\left([0,1], (\mathrm{B}^{\mathfrak{g}}_{R_0}(0) \times \mathrm{B}^{\mathfrak{g}}_{R_0}(0)) \times (\mathfrak{g} \times \mathfrak{g})\right) \longrightarrow C([0,1],\mathfrak{g}) : \eta \mapsto d\mu \circ \eta.$$

The map Φ is a restriction of a bounded linear operator, hence smooth. The map g is a superposition operator and thus smooth (see e.g. of [22, Proposition 3.3.10]).

Hence, the left logarithmic derivative $\delta\colon C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}}_{R_0}(0)\right) \longrightarrow C([0,1],\mathfrak{g})$ is a smooth map.

From now on, we fix the following norm on the space $C^1_*([0,1],\mathfrak{g})$: $\|\gamma\|_{\mathrm{D}} := \|\gamma'\|_\infty$. It generates the usual topology on $C^1_*([0,1],\mathfrak{g})$, because of the estimate:

$$\|\gamma\|_\infty = \sup_{t\in[0,1]} \left\| \int_0^t \gamma'(s) ds \right\| \le \|\gamma\|_{\mathrm{D}}. \tag{$*$}$$

However, it has the advantage, that the linear operator

$$T\colon C^1_*([0,1],\mathfrak{g}) \longrightarrow C([0,1],\mathfrak{g}) : \gamma \mapsto \gamma'$$

becomes an isometric isomorphism. Now, we define the function

$$f\colon V = \mathrm{B}^{C^1_*([0,1],\mathfrak{g})}_{4s_0}(0) \longrightarrow C([0,1],\mathfrak{g}) : \gamma \mapsto \delta\gamma - \gamma'.$$

Our next goal is to show that f is Lipschitz-continuous by estimating the norm of the Fréchet derivative:

$$f'\colon V \longrightarrow \mathcal{L}\left(C^1_*([0,1],\mathfrak{g}), C([0,1],\mathfrak{g})\right).$$

Let $\gamma \in \mathrm{B}^{C^1_*([0,1],\mathfrak{g})}_{4s_0}(0)$, $\eta \in C^1_*([0,1],\mathfrak{g})$ with $\|\eta\|_{\mathrm{D}} = 1$ and $t \in [0,1]$ be given. By $(*)$, this implies that $\|\gamma\|_\infty < 4s_0$. Now

$$\begin{aligned}
&\|(f'(\gamma).\eta)(t)\|_{\mathfrak{g}} = \|(d\delta(\gamma,\eta)-\eta')(t)\|_{\mathfrak{g}} = \|(d\mu)'(\Phi(\gamma)(t))\cdot\Phi(\eta)(t)-\eta'(t)\|_{\mathfrak{g}}\\
&= \|d(d\mu)(\Phi(\gamma)(t),\Phi(\eta)(t))-\eta'(t)\|_{\mathfrak{g}}\\
&= \left\|d(d\mu)\Big((-\gamma(t),\gamma(t),0,\gamma'(t)),(-\eta(t),\eta(t),0,\eta'(t))\Big)-\eta'(t)\right\|_{\mathfrak{g}}\\
&\leq \left\|d^{(2)}\mu\Big((-\gamma(t),\gamma(t)),(0,\gamma'(t)),(-\eta(t),\eta(t))\Big)\right\|_{\mathfrak{g}}\\
&\quad + \left\|d\mu\Big((-\gamma(t),\gamma(t)),(0,\eta'(t))\Big)-\eta'(t)\right\|_{\mathfrak{g}}\\
&= \left\|\mu''\Big(-\gamma(t),\gamma(t)\Big)\Big((0,\gamma'(t)),(-\eta(t),\eta(t))\Big)\right\|_{\mathfrak{g}}\\
&\quad + \|\mu'(-\gamma(t),\gamma(t))(0,\eta'(t))-\alpha_{\mathfrak{g}}(0,\eta'(t))\|_{\mathfrak{g}}\\
&\leq \|\mu''(-\gamma(t),\gamma(t))\|_{\mathrm{op}}\|(0,\gamma'(t))\|\,\|(-\eta(t),\eta(t))\|\\
&\quad + \|\mu'(-\gamma(t),\gamma(t))-\alpha_{\mathfrak{g}}\|_{\mathrm{op}}\|(0,\eta'(t))\|\\
&\leq \|\mu''\|_\infty\|\gamma\|_{\mathrm{D}}\|\eta\|_\infty + \|\mu'(-\gamma(t),\gamma(t))-\alpha_{\mathfrak{g}}\|_{\mathrm{op}}\|\eta\|_{\mathrm{D}} \leq \frac{1}{8}\cdot 4s_0\cdot 1+\frac{1}{2}\cdot 1 < \frac{3}{4}.
\end{aligned}$$

This shows that $f'\colon V \longrightarrow \mathcal{L}(C^1_*([0,1],\mathfrak{g}),C([0,1],\mathfrak{g}))$ is globally bounded by $\frac{3}{4}$ and hence f is $\frac{3}{4}$-Lipschitz.

By the Lipschitz inverse function theorem (the first theorem of [23]), the map $\delta = T+f$ is a homeomorphism of $\mathrm{B}^{C^1_*([0,1],\mathfrak{g})}_{4s_0}(0)$ onto an open subset of $C([0,1],\mathfrak{g})$, containing the ball $\mathrm{B}^{C([0,1],\mathfrak{g})}_{r'}(0)$ with $r' = 4s_0\left(1-\frac{3}{4}\right) = s_0$.

For every fixed $\gamma \in \mathrm{B}^{C^1_*([0,1],\mathfrak{g})}_{4s_0}(0)$, we have $\|\delta'(\gamma)-T\|_{\mathrm{op}} = \|f'(\gamma)\|_{\mathrm{op}} \leq \frac{3}{4}$. Therefore the bounded operator $\delta'(\gamma)$ lies in the open ball with radius $\frac{3}{4} < 1$ around an isometric isomorphism and hence, $\delta'(\gamma)$ is invertible. Using the ordinary inverse function theorem for smooth mappings between Banach spaces, we get that δ is a diffeomorphism between V and $\delta(V) \supseteq \mathrm{B}^{C([0,1],\mathfrak{g})}_{s_0}(0)$. This finishes the proof. ∎

4. Regularity of local and global (LB)-Lie groups

We will only consider direct limits of an ascending sequence of Banach spaces in the category of locally convex topological vector spaces.

Definition 4.1 (Regularity of direct limits). Let $E := \bigcup_{n=1}^\infty E_n$ be a direct limit of Banach spaces. We assume E is Hausdorff.

(i) The sequence $(E_n)_{n\in\mathbb{N}}$ is called *compactly regular*, if every compact subset in E is also a compact set in some E_n.

(ii) The sequence $(E_n)_{n\in\mathbb{N}}$ is called *boundedly regular*, if every bounded subset in E is also a bounded set in some E_n.

Remark 4.2. Compact regularity implies bounded regularity. This fact can be found in [24, Theorem 6.4 and the corresponding Corollary].

Theorem 4.3 (Complex analytic mappings defined on (LB)-spaces)**.** *Let $E_1 \subseteq E_2 \subseteq \cdots$ be an ascending sequence of $\mathbb{C}$-Banach spaces such that all bonding maps are continuous with operator norm at most 1. We assume that the locally convex direct limit $E = \bigcup_n E_n$ is Hausdorff. Let $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}_R^{E_n}(0)$ with a fixed $R > 0$. Let $f\colon U \longrightarrow F$ be a function defined on U with values in a Hausdorff locally convex space F, such that each $f|_{\mathrm{B}_R^{E_n}(0)}$ is $C^\omega_\mathbb{C}$ and* bounded*. Then f is $C^\omega_\mathbb{C}$ as well.*

This theorem and its proof can be found in [5, Theorem A]. Although this result explicitly needs that $\mathbb{K} = \mathbb{C}$, the following easy consequence (also included in [5]) also holds in the real case:

Corollary 4.4 (Continuity of polynomials defined on (LB)-spaces)**.**
Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A polynomial function defined on the direct limit $E = \bigcup_{n\in\mathbb{N}} E_n$ of normed $\mathbb{K}$-vector spaces $E_1 \subseteq E_2 \subseteq \cdots$ with values in a Hausdorff locally convex space is continuous if and only if it is continuous on each step.

We will now use Theorem 4.3, together with the quantitative regularity of local Banach Lie groups (Theorem 4.5) to show the regularity of local (LB)-Lie groups. Afterwards, the regularity of (global) (LB)-Lie groups will follow immediately.

Theorem 4.5 (Regularity of local (LB)-Lie groups)**.** *Let $\mathfrak{g}_1 \subseteq \mathfrak{g}_2 \subseteq \cdots$ be an ascending sequence of Banach-Lie algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with a Hausdorff locally convex direct limit $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$. Assume furthermore, that all inclusion maps and all Lie brackets have operator norm at most 1. As in Proposition 3.2, we have for each $n \in \mathbb{N}$ a local Lie group $\left(\mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0), D_n, *|_{D_n}, 0_{\mathfrak{g}_n}, -\mathrm{id}\right)$, where*

$$D_n := \left\{(x, y) \in \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0) \times \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0) : x * y \in \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right\}.$$

We set $V := \bigcup_n \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)$ and $D := \bigcup_n D_n \subseteq V \times V$.

(a) *The space $\mathfrak{g}$ is again a topological Lie algebra and $(V, D, \mu, 0, \eta_V)$ becomes a local $C^\omega_\mathbb{K}$-Lie group, where μ is the* BCH*-multiplication and $\eta_V = -\mathrm{id}_V$.*

(b) *If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is boundedly regular, then $(V, D, \mu, 0, \eta_V)$ is strongly C^1-regular.*

(c) *If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is compactly regular, then $(V, D, \mu, 0, \eta_V)$ is even strongly C^0-regular.*

(d) *If the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is not boundedly regular, then $(V, D, \mu, 0, \eta_V)$ is not even C^∞-regular.*

Proof. During this proof, we fix the constants $R_0 := \frac{1}{3}\log\frac{3}{2}$ and $C := \log 2$.

(a) The ascending union of Lie algebras clearly is again a Lie algebra. The Lie bracket is continuous by Corollary 4.4, so $\mathfrak{g}$ is a locally convex topological Lie algebra. Now we will show that $(V, D, \mu, 0, \eta_V)$ is a local Lie group. To this end, we treat the cases $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$ separately:

We begin with the case that $\mathbb{K} = \mathbb{C}$. We endow the space $E_n := \mathfrak{g}_n \times \mathfrak{g}_n$ with the norm $\|(x,y)\|_{E_n} := \max\{\|x\|_{\mathfrak{g}_n}, \|y\|_{\mathfrak{g}_n}\}$. Then we have

$$\mathrm{B}^{E_n}_{R_0}(0) = \mathrm{B}^{\mathfrak{g}_n}_{R}(0) \times \mathrm{B}^{\mathfrak{g}_n}_{R}(0)$$

and the BCH-multiplication maps

$$\tilde{\mu}_n \colon \mathrm{B}^{E_n}_{R_0}(0) \longrightarrow \mathrm{B}^{\mathfrak{g}_n}_{C}(0) : (x,y) \mapsto x * y$$

are complex analytic and bounded (Lemma 3.1). It is possible to define the map

$$\tilde{\mu} \colon \bigcup_{n\in\mathbb{N}} \mathrm{B}^{E_n}_{R_0}(0) \longrightarrow V := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{\mathfrak{g}_n}_{C}(0) : (x,y) \mapsto x * y,$$

which is complex analytic by Theorem 4.3. Now, we restrict this map to set D and define: $\mu := \tilde{\mu}|_D \colon D \longrightarrow V$. One verifies easily that $(V, D, \mu, 0, \eta_V)$ becomes a local Lie group with this multiplication.

If $\mathbb{K} = \mathbb{R}$, we may consider the complexifications $(\mathfrak{g}_n)_{\mathbb{C}}$ of the real Banach spaces $(\mathfrak{g}_n, \|\cdot\|_{\mathfrak{g}_n})$. It is well-known that the complexification of a real Banach space becomes a complex Banach space, when equipped with the following norm:

$$\|\tilde{x}\|_{(\mathfrak{g}_n)_{\mathbb{C}}} := \inf\left\{ \sum_j |z_j| \, \|x_j\| \;\middle|\; \tilde{x} = \sum_j z_j x_j \text{ where } x_j \in \mathfrak{g}_n, z_j \in \mathbb{C} \right\}$$

The unique complex extension of a linear or bilinear map between real Banach spaces to the corresponding complexifications always has the same operator norm as the original mapping. (See [2, Section 2] for more details on complexifications.)

Hence, the bonding maps $j_n \colon \mathfrak{g}_n \longrightarrow \mathfrak{g}_{n+1}$ extend to unique continuous $\mathbb{C}$-linear mappings $(j_n)_{\mathbb{C}} \colon (\mathfrak{g}_n)_{\mathbb{C}} \longrightarrow (\mathfrak{g}_{n+1})_{\mathbb{C}}$, still having operator norm at most 1. The Lie bracket on each $\mathfrak{g}_n$ extends uniquely to a continuous Lie bracket on $(\mathfrak{g}_n)_{\mathbb{C}}$, turning it into a complex Lie algebra with compatible norms.

Now, we proceed like in the complex case and obtain a complex analytic BCH-multiplication on a $(0,0)$-neighborhood of $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}}$ which then restricts to a $C^{\omega}_{\mathbb{R}}$-map

$$* = \bigcup_{n\in\mathbb{N}} *_n \colon U \longrightarrow \mathfrak{g},$$

where $U := \bigcup_{n\in\mathbb{N}} \mathrm{B}^{E_n}_{R_0}(0)$.

(b) We start with the continuous linear map

$$\psi \colon C^1([0,1], \bigcup_n \mathfrak{g}_n) \longrightarrow C([0,1], \bigcup_n \mathfrak{g}_n) : \gamma \mapsto \gamma.$$

Every $\gamma \in C^1([0,1], \bigcup_n \mathfrak{g}_n)$ is Lipschtz-continuous, i.e. the set $\left\{\frac{\gamma(s)-\gamma(t)}{s-t} : s \neq t\right\}$ is a bounded subset of $\bigcup_n \mathfrak{g}_n$. Since the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is boundedly regular, this set is bounded in one of the spaces $\mathfrak{g}_n$. If we assume without loss of generality that $\gamma(0) \in \mathfrak{g}_n$, this yields a Lipschitz-continuous curve $\gamma \colon [0,1] \longrightarrow \mathfrak{g}_n$.

Since $\gamma \in C^1([0,1],\bigcup_n \mathfrak{g}_n)$ was arbitrary, we may conclude that the image of ψ is contained in the locally convex direct limit $\bigcup_n C([0,1],\mathfrak{g}_n)$ which is a topological subspace of $C([0,1],\bigcup_n \mathfrak{g}_n)$ by Mujica's Theorem (see [15, Theorem 3']).

This yields a continuous linear map

$$\widetilde{\psi}\colon C^1([0,1],\bigcup_n \mathfrak{g}_n) \longrightarrow \bigcup_n C([0,1],\mathfrak{g}_n) : \gamma \mapsto \gamma.$$

By Theorem 3.6, every local group $\left(\mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0), D_n, *|_{D_n}, 0_{\mathfrak{g}_n}, -\mathrm{id}\right)$ is strongly C^0-regular with the smooth evolution map

$$\mathrm{Evol}_n\colon \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0) \longrightarrow C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) : \delta\gamma \mapsto \gamma.$$

The constant $s_0 > 0$ is chosen as in Lemma 3.5 and is, in particular, independent of n. If $\mathbb{K} = \mathbb{C}$, this allows us to use Theorem 4.3 again to get the complex analyticity of the map

$$\mathrm{Evol}\colon \bigcup_n \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0) \longrightarrow \bigcup_n C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) : \delta\gamma \mapsto \gamma.$$

If $\mathbb{K} = \mathbb{R}$, we complexify the Banach-Lie algebras $\mathfrak{g}_n$ with the norm introduced in part (a). Then, use Theorem 4.3 to this complex setting and then restrict it to get the real analyticity of $\mathrm{Evol} := \bigcup_n \mathrm{Evol}_n$.

By construction, it is clear that $\widetilde{\psi}$ maps the open neighborhood $\bigcup_n \mathrm{B}^{C^1([0,1],\mathfrak{g}_n)}_{s_0}(0)$ in the neighborhood $\bigcup_n \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0)$. Hence, the composition

$$\mathrm{Evol} \circ \widetilde{\psi}\colon \bigcup_n \mathrm{B}^{C^1([0,1],\mathfrak{g}_n)}_{s_0}(0) \longrightarrow \bigcup_n C^1_*\left([0,1], \mathrm{B}^{\mathfrak{g}_n}_{\frac{1}{3}\log\frac{3}{2}}(0)\right) : \delta\gamma \mapsto \gamma$$

is complex analytic. Thus, we have shown that each C^1-curve in the 0-neighborhood $\bigcup_n \mathrm{B}^{C^1([0,1],\mathfrak{g}_n)}_{s_0}(0)$ has a left evolution and that the evolution map is complex analytic. Hence, the local Lie group is strongly C^1-regular.

(c) If we assume that the sequence $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ is even compactly regular, then the Mujica Theorem for compact regular direct limits (see [15, Theorem 3]) tells us that

$$\phi\colon C([0,1],\bigcup_n \mathfrak{g}_n) \longrightarrow \bigcup_n C([0,1],\mathfrak{g}_n) : \gamma \mapsto \gamma$$

is a topological isomorphism and hence we get that

$$\mathrm{Evol} \circ \phi\colon \bigcup_n \mathrm{B}^{C([0,1],\mathfrak{g}_n)}_{s_0}(0) \longrightarrow \bigcup_n \mathrm{B}^{\mathfrak{g}_n}_{C}(0) : \delta\gamma \mapsto \gamma$$

is analytic. Hence, the local Lie group is strongly C^0-regular.

(d) By [17], every C^∞-regular local Lie group has a Mackey complete Lie algebra. But by [7, 1.4.(f)], a countable direct limit of Banach spaces is Mackey complete if and only if the sequence of Banach spaces is boundedly regular. The claim follows. ∎

Theorem 4.6 (Regularity of countable unions of Banach-Lie groups)**.**
Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. *We are given an increasing sequence*

$$G_1 \subseteq G_2 \subseteq \cdots$$

of $\mathbb{K}$*-analytic Banach-Lie groups, such that the inclusion maps* $j_n \colon G_n \longrightarrow G_{n+1}$ *are analytic group homomorphisms. We fix a norm* $\|\cdot\|_{\mathfrak{g}_n}$ *on the Banach-Lie algebra* $\mathfrak{g}_n := \mathbf{L}(G_n)$, *defining its topology. We assume that the locally convex direct limit* $\mathfrak{g} = \bigcup_n \mathfrak{g}_n$ *is Hausdorff and that all inclusion maps and all Lie brackets have operator norm at most* 1. *Also, we assume that the map*

$$\exp_G := \bigcup_n \exp_{G_n} : \mathfrak{g} \longrightarrow \bigcup_n G_n$$

is injective on some 0*-neighborhood in* $\mathfrak{g}$. *Then*

(a) *There exists a unique locally convex Lie group structure on the group* $G := \bigcup_n G_n$, *such that* $\exp_G$ *becomes a local diffeomorphism around* 0.

(b) *If the sequence* $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ *is boundedly regular, then* G *is strongly* C^1*-regular.*

(c) *If the sequence* $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ *is compactly regular, then* G *is even strongly* C^0*-regular.*

(d) *If the sequence* $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ *is not boundedly regular, then* G *is not even* C^∞*-regular.*

Proof. The proof of part (a) can be found in [5, Theorem C].

The Lie group G is locally exponential which means that there is a 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp_G|_U$ is a diffeomorphism onto the open identity-neighborhood $V := \exp_G(U)$. We may assume that V is symmetric. We set $D_V := \{(x, y) \in V \times V : xy \in V\}$.

Then $(V, D_V, \mu, 1_G, \eta_V)$ becomes a local Lie group. After making the 0-neighborhood U smaller, we may assume that the exponential map becomes a local isomorphism between the local Lie groups $W \subseteq G$ and the local Lie group $\left(\mathrm{B}^{\mathfrak{g}}_{\frac{1}{3}\log\frac{3}{2}}(0), D, *, 0_{\mathfrak{g}}, -\mathrm{id}\right)$. By Lemma 2.6 strong C^k-regularity of the Lie group G is equivalent to the regularity of the local group $(V, D, \mu, 1_G, \eta_V)$. Therefore, (b), (c) and (d) follow immediately from Theorem 4.5. ■

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