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Compatible Lie brackets: Towards a Classification

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In memory of my parents

Abstract. We propose an approach to a long-standing problem of classification of pairs of compatible Lie-algebra structures, one of which is semisimple. Any such pair is determined by a linear operator which is defined up to the addition of a derivation. We introduce a special fixing of this operator to get rid of this ambiguity and consider the operators preserving the root decomposition with respect to a Cartan subalgebra. The classification leads to two disjoint classes of pairs depending on the symmetry properties of the corresponding operator with respect to the Killing form. We present a list of known and new examples in each case and conjecture the completeness of these lists.
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1. Introduction

Since its invention by Magri [Mag78] the bihamiltonian property proved to be very useful in the study of integrable systems. A second hamiltonian structure was discovered for the majority of important examples. In most cases it was first found "by hand" but afterwards it was usually recognized as a subcase of some general algebraic scheme. For instance, the bihamiltonian structure of the KdV equation [Mag78] was later understood as the so-called argument shift method [AK98, Ch. VI], [Ke03] on the Virasoro algebra.

Recall that a bihamiltonian structure is a pair of Poisson structures on a manifold which are compatible, i.e. such that their sum is again a Poisson structure. There are many algebraic schemes for bihamiltonian structures. However, using a simple characteristic, the order of the coefficients of the corresponding Poisson bivectors, one can organize these schemes in the following hierarchy:

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- constant+constant;
- linear+constant;
- linear+linear;
- quadratic+linear; etc.

The first case (when the coefficients of both brackets are constant) is rather uninteresting from the point of view of applications. The second one (here the pair of bivectors consists of the Lie–Poisson structure on the dual space to a Lie algebra and a constant bivector related to a cocycle on this Lie algebra) is already very important, for instance, due to the argument shift method mentioned; see also [Bol92]. The third case is the subject of this paper and we will comment on it below. The higher steps in the hierarchy are usually related to different forms of the classical Yang–Baxter equations [KSM89], [KM93, §1–2], [DG93], [GP94], [GR95].

The "linear+linear" case is represented by pairs of compatible Lie–Poisson structures, or, equivalently, pairs of compatible structures of Lie algebras on a vector space (cf. Definition 2.1). The situation here is much more complicated than in the "linear+constant" case (note that it is easy to construct examples of the latter case on any Lie algebra by considering a trivial cocycle). There are no "automatic" "linear+linear" examples for an arbitrary Lie algebra (except the Lie bracket which is the antisymmetrization of an associative multiplication $x \cdot y$: for any fixed element a the operation $x \cdot_a y := x \cdot a \cdot y$ is a new associative product and the corresponding commutator is compatible with the initial one).

One has to distinguish two main fields in which "linear+linear" pairs appeared. In the finite-dimensional context and within the "purely" bihamiltonian theory, the pair $[\,,], [\,,_A]$ of compatible Lie brackets on $\mathfrak{so}(n)$ (here $[\,,]$ is the standard commutator and $[\,,_A]$ is the modified one, $xAy - yAx$, with a diagonal matrix A) was applied by Bolsinov [Bol92], [Bol98] to prove the integrability of the Euler–Manakov top and to show its relation to the so-called Clebsch–Perolomov case (see also [MP96], [Yan00]). Another Lie–Poisson pencil was used to study generalized Steklov–Lyapunov systems on $\mathfrak{so}(n) \times \mathfrak{so}(n)$ [BF92]. A Poisson pencil which incorporates, on one hand, a Lie–Poisson pair (either the $\mathfrak{so}(n)$ pair mentioned above, or another one related to a $\mathbb{Z}_2$-grading on a Lie algebra), and, on the other hand, the argument shift method, was used in the theory of several systems (in particular to build Lax representations) such as Brun–Bogoyavlenskii and Zhukovsky–Volterra systems, the Kovalevski top, etc. [BB02], [BM03, Ch. 2].

Another area where pairs of compatible Lie brackets play an important role is the classical r-matrix formalism. The main idea (which in implicit form is already contained in the paper of Holod [Hol87]) that explains this role is as follows. We say that a Lie algebra $(\hat{\mathfrak{g}}, [\,,])$ with a decomposition $\hat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is *quasigraded* of degree 1 if $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j} \oplus \mathfrak{g}_{i+j+1}$ for any $i, j \in \mathbb{Z}$. Given such a Lie algebra, one observes that $\mathfrak{g}_+ := \bigoplus_{n\geq 0} \mathfrak{g}_n$ and $\mathfrak{g}_- := \bigoplus_{n<0} \mathfrak{g}_n$ are Lie subalgebras, so the difference of the corresponding projectors $P_+ - P_-$ is a classical r-matrix and one can apply the standard Adler–Kostant scheme [RSTS94]. On the other hand, if $\mathfrak{g}$ is a vector space with two compatible Lie brackets $[\,,]_0, [\,,]_1$, one obtains a quasigraded Lie algebra of degree 1 by putting $\hat{\mathfrak{g}} := \mathfrak{g}[\lambda, 1/\lambda]$, $[\,,] := [\,,]_0 + \lambda[\,,]_1$ and extending this bracket by bilinearity to $\hat{\mathfrak{g}}$ (the grading is the usual one, by the degree of a Laurent polynomial). This idea was later developed by Skrypnyk [Skr02], [Skr04], [Skr06], and, in a slightly different form, by Golubchik and Sokolov [GS02], [GS05], generating many new results on finite- and infinite-dimensional integrable systems (see also [LM05], [GVY08, Ch. 16]). Note that the correspondence $\{$compatible Lie brackets on $\mathfrak{g}\} \to \{$Lie algebra structures $(\mathfrak{g}[\lambda, 1/\lambda], [\,,])$ such that $\mathfrak{g}_+, \mathfrak{g}_-$ are Lie subalgebras$\}$ is one-to-one under some reasonable restrictions on the bracket $[\,,]$.

The above mentioned results already motivate considering the classification of pairs of compatible Lie brackets on finite-dimensional spaces. However, this issue (like many other questions of Lie theory) is itself a beautiful mathematical problem which probably inspired Kantor and Persits in their pioneering work in this direction. In [KP88] they announced (unfortunately, the proof never appeared) the following result. Let $\mathfrak{g}$ and V be finite-dimensional vector spaces and let $\{[\,,]^v\}_{v\in V}$ be a family of Lie brackets on $\mathfrak{g}$ parametrized by V (note that any pair of compatible Lie brackets generates such a family with V 1- or 2-dimensional). We say that such a family is *irreducible* if the Lie algebras $(\mathfrak{g}, [\,,]^v)$ do not have common ideals, and is *closed* if for any $a \in \mathfrak{g}$ and $v, w \in V$ there exists $z \in V$ such that $[\,,]^v_{\mathrm{ad}^w a} = [\,,]^z$, where $\mathrm{ad}^w a(b) := [a, b]^w$ and we use notation (3.1.1). Kantor and Persits determined all the closed irreducible $\mathbb{C}$-vector spaces of Lie brackets of dimension greater than one:

(i) $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ is the space of $n \times n$ antisymmetric matrices, V is the space of $n \times n$ symmetric matrices, the commutator $[\,,]^X$, $X \in V$, is given by $[a, b]^X = aXb - bXa =: [a,_X b]$;

(ii) $\mathfrak{g}$ is the space of $n \times n$ symmetric matrices, $V = \mathfrak{so}(n, \mathbb{C}), [\,,]^X = [\,,_X]$;

(iii) $\mathfrak{g}$ is the space of $n \times m$ matrices, V is the space of $m \times n$ matrices, $[\,,]^X = [\,,_X]$;

(iv) $\mathfrak{g} = V$ is an even-dimensional vector space with a nondegenerate skew-symmetric form $(\,,)$, the commmutator is given by $[a, b]^v := (v, a)b-(v, b)a-(a, b)v$;

Compatible Lie brackets from this list were studied in [TF95, §44] and [Yan00].

Another result which should be mentioned here is due to Odesski and Sokolov [OS06] who classified associative multiplications on the space of complex $n\times n$ matrices compatible with the usual one (in particular, by antisymmetrization one can deduce from this result many new examples of compatible Lie brackets on $\mathfrak{gl}(n, \mathbb{C})$; see also [CGM00]).

The main objective of this paper is to outline a systematic approach to the classification of pairs of compatible complex Lie algebras (we call such pairs *bi-Lie structures*) one of which is semisimple. As a byproduct we obtain new examples of such pairs which are not contained in the following list of known examples (KE) of this type (in items 1 and 2, $[\,,]$ stands for the standard commutator of matrices).

KE1 $(\mathfrak{so}(n,\mathbb{C}),[\,,],[\,,_A])$, where A is a symmetric $n\times n$-matrix (cf. the first example from the Kantor–Persits list and Example 2.3).

KE2 $(\mathfrak{sp}(n,\mathbb{C}),[\,,],[\,,_A])$, see Example 2.4 (this example can be derived from the second item of the Kantor–Persits list in the case when the matrix X is the standard antisymmetric matrix of maximal rank equal to $2n$ [TF95, §44]).

KE3 Let $(\mathfrak{g},[\,,])$ be semisimple. There exists a bi-Lie structure related to any $\mathbb{Z}_m$-grading on $(\mathfrak{g},[\,,])$ and to a decomposition of the Lie subalgebra $\mathfrak{g}_0$ into two subalgebras, see Examples 4.10, 12.13, Theorem 13.12 (in the general case it first appeared in [GS02], while for $m=2$ it was known much earlier, see discussion above).

KE4 Let $(\mathfrak{g},[\,,])$ be semisimple. There exists a bi-Lie structure related to any parabolic subalgebra $\mathfrak{g}_0\subset\mathfrak{g}$, see Example 4.9 (in case when $\mathfrak{g}_0$ is a Borel subalgebra this example appeared in [Pan06]).

KE5 Examples on $\mathfrak{sl}(3,\mathbb{C})$ and $\mathfrak{so}(4,\mathbb{C})$ related to $\mathbb{Z}_2\times\mathbb{Z}_2$-gradings [GS02].

Another byproduct of our classification scheme is establishing some non-obvious isomorphisms between certain bi-Lie structures from items KE3 and KE4.

It is known [KSM90], [GS02] that, given a semisimple Lie algebra $(\mathfrak{g},[\,,])$ and a bilinear bracket $[\,,]'$, the triple $(\mathfrak{g},[\,,],[\,,]')$ is a bi-Lie structure if and only if $[\,,]'$ is of the form $[\,,]'=[\,,]_W$ (see (3.1.1)) for some $W\in\operatorname{End}(\mathfrak{g})$ such that there is another operator $P\in\operatorname{End}(\mathfrak{g})$ satisfying the "main identity" $T_W(\,,)=[\,,]_P$, where T_W is the *Nijenhuis torsion* of W (see Section 4). We call an operator W satisfying $T_W(\,,)=[\,,]_P$ for some P a *weak Nijenhuis operator* (WNO for short) and P itself a *primitive* of W. It seems that one of the main difficulties in the theory of compatible brackets lies in the fact that these operators are defined nonuniquely up to the addition of an inner differentiation of the bracket $[\,,]$. Our first main result (see Section 7) proposes a special fixing of these two operators and establishes some of their properties. Namely, among all the WNOs corresponding to a bi-Lie structure $(\mathfrak{g},[\,,],[\,,]')$ we choose one (uniquely defined) orthogonal to $\operatorname{ad}\mathfrak{g}$ in $\operatorname{End}(\mathfrak{g})$ with respect to the "trace form" (we call it *principal*). We further prove that two bi-Lie structures are isomorphic (see Definition 2.6) if and only if so are the corresponding principal WNOs (Theorem 7.12). We also show that the principal operator W of a bi-Lie structure $(\mathfrak{g},[\,,],[\,,]')$ has a unique primitive (called *principal*) which is symmetric with respect to the Killing form on $(\mathfrak{g},[\,,])$ (Theorem 7.6). Thus the problem of classification of bi-Lie structures with $(\mathfrak{g},[\,,])$ semisimple is reduced to the problem of classification of pairs (W,P), where W is a principal WNO on $(\mathfrak{g},[\,,])$ and P is its principal primitive, with respect to the natural action of the automorphisms of $(\mathfrak{g},[\,,])$.

The latter problem seems to be very complicated in general. We try to solve it under the following reasonable restriction: a principal WNO preserves some Γ-grading on $(\mathfrak{g},[\,,])$, where Γ is an abelian group. This restriction is motivated by the fact that the majority of examples from the list above are in fact of this type (for KE3, KE5 this is obvious; a moment's thought is needed to see a $\mathbb{Z}_2\times\cdots\times\mathbb{Z}_2$-grading in KE1, KE2 if A is diagonalizable; for KE4 see Theorem 13.12). We first prove (see Theorem 9.2) that if a principal WNO preserves a grading, then so does its principal primitive. Further on, we make one more restriction and consider principal WNOs W such that:

(1) W preserves the root grading $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ related to some Cartan subalgebra $\mathfrak{h}$ in $(\mathfrak{g}, [\,,])$;

(2) $W|_\mathfrak{h}$ is semisimple.

For such operators the "main identity" $T_N(\,,) = [\,,]_P$ becomes a system of 1-dimensional algebraic equations of second order (see Theorem 10.3), hence solvable in principle. Next we propose a method of solving it.

More precisely, this system consists of two parts. The first is a system of quadratic equations indexed by the roots and relating the eigenvalues of the operators W and P with other invariants, which we call the *times* of the corresponding bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$. These are the complex numbers t_i such that the Lie algebra $(\mathfrak{g}, [\,,]' - t_i[\,,])$ is not semisimple (in the pencil $\{[\,,]' - t[\,,]\}_{t \in \mathbb{C}}$ the generic bracket is isomorphic to $[\,,]$ but for a finite number of values of t the corresponding Lie algebras are nonsemisimple; we call them *exceptional*). The second part, indexed by positive roots (with respect to any basis of roots) relates the eigenvectors and eigenvalues of $W|_\mathfrak{h}$ to the times. As a result we obtain a family of unordered pairs $\{t_{i,\alpha} t_{j,\alpha}\}_{\alpha \in R}$ of complex numbers indexed by the roots (we call such a family a *pair diagram*) and a system of restrictions on the operator $W|_\mathfrak{h}$ (see Section 10). It turns out that the "main identity" implies some rules of behavior of the pairs $\{t_{i,\alpha} t_{j,\alpha}\}$ with respect to addition of roots (see Theorem 10.7(i)). This relates bi-Lie structures to the geometry of the corresponding root systems and makes it possible to classify the former.

Our next result (Theorem 11.1) states that the set of pair diagrams splits into two disjoint classes, I and II. Further on, we study bi-Lie structures corresponding to these classes.

The behavior of unordered pairs $\{t_{i,\alpha} t_{j,\alpha}\}$ in pair diagrams of Class I with respect to addition of roots resembles the operation rules of the well-known pair groupoid and induces (together with the operator $W|_\mathfrak{h}$ subject to the restrictions mentioned) on $(\mathfrak{g}, [\,,])$ a special type of grading which we call an *admissible pairoid quasigrading* (see Section 12). It turns out that this quasigrading together with the grading set $\{t_1, \ldots, t_n\}$ contains all the information about the corresponding bi-Lie structure (see Theorem 12.8; note that the corresponding WNO has a symmetric restriction to $\mathfrak{h}^\perp = \sum_{\alpha \in R} \mathfrak{g}_\alpha$). Thus classification of bi-Lie structures of Class I is equivalent to classification of admissible pairoid quasigradings. The latter problem remains open, but we present a list of examples (known and new) and conjecture that this list is in a sense complete (see Conjecture 12.21). We remark that any such quasigrading is related to a specific type of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$-grading on $(\mathfrak{g}, [\,,])$ (see Lemma 12.10).

In the case of pair diagrams of Class II the corresponding pairoid quasigrading degenerates to the case which can be described essentially by a pair of Lie subalgebras $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ such that their sum is a regular reductive subalgebra which is not a fixed point set of an involutive inner automorphism (we call such pairs *admissible*). However, from admissible pairs of Lie subalgebras we can recover only a part of the information about the initial bi-Lie structures, more precisely, only $W|_{\mathfrak{h}}$ and the symmetric part of $W|_{\mathfrak{h}^\perp}$, where W is the corresponding WNO. The crucial difference here, in contrast to Class I structures, is that the corresponding WNOs also have antisymmetric parts on $\mathfrak{h}^\perp$ which contain the rest of the information about the bi-Lie structures. These antisymmetric parts are subject to some restrictions implied by the "main identity" (see Theorem 10.7(ii)) and in principle can be classified. The problem of classification of bi-Lie structures of Class II is now reduced to the problem of classification of admissible pairs of Lie subalgebras and the corresponding antisymmetric operators. Although this last is not solved in this paper in full extent, we present a list of examples (containing known and new ones) and conjecture the completeness of this list (see Conjecture 13.15). We also prove this conjecture for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, thus obtaining a complete classification of bi-Lie structures of Class II in this case (Theorem 13.16). Note that bi-Lie structures corresponding to pair diagrams of Class II are also related to some gradings on $(\mathfrak{g}, [\,,])$ which are coarsenings of the root grading (see Theorem 13.4).

Let us exhibit the examples which we get within our theory and compare them with those from the list above.

(i) Class I

(a) We recover KE1 in the particular case of $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n, t_{n+1})$ for $\mathfrak{so}(2n+1, \mathbb{C})$ and $A = \mathrm{diag}(t_1, t_1, \ldots, t_n, t_n)$ for $\mathfrak{so}(2n, \mathbb{C})$ (see Examples 12.14, 12.15).

(b) We recover KE2 for $A = \mathrm{diag}(t_1, \ldots, t_n, t_1, \ldots, t_n)$ (see Example 12.16).

(c) Surprisingly, we obtain new examples of bi-Lie structures, analogues of the examples above, for $\mathfrak{sl}(n, \mathbb{C})$, which are not found in the literature (see Examples 12.17, 12.18).

(d) We recover KE3 for $m = 2$ (see Example 12.13).

(ii) Class II

(a) We recover KE3 for $m > 2$ when the $\mathbb{Z}_m$-grading is related to an inner automorphism (see Theorem 13.12).

(b) We recover and generalize KE4 (see Theorem 13.9).

(c) We obtain a new example of a bi-Lie structure on the Lie algebra $\mathfrak{e}_8$ related to a $\mathbb{Z}_3 \times \mathbb{Z}_2$-grading (see Example 13.14).

Moreover, we prove that all the bi-Lie structures in KE4 are isomorphic to special bi-Lie structures in KE3 (see Theorem 13.12). We also deduce from Examples 12.17, 12.18 new examples of real bi-Lie structures on $\mathfrak{su}(n)$ and, in particular, on $\mathfrak{so}(6, \mathbb{R}) \cong \mathfrak{su}(4)$ (see Appendix 16).

Remark that the specific form of the matrix A which appeared in (a) of Class I above is related to the fact that the above restrictions (1), (2) on the WNO imply the following condition on the corresponding bi-Lie structure: the sum of the centres of the exceptional Lie algebras contains the Cartan subalgebra $\mathfrak{h}$ (see Theorem 10.1). In the case of generic A (diagonalizable with simple spectrum) in Example KE1 these centres are trivial and this case does not fit our theory. However, this case is related to a specific pairoid quasigrading which is not admissible (see Remark 12.23).

Summarizing, we can say that although the theory of the second part of the paper (Sections 10–13) only recovers examples related to toral gradings and quasigradings (cf. Appendix 15 and Definition 12.4), or in other words, to inner automorphisms, the results of the first part give a hope that a similar theory can also be built in general, i.e. including gradings coming from outer automorphisms. The author also hopes that some of these results (for instance Theorems 7.5, 7.6, 7.12, 8.3, 8.5, 9.2), which deal with general properties of bi-Lie structures, are new and can be used to further develop the theory of compatible Lie brackets.

Contents

2. Preliminaries and basic examples

We assume throughout (except in Appendix 16) that the ground field is $\mathbb{C}$. The terms "subalgebra" and "Lie subalgebra" are synonymous in this paper.

Definition 2.1. A *bi-Lie structure* is a triple $(\mathfrak{g}, [\,,], [\,,]')$, where $\mathfrak{g}$ is a vector space and $[\,,], [\,,]'$ are two Lie algebra structures on $\mathfrak{g}$ which are *compatible*, i.e. any linear combination $[\,,]^{\lambda,\lambda'} := \lambda[\,,] + \lambda'[\,,]'$ is also a Lie algebra structure.

Note that two Lie brackets $[\,,], [\,,]'$ on $\mathfrak{g}$ are compatible if and only if $[\,,] + [\,,]'$ is a Lie bracket.

Example 2.2. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, and let $A \in \mathfrak{g}$ be a fixed matrix. Put $[x, y]' = xAy - yAx = [x,_A y]$. Then it is easy to see that the triple $(\mathfrak{g}, [\,,], [\,,]')$, where $[\,,]$ is the standard commutator, is a bi-Lie structure.

Example 2.3. Let $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$, and let $A \in \mathrm{Symm}(n, \mathbb{C})$ be a fixed symmetric matrix. Then again $(\mathfrak{g}, [\,,], [\,,]')$, where $[\,,]'$ is given by the formula above, is a bi-Lie structure. Note that the symmetric matrices form the orthogonal complement to $\mathfrak{so}(n, \mathbb{C})$ in $\mathfrak{gl}(n, \mathbb{C})$ with respect to the trace form $\mathrm{Tr}(XY)$.

Example 2.4. Let $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid XJ + JX^T = 0\}$, where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ is the matrix of the standard symplectic form, and let A belong to the orthogonal complement $\mathfrak{sp}(n, \mathbb{C})^\perp$ to the symplectic subalgebra in $\mathfrak{gl}(2n, \mathbb{C})$ with respect to the trace form. This subspace can be described by the formula $\{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid XJ - JX^T = 0\}$. Then again $(\mathfrak{g}, [\,,], [\,,_A])$, is a bi-Lie structure.

The following lemma will be used later.

Lemma 2.5. *Let $(\mathfrak{g}, [\,,])$ and $(\mathfrak{g}, [\,,]')$ be Lie algebras and let $\mathrm{ad}, \mathrm{ad}'$ be the corresponding ad-operators, $\mathrm{ad}\, x(y) = [x, y]$, $\mathrm{ad}'\, x(y) = [x, y]'$. The triple $(\mathfrak{g}, [\,,], [\,,]')$ is a bi-Lie structure if and only if*

$$\mathrm{ad}[x, y]' + \mathrm{ad}'[x, y] = [\mathrm{ad}'\, x, \mathrm{ad}\, y] + [\mathrm{ad}\, x, \mathrm{ad}'\, y]$$

for all $x, y \in \mathfrak{g}$.

Proof. Indeed, $[\,,], [\,,]'$ are compatible if and only if $[\,,]'' := [\,,] + [\,,]'$ is a Lie bracket if and only if $\mathrm{ad}''[x, y]'' = [\mathrm{ad}''\, x, \mathrm{ad}''\, y]$, i.e.

$$(\mathrm{ad} + \mathrm{ad}')([x, y] + [x, y]') = [\mathrm{ad}\, x + \mathrm{ad}'\, x, \mathrm{ad}\, y + \mathrm{ad}'\, y].$$

Expanding the last bracket, taking into account $\mathrm{ad}[x, y] = [\mathrm{ad}\, x, \mathrm{ad}\, y]$ and $\mathrm{ad}'[x, y]' = [\mathrm{ad}'\, x, \mathrm{ad}'\, y]$, we obtain the result. ∎

Definition 2.6. We say that bi-Lie structures $(\mathfrak{g}, [\,,], [\,,]')$ and $(\mathfrak{g}_1, [\,,]_1, [\,,]_1')$ are *strongly isomorphic* (resp. *isomorphic*) if there exists a linear map $\varphi : \mathfrak{g} \to \mathfrak{g}_1$

transforming isomorphically the bracket $[\,,]$ to $[\,,]_1$ and $[\,,]'$ to $[\,,]'_1$ (resp. transforming isomorphically $[\,,]$ and $[\,,]_1$ to some linear combinations of $[\,,]'$ and $[\,,]'_1$).

Remark 2.7. The notion of isomorphism of bi-Lie structures is more natural (than that of strong isomorphism) in the context of studying the *pencils* $\{[\,,]^{\lambda,\lambda'}\}$ and $\{[\,,]_1^{\lambda,\lambda'}\}$ of Lie brackets generated by the pairs $([\,,],[\,,]')$ and $([\,,]_1,[\,,]'_1)$.

3. Lie algebra cohomology

In this section we recall basic definitions from the theory of (low-dimensional) cohomology of a Lie algebra $(\mathfrak{g},[\,,])$ with coefficients in a $\mathfrak{g}$-module $\mathfrak{m}$. The space $C^j(\mathfrak{g},\mathfrak{m})$ of *j-cochains* consists of all j-linear skew-symmetric maps $\mathfrak{g}^j \to \mathfrak{m}$. By definition, $C^0(\mathfrak{g},\mathfrak{m}) = \mathfrak{m}$.

The coboundary maps $\mathfrak{d}^j : C^j(\mathfrak{g},\mathfrak{m}) \to C^{j+1}(\mathfrak{g},\mathfrak{m}), j = 0,1,2$, are defined by

$$\mathfrak{d}^0\alpha(h_1) := h_1\alpha;$$
$$\mathfrak{d}^1\alpha(h_1,h_2) := h_1\alpha(h_2) - h_2\alpha(h_1) - \alpha([h_1,h_2]);$$
$$\mathfrak{d}^2\alpha(h_1,h_2,h_3) := \textstyle\sum_{\text{c.p.}i,j,k}(h_i\alpha(h_j,h_k) + \alpha(h_i,[h_j,h_k]))$$

(here $\alpha \in C^j(\mathfrak{g},\mathfrak{m}), h_i \in \mathfrak{g}$). By definition, the cohomology spaces are: $H^j(\mathfrak{g},\mathfrak{m}) := \ker \mathfrak{d}^j / \operatorname{im} \mathfrak{d}^{j-1}$. The Whitehead lemmata assert that $H^j(\mathfrak{g},\mathfrak{m}) = 0$ if $\mathfrak{g}$ is semisimple and $j = 1$ or 2.

For the *adjoint module* $\mathfrak{m} = \mathfrak{g}$ we will also use the notations $L(\mathfrak{g}) := C^1(\mathfrak{g},\mathfrak{m})$ (later we will also write $\operatorname{End}(\mathfrak{g})$ for $L(\mathfrak{g})$), $L^2(\mathfrak{g}) := C^2(\mathfrak{g},\mathfrak{m})$, and

$$[\cdot,\cdot]_W := \mathfrak{d}W(\cdot,\cdot) = [W\cdot,\cdot] + [\cdot,W\cdot] - W[\cdot,\cdot], \quad W \in L(\mathfrak{g}). \tag{3.1.1}$$

4. Weak Nijenhuis operators

Given two Lie algebras $(\mathfrak{g},[\,,])$ and $(\mathfrak{g},[\,,]')$, we observe that $(\mathfrak{g},[\,,],[\,,]')$ is a bi-Lie structure if and only if $[\,,]'$ is a 2-cocycle with respect to $[\,,]$ with coefficients in the adjoint module. In particular, if $(\mathfrak{g},[\,,],[\,,]')$ is a bi-Lie structure such that $(\mathfrak{g},[\,,])$ is semisimple, then $[\,,]' = [\,,]_W$ for some $W \in L(\mathfrak{g})$ (note that in Theorem 7.5 we will prove the existence of such W without using the Whitehead lemmata).

Vice versa, let $W \in L(\mathfrak{g})$ be such that $[\,,]_W$ is a Lie bracket. Then, since $[\,,]_W$ is a cocycle, it is automatically compatible with $[\,,]$ and $(\mathfrak{g},[\,,],[\,,]_W)$ is a bi-Lie structure.

We come to the following definitions.

Definition 4.1. Let $(\mathfrak{g},[\,,])$ be a Lie algebra. An operator $W \in L(\mathfrak{g})$ is called a *weak Nijenhuis operator* (WNO for short) if $[\cdot,\cdot]_W$ defined by (3.1.1) is a Lie bracket on $\mathfrak{g}$ (to be called the *modified bracket* on $\mathfrak{g}$).

Remark 4.2. A similar structure in the context of Lie algebroids appeared in [CGM04]; we borrowed the terminology from that paper.

Definition 4.3. The *Nijenhuis torsion* of $W \in L(\mathfrak{g})$ is defined as

$$T_W(x, y) := [Wx, Wy] - W[x, y]_W.$$

The following lemma justifies the term "WNO" (in the sense that it generalizes the well known notion of a Nijenhuis operator, i.e. an operator W with T_W vanishing).

Lemma 4.4 ([KSM90]). *An operator $W \in L(\mathfrak{g})$ is a WNO if and only if T_W is a 2-cocycle on $\mathfrak{g}$ with coefficients in the adjoint module.*

Definition 4.5. Let $W \in L(\mathfrak{g})$ be a WNO. An operator $P \in L(\mathfrak{g})$ is said to be a *primitive* of W if $T_W(x, y) = [x, y]_P$, $x, y \in \mathfrak{g}$ (in other words $T_W = \mathfrak{d}P$ in the sense of cohomology with coefficients in the adjoint module).

If $(\mathfrak{g}, [\,,])$ is semisimple, its second cohomology is trivial, hence any WNO possesses a primitive. Below we will prove the existence of such an operator directly (see Theorem 7.6). Note that the primitive of a WNO is defined up to the addition of a derivation of the Lie algebra $(\mathfrak{g}, [\,,])$.

Similarly, given a WNO $W \in L(\mathfrak{g})$ and any derivation $d \in \mathrm{Der}(\mathfrak{g})$ of the Lie algebra $(\mathfrak{g}, [\,,])$, one observes that $W + d$ is again a WNO (since $[\,,]_{W+d} = [\,,]_W$). Assuming W has a primitive P, what is a primitive of $W + d$? Below we will need to know how to express a primitive of $W + d$ in terms of P in the particular case where d is an inner derivation. To find such an expression let us introduce another useful notion.

Definition 4.6. A formal power series $R(t) = I + tR_1 + t^2R_2 + \cdots \in L(\mathfrak{g})[[t]]$ is called a *formal resolving function of order n* for a WNO W if

$$[R(t)\cdot, R(t)\cdot] \overset{n}{=} R(t)([\cdot, \cdot] + t[\cdot, \cdot]_W), \tag{4.6.1}$$

where $\overset{n}{=}$ stands for equality modulo terms of order $> n$ in t (here we understand the left hand and right hand sides as elements of the space $L^2(\mathfrak{g})[[t]]$ of formal power series with coefficients in $L^2(\mathfrak{g})$ and assume that $[\,,]$ is bimultiplicative with respect to t).

We say that $\Phi(t) = I + t\Phi_1 + t^2\Phi_2 + \cdots \in L(\mathfrak{g})[[t]]$ is a *formal automorphism* of the Lie algebra $(\mathfrak{g}, [\,,])$ if

$$[\Phi(t)\cdot, \Phi(t)\cdot] = \Phi(t)[\cdot, \cdot].$$

Clearly, if $\Phi(t) = I + t\Phi_1 + \cdots$ is a formal automorphism, then $\Phi_1 \in \mathrm{Der}(\mathfrak{g})$.

Theorem 4.7. (i) *A series $R(t) = I + tR_1 + \cdots$ is a formal resolving function of order 1 for a WNO W if and only if $R_1 = W + d$ for some $d \in \mathrm{Der}(\mathfrak{g})$.*

(ii) *A series $R(t) = I + t(W + d) + t^2R_2 + \cdots$ is a formal resolving function of order 2 for a WNO W if and only if $-R_2$ is a primitive of the WNO $W + d$.*

(iii) *Let* $R(t)$ *be a formal resolving function of order* n *for a WNO* W. *If* $\Phi(t) = I + t\Phi_1 + t^2\Phi_2 + \cdots$ *is a formal automorphism of the Lie algebra* $(\mathfrak{g}, [\,,\,])$, *then* $\tilde{R}(t) = \Phi(t)R(t)$ *is also a formal resolving function of order* n *for* W.

(iv) *In particular, if* P *is a primitive for* W, *then* $P - \Phi_1 W - \Phi_2$ *is a primitive for* $W + \Phi_1$.

Proof. (i)&(ii) The proof follows by equating the coefficients of t and t^2 on the two sides of (4.6.1).

(iii) The proof follows from the definitions.

(iv) If $R(t) = I + tW + t^2R_2 + \cdots$, then

$$\tilde{R}(t) = I + t(W + \Phi_1) + t^2(R_2 + \Phi_1 W + \Phi_2) + \cdots$$

and the result follows from (ii). ■

Corollary 4.8. *If* P *is a primitive of a WNO* W, *the operator* $P - \operatorname{ad} x_0 W - (\operatorname{ad}^2 x_0)/2$ *is a primitive of the WNO* $W + \operatorname{ad} x_0, x_0 \in \mathfrak{g}$. *In particular, the WNO* $\operatorname{ad} x_0$ *has a primitive* $-(\operatorname{ad}^2 x_0)/2$.

Proof. Use the theorem and the fact that $\Phi(t) := \exp(t \operatorname{ad} x_0) = I + t \operatorname{ad} x_0 + t^2(\operatorname{ad}^2 x_0)/2 + \cdots$ is a formal automorphism of the Lie algebra $(\mathfrak{g}, [\,,\,])$. ■

Example 4.9. Let $\mathfrak{g}$ decompose as a sum of two subalgebras $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Put $W|_{\mathfrak{g}_i} = \lambda_i \operatorname{Id}_{\mathfrak{g}_i}$, $i = 1, 2$, where λ_1 and λ_2 are any scalars. Then it is easy to see that the Nijenhuis torsion of W vanishes (cf. [Pan06]). This way one gets many examples of bi-Lie structures (for instance, taking $\mathfrak{g}$ to be semisimple, $\mathfrak{g}_1$ to be a parabolic subalgebra and $\mathfrak{g}_2$ its "complement", cf. Theorem 13.9).

Example 4.10. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{n-1}$ be a $\mathbb{Z}_n$-grading on $\mathfrak{g}$. Put $W|_{\mathfrak{g}_i} = i \operatorname{Id}_{\mathfrak{g}_i}, i = 0, \ldots, n-1$. We will show that W is a WNO by calculating explicitly a primitive P of W.

Put $P|_{\mathfrak{g}_i} = \frac{1}{2}i(n-i) \operatorname{Id}_{\mathfrak{g}_i}$. To show that it is indeed a primitive of W, let $x_i \in \mathfrak{g}_i, x_j \in \mathfrak{g}_j$. Then, if $i + j \le n - 1$, we get $T_W(x_i, x_j) = [Wx_i, Wx_j] = ij[x_i, x_j]$ (the term $W[x_i, x_j]_W$ vanishes). On the other hand, we have $[x_i, x_j]_P = \frac{1}{2}(i(n-i) + j(n-j) - (i+j)(n-i-j))[x_i, x_j] = ij[x_i, x_j]$. If $i + j \ge n$, we get $T_W(x_i, x_j) = [Wx_i, Wx_j] - W[x_i, x_j]_W = (ij - (n-i-j)(i+j-(n-i-j)))[x_i, x_j] = (ij + (n-i-j)n)[x_i, x_j]$ and $[x_i, x_j]_P =$ $\frac{1}{2}(i(n-i) + j(n-j) - (i+j-n)(2n-i-j))[x_i, x_j] = (ij - (i+j-n)n)[x_i, x_j]$.

Example 4.11. Consider $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ and the operator L_A, $L_A x := Ax$, of left multiplication by a fixed matrix $A \in \mathfrak{g}$. Then the Nijenhuis torsion of L_A vanishes, i.e. it is a WNO, and the modified bracket $[\,,\,]_{L_A}$ coincides with the bracket $[\,,_A]$ from Example 2.2.

Example 4.12. Consider $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $A \in \mathrm{Symm}(n, \mathbb{C})$. The operator L_A is not correctly defined on $\mathfrak{g}$ (since $L_A\mathfrak{g} \not\subset \mathfrak{g}$). However, the operator $W := W'|_{\mathfrak{g}}$, where $W' := (1/2)(L_A + R_A) \in \mathrm{End}(\mathfrak{gl}(n, \mathbb{C}))$ and R_A stands for the operator of right multiplication by A, is a WNO and it is easy to see that again $[\,,]_W = [\,,_A]$ (see Example 2.3). The torsion of W does not vanish, but it is a cocycle. Indeed, since $W' = L_A - (1/2)(L_A - R_A) = L_A - \mathrm{ad}(A/2)$ and 0 is a primitive of L_A, by Corollary 4.8 we conclude that the operator $\mathrm{ad}(A/2) \circ L_A - (1/8)\,\mathrm{ad}^2 A$ is a primitive of W'. Thus $T_{W'}$ is a cocycle and so is T_W.

Using Lemma 6.1 below one can show that $(1/8)\,\mathrm{ad}^2 A|_{\mathfrak{g}}$ is a primitive of W.

5. Partial Nijenhuis operators

Let us start from another look at Example 4.12. Put $\mathcal{G} := \mathfrak{gl}(n, \mathbb{C})$ and consider the operator L_A as a map acting from $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ to $\mathcal{G}$. It is easy to see that the formula

$$[\cdot,\cdot]_{L_A} := [L_A\cdot,\cdot] + [\cdot, L_A\cdot] - L_A[\cdot,\cdot]$$

(cf. (3.1.1)), where the brackets on the right denote the commutator of matrices in $\mathcal{G}$, still defines a bilinear operation on $\mathfrak{g}$ (in fact $[x, y]_{L_A} = [x,_A y]$), and moreover $T_{L_A}(x, y) = [L_A x, L_A y] - L_A[x, y]_{L_A}$ vanishes for any $x, y \in \mathfrak{g}$. This motivates the following definition.

Definition 5.1. Let $\mathcal{G}$ be a Lie algebra and $\mathfrak{g} \subset \mathcal{G}$ a Lie subalgebra. We say that a pair $(\mathfrak{g}, N)$, where $N\colon \mathfrak{g} \to \mathcal{G}$ is a linear map, is a *partial Nijenhuis operator on* $\mathcal{G}$ (PNO for short) if the following two conditions hold:

(i) $[x, y]_N \in \mathfrak{g}$ for any $x, y \in \mathfrak{g}$;

(ii) $T_N(x, y) = 0$ for any $x, y \in \mathfrak{g}$.

Remark 5.2. A similar structure in the context of Lie algebroids appeared in [CGM04] under the name of "outer Nijenhuis tensor".

Lemma 5.3. *If $(\mathfrak{g}, N)$ is a PNO on $\mathcal{G}$, then:*

(i) *$N\mathfrak{g}$ is a Lie subalgebra in $\mathcal{G}$;*

(ii) *$(\mathfrak{g}, N^t)$, $N^t := N - tI$, is a PNO on $\mathcal{G}$ for any $t \in \overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, where $I\colon \mathfrak{g} \to \mathcal{G}$ is the natural embedding and $N^\infty = I$ by definition;*

(iii) *$N^t\mathfrak{g}$ is a Lie subalgebra in $\mathcal{G}$ for any t;*

(iv) *for any $t \in \overline{\mathbb{C}}$ the operation $[\,,]_{N^t}$ is a Lie algebra structure on $\mathfrak{g}$ and $N^t\colon \mathfrak{g} \to \mathcal{G}$ is a homomorphism from $(\mathfrak{g}, [\,,]_{N^t})$ to $(\mathcal{G}, [\,,])$;*

(v) *the Lie bracket $[\,,]_N$ is compatible with $[\,,]$.*

Proof. Indeed, (i) follows from the definition of a PNO; (ii) comes from the equalities $[\,,]_{N^t} = [\,,]_N - t[\,,]$ and $T_{N-tI} = T_N$; and (iii) follows from (ii).

Now (iv) follows easily from the equality

$$[x, y]_{N-tI} = (N - tI)^{-1}[(N - tI)x, (N - tI)y],$$

which makes sense for almost all t, and (v) is a consequence of (iv). ■

Example 5.4. The basic example here is $L_A : \mathfrak{so}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$, $A \in \mathrm{Symm}(n, \mathbb{C})$, $[\,,]_{L_A} = [\,,_A]$.

Below we will associate a PNO with any bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ with a semisimple bracket $[\,,]$.

6. PNOs and WNOs in the presence of orthogonal decomposition

Let $\mathcal{G}$ be a Lie algebra, $\tilde{\mathfrak{g}} \subset \mathcal{G}$ a Lie subalgebra and let B be an invariant form on $\mathcal{G}$ (we put a tilde over $\mathfrak{g}$ for consistency with the notation of the next section).

Lemma 6.1. *$[\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1$, where $\mathfrak{g}_0 := \tilde{\mathfrak{g}}$ and $\mathfrak{g}_1 := \tilde{\mathfrak{g}}^\perp$ is the orthogonal complement to $\tilde{\mathfrak{g}}$ with respect to B.*

Proof. Let $x, z \in \mathfrak{g}_0$ and $y \in \mathfrak{g}_1$. Then $B([x, y], z) = -B(y, [x, z]) = 0$. Since $z \in \mathfrak{g}_0$ is arbitrary, we get the result. ■

From now on we will assume that

$$\mathfrak{g}_0 \oplus \mathfrak{g}_1 = \mathcal{G}. \tag{6.1.1}$$

Such a decomposition occurs for instance when B is nondegenerate and so is $B|_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$. The main example which we have in mind is the pair $\tilde{\mathfrak{g}} = \mathrm{ad}\,\mathfrak{g} \subset \mathrm{End}(\mathfrak{g})$, where $\mathfrak{g}$ is semisimple and B is the trace form on $\mathrm{End}(\mathfrak{g})$ (see Section 7; cf. also Example 6.3 for an instance of a different kind).

Now let $N : \tilde{\mathfrak{g}} \to \mathcal{G}$, be a linear map. Write $p_i : \mathcal{G} \to \mathfrak{g}_i$ for the corresponding orthogonal projection, $N_i := p_i N$, $i = 1, 2$. Then $N = N_0 + N_1$ and $[\,,]_N = [\,,]_{N_0} + [\,,]_{N_1}$. We obviously have $[\mathfrak{g}_0, \mathfrak{g}_0]_{N_0} \subset \mathfrak{g}_0$ whereas from the lemma we conclude that

$$[\mathfrak{g}_0, \mathfrak{g}_0]_{N_1} = \{[N_1 x, y] + [x, N_1 y] - N_1[x, y] \mid x, y \in \mathfrak{g}_0\} \subset \mathfrak{g}_1.$$

Assuming that $[\mathfrak{g}_0, \mathfrak{g}_0]_N \subset \mathfrak{g}_0$ (the first condition of the definition of PNO) we get the equality $[\,,]_N = [\,,]_{N_0}$ on $\mathfrak{g}_0$. Hence, if also the second condition of the definition of PNO holds for N, the map N_0 is a WNO on $\tilde{\mathfrak{g}}$ (by Lemma 5.3(iv)), which is uniquely (!) defined by N and which gives the same modified bracket on $\tilde{\mathfrak{g}}$ as N. Let us summarize the above in the following lemma.

Lemma 6.2. *Let $N : \tilde{\mathfrak{g}} \to \mathcal{G}$ be a PNO on $\mathcal{G}$. Assume that an invariant form B on $\mathcal{G}$ is given and condition (6.1.1) is satisfied, where $\mathfrak{g}_0 = \tilde{\mathfrak{g}}$ and $\mathfrak{g}_1 = \tilde{\mathfrak{g}}^\perp$. Then the map $N_0 := p_0 N$ (here $p_0 : \mathcal{G} \to \mathfrak{g}_0$ is the orthogonal projection) is a WNO on $\tilde{\mathfrak{g}}$ and moreover $[\,,]_N = [\,,]_{N_0}$.*

Example 6.3. Let $\tilde{\mathfrak{g}} = \mathfrak{g}_0 = \mathfrak{so}(n,\mathbb{C})$, $\mathcal{G} = \mathfrak{gl}(n,\mathbb{C})$, $\mathfrak{g}_1 = \mathrm{Symm}(n,\mathbb{C})$, $N = L_A|_{\tilde{\mathfrak{g}}}$, $A \in \mathrm{Symm}(n,\mathbb{C})$ (see Example 5.4), and let B be the trace form on $\mathcal{G}$. Then $N_0 = (1/2)(L_A + R_A)|_{\tilde{\mathfrak{g}}}, N_1 = (1/2)(L_A - R_A)|_{\tilde{\mathfrak{g}}}$. Thus we get the WNO $W = N_0$ from Example 4.12.

Let us study the pair of maps (N, N_0), where N is a PNO, in more detail. First, we can express the Nijenhuis torsion of the WNO N_0 by means of the map N_1. Indeed, using Lemma 6.1 we get $0 = p_0 T_N(x,y) = p_0([Nx,Ny] - N[x,y]_N) = [N_0x, N_0y] + p_0[N_1x, N_1y] - N_0([x,y]_{N_0}) = T_{N_0} + p_0[N_1x,N_1y]$ for $x, y \in \mathfrak{g}_0$, hence

$$T_{N_0}(x,y) = -p_0[N_1x, N_1y], \quad x, y \in \tilde{\mathfrak{g}}. \tag{6.3.1}$$

Second, the equality $[\,,]_N = [\,,]_{N_0}$ shows that $[\,,]_{N_1} = 0$ on $\tilde{\mathfrak{g}}$. In other words, the map $N_1 : \tilde{\mathfrak{g}} \to \mathfrak{g}_1$ is a 1-cocycle on $\tilde{\mathfrak{g}}$ with coefficients in the $\tilde{\mathfrak{g}}$-module $\mathfrak{g}_1$. Thus, if we assume additionally that $\tilde{\mathfrak{g}}$ is semisimple, by the Whitehead lemma $N_1 = \mathfrak{d}w$ for a (unique) $w \in \mathfrak{g}_1$, i.e.

$$N_1 = -(\mathrm{ad}_{\mathcal{G}}\, w)|_{\tilde{\mathfrak{g}}}.$$

By Corollary 4.8 we have $[\mathrm{ad}_{\mathcal{G}}\, w(x), \mathrm{ad}_{\mathcal{G}}\, w(y)] = T_{\mathrm{ad}_{\mathcal{G}}\, w}(x,y) = [x,y]_{-(\mathrm{ad}^2_{\mathcal{G}}\, w)/2}$, hence

$$T_{N_0}(x,y) = -p_0[x,y]_{-(\mathrm{ad}^2_{\mathcal{G}}\, w)/2} = [x,y]_{p_0(\mathrm{ad}^2_{\mathcal{G}}\, w)/2}, \quad x, y \in \tilde{\mathfrak{g}}.$$

In other words, $P := p_0(\mathrm{ad}^2_{\mathcal{G}}\, w)/2$ is a primitive of the WNO N_0. Note that P is a symmetric operator with respect to the form $B|_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$. Indeed, $\mathrm{ad}_{\mathcal{G}}\, w$ is anti-symmetric, hence, given $x, y \in \tilde{\mathfrak{g}}$, we have $B(Px,y) = (1/2)B((\mathrm{ad}^2_{\mathcal{G}}\, w)(x), y) = (1/2)B(x, (\mathrm{ad}^2_{\mathcal{G}}\, w)(y)) = B(x, Py)$. Let us summarize this in

Lemma 6.4. *Retain the hypotheses of Lemma 6.2 and assume moreover that $\tilde{\mathfrak{g}}$ is semisimple. Then there exists a unique $w \in \mathfrak{g}_1$ such that $N_1 := p_1 N = -\mathrm{ad}_{\mathcal{G}}\, w$ (here $p_1 : \mathcal{G} \to \mathfrak{g}_1$ is the orthogonal projection). Moreover the operator $P := (1/2)p_0(\mathrm{ad}^2_{\mathcal{G}}\, w) : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is symmetric with respect to $B|_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$ and is a primitive of the WNO N_0.*

7. Semisimple bi-Lie structures and their principal WNOs

Definition 7.1. We say that a bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ is *semisimple* if $(\mathfrak{g}, [\,,])$ is a semisimple Lie algebra (which will be called the *underlying Lie algebra* of the bi-Lie structure).

Lemma 7.2. *Let $(\mathfrak{g}, [\,,])$ be a semisimple Lie algebra. Put $\mathcal{G} := \mathrm{End}(\mathfrak{g})$ (with the commutator bracket), $\mathfrak{g}_0 := \tilde{\mathfrak{g}} := \mathrm{ad}\,\mathfrak{g}$, $\mathfrak{g}_1 = (\mathrm{ad}\,\mathfrak{g})^\perp$ (the orthogonal complement with respect to the trace form B). Then*

(i) *condition (6.1.1) is satisfied;*

(ii) *for any semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ there exists a unique WNO W such that $[\,,]' = [\,,]_W$ and $W \in \mathfrak{g}_1$;*

(iii) *for any WNO W there exists a unique operator P such that $T_W(\,,\,) = [\,,]_P$ and $P \in \mathfrak{g}_1$.*

Proof. (i) It is easy to see that $B|_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$ is the Killing form of $(\tilde{\mathfrak{g}}, [\,,])$ (here $[\,,]$ is the commutator of endomorphisms). It is nondegenerate (due to the semisimplicity of $\mathfrak{g}$) and, moreover, B is nondegenerate itself, hence condition (6.1.1) is satisfied.

(ii)&(iii) The existence of the operators W and P satisfying the equalities $[\,,]' = [\,,]_W$ and $T_W(\,,\,) = [\,,]_P$ was discussed in Section 4. Let us consider the uniqueness matter.

If $V, V_1 : \mathfrak{g} \to \mathfrak{g}$ are two operators satisfying $[\,,]_{V_1} = [\,,]_V$, then the difference $V - V_1$ obviously should be a derivation of the bracket $[\,,]$. Since all the derivations of $[\,,]$ are inner, V and V_1 should differ by some $\operatorname{ad} x, x \in \mathfrak{g}$, i.e. by an element of $\mathfrak{g}_0$. Now, if moreover $V, V_1 \in \mathfrak{g}_1$, the decomposition property (6.1.1) implies $V = V_1$. ■

This lemma motivates the following

Definition 7.3. Let $(\mathfrak{g}, [\,,])$ be a semisimple Lie algebra. An operator $V \in \operatorname{End}(\mathfrak{g})$ is said to be *principal* if $V \in (\operatorname{ad}\mathfrak{g})^{\perp}$.

Given a semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$, any $W \in \operatorname{End}(\mathfrak{g})$ such that $[\,,]' = [\,,]_W$ will be called a *WNO of the bi-Lie structure* and the unique (in view of Lemma 7.2) WNO of the bi-Lie structure belonging to $(\operatorname{ad}\mathfrak{g})^{\perp}$ will be called the *principal WNO* of this bi-Lie structure.

Finally, if W is the principal WNO of a a semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$, the unique primitive of W belonging to $(\operatorname{ad}\mathfrak{g})^{\perp}$ will be called the *principal primitive* of the principal WNO W.

Below we will give a constructive description of the pricipal WNO and its principal primitive which will allow to obtain some useful properties of these operators. To this end we will apply Lemma 6.2 to a situation in which:

- $(\mathfrak{g}, [\,,], [\,,]')$ is a semisimple bi-Lie structure;
- $\mathcal{G} := \operatorname{End}(\mathfrak{g})$ (with the commutator bracket);
- $\tilde{\mathfrak{g}} := \operatorname{ad}\mathfrak{g} \subset \mathcal{G}$;
- $N := \operatorname{ad}'(\operatorname{ad})^{-1} : \tilde{\mathfrak{g}} \to \mathcal{G}$ (here $\operatorname{ad} x(y) := [x, y], \operatorname{ad}' x(y) = [x, y]', x, y \in \mathfrak{g}$);
- $B(a, b) := \operatorname{Tr}(ab), \quad a, b \in \mathcal{G}$.

The following lemma shows that N is a PNO on $\mathcal{G}$.

Lemma 7.4. (i) $[\operatorname{ad} x, \operatorname{ad} y]_N = \operatorname{ad}[x, y]'$, $x, y \in \mathfrak{g}$;

(ii) $T_N(\operatorname{ad} x, \operatorname{ad} y) = 0$, $x, y \in \mathfrak{g}$.

Proof. (i) By definition, $N(\operatorname{ad} x) = \operatorname{ad}' x$. Thus $[\operatorname{ad} x, \operatorname{ad} y]_N = [N(\operatorname{ad} x), \operatorname{ad} y] + [\operatorname{ad} x, N(\operatorname{ad} y)] - N[\operatorname{ad} x, \operatorname{ad} y] = [\operatorname{ad}' x, \operatorname{ad} y] + [\operatorname{ad} x, \operatorname{ad}' y] - N \operatorname{ad}[x,y] = [\operatorname{ad}' x, \operatorname{ad} y] + [\operatorname{ad} x, \operatorname{ad}' y] - \operatorname{ad}'[x,y]$. The last expression is equal to $\operatorname{ad}[x,y]'$ due to the compatibility of the brackets $[\,,], [\,,]'$ (see Lemma 2.5).

(ii) $T_N(\operatorname{ad} x, \operatorname{ad} y) = [N \operatorname{ad} x, N \operatorname{ad} y] - N[\operatorname{ad} x, \operatorname{ad} y]_N = [\operatorname{ad}' x, \operatorname{ad}' y] - N \operatorname{ad}[x,y]' = [\operatorname{ad}' x, \operatorname{ad}' y] - \operatorname{ad}'[x,y]' = 0$. ■

The construction from the previous section gives a WNO $\widetilde{W} : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$, $\widetilde{W} := N_0 = p_0 \circ N$. Let us denote by W the corresponding WNO on $\mathfrak{g}$ induced by $\widetilde{W}$ and by the Lie algebra isomorphism $\operatorname{ad} : (\mathfrak{g}, [\,,]) \to (\tilde{\mathfrak{g}}, [\,,])$, in other words, $\widetilde{W}(\operatorname{ad} x) = \operatorname{ad} Wx$, $x \in \mathfrak{g}$. It follows from the lemma above that under the identification "ad" the second bracket $[\,,]'$ corresponds to the bracket $[\,,]_N = [\,,]_{\widetilde{W}}$, hence $[\,,]' = [\,,]_W$. Summarizing we get the following theorem.

Theorem 7.5. *Let a semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ be given. Then the operator*

$$W = \operatorname{ad}^{-1} \circ p_0 \circ \operatorname{ad}' \in \operatorname{End}(\mathfrak{g}),$$

where $p_0 : \mathcal{G} \to \tilde{\mathfrak{g}}$ is the orthogonal projection with respect to B, is a WNO of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.

Let W be the WNO built above. The next result gives an explicit expression of the orthogonal decomposition $N = N_0 + N_1$ (cf. Section 6) for $N = \operatorname{ad}'(\operatorname{ad})^{-1}$ in terms of the operator W and calculates one of the primitives of the WNO W.

Theorem 7.6. (i) *The action of the map $N : \tilde{\mathfrak{g}} \to \mathcal{G}$ can be given by the following expression: $N(\operatorname{ad} x) = \operatorname{ad} Wx + [\operatorname{ad} x, W]$; moreover, $N_0(\operatorname{ad} x) = \widetilde{W}(\operatorname{ad} x) = \operatorname{ad} Wx, N_1(\operatorname{ad} x) = [\operatorname{ad} x, W]$, $x \in \mathfrak{g}$ (in other words $N_1 = L|_{\tilde{\mathfrak{g}}}$, where $L : \mathcal{G} \to \mathcal{G}$ is equal to $-\operatorname{ad}_{\mathcal{G}} W$).*

(ii) *The operator $\widetilde{P} := (1/2) p_0 \circ \operatorname{ad}^2_{\mathcal{G}} W : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is symmetric with respect to the Killing form $B_{\tilde{\mathfrak{g}}}$ and is a primitive of the WNO $\widetilde{W}$. The operator $P = \operatorname{ad}^{-1} \circ \widetilde{P} \circ \operatorname{ad}$ is a primitive of the WNO W, which is symmetric with respect to the Killing form $B_{\mathfrak{g}}$.*

Proof. By the definition of N for any $x, y \in \mathfrak{g}$ we have $(N(\operatorname{ad} x))y = \operatorname{ad}' x(y) = [x,y]'$. Moreover, we know that $[\,,]' = [\,,]_W$, hence $(N(\operatorname{ad} x))y = [Wx, y] + [x, Wy] - W[x,y]$. Rewrite the right hand side of this equality as $\operatorname{ad} Wx(y) + [\operatorname{ad} x, W]y$. Since the first term of this expression is equal to $(\widetilde{W}(\operatorname{ad} x))y$ (by the definition of W), the second one has to equal $((N - \widetilde{W})(\operatorname{ad} x))y = (N_1(\operatorname{ad} x))y$. The rest follows from Lemma 6.4. ■

The following two statements imply that the operators W and P built in Theorems 7.5, 7.6 are in fact the principal WNO of the corresponding semisimple bi-Lie structure and its principal primitive, respectively.

Corollary 7.7. *The endomorphism $W \in \operatorname{End}(\mathfrak{g})$ defined in Theorem 7.5 belongs to $\mathfrak{g}_1 = \tilde{\mathfrak{g}}^\perp$. In particular, W is the principal WNO of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.*

Proof. We know from Theorem 7.6 that $[\operatorname{ad} x, W] \in \mathfrak{g}_1$ for any $x \in \mathfrak{g}$, i.e. $B(\operatorname{ad} y, [\operatorname{ad} x, W]) = 0$ for any $x, y \in \mathfrak{g}$. By the invariance property of B we have $B(W, [\operatorname{ad} x, \operatorname{ad} y]) = 0$. However, $(\mathfrak{g}, [\,,])$ is semisimple and coincides with its commutant, hence the last equality means that $W \in \mathfrak{g}_1$. ∎

Lemma 7.8. *Let an operator $V \in \operatorname{End}(\mathfrak{g})$ on a semisimple Lie algebra $(\mathfrak{g}, [\,,])$ be symmetric with respect to the Killing form $B_\mathfrak{g}$. Then V is orthogonal to any antisymmetric operator with respect to the trace form B on $\operatorname{End}(\mathfrak{g})$, in particular V is principal.*

Consequently, the operator P defined in Theorem 7.6 is the principal primitive of the principal WNO W.

Proof. Let $A : \mathfrak{g} \to \mathfrak{g}$ be an antisymmetric operator. If $e_1, \ldots, e_n$ is a basis in $\mathfrak{g}$ orthonormal with respect to $B_\mathfrak{g}$, the trace of any operator L can be calculated as $\operatorname{Tr}(L) = \sum_{i=1}^n B_\mathfrak{g}(Le_i, e_i)$. Now, $\operatorname{Tr}(VA) = \sum_{i=1}^n B_\mathfrak{g}(VAe_i, e_i) = -\sum_{i=1}^n B_\mathfrak{g}(e_i, AVe_i) = -\operatorname{Tr}(AV) = -\operatorname{Tr}(VA)$. Hence $\operatorname{Tr}(VA) = 0$. ∎

The following two corollaries of Theorem 7.6 show a significance of the principal WNO of a bi-Lie structure.

Corollary 7.9. *An element $x \in \mathfrak{g}$ belongs to the centre of the Lie bracket $[\,,]'$ if and only if $x \in \ker W$ and $[\operatorname{ad} x, W] = 0$, where W is the principal WNO of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.*

Proof. Indeed, by definition the images of the maps $\widetilde{W}$ and N_1 are mutually orthogonal. Thus $0 = \operatorname{ad}' x = N(\operatorname{ad} x)$ if and only if $\widetilde{W}(\operatorname{ad} x) = 0$, and $N_1(\operatorname{ad} x) = 0$ if and only if $Wx = 0$ and $[\operatorname{ad} x, W] = 0$. ∎

Corollary 7.10. *Let $(\mathfrak{g}, [\,,], [\,,]_1')$ and $(\mathfrak{g}, [\,,], [\,,]_2')$ be two bi-Lie structures with $(\mathfrak{g}, [\,,])$ semisimple and let W_1, W_2 be the corresponding principal WNOs on $\mathfrak{g}$. Then the bi-Lie structures are strongly isomorphic (see Definition 2.6) if and only if there exists an automorphism φ of the Lie algebra $(\mathfrak{g}, [\,,])$ with $\varphi \circ W_1 = W_2 \circ \varphi$.*

Proof. Assume first that φ is an automorphism with the given property. Then direct calculation shows that it transforms the bracket $[\,,]_{W_1}$ to $[\,,]_{W_2}$.

Now assume φ is an automorphism of $(\mathfrak{g}, [\,,])$ with $\varphi[\cdot, \cdot]_{W_1} = [\varphi\,\cdot, \varphi\,\cdot]_{W_2}$.

Note that, given Lie algebras $(\mathfrak{g}, [\,,]_1)$ and $(\mathfrak{g}, [\,,]_2)$, a linear automorphism $\varphi \in \operatorname{End}(\mathfrak{g})$ transforms $[\,,]_1$ to $[\,,]_2$ if and only if $\varphi \circ \operatorname{ad}_1 x \circ \varphi^{-1} = \operatorname{ad}_2 \varphi x$ for any $x \in \mathfrak{g}$, where $\operatorname{ad}_i x(y) := [x, y]_i$.

Thus, by the assumption, $\varphi \circ \operatorname{ad} x \circ \varphi^{-1} = \operatorname{ad} \varphi x$ and

$$\varphi \circ \operatorname{ad}^{W_1} x \circ \varphi^{-1} = \operatorname{ad}^{W_2} \varphi x, \tag{7.10.1}$$

where $\operatorname{ad}^{W_i} x(y) := [x,y]_{W_i}$.

We know that $\operatorname{ad}^{W_2} \varphi x = \operatorname{ad} W_2\varphi x + [\operatorname{ad}\varphi x, W_2]$, where the first term in the right hand side belongs to $\tilde{\mathfrak{g}}$ and the second one to $\tilde{\mathfrak{g}}^{\perp}$.

On the other hand, $\varphi\circ\operatorname{ad}^{W_1} x\circ\varphi^{-1} = \varphi\circ\operatorname{ad} W_1 x\circ\varphi^{-1}+\varphi\circ[\operatorname{ad} x, W_1]\circ\varphi^{-1} = \operatorname{ad}\varphi W_1 x+[\operatorname{ad}\varphi x, \varphi\circ W_1\circ\varphi^{-1}]$. We claim that the first term in the last expression belongs to $\tilde{\mathfrak{g}}$ (which is obvious) and the second one to $\tilde{\mathfrak{g}}^{\perp}$. Indeed, $B(\operatorname{ad} x, \varphi\circ W_1\circ\varphi^{-1}) = \operatorname{Tr}(\operatorname{ad} x\circ\varphi\circ W_1\circ\varphi^{-1}) = \operatorname{Tr}(\varphi^{-1}\circ\operatorname{ad} x\circ\varphi\circ W_1) = B(\operatorname{ad}\varphi^{-1}x, W_1)$. The last expression is equal to zero by Corollary 7.7. Thus $\varphi\circ W_1\circ\varphi^{-1}\in\tilde{\mathfrak{g}}^{\perp}$ and by Lemma 6.1, $[\operatorname{ad}\varphi x, \varphi\circ W_1\circ\varphi^{-1}]\in\tilde{\mathfrak{g}}^{\perp}$.

Hence (7.10.1) implies the equality of 0-components, $\operatorname{ad} W_2\varphi x = \operatorname{ad}\varphi W_1 x$, for any $x\in\mathfrak{g}$, which completes the proof by injectivity of ad. ■

We introduce a definition and summarize the preceding discussion.

Definition 7.11. Let $(\mathfrak{g}, [\,,])$ be a semisimple Lie algebra and let $W, W_1 \in \operatorname{End}(\mathfrak{g})$ be two principal WNOs. We say that they are *strongly equivalent* (resp. *equivalent*) if there exists an automorphism φ of the Lie algebra $(\mathfrak{g}, [\,,])$ such that $\varphi\circ W\circ\varphi^{-1} = W_1$ (resp. $\varphi\circ W\circ\varphi^{-1} = \lambda W_1 + \lambda' \operatorname{Id}_{\mathfrak{g}}$ for some $\lambda, \lambda'\in\mathbb{C}$).

Theorem 7.12. *There is a one-to-one correspondence between semisimple bi-Lie structures* $(\mathfrak{g}, [\,,], [\,,]')$ *and principal WNOs* W *on* $(\mathfrak{g}, [\,,])$ *given by* $[\,,]' = [\,,]_W$. *Two semisimple bi-Lie structures* $(\mathfrak{g}, [\,,], [\,,]')$ *and* $(\mathfrak{g}, [\,,], [\,,]_1')$ *with the same underlying Lie algebra are (strongly) isomorphic (see Definition 2.6) if and only if the corresponding principal WNOs* W, W_1 *are (strongly) equivalent.*

We conclude this section by an example of a principal WNO.

Example 7.13. Consider the operator W from Example 4.12. We claim that it is the principal WNO for the bi-Lie structure $(\mathfrak{so}(n,\mathbb{C}), [\,,], [\,,_A])$. Indeed, $\operatorname{Tr}(W(X)Y) = (1/2)\operatorname{Tr}((AX+XA)Y) = (1/2)\operatorname{Tr}(X(AY+YA)) = \operatorname{Tr}(XW(Y))$, i.e. W is symmetric with respect to the Killing form $B_{\mathfrak{so}(n,\mathbb{C})}$ which is proportional to the trace form. It remains to use Lemma 7.8.

8. Kernels of Killing forms of exceptional Lie algebras and their centres

In this section we assume that $(\mathfrak{g}, [\,,], [\,,]')$ is a semisimple bi-Lie structure and $W:\mathfrak{g}\to\mathfrak{g}$ is its principal WNO. As usual, we will write B for the trace form on $\mathcal{G} := \operatorname{End}(\mathfrak{g})$ and $B_{\mathfrak{g}}$ for the Killing form of the Lie algebra $(\mathfrak{g}, [\,,])$.

Consider the bracket $[\,,]^t := [\,,]' - t[\,,] = [\,,]_{W^t}$, $t\in\mathbb{C}$, where $W^t := W - t\operatorname{Id}_{\mathfrak{g}}$. Let $\operatorname{ad}^t:\mathfrak{g}\to\mathfrak{g}$ stand for the corresponding ad-operator, $\operatorname{ad}^t x(y) = [x,y]^t$. It can also be written in the form $\operatorname{ad}^t x = \operatorname{ad} W^t x + [\operatorname{ad} x, W]$ (since $[\operatorname{ad} x, W^t] = [\operatorname{ad} x, W]$ for any $x\in\mathfrak{g}$, cf. the proof of Theorem 7.6).

Theorem 8.1. *The Killing form B^t of the Lie algebra $(\mathfrak{g}, [\,,]^t)$ is given by*

$$B^t(x,y) = B_{\mathfrak{g}}(W^t x, W^t y) + b(x,y), \quad x, y \in \mathfrak{g},$$

where b is a symmetric form on $\mathfrak{g}$ defined by

$$b(x,y) = B([\operatorname{ad} x, W], [\operatorname{ad} y, W]) = -2B_{\mathfrak{g}}(Px, y);$$

here P is the principal primitive of the principal WNO W (see Definition 7.3).

Proof. By definition $B^t(x,y) = B(\operatorname{ad}^t x, \operatorname{ad}^t y) = B(\operatorname{ad} W^t x + [\operatorname{ad} x, W], \operatorname{ad} W^t y + [\operatorname{ad} x, W]) = B(\operatorname{ad} W^t x, \operatorname{ad} W^t y) + B([\operatorname{ad} x, W], [\operatorname{ad} y, W]) = B_{\mathfrak{g}}(W^t x, W^t y) + b(x,y)$. Here we used the fact that the endomorphisms $\operatorname{ad} W^t z$, $z \in \mathfrak{g}$, and $[\operatorname{ad} z, W]$ are mutually orthogonal with respect to B (see Corollary 7.7 and Lemma 6.1). The equality $b(x,y) = -2B_{\mathfrak{g}}(Px, y)$ follows from the characterization of P given by Lemma 7.8 and from the invariance of the form B. ■

The theorem above implies (by nondegeneracy of $B_{\mathfrak{g}}$) that $x \in \mathfrak{g}$ belongs to $\ker B^t$ for some $t \in \mathbb{C}$ if and only if it belongs to the kernel of the operator $\hat{B}^t := (W^t)^* W^t - 2P \in \operatorname{End}(\mathfrak{g})$, or in more detail,

$$\hat{B}^t = W^* W - 2P - t(W + W^*) + t^2 \operatorname{Id}_{\mathfrak{g}}; \tag{8.1.1}$$

here L^* stands for the operator adjoint to L with respect to $B_{\mathfrak{g}}$. In particular, the set $T = \{t \in \mathbb{C} \mid \ker B^t \neq \{0\}\}$ is finite.

Definition 8.2. The elements of the set $T = \{t_1, t_2, \ldots\}$ will be called the *times* of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$, and the corresponding Lie algebras $(\mathfrak{g}, [\,,]^{t_i})$ will be called *exceptional*.

The following result describes the centres of the exceptional Lie algebras.

Theorem 8.3. (i) *An element $x \in \mathfrak{g}$ belongs to the centre $\mathfrak{z}^t$ of the Lie algebra $[\,,]^t := [\,,]' - t[\,,] = [\,,]_{W^t}$ if and only if x is an eigenvector of the principal WNO W corresponding to the eigenvalue t and $[\operatorname{ad} x, W] = 0$;*

(ii) *The subset $\mathfrak{z}^t$ is a subalgebra in $(\mathfrak{g}, [\,,])$ for any t;*

(iii) *$\mathfrak{z}^{t_1} \cap \mathfrak{z}^{t_2} = \{0\}$ if $t_1 \neq t_2$;*

(iv) *$[\mathfrak{z}^{t_1}, \ker W^{t_2}] \subset \ker W^{t_2}$ for any $t_1, t_2 \in \mathbb{C}$;*

(v) *$[\mathfrak{z}^{t_1}, \mathfrak{z}^{t_2}] = 0$ if $t_1 \neq t_2$; in particular, the set $\mathfrak{z} := \mathfrak{z}^{\theta_1} \oplus \cdots \oplus \mathfrak{z}^{\theta_m}$ is a subalgebra in $(\mathfrak{g}, [\,,])$ which is a direct sum of its ideals $\mathfrak{z}^{\theta_i}$; here $\Theta := \{\theta_1, \ldots, \theta_m\} = \{\theta \in \mathbb{C} \mid \mathfrak{z}^\theta \neq \{0\}\} \subset T$.*

Proof. (i) is a direct consequence of Corollary 7.9 and the fact that $[\operatorname{ad} x, W^t] = [\operatorname{ad} x, W]$.

(ii) For any $x, y \in \mathfrak{z}^\theta \subset \ker W^\theta$ we have $0 = [x,y]_{W^\theta} = -W^\theta [x,y]$, hence $[x,y] \in \ker W^\theta$. On the other hand, since $[\operatorname{ad} x, W] = 0 = [\operatorname{ad} y, W]$, then

$[\operatorname{ad}[x,y],W] = [[\operatorname{ad}x,\operatorname{ad}y],W] = -[[W,\operatorname{ad}x],\operatorname{ad}y] - [[\operatorname{ad}y,W],\operatorname{ad}x] = 0$. Thus by Item (i) $[x,y] \in \mathfrak{z}^\theta$.

(iii) follows from the inclusion $\mathfrak{z}^t \subset \ker W^t$.

(iv) Let $x \in \mathfrak{z}^{t_1}, y \in \ker W^{t_2}$. Then $0 = [x,y]_{W^{t_1}} = [x, W^{t_1}y] - W^{t_1}[x,y] = [x, W^{t_2}y] + (t_2 - t_1)[x,y] - W^{t_1}[x,y] = (t_2-t_1)[x,y] - W^{t_1}[x,y] = t_2[x,y] - W[x,y]$. Hence $[x,y] \in \ker W^{t_2}$.

(v) We have $[\mathfrak{z}^{t_1},\mathfrak{z}^{t_2}] \subset [\mathfrak{z}^{t_1}, \ker W^{t_2}] \subset \ker W^{t_2}$ by (iv) and, analogously, $[\mathfrak{z}^{t_1},\mathfrak{z}^{t_2}] \subset [\ker W^{t_1}, \mathfrak{z}^{t_2}] \subset \ker W^{t_1}$. Hence $[\mathfrak{z}^{t_1},\mathfrak{z}^{t_2}] \subset \ker W^{t_1} \cap \ker W^{t_2} = \{0\}$. The inclusion $\Theta \subset T$ is obvious, since the centre of any Lie algebra is contained in the kernel of the Killing form. ■

Part (ii) was known [TF95, §44].

Definition 8.4. The subalgebra $\mathfrak{z} \subset \mathfrak{g}$ equal to the sum of the centres of the exceptional Lie algebras will be called the *central subalgebra* of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.

We conclude this section by introducing a series of subalgebras of $(\mathfrak{g}, [\,,])$ invariantly defined by the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$.

Theorem 8.5. *Let $(\mathfrak{g}, [\,,], [\,,]')$ be a semisimple bi-Lie structure, $W : \mathfrak{g} \to \mathfrak{g}$ its principal WNO, and P its pricipal primitive.*

(i) *The sets $\mathfrak{g}_P := \{x \in \mathfrak{g} \mid [\operatorname{ad}x, P] = 0\}$ and $\hat{\mathfrak{z}} := \mathfrak{g}_P \cap \ker P$ are subalgebras in $(\mathfrak{g}, [\,,])$ and $\mathfrak{z} \subset \hat{\mathfrak{z}} \subset \mathfrak{g}_P$.*

(ii) *If W is diagonalizable, then $\hat{\mathfrak{z}}^t := \hat{\mathfrak{z}} \cap \ker W^t$ is also a subalgebra for any $t \in \mathbb{C}$, $\mathfrak{z}^t \subset \hat{\mathfrak{z}}^t$ and $\hat{\mathfrak{z}}^{t_1} \cap \hat{\mathfrak{z}}^{t_2} = \{0\}$ if $t_1 \neq t_2$. If, moreover, $W\hat{\mathfrak{z}} \subset \hat{\mathfrak{z}}$, then $\hat{\mathfrak{z}} = \hat{\mathfrak{z}}^{t_1} \oplus \cdots \oplus \hat{\mathfrak{z}}^{t_k}$ for some $t_i \in T$ and $\hat{\mathfrak{z}}^{t_i} + \hat{\mathfrak{z}}^{t_j}$ is a subalgebra for any i, j.*

Proof. (i) By the Jacobi identity $[\operatorname{ad}[x,y], P] = [[\operatorname{ad}x, P], \operatorname{ad}y] - [[\operatorname{ad}y, P], \operatorname{ad}x]$. Thus $x, y \in \mathfrak{g}_P$ implies $[x,y] \in \mathfrak{g}_P$. If $x, y \in \hat{\mathfrak{z}}$, then $P[x,y] = P\operatorname{ad}x(y) = \operatorname{ad}x(Py) = 0$ and $[x,y] \in \hat{\mathfrak{z}}$.

To prove the inclusion $\mathfrak{z} \subset \hat{\mathfrak{z}}$ we rewrite the equality $T_W(x,y) = [x,y]_P$ in a different way. Namely, fix x and consider the left and right hand sides of this equality as operators in the second variable. Then we get $T_W(x,\cdot) = [Wx, W\cdot] - W([x,\cdot]_W) = (\operatorname{ad}Wx \circ W - W \circ (\operatorname{ad}Wx + [\operatorname{ad}x, W]))(\cdot) = ([\operatorname{ad}Wx, W] - W \circ [\operatorname{ad}x, W])(\cdot)$. Analogously, $[x,\cdot]_P = (\operatorname{ad}Px + [\operatorname{ad}x, P])(\cdot)$. Finally,

$$[\operatorname{ad}Wx, W] - W \circ [\operatorname{ad}x, W] = \operatorname{ad}Px + [\operatorname{ad}x, P]. \qquad (8.5.1)$$

Note also that $T_W = T_{W^t}$ for any $t \in \mathbb{C}$, hence $[\operatorname{ad}W^t x, W] - W^t \circ [\operatorname{ad}x, W] = \operatorname{ad}Px + [\operatorname{ad}x, P]$. If $x \in \mathfrak{z}$, then by Theorem 8.3, $x \in \ker W^t$ for some t and $[\operatorname{ad}x, W] = 0$, whence $\operatorname{ad}Px + [\operatorname{ad}x, P] = 0$. Now it is enough to use Lemmata 6.1 and 7.8 to deduce that $\operatorname{ad}Px \in \operatorname{ad}\mathfrak{g}$ and $[\operatorname{ad}x, P] \in (\operatorname{ad}\mathfrak{g})^\perp$ have to be zero, and $x \in \hat{\mathfrak{z}}$. The same argument proves $\mathfrak{z}^t \subset \hat{\mathfrak{z}}^t$.

(ii) Let $x, y \in \hat{\mathfrak{z}}^t$. Then $T_W(x,y) = t^2[x,y] - W(2t[x,y] - W[x,y]) = (W^t)^2[x,y]$ (since $x, y \in \ker W^t$). On the other hand, $T_W(x,y) = \operatorname{ad} Px(y) + [\operatorname{ad} x, P]y = 0$ (since $x \in \hat{\mathfrak{z}}$). The diagonalizability of W implies $[x,y] \in \ker W^t$.

The equality $\hat{\mathfrak{z}}^{t_1} \cap \hat{\mathfrak{z}}^{t_2} = \{0\}$ follows from $\ker W^{t_1} \cap \ker W^{t_2} = \{0\}$.

If $\hat{\mathfrak{z}}$ is W-invariant, we have a direct decomposition $\hat{\mathfrak{z}} = \hat{\mathfrak{z}}^{t_1} \oplus \cdots \oplus \hat{\mathfrak{z}}^{t_k}$. To prove the last assertion it is enough to observe that by (8.5.1), $W|_{\hat{\mathfrak{z}}}$ has zero torsion, i.e. is Nijenhuis. The desired property is a feature of any diagonalizable Nijenhuis operator [GS00], [Pan06]. ∎

9. Operators preserving a grading

We will consider Γ-*gradings* of a Lie algebra $\mathfrak{g}$, i.e. direct decompositions $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for any $i, j \in \Gamma$, where Γ is an abelian group and we use the additive notation.

Definition 9.1. We say that a linear operator $W \in \operatorname{End}(\mathfrak{g})$ *preserves a* Γ-*grading* $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ if $W\mathfrak{g}_i \subset \mathfrak{g}_i$ for any $i \in \Gamma$.

The first two items of the following theorem are standard.

Theorem 9.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Killing form $B_{\mathfrak{g}}$ and let $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ be a Γ-grading.*

(i) $B_{\mathfrak{g}}(\mathfrak{g}_i, \mathfrak{g}_j) = 0$ *if* $i + j \neq 0$.

(ii) *The restriction of $B_{\mathfrak{g}}$ to $\mathfrak{g}_i \times \mathfrak{g}_j$ is a nondegenerate pairing for all $i, j \in \Gamma$ with $i + j = 0$. In particular so is the restriction of $B_{\mathfrak{g}}$ to $\mathfrak{g}_0$.*

(iii) *If $W \in \mathcal{G} = \operatorname{End}(\mathfrak{g})$ preserves the grading, then the operator $\widetilde{P} \in \operatorname{End}(\tilde{\mathfrak{g}})$, $\tilde{\mathfrak{g}} := \operatorname{ad} \mathfrak{g}$, given by $\widetilde{P} := (1/2)p_0 \circ \operatorname{ad}^2_{\mathcal{G}} W$ preserves the induced grading on $\tilde{\mathfrak{g}}$; here $p_0 : \mathcal{G} \to \tilde{\mathfrak{g}}$ is the orthogonal projection onto $\operatorname{ad} \mathfrak{g}$ with respect to the trace form B. Consequently, the operator $P := \operatorname{ad}^{-1} \circ \widetilde{P} \circ \operatorname{ad}$ preserves the initial grading on $\mathfrak{g}$.*

(iv) *If $W \in \operatorname{End}(\mathfrak{g})$ and $W|_{\mathfrak{g}_i} = \lambda_i \operatorname{Id}_{\mathfrak{g}_i}$, $\lambda_i \in \mathbb{C}$, for any $i \in \Gamma$, then the operator W^* adjoint with respect to $B_{\mathfrak{g}}$ is given by $W^*|_{\mathfrak{g}_i} = \lambda_{-i} \operatorname{Id}_{\mathfrak{g}_i}$.*

Proof. (i) Let $x_i \in \mathfrak{g}_i, x_j \in \mathfrak{g}_j$. Consider a basis of $\mathfrak{g}$ which can be divided into parts each of which forms a basis of one of the subspaces $\mathfrak{g}_k$. Let $e^1, \dots, e^m$ be such a part which is a basis of $\mathfrak{g}_k$. Then for $A := \operatorname{ad} x_i \circ \operatorname{ad} x_j$ we have $A(e^l) \notin \mathfrak{g}_k$ as soon as $i + j \neq 0$. Hence the matrix of A in such a basis has zero diagonal entries and $\operatorname{Tr}(A) = B_{\mathfrak{g}}(x_i, x_j) = 0$.

(ii) Now let $x \in \mathfrak{g}_i$ and $i + j = 0$. Then there exists $y \in \mathfrak{g}_j$ such that $B_{\mathfrak{g}}(x, y) \neq 0$. Indeed, otherwise x would be orthogonal to all the space $\mathfrak{g}$ by (i).

(iii) Let $x_{i'} \in \mathfrak{g}_{i'}$ and $x_{i''} \in \mathfrak{g}_{i''}$. Since W preserves the grading, it follows that $\operatorname{ad}^2_{\mathcal{G}} W(\operatorname{ad} x_{i'})(x_{i''}) = (W^2 \circ \operatorname{ad} x_{i'} - 2W \circ \operatorname{ad} x_{i'} \circ W + \operatorname{ad} x_{i'} \circ W^2)(x_{i''}) \in \mathfrak{g}_{i'+i''}$.

By (i) and (ii) we can choose a hyperbolic basis $e_i^1,\dots,e_i^{m_i},e_{-i}^1,\dots,e_{-i}^{m_i}$, $e_i^j\in\mathfrak{g}_i$, in each $\mathfrak{g}_i+\mathfrak{g}_{-i}$ so that $2i\neq 0$. In $\mathfrak{g}_i$ with $2i=0$ we will choose an orthonormal basis $e_i^1,\dots,e_i^{m_i}$. The union of all these bases forms a basis of $\mathfrak{g}$. If $X\in\mathcal{G}$, then $p_0X=\sum_{ij}x_i^j\operatorname{ad}e_i^j$, where $x_i^j=B(\operatorname{ad}e_{-i}^j,X)$. Put $X_{i'}^{j'}=\operatorname{ad}_{\mathfrak{g}}^2W(\operatorname{ad}e_{i'}^{j'})$. Then $\operatorname{ad}e_{-i}^j\circ X_{i'}^{j'}(e_{i''}^{j''})\in\mathfrak{g}_{-i+i'+i''}$. Nonzero diagonal entries of the matrix of $\operatorname{ad}e_{-i}^j\circ X_{i'}^{j'}$ in the basis $\{e_i^j\}$ can appear only if $i'=i$. Thus $x_{ii'}^{jj'}:=B(\operatorname{ad}e_{-i}^j,X_{i'}^{j'})=0$ if $i'\neq i$.

Summarizing, we get $\widetilde{P}(\operatorname{ad}e_{i'}^{j'})=(1/2)\sum_{ij}x_{ii'}^{jj'}\operatorname{ad}e_i^j$, where $x_{ii'}^{jj'}=0$ for $i\neq i'$, which shows the invariance of $\mathfrak{g}_i$ with respect to $\widetilde{P}$.

(iv) It is enough to consider W in the basis built in the proof of (iii). ■

Corollary 9.3. *Let W be a principal WNO of a semisimple bi-Lie structure (see Definition 7.3).*

(i) *Assume that W preserves a Γ-grading on $\mathfrak{g}$. Then its principal primitive P (see Definition 7.3 and Lemma 7.8) also preserves this grading.*

(ii) *If the initial grading is the root decomposition grading with respect to a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$,*

$$\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in R}\mathfrak{g}_\alpha, \tag{9.3.1}$$

(here R stands for the set of roots) and W preserves this grading, then $P|_{\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}}=\pi_\alpha\operatorname{Id}_{\mathfrak{g}_\alpha+\mathfrak{g}_{-\alpha}}$, $\alpha\in R$, for some $\pi_\alpha\in\mathbb{C}$ and $P\mathfrak{h}\subset\mathfrak{h}$.

Theorem 9.4. *Let $W\in\mathcal{G}=\operatorname{End}(\mathfrak{g})$ preserve the root grading (9.3.1), $W|_{\mathfrak{g}_\alpha}=\lambda_\alpha\operatorname{Id}_{\mathfrak{g}_\alpha}$, and $\kappa_\alpha:=(1/2)(\lambda_\alpha-\lambda_{-\alpha})$. Then the operator $\operatorname{pr}(W):=p_1(W)\in\operatorname{End}(\mathfrak{g})$, where $p_1:\mathcal{G}\to\tilde{\mathfrak{g}}^\perp$ is the orthogonal projection onto $\tilde{\mathfrak{g}}^\perp$ along $\tilde{\mathfrak{g}}$, is given by*

$$\operatorname{pr}(W)=W-\operatorname{ad}(\sum_{\alpha\in R}\lambda_\alpha H_\alpha)=W-\operatorname{ad}(2\sum_{\alpha\in R^+}\kappa_\alpha H_\alpha),$$

where R^+ stands for the set of positive roots with respect to any basis and $H_\alpha\in\mathfrak{h}$ is given by $B_{\mathfrak{g}}(H_\alpha,H)=\alpha(H)$, $H\in\mathfrak{h}$. The operator $\operatorname{pr}(W)$ will be called the principal projection *of W. In particular, $\operatorname{pr}(W)=W$ (i.e. W is the principal operator in the sense of Definition 7.3) if and only if $\sum_{\alpha\in R^+}\kappa_\alpha\alpha=0$.*

Proof. We have to prove that $\operatorname{ad}(2\sum_{\alpha\in R^+}\kappa_\alpha H_\alpha)=p_0(W)$, where $p_0:\mathcal{G}\to\tilde{\mathfrak{g}}$ is the orthogonal projection. To this end choose a basis in $\mathfrak{g}$ as in the proof of Theorem 9.2(iii), i.e. an orthonormal basis $e_1,\dots,e_n$ in $\mathfrak{h}$ and an element $E_\alpha\in\mathfrak{g}_\alpha$ for any $\alpha\in R$ such that $B_{\mathfrak{g}}(E_\alpha,E_{-\alpha})=1$. Moreover, let $f_1,\dots,f_k$ be any basis in $\tilde{\mathfrak{g}}^\perp$. Then we have to calculate the coefficients a_i,b_α of the decomposition $W=\sum_i a_i\operatorname{ad}e_i+\sum_\alpha b_\alpha\operatorname{ad}E_\alpha+\sum_j c_jf_j$. Since $\operatorname{ad}e_i,\operatorname{ad}E_\alpha,f_j$ are mutually orthogonal, we have $a_i=B(\operatorname{ad}e_i,W),b_\alpha=B(\operatorname{ad}E_{-\alpha},W)$. The operator $\operatorname{ad}E_\alpha$ does not preserve any of the subspaces $\mathfrak{h},\mathfrak{g}_\alpha$, hence $b_\alpha=\operatorname{Tr}(\operatorname{ad}E_{-\alpha}\circ W)=0$.

To calculate a_i, note that $\operatorname{ad} e_i|_{\mathfrak{h}} = 0, \operatorname{ad} e_i|_{\mathfrak{g}_\alpha} = \alpha(e_i)\operatorname{Id}_{\mathfrak{g}_\alpha}$, whence $a_i = \operatorname{Tr}(\operatorname{ad} e_i \circ W) = \sum_{\alpha\in R}\alpha(e_i)\lambda_\alpha$. Thus $p_0(W) = \sum_i a_i \operatorname{ad} e_i = \sum_i \sum_{\alpha\in R}\alpha(e_i)\lambda_\alpha \operatorname{ad} e_i = \operatorname{ad}\sum_{\alpha\in R}\lambda_\alpha \sum_i B_{\mathfrak{g}}(H_\alpha, e_i)e_i = \operatorname{ad}\sum_{\alpha\in R}\lambda_\alpha H_\alpha = 2\operatorname{ad}\sum_{\alpha\in R^+}\kappa_\alpha H_\alpha$. The last equality is due to the fact that $H_\alpha = -H_{-\alpha}$. ■

10. Regular semisimple bi-Lie structures

Let $(\mathfrak{g}, [\,,], [\,,]')$ be a semisimple bi-Lie structure and $W \in \operatorname{End}(\mathfrak{g})$ its principal WNO (see Definition 7.3). Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of the semisimple Lie algebra $(\mathfrak{g}, [\,,])$ and $R = R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*_{\mathbb{R}}$ the corresponding root system.

Theorem 10.1. *The following two conditions are equivalent:*

(i) *The operator $W \in \operatorname{End}(\mathfrak{g})$ preserves the root grading (9.3.1) and the operator $W|_{\mathfrak{h}}$ is diagonalizable.*

(ii) *$\mathfrak{h} \subset \mathfrak{z}$, where $\mathfrak{z}$ is the central subalgebra (see Definition 8.4).*

If the conditions above are satisfied, the central subalgebra $\mathfrak{z} \subset \mathfrak{g}$ is homogeneous with respect to the decomposition (9.3.1), i.e.

$$\mathfrak{z} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{z} \cap \mathfrak{g}_\alpha.$$

Proof. Assume first that $\mathfrak{h} \subset \mathfrak{z}$. Theorem 8.3(i) implies that $[\operatorname{ad} x, W] = 0$ for any $x \in \mathfrak{h}$. Let $R = \{\alpha_1, \dots, \alpha_k\}$ (with an arbitrary ordering of the nonzero roots). Choose a basis in $\mathfrak{g}$ of the form $e_1, \dots, e_l, E_{\alpha_1}, \dots, E_{\alpha_k}$, $E_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, where $e_1, \dots, e_l$ is any basis in $\mathfrak{h}$. The matrix of $\operatorname{ad} x$ in this basis has the block form

$$\begin{bmatrix} 0 & 0 \\ 0 & D(x) \end{bmatrix},$$

where $D(x) := \operatorname{diag}(\alpha_1(x), \dots, \alpha_k(x))$. For generic $x \in \mathfrak{h}$ the numbers $\alpha_1(x), \dots, \alpha_k(x)$ are pairwise distinct, hence the matrix of W in the same basis is of the form

$$\begin{bmatrix} M & 0 \\ 0 & D \end{bmatrix},$$

where M is an $l \times l$ matrix and D is a diagonal $k \times k$ matrix.

On the other hand, Theorem 8.3(i) implies that $\mathfrak{z}^{\theta_i} \subset \ker W^{\theta_i}$, i.e. $\mathfrak{z} = \mathfrak{z}^{\theta_1} \oplus \cdots \oplus \mathfrak{z}^{\theta_m}$ is a sum of eigenspaces of W containing a W-invariant subspace $\mathfrak{h}$. Hence $\mathfrak{h}$ is also a direct sum of eigenspaces of W.

Conversely, assume (i) holds. Then, obviously, $[\operatorname{ad} x, W] = 0$ for any $x \in \mathfrak{h}$ and $\mathfrak{h}$ is a direct sum of eigenspaces of W, hence by Theorem 8.3, $x \in \mathfrak{z}$.

The proof of the last statement also follows from that theorem. ■

Definition 10.2. A semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ with central subalgebra $\mathfrak{z}$ will be called *regular* if there exists a Cartan subalgebra $\mathfrak{h}$ of the semisimple

Lie algebra $(\mathfrak{g},[\,,])$ such that $\mathfrak{h}\subset\mathfrak{z}$. (This terminology is motivated by the fact that the central subalgebra of a regular bi-Lie structure is a regular reductive Lie subalgebra, see Theorem 10.3.)

Fix a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and assume that the equivalent conditions of Theorem 10.1 are satisfied for a semisimple bi-Lie structure $(\mathfrak{g},[\,,],[\,,]')$ with the principal WNO W.

Choose $E_\alpha\in\mathfrak{g}_\alpha$ for any $\alpha\in R$ such that $B_\mathfrak{g}(E_\alpha,E_{-\alpha})=1$, in particular, $[E_\alpha,E_{-\alpha}]=H_\alpha$, where $B_\mathfrak{g}(H_\alpha,H)=\alpha(H)$ for any $H\in\mathfrak{h}$. Let $\lambda_\alpha\in\mathbb{C}$ be such that $W(E_\alpha)=\lambda_\alpha E_\alpha$. We conclude from Corollary 9.3 that for any $\alpha\in R$ there exists $\pi_\alpha\in\mathbb{C}$ such that $\pi_\alpha=\pi_{-\alpha}$ and $P(E_\alpha)=\pi_\alpha E_\alpha$, where P is the principal primitive of the principal WNO W. Moreover, since by Corollary 7.9 $[\operatorname{ad}x,W]=0$ for any $x\in\mathfrak{z}$, the characterization of P given by Lemma 7.8 implies that $P|_\mathfrak{z}=0$, in particular $P|_\mathfrak{h}=0$.

If W^* stands for the operator adjoint to W with respect to the Killing form $B_\mathfrak{g}$, then $W^*E_\alpha=\lambda_{-\alpha}E_\alpha$ for any $\alpha\in R$ by Theorem 9.2(iv). Put $\sigma_\alpha:=(1/2)(\lambda_\alpha+\lambda_{-\alpha})$ and $\kappa_\alpha:=(1/2)(\lambda_\alpha-\lambda_{-\alpha})$.

Theorem 10.3. *Retain the assumptions and notations above. Then*

(i) *For any $\alpha\in R$ the set of times $\{t\in T\mid E_\alpha\in\ker B^t\}$ (see Definition 8.2) coincides with the set $T_\alpha=\{t_{1,\alpha},t_{2,\alpha}\}$ of solutions of the quadratic equation*

$$(t-\lambda_\alpha)(t-\lambda_{-\alpha})-2\pi_\alpha=0. \tag{10.3.1}$$

Moreover, $T_\alpha=T_{-\alpha}$.

(ii) *For any $\alpha\in R$,*

$$\sigma_\alpha=(1/2)(t_{1,\alpha}+t_{2,\alpha}),\kappa_\alpha\in\sqrt{\zeta_\alpha-2\pi_\alpha},$$

where $\zeta_\alpha:=((t_{1,\alpha}-t_{2,\alpha})/2)^2$ and $\sqrt{a}$ stands for the set of square roots of a.

(iii) *For any $\alpha\in R$,*

$$W^{t_{1,\alpha}}W^{t_{2,\alpha}}H_\alpha=0$$

(recall that $W^t:=W-t\operatorname{Id}_\mathfrak{g}$). Consequently, H_α is either an eigenvector of W corresponding to the eigenvalue $t_{1,\alpha}$, or to $t_{2,\alpha}$, or a sum of such eigenvectors.

(iv) *The centre $\mathfrak{z}^{\theta_i}$ (see Section 8) of the exceptional Lie algebra $[\,,]^{\theta_i}$ is a subalgebra regular with respect to $\mathfrak{h}$ (i.e. stabilized by $\operatorname{ad}\mathfrak{h}$) and reductive in $(\mathfrak{g},[\,,])$, in particular, $E_\alpha\in\mathfrak{z}^{\theta_i}\implies E_{-\alpha}\in\mathfrak{z}^{\theta_i}$. Moreover, $\mathfrak{z}^{\theta_i}=\mathfrak{h}\cap\ker W^{\theta_i}+\bigoplus_{\alpha\in R^{\theta_i}}\mathfrak{g}_\alpha$, where $R^{\theta_i}\subset R$ is the set of roots α such that $\lambda_\alpha=\lambda_{-\alpha}=t_{1,\alpha}=t_{2,\alpha}=\theta_i$ and $\alpha(\mathfrak{z}^{\theta_j}\cap\mathfrak{h})=0$ for any $\theta_j\in\Theta,\theta_j\neq\theta_i$.*

In particular, the central subalgebra $\mathfrak{z}=\mathfrak{z}^{\theta_1}\oplus\cdots\oplus\mathfrak{z}^{\theta_m}$ is regular with respect to $\mathfrak{h}$ and reductive in $(\mathfrak{g},[\,,])$.

(v) *The set $\Theta=\{\theta_1,\dots,\theta_m\}$ of times corresponding to nontrivial centres is equal to the spectrum of the operator $W|_\mathfrak{h}$.*

(vi) *The operator W preserves the irreducible components of the representation $x \mapsto \mathrm{ad}_{\mathfrak{g}} x$ of the Lie algebra $\mathfrak{z}$ in $\mathfrak{z}^{\perp} \subset \mathfrak{g}$, and the restriction of W to any of them is a scalar operator (here $\mathfrak{z}^{\perp}$ is the orthogonal complement to $\mathfrak{z}$ in $\mathfrak{g}$ with respect to the Killing form).*

(vii) *The set T of times of the bi-Lie structure is exhausted by the numbers θ_i and $t_{j,\alpha}$, i.e. $T = \Theta \cup \bigcup_{\alpha \in R} T_\alpha$.*

Proof. (i) Indeed, since $WE_\alpha = \lambda_\alpha E_\alpha, W^* E_\alpha = \lambda_{-\alpha} E_\alpha, PE_\alpha = \pi_\alpha E_\alpha$, we have $\hat{B}^t E_\alpha = [\lambda_{-\alpha}\lambda_\alpha - 2\pi_\alpha - t(\lambda_\alpha + \lambda_{-\alpha}) + t^2]E_\alpha = [(t-\lambda_\alpha)(t-\lambda_{-\alpha}) - 2\pi_\alpha]E_\alpha$ (see (8.1.1)).

(ii) The Viète formulae and (i) give

$$t_{1,\alpha} + t_{2,\alpha} = \lambda_\alpha + \lambda_{-\alpha}, \quad t_{1,\alpha} t_{2\alpha} = \lambda_\alpha \lambda_{-\alpha} - 2\pi_\alpha,$$

which implies the formula for σ_α and the formula $(\lambda_\alpha - \lambda_{-\alpha})^2 = (t_{1,\alpha} - t_{2\alpha})^2 - 8\pi_\alpha$, hence the formula for κ_α.

(iii) Recall the basic equality $T_W(\cdot,\cdot) = [\cdot,\cdot]_P$ relating the WNO W to its primitive P. Calculate this expression on the pair $E_\alpha, E_{-\alpha}$:

$$\lambda_\alpha \lambda_{-\alpha} H_\alpha - W(\lambda_\alpha + \lambda_{-\alpha} - W)H_\alpha = 2\pi_\alpha H_\alpha$$

(we used the fact that $PH_\alpha = 0$). Taking into account the Viète formulae above we get $W(t_{1,\alpha} + t_{2\alpha} - W)H_\alpha = t_{1,\alpha} t_{2\alpha} H_\alpha$, or finally $(W - t_{1,\alpha})(W - t_{2\alpha})H_\alpha = 0$.

(iv) Recall that by Theorem 8.3, $E_\alpha \in \mathfrak{z}$ if and only if $[\mathrm{ad}\, E_\alpha, W] = 0$. To show that $\mathfrak{z}$ is reductive in $\mathfrak{g}$, in view of [Bou75, VIII, §3, Prop. 2] it is sufficient to prove that $E_{-\alpha} \in \mathfrak{z}$. Since $WE_{-\alpha} = \lambda_{-\alpha} E_{-\alpha}$, the latter is equivalent to $[\mathrm{ad}\, E_{-\alpha}, W] = 0$ again by Theorem 8.3.

The equality $[\mathrm{ad}\, E_\alpha, W] = 0$ is equivalent to the following list of conditions:

a) $0 = [\mathrm{ad}\, E_\alpha, W]E_\beta = N_{\alpha,\beta}(\lambda_\beta - \lambda_{\alpha+\beta})E_{\alpha+\beta}$ for any $\beta \in R$ such that $\alpha + \beta \in R$;

b) $0 = [\mathrm{ad}\, E_\alpha, W]E_{-\alpha} = (\lambda_{-\alpha} - W)H_\alpha = -W^{\lambda_{-\alpha}} H_\alpha$;

c) $0 = [\mathrm{ad}\, E_\alpha, W]H = (\lambda_\alpha \alpha(H) - \alpha(WH))E_\alpha = -\alpha(W^{\lambda_\alpha} H)E_\alpha$ for any $H \in \mathfrak{h}$;

here $N_{\alpha,\beta} \neq 0$ is given by $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$.

On the other hand, the equality $[\mathrm{ad}\, E_{-\alpha}, W] = 0$ is equivalent to a similar list:

a') $0 = [\mathrm{ad}\, E_{-\alpha}, W]E_{\beta'} = N_{-\alpha,\beta'}(\lambda_{\beta'} - \lambda_{-\alpha+\beta'})E_{-\alpha+\beta'}$ for any $\beta' \in R$ such that $-\alpha + \beta' \in R$;

b') $0 = [\mathrm{ad}\, E_{-\alpha}, W]E_\alpha = W^{\lambda_\alpha} H_\alpha$;

c') $0 = [\mathrm{ad}\, E_{-\alpha}, W]H = \alpha(W^{\lambda_{-\alpha}} H)E_\alpha$ for any $H \in \mathfrak{h}$.

It is clear that a') follows from a) (by putting $\beta := -\alpha + \beta'$). Now we will prove that b), c) imply b'), c').

Indeed, by b) we have $WH_\alpha = \lambda_{-\alpha} H_\alpha$, whence $W^{\lambda_\alpha} H_\alpha = (\lambda_{-\alpha} - \lambda_\alpha) H_\alpha$, from which by c) we deduce that $\lambda_\alpha = \lambda_{-\alpha}$ (since $\alpha(H_\alpha) \neq 0$). Thus b), c) give $W^{\lambda_\alpha} = W^{\lambda_{-\alpha}}$ and b'), c') follow, and we have proven that $\mathfrak{z}$ is reductive in $\mathfrak{g}$.

Now let $E_\alpha \in \mathfrak{z}^{\theta_i}$, i.e. $E_\alpha \in \mathfrak{z} \cap \ker W^{\theta_i}$, in particular, $\theta_i = \lambda_\alpha$. We have shown that $\lambda_\alpha = \lambda_{-\alpha}$ and $E_\alpha, E_{-\alpha}, H_\alpha \in \ker W^{\lambda_\alpha}$. Also, since $\pi_\alpha = 0$, (i) implies that $T_\alpha = \{\lambda_\alpha\}$. The condition $\alpha(\mathfrak{z}^{\theta_j} \cap \mathfrak{h}) = 0$, $\theta_j \neq \theta_i$, follows from c).

(v) The fact that the spectrum of $W|_\mathfrak{h}$ lies in Θ follows from Theorem 8.3(i) and the inclusion $\mathfrak{h} \subset \{x \mid [\operatorname{ad} x, W] = 0\}$. Conversely, any $\mathfrak{z}^{\theta_i}$ is regular with respect to $\mathfrak{h}$ and reductive, hence $\mathfrak{z}^{\theta_i} \cap \mathfrak{h} = \ker W^{\theta_i} \cap \mathfrak{h} \neq \{0\}$.

(vi) Note that any irreducible component of the representation $\operatorname{ad}_\mathfrak{g}|_\mathfrak{z}$ in $\mathfrak{z}^\perp$ is spanned by some root spaces $\mathfrak{g}_\alpha$, hence is preserved by W. Moreover, due to the equality $[\operatorname{ad}_\mathfrak{g} x, W] = 0, x \in \mathfrak{z}$, (see Theorem 8.3) the operator W is intertwining for the $\operatorname{ad}_\mathfrak{g}$ representation of $\mathfrak{z}$. Now the result follows by the Schur lemma (another argument is to use Theorem 8.3(iv), saying that the eigenspaces of W are invariant for $\mathfrak{z}^{\theta_i}$).

(vii) Taking into account the block structure of W, P and the fact that $P|_\mathfrak{h} = 0$ we have $\det \hat{B}^t = \det((W^*)^t W^t - 2P) = \det(W|_\mathfrak{h} - t \operatorname{Id}_\mathfrak{h})^2 \prod_{\alpha \in R}((t - \lambda_\alpha)(t - \lambda_{-\alpha}) - 2\pi_\alpha)$. It remains to use (v). ■

Identify for a moment $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the Killing form. Then each vector H_α will be identified with the root α itself and the root system R induces a reduced root system on $\mathfrak{h}$ which will be denoted also by R. Item (iii) of the theorem above shows that $W|_\mathfrak{h}$ is an R-admissible operator in the following sense.

Definition 10.4. Let V be a vector space over $\mathbb{R}$ and let $R \subset V$ be a reduced root system in V. A diagonalizable linear operator $U : V^\mathbb{C} \to V^\mathbb{C}$ (here $V^\mathbb{C}$ stands for the complexification of the vector space V) will be called *R-admissible* if for any $\alpha \in R \subset V^\mathbb{C}$ either

(i) there exist two eigenvectors $w_{1,\alpha}, w_{2,\alpha} \in V^\mathbb{C}$ corresponding to different eigenvalues $t_{1,\alpha}, t_{2,\alpha}$ of U such that $\alpha = w_{1,\alpha} + w_{2,\alpha}$; or

(ii) α itself is an eigenvector of U corresponding to an eigenvalue t_α.

We say that a root $\alpha \in R$ is *complete* with respect to an R-admissible operator U if condition (i) holds. An R-admissible operator U is called *complete* if any $\alpha \in R$ is complete with respect to U.

Remark 10.5. Note that, since eigenvectors corresponding to different eigenvalues are linearly independent, the eigenvectors $w_{1,\alpha}, w_{2,\alpha}$ if they exist are defined uniquely. Put $V_\alpha := \langle w_{1,\alpha}, w_{2,\alpha} \rangle$ and $U_\alpha := \{t_{1,\alpha}, t_{2,\alpha}\}$ if α is complete, and $V_\alpha := \langle \alpha \rangle$ and $U_\alpha := \{t_\alpha\}$ otherwise (here $\langle \cdot \rangle$ stands for linear span).

We see that to classify regular semisimple bi-Lie structures $(\mathfrak{g},[\,,],[\,,]')$ one needs, in particular, to classify R-admissible operators on $\mathfrak{h}$, where $R=R(\mathfrak{g},\mathfrak{h})$ is the root system of the semisimple Lie algebra $(\mathfrak{g},[\,,])$ transported to $\mathfrak{h}$ by means of the Killing form, up to conjugations by elements of $\{\varphi|_{\mathfrak{h}}\mid\varphi\in\mathrm{Aut}(\mathfrak{g})\}$.

Another piece of data which can be extracted from the principal WNO of a semisimple bi-Lie structure with the help of Theorem 10.3 and which will be important in classification issues is assigning a pair T_α of times to any root $\alpha\in R$. This will be formalized in the following definition (the "times selection rules" will be justified below in Theorem 10.7).

Definition 10.6. Let R be a reduced root system on a vector space V. A collection $\{T_\alpha\}_{\alpha\in R}$ of unordered pairs $T_\alpha=\{t_{1,\alpha}t_{2,\alpha}\}$ of complex numbers is called a *diagram of pairs of times* (or simply a *pair diagram*) if $T_\alpha=T_{-\alpha}$ for any $\alpha\in R$ and for any triple $\alpha,\beta,\gamma\in R$ such that $\alpha+\beta+\gamma=0$ (such triples will be called *triangles*) the pairs $T_\alpha,T_\beta,T_\gamma$ obey the following "times selection rules":

(i) either there exist $t_1,t_2,t_3\in\mathbb{C}$ such that

$$T_\alpha=\{t_1t_2\},\quad T_\beta=\{t_2t_3\},\quad T_\gamma=\{t_3t_1\};$$

(ii) or there exist $t_1,t_2\in\mathbb{C},t_1\neq t_2$, such that

$$T_\alpha=T_\beta=T_\gamma=\{t_1t_2\}.$$

We write (T_α) if we understand T_α as a set. A pair $(\mathcal{T},U)$, where $\mathcal{T}:=\{T_\alpha\}_{\alpha\in R}$ is a pair diagram and $U:V^{\mathbb{C}}\to V^{\mathbb{C}}$ is an R-admissible operator such that $U_\alpha\subset(T_\alpha)$ (see Remark 10.5) for any $\alpha\in R$, will be called an *augmented pair diagram.*

We put $T_{\mathcal{T}}:=\bigcup_{\alpha\in R}(T_\alpha)$ and call $T_{\mathcal{T}}$ the *set of times* of the diagram $\mathcal{T}$.

Let $(\mathcal{T},U)$ be an augmented pair diagram. Note that, if $\alpha\in R$ is complete, the set of eigenvalues U_α coincides with the set of times (T_α). In general, a root α itself can be an eigenvector of U and the corresponding eigenvalue t_α is one of the elements of (T_α). The second element is called *virtual* and cannot be read off from U (cf. Example 12.14).

The next theorem elucidates further properties of regular semisimple bi-Lie structures, in particular, it shows that the set consisting of the pairs T_α from Theorem 10.3 forms a pair diagram (if (10.3.1) has a root t_α of multiplicity 2 we understand T_α as the unordered pair $\{t_\alpha t_\alpha\}$). We will refer to the "times selection rules" of Definition 10.6 as TSR 1 and TSR 2.

Theorem 10.7. *Retain the assumptions and notations of Theorem 10.3. Then*

(i) *Given a triangle $\alpha,\beta,\gamma\in R$, the sets $T_\alpha,T_\beta,T_\gamma$ from Theorem 10.3(i) understood as unordered pairs obey the "times selection rules" of Definition 10.6.*

(ii) *If a triangle* $\alpha, \beta, \gamma \in R$ *is such that TSR 1 (respectively TSR 2) is satisfied, then*

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0,$$

respectively

$$\kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm(t_1 - t_2)/2.$$

(iii) *The set* $\mathfrak{g}_0^t := \mathfrak{h} \cap \ker W^t + \bigoplus_{\alpha \in R_0^t} \mathfrak{g}_\alpha$, *where* $R_0^t \subset R$ *is the set of roots* α *with* $t_{1,\alpha} = t_{2,\alpha} = t$, *forms a subalgebra regular with respect to* $\mathfrak{h}$ *and reductive in* $(\mathfrak{g}, [\,,])$ *such that* $\mathfrak{g}_0^t \supset \mathfrak{z}^t$, *where* $\mathfrak{z}^t$ *is the centre of the corresponding Lie algebra* $[\,,]^t$. *The subalgebra* $\mathfrak{g}_0^t$ *is nontrivial if and only if so is* $\mathfrak{z}^t$.

(iv) *(In items (iv)–(vii) assume additionally that* R *is irreducible, i.e.* $(\mathfrak{g}, [\,,])$ *is simple.) For any* $\alpha \in R_0^t$ *we have* $\kappa_\alpha = 0$ *and* $\pi_\alpha = 0$.

(v) *For any* $t \in \mathbb{C}$ *we have* $\mathfrak{g}_0^t \subset \ker W^t$. *In particular,* $\mathfrak{g}_0^{t_1} \cap \mathfrak{g}_0^{t_2} = \{0\}$ *for* $t_1 \neq t_2$.

(vi) *The set* $\mathfrak{g}_0 = \mathfrak{g}_0^{\theta_1} \oplus \cdots \oplus \mathfrak{g}_0^{\theta_m}$ *(here* $\{\theta_1, \ldots, \theta_m\} = \Theta$, *see Theorems 8.3, 10.3) is a subalgebra reductive in* $(\mathfrak{g}, [\,,])$ *such that* $\mathfrak{g}_0 \supset \mathfrak{z}$. *The subspace* $\mathfrak{g}_0^{\theta_i} \oplus \mathfrak{g}_0^{\theta_j}$ *is a subalgebra in* $(\mathfrak{g}, [\,,])$ *for any* i, j. *The subalgebra* $\mathfrak{g}_0$ *will be called the* basic subalgebra *of a regular semisimple bi-Lie structure.*

(vii) *The antisymmetric part* W_{a} *of the operator* $W|_{\mathfrak{h}^\perp}$ *preserves the irreducible components of the representation* $x \mapsto \mathrm{ad}_{\mathfrak{g}}\, x$ *of the Lie algebra* $\mathfrak{g}_0$ *in* $\mathfrak{g}_0^\perp \subset \mathfrak{g}$, *and the restriction of* W_{a} *to any of them is a scalar operator; here the orthogonal complement is taken with respect to the Killing form.*

Remark 10.8. It can be shown that $\mathfrak{g}_0 \subset \hat{\mathfrak{z}}$, where $\hat{\mathfrak{z}}$ is the subalgebra from Theorem 8.5. We conjecture that in fact $\mathfrak{g}_0 = \hat{\mathfrak{z}}$.

Remark 10.9. The assumption that $(\mathfrak{g}, [\,,])$ is simple in (iv)–(vii) reduces the complexity of formulations. It can be replaced by the less restrictive assumption that $(\mathfrak{g}, [\,,])$ is semisimple and the subalgebras $\mathfrak{g}_0^t$ do not contain any of its simple components.

The proof of the theorem will use the following lemma.

Lemma 10.10. *Retain the assumptions of item (i) of the theorem. Let* $T_\alpha := \{t_1 t_2\}$, $T_\beta := \{t_3 t_4\}$, $T_\gamma := \{t_5 t_6\}$. *Then the following equalities hold:*

$$(t_5 + t_6)(t_1 + t_2 - t_3 - t_4) + (t_3^2 + t_4^2 - t_1^2 - t_2^2) = 0,$$
$$(t_1 + t_2)(t_3 + t_4 - t_5 - t_6) + (t_5^2 + t_6^2 - t_3^2 - t_4^2) = 0,$$
$$(t_3 + t_4)(t_5 + t_6 - t_1 - t_2) + (t_1^2 + t_2^2 - t_5^2 - t_6^2) = 0.$$

Proof. The proof will be deduced from the basic equality $T_W(\cdot,\cdot) = [\cdot,\cdot]_P$. Taking E_α, E_β as arguments, we get $\lambda_\alpha\lambda_\beta - \lambda_{-\gamma}(\lambda_\alpha + \lambda_\beta - \lambda_{-\gamma}) = \pi_\alpha + \pi_\beta - \pi_{-\gamma}$, hence $(\lambda_{-\gamma} - \lambda_\alpha)(\lambda_{-\gamma} - \lambda_\beta) = \pi_\alpha + \pi_\beta - \pi_{-\gamma}$. Substituting here $-\alpha, -\beta, -\gamma$ instead of α, β, γ respectively and recalling that $\pi_\alpha = \pi_{-\alpha}$ for any $\alpha \in R$ we obtain

$$(\lambda_{-\gamma} - \lambda_\alpha)(\lambda_{-\gamma} - \lambda_\beta) = (\lambda_\gamma - \lambda_{-\alpha})(\lambda_\gamma - \lambda_{-\beta}).$$

Recalling that $\lambda_{\pm\alpha} = \sigma_\alpha \pm \kappa_\alpha$ (and $\sigma_{-\alpha} = \sigma_\alpha, \kappa_{-\alpha} = -\kappa_\alpha$) we get

$$\begin{aligned}((\sigma_\gamma - \sigma_\alpha) - (\kappa_\gamma + \kappa_\alpha))&((\sigma_\gamma - \sigma_\beta) - (\kappa_\gamma + \kappa_\beta))\\ &= ((\sigma_\gamma - \sigma_\alpha) + (\kappa_\gamma + \kappa_\alpha))((\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\beta)).\end{aligned}$$

Put $A := \sigma_\gamma - \sigma_\alpha, B := \kappa_\gamma + \kappa_\alpha, C := \sigma_\gamma - \sigma_\beta, D := \kappa_\gamma + \kappa_\beta$. Then the above equality can be rewritten as $(A - B)(C - D) = (A + B)(C + D)$, whence $AD + BC = 0$ and $(A - B)(C - D) = AC + BD$. Thus

$$(\sigma_\gamma - \sigma_\alpha)(\kappa_\gamma + \kappa_\beta) + (\kappa_\gamma + \kappa_\alpha)(\sigma_\gamma - \sigma_\beta) = 0, \tag{10.10.1}$$

$$(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\alpha)(\kappa_\gamma + \kappa_\beta) = \pi_\alpha + \pi_\beta - \pi_\gamma. \tag{10.10.2}$$

Now we note that Theorem 10.3(ii) implies $\pi_\alpha = (1/2)(\zeta_\alpha - \kappa_\alpha^2)$, in view of which equality (10.10.2) gives

$$(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + (\kappa_\gamma + \kappa_\alpha)(\kappa_\gamma + \kappa_\beta) = \frac{1}{2}(\zeta_\alpha + \zeta_\beta - \zeta_\gamma - (\kappa_\alpha^2 + \kappa_\beta^2 - \kappa_\gamma^2)),$$

or, equivalently,

$$(\sigma_\gamma - \sigma_\alpha)(\sigma_\gamma - \sigma_\beta) + \frac{1}{2}(\kappa_\alpha + \kappa_\beta + \kappa_\gamma)^2 = \frac{1}{2}(\zeta_\alpha + \zeta_\beta - \zeta_\gamma). \tag{10.10.3}$$

Cyclic permutations of α, β, γ give

$$(\sigma_\beta - \sigma_\gamma)(\sigma_\beta - \sigma_\alpha) + \frac{1}{2}\kappa = \frac{1}{2}(\zeta_\gamma + \zeta_\alpha - \zeta_\beta) \tag{10.10.4}$$

$$(\sigma_\alpha - \sigma_\beta)(\sigma_\alpha - \sigma_\gamma) + \frac{1}{2}\kappa = \frac{1}{2}(\zeta_\beta + \zeta_\gamma - \zeta_\alpha), \tag{10.10.5}$$

where we put $\kappa := (\kappa_\alpha + \kappa_\beta + \kappa_\gamma)^2$.

Adding respectively (10.10.3) and (10.10.4), (10.10.3) and (10.10.5), and (10.10.4) and (10.10.5), we get

$$\begin{aligned}(\sigma_\beta - \sigma_\gamma)^2 + \kappa &= \zeta_\alpha,\\ (\sigma_\gamma - \sigma_\alpha)^2 + \kappa &= \zeta_\beta,\\ (\sigma_\alpha - \sigma_\beta)^2 + \kappa &= \zeta_\gamma.\end{aligned} \tag{10.10.5}$$

This yields $(\sigma_\beta - \sigma_\gamma)^2 - \zeta_\alpha = (\sigma_\gamma - \sigma_\alpha)^2 - \zeta_\beta$, or, equivalently (using Theorem 10.3),

$$(t_5 + t_6 - (t_3 + t_4))^2 - (t_1 - t_2)^2 = (t_5 + t_6 - (t_1 + t_2))^2 - (t_3 - t_4)^2.$$

Elementary calculations show that this is equivalent to the first equality of the lemma. The remaining equalities are proved analogously. ■

Proof of Theorem 10.7. (i) We have to consider several cases which exhaust all possibilities. Recall that $W|_{\mathfrak{h}}$ is R-admissible in the sense of Definition 10.4 and we use the notation from Remark 10.5. Recall that we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ (in particular, H_α with α) by means of the Killing form $B_{\mathfrak{g}}$.

Case (a): α, β, γ are complete. Assume first that $V_\alpha \cap V_\beta = \{0\}$. Then, necessarily, $T_\alpha = T_\beta = \{t_1 t_2\}$, where $t_1 \neq t_2$ (otherwise the vectors $w_{1,\alpha}, w_{2,\alpha}, w_{1,\beta}, w_{2,\beta}$ would be linearly independent and γ would depend on more than two eigenvectors), and $T_\gamma = \{t_1 t_2\}$ (TSR 2). The cases $V_\alpha \cap V_\gamma = \{0\}$ and $V_\gamma \cap V_\beta = \{0\}$ are analogous.

Now assume that $V_\alpha \cap V_\beta \neq \{0\}$, $V_\alpha \cap V_\gamma \neq \{0\}$, $V_\gamma \cap V_\beta \neq \{0\}$. Then either $V_\alpha = V_\beta = V_\gamma = \langle w_1, w_2 \rangle$, where w_i is an eigenvector corresponding to an eigenvalue t_i, $i = 1, 2$, and $T_\alpha = T_\beta = T_\gamma = \{t_1 t_2\}$, $t_1 \neq t_2$ (TSR 2), or $\alpha = w_1 - w_2$, $\beta = w_2 - w_3$, $\gamma = w_3 - w_1$, where w_i is an eigenvector corresponding to an eigenvalue t_i, $i = 1, 2, 3$, and $T_\alpha = \{t_1 t_2\}$, $T_\beta = \{t_2 t_3\}$, $T_\gamma = \{t_3 t_1\}$ with $t_1 \neq t_2$, $t_2 \neq t_3$, $t_3 \neq t_1$ (TSR 1).

Case (b): one of the roots, say γ, is not complete, but the other two are complete. Then $V_\alpha = V_\beta = \langle w_1, w_2 \rangle$, where w_i is an eigenvector corresponding to an eigenvalue t_i, $i = 1, 2$, $t_1 \neq t_2$, and $\alpha = w_1 + w_2$, $\beta = -w_1 + a w_2$ for some $a \neq -1$, i.e. $V_\gamma = \langle w_2 \rangle$, $T_\alpha = T_\beta = \{t_1 t_2\}$, $T_\gamma = \{t_2 (t_6)\}$ for some $t_6 \in \mathbb{C}$ (here the parentheses ( ) mean that the enclosed element is virtual, see discussion after Definition 10.6). A priori t_6 can be arbitrary, but in fact it is not so. Indeed, the substitution $t_3 = t_1$, $t_4 = t_2$, $t_5 = t_2$ reduces the equalities of Lemma 10.10 to one equality $(t_6 - t_1)(t_2 - t_6) = 0$. Hence $t_6 = t_1$ (TSR 2) or $t_6 = t_2$ (TSR 1).

Case (c): two of the roots, say α, β, are not complete, but the remaining one is complete. Then $V_\alpha = \langle \alpha \rangle$, $V_\beta = \langle \beta \rangle$, $V_\gamma = \langle \alpha, \beta \rangle$, the roots α, β are the eigenvectors corresponding to eigenvalues t_1, t_3 respectively with $t_1 \neq t_3$. Moreover $T_\alpha = \{t_1 (t_2)\}, T_\beta = \{t_3 (t_4)\}, T_\gamma = \{t_1 t_3\}$ for some $t_2, t_4 \in \mathbb{C}$. The second and third equalities of the lemma give $(t_4 - t_1)(t_2 - t_4) = 0$ and $(t_3 - t_2)(t_4 - t_2) = 0$ respectively. Thus either $t_2 = t_4$ (TSR 1), or $t_2 \neq t_4$, but $t_4 = t_1$ and $t_3 = t_2$ (TSR 2).

Case (d): none of the three roots α, β, γ is complete. Then $V_\alpha = \langle \alpha \rangle$, $V_\beta = \langle \beta \rangle$, $V_\gamma = \langle \gamma \rangle$ and the roots α, β, γ are eigenvectors corresponding to the same eigenvalue t_1. Then $T_\alpha = \{t_1 (t_2)\}$, $T_\beta = \{t_1 (t_4)\}$, $T_\gamma = \{t_1 (t_6)\}$. The first two equalities of the lemma give:

$$(t_4 - t_2)(t_4 + t_2 - t_1 - t_6) = 0,$$
$$(t_6 - t_4)(t_6 + t_4 - t_1 - t_2) = 0.$$

Hence, either $t_4 = t_2$, which implies $(t_6 - t_2)(t_6 - t_1) = 0$ and t_6 equals t_1 (TSR 1) or t_2 (TSR 2 or 1 with all times equal), or $t_4 = -t_2 + t_1 + t_6$, which implies $(t_6 - t_4)(t_6 - t_2) = 0$. In the latter case we have either $t_6 = t_4$, which implies $t_1 = t_2$ (TSR 1), or $t_6 = t_2$, which implies $t_4 = t_1$ (TSR 1).

This finishes the list of possible cases and the proof of the first item of the theorem.

(ii) is proved by direct check using (10.10.5) and the definition of $\sigma_\alpha, \zeta_\alpha$.

(iii) The fact that R_0^t is a closed set of roots follows from the times selection rules (TSR 1). The symmetry of R_0^t (which will imply reductivity) comes from

$T_\alpha = T_{-\alpha}$. The fact that $\mathfrak{g}_0^t$ is a subalgebra follows from the closedness of R_0^t and the condition $H_\alpha = [E_\alpha, E_{-\alpha}] \in \ker W^t, \alpha \in R_0^t$, which is due to Theorem 10.3(iii).

The inclusion $\mathfrak{g}_0^t \supset \mathfrak{z}^t$ follows from Theorem 10.3(iv). Thus the nontriviality of $\mathfrak{z}^t$ implies that of $\mathfrak{g}_0^t$. Finally, if $\mathfrak{g}_0^t$ is nontrivial, then $\mathfrak{h} \cap \ker W^t \neq \{0\}$ and $\mathfrak{z}^t$ is nontrivial by Theorem 10.3(v).

(iv) Assume $R_0^t \neq \emptyset$. We will first prove that

$$\forall \alpha \in R_0^t \quad \exists \beta \in R \setminus R_0^t : \quad \alpha + \beta \in R. \tag{10.10.6}$$

Let $R_{\max}$ be the maximal symmetric closed proper root set containing R_0^t. We will show that

$$\forall \alpha \in R_{\max} \quad \exists \beta, \gamma \in R \setminus R_{\max} : \quad \alpha = \beta + \gamma, \tag{10.10.7}$$

which will prove assertion (10.10.6).

All maximal symmetric closed root sets, which are the root sets of maximal reductive regular subalgebras of simple Lie algebras, are known [GOV94, Ch. 6]. Each of these subalgebras is the fixed point subalgebra $\mathfrak{g}_{\mathrm{fix}}$ of an inner automorphism of order 2, 3 or 5. Obviously, (10.10.7) is equivalent to $[\mathfrak{m}, \mathfrak{m}] \cap \mathfrak{g}_{\mathrm{fix}} = \mathfrak{g}_{\mathrm{fix}}$, where $\mathfrak{m} = (\mathfrak{g}_{\mathrm{fix}})^\perp$ is the orthogonal complement to $\mathfrak{g}_{\mathrm{fix}}$ with respect to the Killing form.

The last assertion follows from the fact that $[\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is an ideal in the initial simple Lie algebra, which can be proved directly using the commutation relations of the corresponding $\mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_5$ gradings.

Now let $\alpha \in R_0^t$ and $\beta \in R \setminus R_0^t$ be such that $-\gamma := \alpha + \beta \in R$. Then by the times selection rules there exists $t' \in \mathbb{C}$, $t' \neq t$, such that $T_\beta = T_\gamma = \{tt'\}$ (recall that $T_\alpha = \{tt\}$). By (10.10.1) we have $((t' - t)/2)(\kappa_\gamma + \kappa_\beta) = 0$, whence $\kappa_\gamma = -\kappa_\beta$. Using (ii), we conclude that $\kappa_\alpha = 0$.

The equality $\pi_\alpha = 0$ now follows from the formula $\kappa_\alpha^2 = \zeta_\alpha - 2\pi_\alpha$ (see Theorem 10.3(ii)).

(v) The previous item and Theorem 10.3(i) imply that $\lambda_\alpha = \lambda_{-\alpha} = t$ for any $\alpha \in R_0^t$, hence $E_\alpha, E_{-\alpha} \in \ker W^t$.

(vi) The proof of the fact that $\mathfrak{g}_0^{\theta_i} \oplus \mathfrak{g}_0^{\theta_j}$ is a subalgebra in $(\mathfrak{g}, [\,,])$ follows from the definition of $\mathfrak{g}_0^t$. The rest follows from (iii) and (v).

(vii) Any irreducible component of the representation $\mathrm{ad}_\mathfrak{g}|_{\mathfrak{g}_0}$ in $\mathfrak{g}_0^\perp$ is spanned by some root spaces $\mathfrak{g}_\alpha$, hence is preserved by W_a. If $E_\alpha \in \mathfrak{g}_0$ and $E_\beta \in \mathfrak{g}_0^\perp$, we have $[\mathrm{ad}\, E_\alpha, W_\mathrm{a}]E_\beta = N_{\alpha,\beta}(\kappa_\beta - \kappa_{\alpha+\beta})E_{\alpha+\beta} = 0$ (cf. the proof of Theorem 10.3(iv)), where the last equality is due to the first equality of (ii) and the fact that $\kappa_\alpha = 0$ (item (iv)). Moreover, $[\mathrm{ad}\, H, W_\mathrm{a}]E_\beta = \beta(H)\kappa_\beta E_\beta - \kappa_\beta \beta(H) E_\beta = 0$ for any $H \in \mathfrak{h}$. Thus the operator W_a is intertwining for the corresponding representation and the Schur lemma can be applied. ∎

Definition 10.11. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra and $R = R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}_\mathbb{R}^*$ the corresponding root system. Let $R_0 \subset R$ be a closed symmetric root set (see Section 13) and $S \in \mathrm{End}(\mathfrak{h}^\perp)$ an antisymmetric operator preserving the root spaces $\mathfrak{g}_\alpha$ of the corresponding root decomposition, $S|_{\mathfrak{g}_\alpha} = \kappa_\alpha \operatorname{Id}_{\mathfrak{g}_\alpha}$, $\kappa_\alpha \in \mathbb{C}$, $\kappa_\alpha = -\kappa_{-\alpha}$ (the last equality follows from Theorem 9.2(iv)). Given a constant $a \in \mathbb{C}$, we say that S obeys the *a-triangle rule with respect to the root set* R_0 if $\kappa_\alpha = 0$ for any $\alpha \in R_0$ and for any triangle $\alpha, \beta, \gamma \in R$ (see Definition 10.6):

(i) $\kappa_\alpha + \kappa_\beta + \kappa_\gamma = 0$ whenever two of the three roots belong to $R \setminus R_0$ and the third one to R_0, or all the three roots belong to R_0;

(ii) $\kappa_\alpha + \kappa_\beta + \kappa_\gamma = \pm a$ whenever $\alpha, \beta, \gamma \in R \setminus R_0$.

We then say that $0, +$ or $-$ is the *label* of the corresponding triangle. The whole family of labels is called the *system of labels* corresponding to the operator S and is denoted by $\mathcal{L}_S$.

Theorem 10.7(ii)&(iv) implies the following result.

Corollary 10.12. *Assume $(\mathfrak{g}, [\,,])$ is simple. Then the antisymmetric part of the operator $W|_{\mathfrak{h}^\perp}$ obeys the $((t_1 - t_2)/2)$-triangle rule with respect to the root set $R_0 \subset R$, where $R_0 := R_0^{\theta_1} \cup \cdots \cup R_0^{\theta_m}$ (see Theorem 10.7(iii)&(vi)).*

In fact due to Theorem 10.7(v)–(vii) we can say more about the antisymmetric part W_{a} of $W|_{\mathfrak{h}^\perp}$.

Definition 10.13. Let a Γ-grading $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$ of a semisimple Lie algebra $\mathfrak{g}$ be given, where Γ is an abelian group. Put $\overline{\Gamma} := \{i \in \Gamma \setminus \{0\} \mid \mathfrak{g}_i \neq \{0\}\}$ and call the elements of this set *quasiroots*. We define a *triangle* to be any triple $i, j, k \in \overline{\Gamma}$ such that $i + j + k = 0$. Given an antisymmetric operator $S \in \operatorname{End}(\mathfrak{g})$ which is scalar on each $\mathfrak{g}_i$, i.e. $S|_{\mathfrak{g}_i} = \kappa_i \operatorname{Id}_{\mathfrak{g}_i}$ for some $\kappa_i \in \mathbb{C}$ (then necessarily $\kappa_i = -\kappa_{-i}$, cf. Theorem 9.2(iv)), and a constant $a \in \mathbb{C}$, we say that S obeys the *a-triangle rule with respect to the grading* if

$$\kappa_i + \kappa_j + \kappa_k = \pm a$$

for any triangle i, j, k. We then say that $+$ or $-$ is the *label* of the corresponding triangle. The whole family of labels is called the *system of labels* corresponding to the operator S and is denoted by $\mathcal{L}_S$. (We use the same notation as in Definition 10.11. It will be clear from context which type of system of labels is used.)

Remark 10.14. Note that if $S \in \operatorname{End}(\mathfrak{h}^\perp)$ satisfies the conditions of Definition 10.11 with $R_0 = \emptyset$, then its extension by zero on $\mathfrak{h}$ will obey the a-triangle rule with respect to the root grading in the sense of Definition 10.13.

Items (ii) and (iv)–(vii) of Theorem 10.7 imply the following result.

Corollary 10.15. *Assume $(\mathfrak{g}, [\,,])$ is simple. The antisymmetric part of the operator $W|_{\mathfrak{h}^\perp}$ extended by zero on $\mathfrak{h}$ obeys the $((t_1 - t_2)/2)$-triangle rule with* *respect to the irreducible toral $\Gamma(\mathfrak{g}_0)$-grading on $(\mathfrak{g}, [\,,])$ (see Definition 15.3) corresponding to the reductive Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ (the basic subalgebra).*

11. Two classes of pair diagrams

The aim of this section is to prove the following theorem.

Theorem 11.1. *Let R be a reduced irreducible root system in a vector space V over $\mathbb{R}$ and let $\mathcal{T} := \{T_\alpha\}_{\alpha\in R}$ be a pair diagram (see Definition 10.6). Assume that there exists a triangle $\alpha, \beta, \gamma \in R$ such that*

$$T_\alpha = T_\beta = T_\gamma = \{t_1 t_2\}$$

for some $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$. Then the set of times $T_{\mathcal{T}}$ is equal to $\{t_1, t_2\}$.

Definition 11.2. Let R be a reduced root system. We say that a pair diagram $\{T_\alpha\}_{\alpha\in R}$ is of *Class II*, if there exist $\alpha, \beta, \gamma \in R$ satisfying the hypotheses of the theorem, and of *Class I*, if such roots do not exist.

We start from auxiliary results.

Let R be a reduced root system. We define on R an equivalence relation by putting $\alpha \sim \alpha$ and $\alpha \sim -\alpha$ for any $\alpha \in R$. Write $\widetilde{R} := (R/\sim) = \{[\alpha] \mid \alpha \in R\}$. Obviously, any pair diagram is in fact indexed by the set $\widetilde{R}$. An unordered triple $\{a, b, c\} \subset \widetilde{R}$ is called a *triangle* if there exist $\alpha \in a, \beta \in b, \gamma \in c$ such that $\alpha + \beta + \gamma = 0$. Given a pair diagram $\{T_a\}_{a\in\widetilde{R}}$, a triangle $\{a, b, c\}$ is said to be a $\{t_1 t_2\}$-*triangle*, if

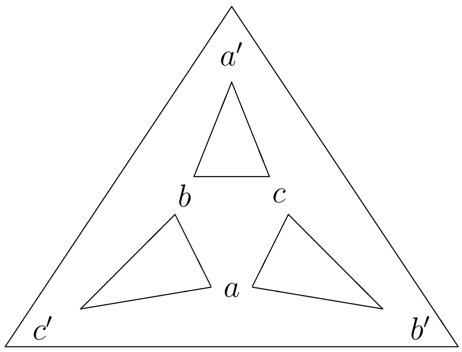
$$T_a = T_b = T_c = \{t_1 t_2\}$$

for some $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$.

Lemma 11.3. *Let R be a reduced irreducible root system in a vector space V and let $\{T_a\}_{a\in\widetilde{R}}$ be a pair diagram. Let $a, b, c, a', b', c' \in \widetilde{R}$ be pairwise distinct elements such that $\{a, b, c'\}, \{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\}$ are triangles (see the figure below). Then, if $\{a, b, c'\}$ is a $\{t_1 t_2\}$-triangle, the pairs of times $T_{a'}, T_{b'}, T_c$ are equal to one of the pairs*

$$\{t_1 t_2\}, \{t_1 t_1\}, \{t_2 t_2\}$$

and moreover at least one of the triangles $\{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\}$ is a $\{t_1 t_2\}$-triangle.



Proof. The times selection rules (TSR for short) applied to the triangle a, c, b' imply, that each of the pairs $T_c, T_{b'}$ should contain at least one of the times t_1, t_2. Thus, a priori, we have the following possibilities:

(i) $T_a = \{t_1t_2\}, T_c = \{t_1t_2\}, T_{b'} = \{t_1t_2\}$;

(ii) $T_a = \{t_1t_2\}, T_c = \{t_2t_1\}, T_{b'} = \{t_1t_1\}$;

(iii) $T_a = \{t_2t_1\}, T_c = \{t_1t_1\}, T_{b'} = \{t_1t_2\}$;

(iv) $T_a = \{t_2t_1\}, T_c = \{t_1t_2\}, T_{b'} = \{t_2t_2\}$;

(v) $T_a = \{t_1t_2\}, T_c = \{t_2t_2\}, T_{b'} = \{t_2t_1\}$;

(vi) $T_a = \{t_1t_2\}, T_c = \{t_2t_3\}, T_{b'} = \{t_3t_1\}$ for some $t_3 \neq t_1, t_2$;

(vii) $T_a = \{t_2t_1\}, T_c = \{t_1t_3\}, T_{b'} = \{t_3t_2\}$ for some $t_3 \neq t_1, t_2$.

However the last two do not occur. Indeed, assume possibility (vi) occurs. Then the TSR for the triangle b, c, a' would imply $T_{a'} = \{t_1t_3\}$ which contradicts the TSR for the triangle c', b', a'. Analogous considerations show impossibility of (vii).

Thus we have proved that pairs of times $T_c, T_{b'}$ are equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$. By the symmetry of the triangles a, c, b' and b, c, a' one comes to the same conclusion about $T_{a'}$.

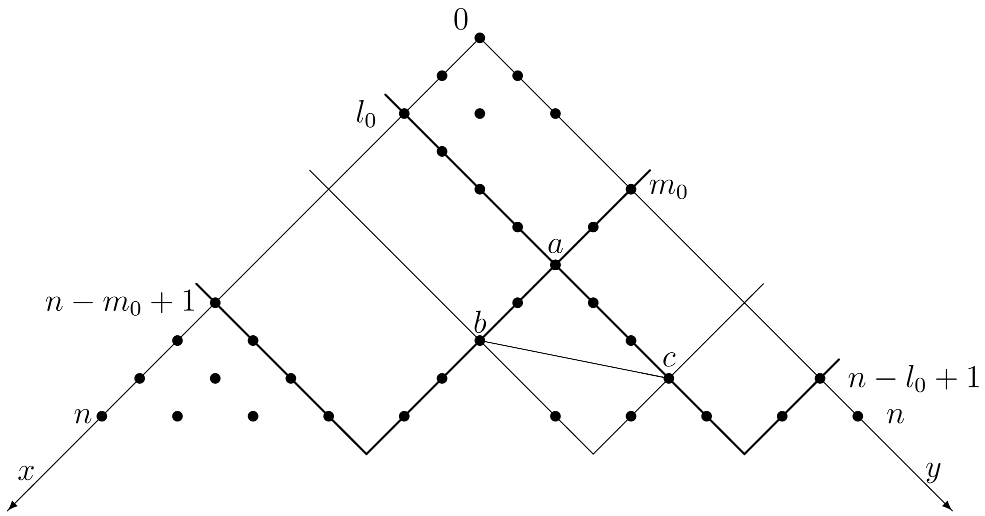
To finish the proof note that the TSR imply that the pairs $\{t_1t_1\}$ or $\{t_2t_2\}$ can appear only once among $T_c, T_{b'}, T_{a'}$ and the rest should be equal to $\{t_1t_2\}$. ■

Theorem 11.4. *Theorem 11.1 holds for the root system $R := \mathfrak{a}_n$.*

Proof. We will give a "geometrical" proof of this theorem. The elements of the set $\widetilde{R}$ will be represented as points of the xy-coordinate plane with integer coordinates (l, m) with $l, m \geq 0, l+m \leq n$. The points $(n, 0), (n-1, 1), \ldots, (0, n)$ represent the elements $[\alpha_1], \ldots, [\alpha_n]$, where $B := \{\alpha_1, \ldots, \alpha_n\}$ is a basis of the root system $\mathfrak{a}_n$. Recall [Bou68, VI, §1, Cor. 3 of Prop. 19] that, B is identified with the set of vertices of the graph of the corresponding Coxeter system (which will be denoted by $\Gamma(R)$ and called the *Coxeter graph* of R) and that for any subset $Y \subset B$ which is connected as a subset in $\Gamma(R)$ we have $\sum_{\beta \in Y} \beta \in R$. On the other hand, for the root system $\mathfrak{a}_n$ all the positive roots are obtained this way. From this we can get the following lemma.

Lemma 11.5. *Let $a = (l_0, m_0) \in \widetilde{\mathfrak{a}}_n$. Then the elements $b \in \widetilde{\mathfrak{a}}_n$ such that there exists a triangle a, b, c for some $c \in \widetilde{\mathfrak{a}}_n$ lie on the "X-shape" $X(a)$, i.e. the union of lines $x = l_0, y = m_0, x = n - m_0 + 1, y = n - l_0 + 1$ intersected with $\widetilde{\mathfrak{a}}_n$ (see the figure below) with the crossing at a.*

Given $a \in \widetilde{\mathfrak{a}}_n$ and $b \in X(a)$, the element $c \in \widetilde{\mathfrak{a}}_n$ such that a, b, c is a triangle is uniquely defined by $c = X(a) \cap X(b)$.



Lemma 11.6. *Let $R \neq \mathfrak{g}_2, \mathcal{T}$ and $\alpha, \beta, \gamma \in R$ satisfy the hypotheses of Theorem 11.1. Then there exists a basis $\alpha_1, \ldots, \alpha_n$ of the root system R and two of its elements α_i, α_j such that $[\alpha_i], [\alpha_j]$ belong to the triangle $\{[\alpha_i], [\alpha_j], [\alpha_i + \alpha_j]\}$ (in particular, α_i, α_j form a connected subgraph of the Coxeter graph of R) and $T_{[\alpha_i]} = T_{[\alpha_j]} = T_{[\alpha_i+\alpha_j]} = \{t_1 t_2\}$.*

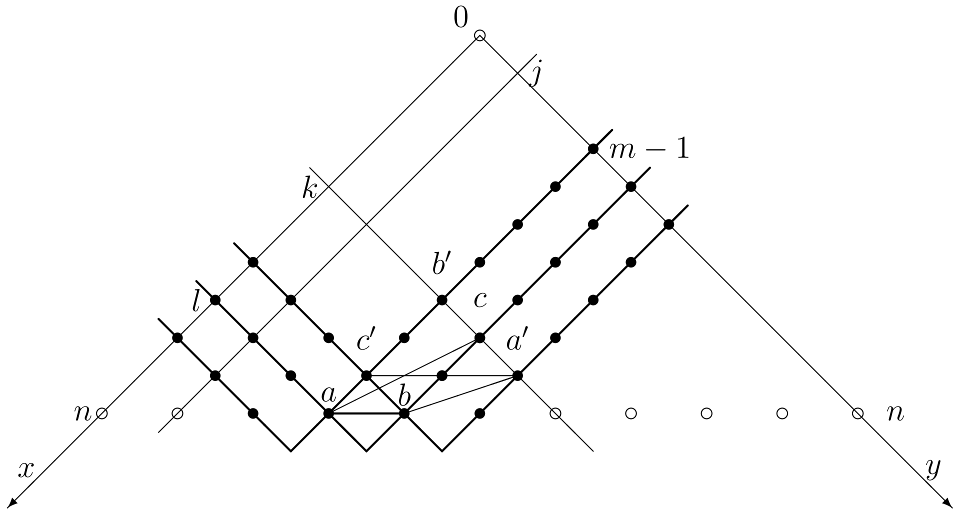
Proof. Let $V' \subset V$ be a (2-dimensional) subspace generated by α, β, γ and let $R' := R \cap V'$. Then R' is a root system in V' by [Bou68, VI, §1, Cor. of Prop. 4] and is equal to one of the root systems $\mathfrak{a}_2, \mathfrak{b}_2$. Indeed, R' is an irreducible reduced root system of rank 2; the list of all such systems is $\mathfrak{a}_2, \mathfrak{b}_2, \mathfrak{g}_2$ (cf. [Hel00, Ch. X, Exc. B1]). However the case $\mathfrak{g}_2$ is excluded by the assumption $R \neq \mathfrak{g}_2$ as we will show below.

Let B' be a basis of R'. Then by [Bou68, VI, §1, Prop. 24] there exists a basis B of R such that $B' \subset B$. In particular, this shows that $R' \neq \mathfrak{g}_2$, since neither of the Coxeter graphs of reduced irreducible root systems R contains the graph of $\mathfrak{g}_2$ except $R = \mathfrak{g}_2$ itself.

Now, the direct inspection of the systems $\mathfrak{a}_2, \mathfrak{b}_2$ shows that among the roots $\alpha, \beta, \gamma, -\alpha, -\beta, -\gamma$ there exist two roots α', β' forming a basis B' of R' and the proposition cited gives the result. ■

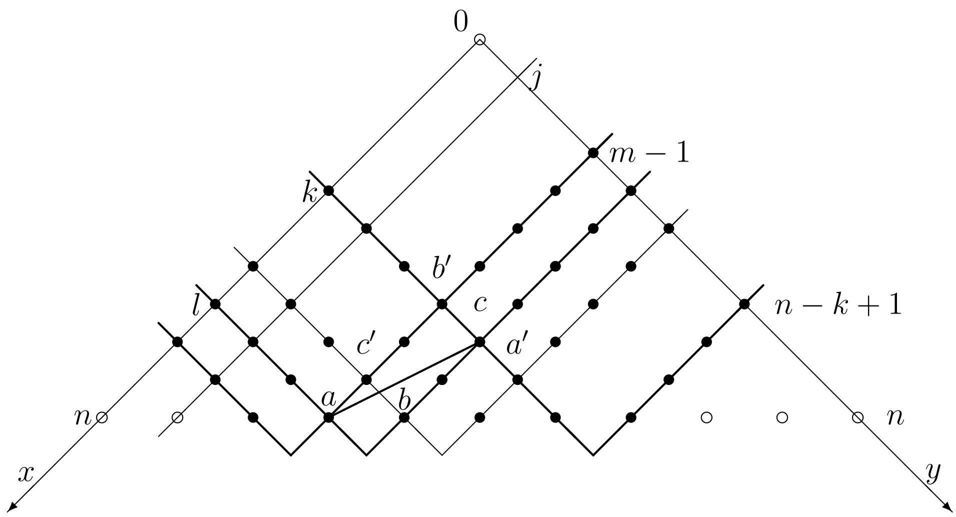
Now we are able to prove Theorem 11.4. Let $[\alpha_i], [\alpha_j]$ be as in Lemma 11.6. Then they have to be neighbours in the first row of the coordinate representation of $\widetilde{R}$. Denote them a, b and let $a = (l, m-1), b = (l-1, m), l + m - 1 = n$. The element $c' := (l-1, m-1)$ corresponds to the third element $[\alpha_i + \alpha_j]$ of the

$\{t_1t_2\}$-triangle, which will be called *basic* for a moment.



We claim that for any $e \in X(a) \cup X(b) \cup X(c')$ (the union of the "X-shapes" with crossings at a, b, c', see Lemma 11.5) the pair T_e is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$. Indeed, let e lie on the line $x = k$ with $0 \le k \le l-2$, i.e. e is one of the points $b' := (k, m-1), c := (k, m), a' := (k, m+1)$. Then the elements a, b, c, a', b', c' satisfy hypotheses of Lemma 11.3 and our claim follows (analogous arguments work if e lies on the line $y = j$ with $0 \le j \le m-2$).

Moreover, among the triangles $\{a, c, b'\}, \{b, c, a'\}, \{a', b', c'\}$ at least one is a $\{t_1t_2\}$-triangle. In any of these cases we can repeat the above considerations taking this triangle as a basic one. Irrespectively of which of these cases is considered we will obtain new points e such that T_e is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$. These points will lie on on $X(a') \cap X(b') \cap X(c)$, i.e. on the intersection of the lines $x = k, y = n - k + 1$ with $\widetilde{R}$, see the figure below (which corresponds to the new basic triangle $\{a, c, b'\}$).



Varying k from 0 to $l-2$ and j from 0 to $m-2$ we will cover the whole set $\widetilde{R}$. ■

Proof. *Proof of Theorem 11.1.* Let $R \neq \mathfrak{g}_2$ be a reduced irreducible root system with the Coxeter graph $\Gamma(R)$ (see the beginning of the proof of Theorem 11.4). We say that a connected subgraph $\Gamma' \subset \Gamma(R)$ is a *chain* if it does not have ramification points (i.e. vertices connected with at least three other vertices, see [Bou68, IV, App.]). In particular, the graphs of $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{f}_4$ are chains in themselves.

Let α_i, α_j be as in Lemma 11.6. We can consider any chain $\Gamma' \subset \Gamma(R)$ containing α_i, α_j and proceed as in the proof of Theorem 11.4 to prove that T_e is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$ for any e of the form $[\sum_{\beta\in Y}\beta]$, where Y is a subset of vertices of Γ' connected as a subgraph. Taking all maximal chains Γ' containing α_i, α_j we will prove in this way that $T_{[\alpha]}$ is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$ for all the elements α of the basis B (and also for all positive roots which are the combinations of the simple roots with coefficients not greater than 1).

To prove this for all positive roots we will use the fact that any such root can be written as $\beta = \beta_1 + \cdots + \beta_k$, where all β_j (not necessarily distinct) belong to B and each partial sum $\beta_1 + \cdots + \beta_j \in R$ (see [Hel00, Lem. 3.10, Ch. X]). Proceed by induction. We already know that $T_{[\beta_j]}$ is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$. Assume this is true for $T_{[\beta_1+\cdots+\beta_{j-1}]}$. Then the times selection rules applied to the triangle $\{[\beta_1 + \cdots + \beta_{j-1}], [\beta_j], [\beta_1 + \cdots + \beta_j]\}$ imply that also $T_{[\beta_1+\cdots+\beta_j]}$ is equal to one of the pairs $\{t_1t_2\}, \{t_1t_1\}, \{t_2t_2\}$.

To complete the proof consider the case $R = \mathfrak{g}_2$ and proceed by direct inspection. ■

12. Bi-Lie structures of Class I

Let $(\mathfrak{g}, [\,,], [\,,]')$ be a regular semisimple bi-Lie structure with a semisimple Lie algebra $(\mathfrak{g}, [\,,])$ and let $W : \mathfrak{g} \to \mathfrak{g}$ be the corresponding principal WNO. Then the results of Section 10 show that W defines a pair diagram $\mathcal{T} = \mathcal{T}_W$ (see Definition 10.6).

Definition 12.1. We say that a regular semisimple bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ with the principal WNO W is of *Class I or II* depending on the class of the pair diagram $\mathcal{T}_W$ (see Definition 11.2).

An algebraic structure which will be introduced below is essentially what we obtain from the structure of the pair groupoid (see eg. [Wei96]) if we substitute unordered pairs to ordered ones.

Definition 12.2. Let X be a finite set and let $X^{\hat{2}} := \{\{xy\} \mid x, y \in X\}$ be the set of unordered pairs of elements from X. Introduce a partial binary operation $X^{\hat{2}} \times X^{\hat{2}} \to X^{\hat{2}}, (\{xy\}, \{zu\}) \mapsto \{xy\} \circ \{zu\}$ by the following rules:

- $\{xy\} \circ \{zu\}$ is defined if and only if the sets $\{x, y\}, \{z, u\}$ have a common element;
- $\{xy\} \circ \{yu\} = \{xu\}$ if $x \neq u$;

- $\{xy\} \circ \{xy\}$ is either equal to $\{xx\}$ or to $\{yy\}$.

The pair $(X^{\hat{2}}, \circ)$ will be called a *pairoid with the base* X.

Remark 12.3. Note that unlike the pair groupoid structure on $X \times X$ a pairoid structure is not defined uniquely by the set X due to uncertainty of Condition 3.

From now on we assume that $(\mathfrak{g}, [\,,])$ is a simple Lie algebra (cf. Remark 10.9), $\mathfrak{h} \subset \mathfrak{g}$ a fixed Cartan subalgebra. Let $R = R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*_{\mathbb{R}}$ be the corresponding reduced irreducible root system and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ the root grading.

Definition 12.4. Let X be a finite set. A decomposition $\mathfrak{g} = \bigoplus_{\{xy\} \in X^{\hat{2}}} \mathfrak{g}_{\{xy\}}$ is a *pairoid quasigrading on* $\mathfrak{g}$ *with the base* X if

- $[\mathfrak{g}_{\{xy\}}, \mathfrak{g}_{\{yu\}}] \subset \mathfrak{g}_{\{xu\}} \quad \forall x, y, u \in X, x \neq u$;
- $[\mathfrak{g}_{\{xy\}}, \mathfrak{g}_{\{xy\}}] \subset \mathfrak{g}_{\{xx\}} \oplus \mathfrak{g}_{\{yy\}} \quad \forall x, y \in X$.

A pairoid quasigrading is *toral* (with respect to $\mathfrak{h}$, cf. Definition 15.1) if each $\mathfrak{g}_{\{xy\}}$ is the sum of the spaces $\mathfrak{g}_\alpha$ and some subspaces of $\mathfrak{h}$ and, moreover, $\mathfrak{h} \subset \bigoplus_{x \in X} \mathfrak{g}_{xx}$. A toral pairoid quasigrading is *symmetric* if $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}_{\{xy\}}$ whenever $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\{xy\}}$. We say that a pairoid quasigrading is *admissible* if it is toral, symmetric and obeys the following nondegeneracy property:

- $[\mathfrak{g}_{\{xy\}} \cap \mathfrak{h}^\perp, \mathfrak{g}_{\{zu\}} \cap \mathfrak{h}^\perp] = \{0\}$ whenever $\{xy\} \circ \{zu\}$ is not defined;

here $\mathfrak{h}^\perp = \sum_{\alpha \in R} \mathfrak{g}_\alpha$ is the orthogonal complement to $\mathfrak{h}$ with respect to the Killing form.

The elements of the set X are called the *times* of the quasigrading. A time $t \in X$ is *virtual* if $\mathfrak{g}_{\{tt\}} = \{0\}$. If we want to distinguish virtual times we denote them in parentheses (cf. the proof of Theorem 10.7 and examples below).

Definition 12.5. Two pairoid quasigradings, $\mathfrak{g} = \bigoplus_{\{xy\} \in X^{\hat{2}}} \mathfrak{g}_{\{xy\}}$ and $\mathfrak{g} = \bigoplus_{\{x'y'\} \in (X')^{\hat{2}}} \mathfrak{g}'_{\{x'y'\}}$, with the bases X and X' are *equivalent* if there exists $\varphi \in \operatorname{Aut}(\mathfrak{g})$ and a bijection $\chi : X \to X'$ such that $\varphi(\mathfrak{g}_{\{ab\}}) = \mathfrak{g}'_{\{\chi(a)\chi(b)\}}$ for any $a, b \in X$ (here $\operatorname{Aut}(\mathfrak{g})$ denotes the set of automorphisms of the Lie algebra $(\mathfrak{g}, [\,,])$).

For the sake of simplicity we will omit the braces in the notations of unordered pairs indexing grading subspaces.

Example 12.6. Let $X := \{x, y\}$, where $x \neq y$, and let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a $\mathbb{Z}_2$-grading on $\mathfrak{g}$. Putting $\mathfrak{g}_{xx} := \mathfrak{g}_0, \mathfrak{g}_{yy} := \{0\}, \mathfrak{g}_{x(y)} := \mathfrak{g}_1$ we get a pairoid quasigrading on $\mathfrak{g}$ with the base X and a virtual time y. An equivalent quasigrading can be given by $\mathfrak{g}_{xx} := \{0\}, \mathfrak{g}_{yy} := \mathfrak{g}_0, \mathfrak{g}_{(x)y} := \mathfrak{g}_1$. An example of quasigrading nonequivalent to that of the previous examples is obtained when $\mathfrak{g}_0$ is a sum of two nontrivial subalgebras, $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$: $\mathfrak{g}_{xx} := \mathfrak{g}_0^1, \mathfrak{g}_{yy} := \mathfrak{g}_0^2, \mathfrak{g}_{xy} := \mathfrak{g}_1$. These examples exhaust up to equivalence all nontrivial pairoid quasigradings with a two element base, (they are admissible if and only if $\mathfrak{g}_0 \supset \mathfrak{h}$, i.e. the involutive automorphism corresponding to the $\mathbb{Z}_2$-grading is inner).

Let us mention some properties of an admissible pairoid quasigrading. Obviously, the notion of pairoid quasigrading on $\mathfrak{g}$ with the base X is independent of the choice of pairoid structure on $X^{\hat{2}}$ (see Remark 12.3). Moreover, the subspaces $\mathfrak{g}_{xx}$ are subalgebras and if the pairoid quasigrading is admissible these subalgebras are reductive in $\mathfrak{g}$ (by [Bou75, VIII, §3], see also the beginning of Section 13). The sum of subalgebras $\mathfrak{g}_{xx} \oplus \mathfrak{g}_{yy}$ is also a subalgebra for any $x, y \in X$.

The following lemma is a direct consequence of the definitions.

Lemma 12.7. *Let $X := \{t_1, \dots, t_n\}$ be a set of pairwise distinct complex numbers.*

Let $(\mathcal{T}, U)$, be an augmented pair diagram (see Definition 10.6) such that $T_{\mathcal{T}} = X, U \in \operatorname{End}(\mathfrak{h})$ and $\mathcal{T} = \{T_\alpha\}_{\alpha \in R}$ is of Class I (we transport the root system R to $\mathfrak{h}$ by means of the Killing form, see the discussion before Definition 10.4). Write $\mathfrak{h}_{t_i}$ for the eigenspace of the operator U corresponding to the eigenvalue t_i.

Put $\mathfrak{g}_{t_it_j} := \bigoplus_{\alpha, T_\alpha = \{t_it_j\}} \mathfrak{g}_\alpha$ if $i \neq j$ and $\mathfrak{g}_{t_it_i} := \mathfrak{h}_{t_i} \oplus \bigoplus_{\alpha, T_\alpha = \{t_it_i\}} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{t_it_j}$ is an admissible pairoid quasigrading with the base X.

Vice versa, given an admissible pairoid quasigrading $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{t_it_j}$ with the base X, one obtains an augmented pair diagram $(\mathcal{T}, U)$ with $T_{\mathcal{T}} = X$ and a pair diagram of Class I by putting $T_\alpha := \{t_it_j\}$ for all $\alpha \in R$ such that $\mathfrak{g}_\alpha \subset \mathfrak{g}_{t_it_j}$ and $U|_{\mathfrak{g}_{t_it_i} \cap \mathfrak{h}} = t_i \operatorname{Id}_{\mathfrak{g}_{t_it_i} \cap \mathfrak{h}}$.

The described correspondence is one-to-one.

In the following theorem we will build a WNO from a given admissible pairoid quasigrading, or, in view of the lemma above, from an augmented pair diagram. This theorem shows that the necessary conditions for an operator to be a WNO obtained in Section 10 are in fact sufficient in the case of Class I bi-Lie structures.

Theorem 12.8. *Let $(\mathfrak{g}, [\,,\,])$ be a simple Lie algebra.*

(i) *Let $X := \{t_1, \dots, t_n\}$ be a set of pairwise distinct complex numbers and let $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{t_it_j}$ be an admissible pairoid quasigrading with the base X. Define an operator $W \in \operatorname{End}(\mathfrak{g})$ by*

$$W|_{\mathfrak{g}_{t_it_j}} := \frac{t_i + t_j}{2} \operatorname{Id}_{\mathfrak{g}_{t_it_j}}.$$

Then the triple $(\mathfrak{g}, [\,,\,], [\,,\,]_W)$ is a semisimple bi-Lie structure of Class I such that

(a) *X is its set of times, the subalgebras $\mathfrak{g}_{t_it_i}$ coincide with the subalgebras $\mathfrak{g}_0^{t_i}$ from Theorem 10.7, in particular, the basic subalgebra is equal to $\mathfrak{g}_0 := \bigoplus_{i=1}^n \mathfrak{g}_{t_it_i}$;*

(b) *its central subalgebra* $\mathfrak{z}$ *contains* $\mathfrak{h}$ *and, moreover,* $\mathfrak{z} = \bigoplus_{i=1}^n \mathfrak{z}^{t_i}$ *(recall that* $\mathfrak{z}^{t_i}$ *is the centre of the exceptional Lie algebra* $(\mathfrak{g}, [\,,]' - t_i[\,,])$*),* $\mathfrak{z}^{t_i} = \mathfrak{g}_{t_it_i} \cap \mathfrak{h} + \bigoplus_{\alpha \in \widetilde{R}^i} \mathfrak{g}_\alpha$*, where* $\widetilde{R}^i := \{\alpha \in R^i \mid \alpha(\bigoplus_{j\neq i} \mathfrak{g}_{t_jt_j} \cap \mathfrak{h}) = 0\}$*,* R^i *being the closed symmetric root set corresponding to the subalgebra* $\mathfrak{g}_{t_it_i}$ *(see the beginning of Section 13);*

(c) *the operator* $W|_{\mathfrak{h}^\perp}$ *is symmetric, in particular,* W *is the principal WNO of the constructed bi-Lie structure.*

(ii) *Any semisimple bi-Lie structure* $(\mathfrak{g}, [\,,], [\,,]')$ *of Class I is of the form above.*

(iii) *If* W, W' *are the operators built by two admissible pairoid quasigradings with the bases* X, X'*, then the corresponding bi-Lie structures are strongly isomorphic if and only if the quasigradings are equivalent,* $X = X'$*, and* $\chi(x) = x$ *for any* $x \in X$*; here* χ *is the bijection from Definition 12.5. The bi-Lie structures are isomorphic if and only if the quasigradings are equivalent and there exist* $\lambda, \lambda' \in \mathbb{C}$ *such that* $\chi(x) = \lambda x + \lambda'$ *for any* $x \in X$*.*

Proof. (i) We will first prove that the operator W satisfies the basic equality $T_W(x,y) = [x,y]_P$, where $P \in \operatorname{End}(\mathfrak{g})$ is a symmetric operator preserving the root grading, $P|_{\mathfrak{h}} \equiv 0, P|_{\mathfrak{g}_\alpha} = \pi_\alpha$, where π_α are some complex numbers, $\pi_\alpha = \pi_{-\alpha}$.

Let $\{t_{1,\alpha}t_{2,\alpha}\} = \{t_it_j\}$ if $\mathfrak{g}_\alpha \subset \mathfrak{g}_{t_it_j}$. Put $\lambda_\alpha := (t_{1,\alpha} + t_{2,\alpha})/2, \pi_\alpha := (t_{1,\alpha} - t_{2,\alpha})^2/8$ (the last choice is suggested by Theorem 10.3(ii) since $\lambda_\alpha = \lambda_{-\alpha}$ and $\kappa_\alpha = (\lambda_\alpha - \lambda_{-\alpha})/2 = 0$). Then, obviously,

$$\lambda_\alpha + \lambda_{-\alpha} = t_{1,\alpha} + t_{2,\alpha}, \lambda_\alpha\lambda_{-\alpha} = t_{1,\alpha}t_{2,\alpha} + 2\pi_\alpha. \tag{12.8.1}$$

We claim that

$$(\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma) = \pi_\alpha + \pi_\beta - \pi_\gamma \tag{12.8.2}$$

whenever $\alpha + \beta = \gamma$ for some $\alpha, \beta, \gamma \in R$. To prove this take into account the commutation relations of pairoid quasigrading and observe that the equality $\alpha + \beta = \gamma$ implies the inclusions

$$\mathfrak{g}_\alpha \subset \mathfrak{g}_{t_it_j}, \mathfrak{g}_\beta \subset \mathfrak{g}_{t_jt_k}, \mathfrak{g}_\gamma \subset \mathfrak{g}_{t_kt_i},$$

where some of the indices i, j, k can be equal. Now we have $\lambda_\alpha = (t_i + t_j)/2, \lambda_\beta = (t_j + t_k)/2, \lambda_\gamma = (t_k + t_i)/2, \pi_\alpha = (t_i - t_j)^2/8, \pi_\beta = (t_j - t_k)^2/8, \pi_\gamma = (t_k - t_i)^2/8$ and we can make a direct inspection.

To prove the main equality we have to consider several cases. Let $x, y \in \mathfrak{h}$. Then, obviously, $T_W(x,y) = 0 = [x,y]_P$.

If $x \in \mathfrak{h}, y \in \mathfrak{g}_\alpha$ (recall that $[x,y] = \alpha(x)y$), we have $T_W(x,y) = [Wx, \lambda_\alpha y] - W([Wx,y] + [x, \lambda_\alpha y] - W(\alpha(x)y)) = \lambda_\alpha\alpha(Wx)y - W(\alpha(Wx)y + \lambda_\alpha\alpha(x)y - \lambda_\alpha\alpha(x)y) = 0$. On the other hand, $[x,y]_P = [x, \pi_\alpha y] - P[x,y] = \pi_\alpha\alpha(x)y - P\alpha(x)y = 0$.

Now, let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ be such that $B(x,y) = 1$. Then (recall that $[x,y] = H_\alpha$) $T_W(x,y) = (\lambda_\alpha\lambda_{-\alpha}\operatorname{Id} - (\lambda_\alpha + \lambda_{-\alpha})W + W^2)H_\alpha = (W - \lambda_\alpha\operatorname{Id})(W - \lambda_{-\alpha}\operatorname{Id})H_\alpha$. By formulae (12.8.1) the last expression is equal to $(W - t_{1,\alpha}\operatorname{Id})(W - t_{2,\alpha}\operatorname{Id})H_\alpha + 2\pi_\alpha H_\alpha = 2\pi_\alpha H_\alpha$ (here we used the fact that

H_α is a sum of eigenvectors of W corresponding to the eigenvalues $t_{1,\alpha}, t_{2,\alpha}$ or is an eigenvector corresponding to one of them). On the other hand, $[x,y]_P = [\pi_\alpha x, y] + [x, \pi_\alpha y] - PH_\alpha = 2\pi_\alpha H_\alpha$.

If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ with $\alpha + \beta \notin R \cup \{0\}$, then $T_W(x,y) = 0 = [x,y]_P$. Finally, assume $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ with $\alpha + \beta = \gamma \in R$. Then $T_W(x,y) = (\lambda_\alpha\lambda_\beta - \lambda_\gamma(\lambda_\alpha + \lambda_\beta - \lambda_\gamma))[x,y] = (\lambda_\alpha - \lambda_\gamma)(\lambda_\beta - \lambda_\gamma)[x,y] = (\pi_\alpha + \pi_\beta - \pi_\gamma)[x,y] = [x,y]_P$, where we used (12.8.2).

We have proven that $(\mathfrak{g}, [\,,], [\,,]_W)$ is a (semisimple) bi-Lie structure which by construction is of Class I.

Properties (a), (b) follow from Theorems 10.3, 10.7. Property (c) is a consequence of Theorem 9.2(iv) and the symmetric property of the quasigrading.

(ii) follows from Theorems 10.3, 10.7.

(iii) is a consequence of Theorem 7.12. ■

Now we will show that any pairoid quasigrading induces a special type of $\mathbb{Z}_2^{m-1}$-grading on $(\mathfrak{g}, [\,,])$. Recall that a *basis* of a finite abelian group G is a set of elements $e_1, \ldots, e_n \in G$ such that any element $g \in G$ has a unique decomposition of the form $g = k_1 e_1 + \cdots + k_n e_n$ (we use the additive notation) with $0 \le k_i < o_i$, where o_i is the order of the element e_i. Clearly, given a basis as above we can define an isomorphism $\varphi : G \to \mathbb{Z}_{o_1} \times \cdots \times \mathbb{Z}_{o_n}$ by the formula $g \mapsto (k_1, \ldots, k_n)$. If $G = G_{m-1}$ is a group isomorphic to $\mathbb{Z}_2^{m-1}$, the cardinality of all the bases of G is the same.

Let $\mathfrak{g} = \bigoplus_{i \in G_{m-1}} \mathfrak{g}_i$ be a G_{m-1}-grading. An element $i \in G_{m-1}$ will be called a *quasiroot* if $\mathfrak{g}_i \ne \{0\}$.

Definition 12.9. Given $k, l \in \{1, \ldots, m\}, k < l$, put $I_{kl} := (i_1, \ldots, i_{m-1})$, where $i_1 = \cdots = i_{k-1} = i_l = \cdots = i_{m-1} = 0, i_k = \cdots = i_{l-1} = 1$.

We say that a toral (see Appendix 15) G_{m-1}-grading $\mathfrak{g} = \bigoplus_{i \in G_{m-1}} \mathfrak{g}_i$ is *admissible* if there exists a basis of G_{m-1} such that the element $(0, \ldots, 0)$ and all the elements $I_{kl}, k \in \{1, \ldots, m-1\}, l \in \{2, \ldots, m\}, k < l$, are the only quasiroots of the induced $\mathbb{Z}_2^{m-1}$-grading

$$\mathfrak{g} = \bigoplus_{(i_1,\ldots,i_{m-1}) \in \mathbb{Z}_2^{m-1}} \mathfrak{g}_{(i_1,\ldots,i_{m-1})}.$$

The following lemma is a direct consequence of the definition.

Lemma 12.10. *Let $X := \{t_1, \ldots, t_n\}$ be a set of pairwise distinct complex numbers and let $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}_{\{t_i t_j\}}$ be an admissible pairoid quasigrading with the base X. Then the formula*

$$\mathfrak{g}_{(i_1,\ldots,i_{m-1})} := \begin{cases} \bigoplus_{i=1}^n \mathfrak{g}_{\{t_i t_i\}} & \text{if } (i_1, \ldots, i_{m-1}) = (0, \ldots, 0); \\ \mathfrak{g}_{\{t_k t_l\}} & \text{if } (i_1, \ldots, i_{m-1}) = I_{kl}; \\ \{0\} & \text{in other cases.} \end{cases}$$

gives an admissible $\mathbb{Z}_2^{m-1}$-grading on $\mathfrak{g}$.

Note that in fact the grading above is determined by the corresponding pair diagram $\mathcal{T}$ (without participation of the R-admissible operator U, cf. Lemma 12.7).

Below we will construct a series of examples of pairoid quasigradings. In order to do this one should start from constructing admissible $\mathbb{Z}_2^{m-1}$-gradings. By definition any toral $\mathbb{Z}_2^{m-1}$-grading with $m = 2, 3$ is admissible, hence in principle this is a nontrivial problem only for $m > 3$. However the first of the following examples shows that the correspondence from the preceding lemma is not one-to-one, i.e. finding an admissible toral $\mathbb{Z}_2^{m-1}$-grading could be insufficient for building a pairoid quasigrading (even a pair diagram).

It is clear that any toral symmetric $\mathbb{Z}_2^{m-1}$-grading is defined by $m-1$ commuting inner automorphisms of order 2. In the examples below we will use the model diffeomorphisms of different types (see [Hel00, Ch. X] and Appendix 14). We will also use the notations from [Bou68, Tables I-IV].

Example 12.11. Let $\mathfrak{g} = \mathfrak{d}_4$ and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the standard basis of the root system. Here $\alpha_1, \alpha_3, \alpha_4$ have labels[1] 1 and α_2 has label 2. Consider the $\mathbb{Z}_2^2$-grading defined by the model automorphisms of type $(0,1,0,0,1;1), (0,0,0,1,1;1)$. Then we have $\mathfrak{g}_{\alpha_1} \subset \mathfrak{g}_{(1,0)}, \mathfrak{g}_{\alpha_2} \subset \mathfrak{g}_{(0,0)}, \mathfrak{g}_{\alpha_3} \subset \mathfrak{g}_{(0,1)}, \mathfrak{g}_{\alpha_4} \subset \mathfrak{g}_{(1,1)}$, or, by the hypothetical inverse correspondence to that of Lemma 12.10, $\mathfrak{g}_{\alpha_1} \subset \mathfrak{g}_{t_1t_2}, \mathfrak{g}_{\alpha_3} \subset \mathfrak{g}_{t_2t_3}, \mathfrak{g}_{\alpha_4} \subset \mathfrak{g}_{t_1t_3}$. However, there is a problem: in which of the three components $\mathfrak{g}_{t_1t_1}, \mathfrak{g}_{t_2t_2}$ or $\mathfrak{g}_{t_3t_3}$ should $\mathfrak{g}_{\alpha_2}$ lie in order that the axioms of the pairoid quasigrading be satisfied? This problem is related to the ramification point of the Dynkin diagram.

The next example shows that nonadmissible toral $\mathbb{Z}_2^{m-1}$-gradings with $m > 3$ exist.

Example 12.12. Let $\mathfrak{g} = \mathfrak{d}_4$. Consider the model automorphisms of types $(1,1,0,0,0;1), (1,0,0,1,0;1), (1,0,0,0,1;1)$. They define a $\mathbb{Z}_2^3$-grading on $\mathfrak{g}$ which is not admissible. Indeed, it has eight quasiroots: $\mathfrak{g}_{\alpha_1} \subset \mathfrak{g}_{(100)}, \mathfrak{g}_{\alpha_2} \subset \mathfrak{g}_{(000)}, \mathfrak{g}_{\alpha_3} \subset \mathfrak{g}_{(010)}, \mathfrak{g}_{\alpha_4} \subset \mathfrak{g}_{(001)}, \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3} \subset \mathfrak{g}_{(110)}, \mathfrak{g}_{\alpha_2+\alpha_3+\alpha_4} \subset \mathfrak{g}_{(011)}, \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_4} \subset \mathfrak{g}_{(101)}, \mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \subset \mathfrak{g}_{(111)}$. However, an admissible $\mathbb{Z}_2^3$-grading has only seven quasiroots. Again the ramification point plays a crucial role.

Below we give a series of examples of admissible pairoid quasigradings. The first of these examples is universal, i.e. it appears in any semisimple Lie algebra.

Example 12.13. Consider pairoid quasigradings from Example 12.6 such that the corresponding $\mathbb{Z}_2$-grading is induced by an inner automorphism. Then the quasigradings are admissible and by Theorem 12.8 we get a number of regular semisimple bi-Lie structures of Class I with two times.

Now we will construct a series of (in a sense canonical, see Conjecture 12.21)

[1] These labels are introduced in Appendix 14 and should not be confused with those introduced in Definitions 10.11, 10.13.

admissible pairoid quasigradings on the classical Lie algebras. First three series were known in the literature. The forth an fifth ones, related to the $\mathfrak{a}_n$-series, are new.

We will describe the quasigradings by means of augmented pair diagrams $(\mathcal{T}, U)$ (cf. Lemma 12.7).

Example 12.14. Let $\mathfrak{g} = \mathfrak{b}_n = \mathfrak{so}(2n+1, \mathbb{C})$, then the roots are $\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j$, $1 \le i < j \le n$, where ε_i are the elements of an orthonormal basis in $\mathfrak{h}^*_{\mathbb{R}}$. Put $U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm\varepsilon_i \pm \varepsilon_j} := \{t_i t_j\}$ and $T_{\pm\varepsilon_i} := \{t_i(t_{n+1})\}$ (recall that the parentheses mean that the corresponding time is virtual, cf. Definition 10.6, discussion after it, and Definition 12.4). The set of times is $\{t_1, \ldots, t_{n+1}\}$. The labels of the standard basis $\alpha_1, \ldots, \alpha_n$ are $1, 2, \ldots, 2$. The corresponding admissible $\mathbb{Z}_2^n$-grading is defined by the model automorphisms of types $(1,1,0,0,\ldots,0;1), (0,0,1,0,\ldots,0;1)$, $(0,0,0,1,\ldots,0;1), \ldots, (0,0,0,0,\ldots,1;1)$. This example corresponds to Example 2.3 with the diagonal matrix $A = \mathrm{diag}(t_1, t_1, t_2, t_2, \ldots, t_n, t_n, t_{n+1})$.

Example 12.15. Let $\mathfrak{g} = \mathfrak{d}_n = \mathfrak{so}(2n, \mathbb{C})$; the roots are $\pm\varepsilon_i \pm \varepsilon_j, 1 \le i < j \le n$. Put $U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm\varepsilon_i \pm \varepsilon_j} := \{t_i t_j\}$. The set of times is $\{t_1, \ldots, t_n\}$ and there are no virtual times. The labels of the standard basis $\alpha_1, \ldots, \alpha_n$ are $1, 2, \ldots, 2, 1, 1$. The corresponding admissible $\mathbb{Z}_2^{n-1}$-grading is defined by the model automorphisms of types $(1,1,0,0,\ldots,0,0,0;1), (0,0,1,0,\ldots,0,0,0;1)$, $(0,0,0,1,\ldots,0,0,0;1), \ldots,$
$(0,0,0,0,\ldots,1,0,0;1), (0,0,0,0,\ldots,0,1,1;1)$. This example corresponds to Example 2.3 with the diagonal matrix $A = \mathrm{diag}(t_1, t_1, t_2, t_2, \ldots, t_n, t_n)$.

Example 12.16. Let $\mathfrak{g} = \mathfrak{c}_n = \mathfrak{sp}(n, \mathbb{C})$, then the roots are $\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j$, $1 \le i < j \le n$. Put $U(H_{\varepsilon_i}) := t_i H_{\varepsilon_i}, T_{\pm\varepsilon_i \pm \varepsilon_j} := \{t_i t_j\}$ and $T_{\pm 2\varepsilon_i} = \{t_i t_i\}$. The set of times is $\{t_1, \ldots, t_n\}$ and there are no virtual times. The labels of the standard basis $\alpha_1, \ldots, \alpha_n$ are $2, 2, \ldots, 2, 1$. The corresponding admissible $\mathbb{Z}_2^{n-1}$-grading is defined by the model automorphisms of types $(0,1,0,\ldots,0,0;1)$, $(0,0,1,\ldots,0,0;1), \ldots, (1,0,0,\ldots,1,0;1)$. This example corresponds to Example 2.4 with the diagonal matrix $A = \mathrm{diag}(t_1, t_2, \ldots, t_n, t_1, t_2, \ldots, t_n)$.

Example 12.17. Let $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{sl}(n+1, \mathbb{C})$, then the roots are $\pm(\varepsilon_i - \varepsilon_j)$, $1 \le i < j \le n+1$, where ε_i are the elements of an orthonormal basis in a $(n+1)$-dimensional euclidean space in which $\mathfrak{h}^*_{\mathbb{R}}$ is embedded as the hyperplane orthogonal to the vector $(1,1,\ldots,1)$. Put $U(H_{\varepsilon_i - \varepsilon_{n+1}}) := t_i H_{\varepsilon_i - \varepsilon_{n+1}}, i = 1, \ldots, n, T_{\pm(\varepsilon_i - \varepsilon_j)} :=$ $\{t_i t_j\}$, if $i < j < n+1$ and $T_{\pm(\varepsilon_i - \varepsilon_{n+1})} = \{t_i(t_{n+1})\}$. The set of times is $\{t_1, \ldots, t_{n+1}\}$ and the time t_{n+1} is virtual.

The labels of the standard basis $\alpha_1, \ldots, \alpha_n$ are $1, 1, \ldots, 1$. The corresponding admissible $\mathbb{Z}_2^n$-grading is defined by the model automorphisms of types $(1,1,0,\ldots,0;1), (1,0,1,\ldots,0;1), \ldots, (1,0,0,\ldots,1;1)$. The WNO constructed via Theorem 12.8 by these data has the form $WX = (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A +$

$R_A)X)B$, where $X \in \mathfrak{sl}(n+1), A = \mathrm{diag}(t_1, t_2, \ldots, t_{n+1}), B = \mathrm{diag}(0, 0, \ldots, 0, 1)$ (cf. Example 4.12). Explicitly, if $X = ||x_{ij}|| \in \mathfrak{sl}(n+1, \mathbb{C})$, then $WX = ||y_{ij}||$, where $y_{ij} = x_{ij}(t_i + t_j)/2$ for $i \neq j$, $y_{ii} = x_{ii}t_i$ for $i = 1, \ldots, n$, and $y_{n+1,n+1} = -\sum_{j=1}^{n} x_{jj}t_j$.

The proof of Theorem 12.8 suggests that the WNO W will have the principal primitive equal to that of the operator $Z := (1/2)(L_A + R_A)$ (considered as an element of $\mathrm{End}(\mathfrak{gl}(n+1))$). In other words, the following fact is true: $T_W = T_Z$ (here also W is considered as an operator on $\mathfrak{gl}(n+1)$). Its direct proof surprisingly is not evident, and we present it below.

We have to prove that $T_{Z-V} = T_Z$, where $VX := (Z - W)X = \mathrm{Tr}(ZX)B$. We will use the following simple observations (which are true for any $X, Y \in \mathfrak{gl}(n+1)$): 1) $Z[B, X] = [B, ZX]$ (since the matrices A and B commute); 2) $Z[VX, Y] = [VX, ZY]$ (as a consequence of 1)); 3) $V^2X = t_{n+1}VX$ and $ZVX = t_{n+1}VX$ (by definitions); 4) $\mathrm{Tr}(Z[X, Y]_Z) = 0$ (this follows from the "main identity" $T_Z(X, Y) = [X, Y]_P$, where $P = \mathrm{ad}(A/2) \circ L_A - (1/8)\,\mathrm{ad}^2(A)$, see Example 4.12, since $Z[X, Y]_Z = [ZX, ZY] - [X, Y]_P = [ZX, ZY] - ([PX, Y] + [X, PY] - [A/2, A[X, Y]] + (1/8)[A, [A, [X, Y]]])$ is a combination of commutators); 5) $[VX, VY] = 0$ (obvious); 6) $\mathrm{Tr}(Z[B, X]) = 0$ (the matrices $[B, X]$ and $Z[B, X]$ has zeroes on the diagonal); 7) $\mathrm{Tr}(Z[VX, Y]) = 0$ (as a consequence of 6)).

Now we have $T_{Z-V}(X, Y) = T_Z(X, Y) - [ZX, VY] - [VX, ZY] + [VX, VY] + Z[X, Y]_V + V[X, Y]_Z - V[X, Y]_V = T_Z(X, Y) - [ZX, VY] - [VX, ZY] + [VX, VY] + ([VX, ZY] + [ZX, VY] - t_{n+1}V[X, Y]) + \mathrm{Tr}(Z[X, Y]_Z)B - (\mathrm{Tr}(Z[VX, Y])B + \mathrm{Tr}(Z[X, VY])B - t_{n+1}V[X, Y]) = T_Z(X, Y)$.

The next two examples contain an additional continuous parameter (i.e. moduli of bi-Lie structures with a fixed set of times are appearing here).

Example 12.18. Let $\mathfrak{g} = \mathfrak{a}_n = \mathfrak{sl}(n+1, \mathbb{C})$. Put $T_{\pm(\varepsilon_i - \varepsilon_j)} := \{t_i t_j\}$, if $i < j < n+1$, and $T_{\pm(\varepsilon_i - \varepsilon_{n+1})} = \{t_i t_n\}, i = 1, \ldots, n$. The corresponding admissible $\mathbb{Z}_2^{n-1}$-grading is defined by the model automorphisms of types $(1, 1, 0, \ldots, 0; 1)$, $(1, 0, 1, \ldots, 0; 1), \ldots, (1, 0, 0, \ldots, 1, 0; 1)$. Put $w_n := aH_{\alpha_n}, w_{n-1} := w_n + H_{\alpha_{n-1}}, \ldots$, $w_1 := w_2 + H_{\alpha_1}$, where $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$ are the elements of the standard basis of the root system and a is a complex parameter, and $U(w_i) := t_i w_i$. It is easy to see that any root vector is a linear combination of not more than two eigenvectors of U, i.e. the operator U is R-admissible, and that $(U, \mathcal{T})$ is an augmented pair diagram. By Theorem 12.8 we get a family of semisimple bi-Lie structures with the set of times $X := \{t_1, \ldots, t_n\}$ (there are no virtual times) depending on the parameter a. It turns out that these structures are pairwise nonisomorphic when the parameter a belongs to a small neighbourhood of 1.

Indeed, in order for two such structures related with quasigradings $\mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}^a_{\{t_i t_j\}}, \mathfrak{g} = \bigoplus_{t_i, t_j \in X} \mathfrak{g}^{a'}_{\{t_i t_j\}}$, corresponding to parameters a and a', to be isomorphic it is necessary that there exists an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{g}^a_{t_n t_n}) = \mathfrak{g}^{a'}_{t_n t_n}$ and $\varphi(\mathfrak{g}^a_{t_i t_i}) = \mathfrak{g}^{a'}_{t_{\sigma(i)} t_{\sigma(i)}}, i = 1, \ldots, n-1$, for some permutation $\sigma \in S_{n-1}$. By construction the subalgebra $\mathfrak{g}^a_{t_n t_n}$ is a 3-dimensional subalgebra generated by the subspace $\mathfrak{g}_{\alpha_n}$, while the remaining subalgebras $\mathfrak{g}^a_{t_i t_i}$ are one-dimensional subalgebras in $\mathfrak{h}$ of specific form. It can be shown that such φ has to preserve not only the subalgebra $\mathfrak{h} + \mathfrak{g}_{\alpha_n} + \mathfrak{g}_{-\alpha_n}$ (which is the zero subalgebra of the admissible $\mathbb{Z}_2^{n-1}$-grading), but also the Cartan subalgebra $\mathfrak{h}$ itself. Thus the automorphisms which could realize an isomorphism between these bi-Lie structures belong to a discrete group, while the parameter a gives a "continuous degree of freedom".

Example 12.19. Let $\mathfrak{g} = \mathfrak{b}_n = \mathfrak{so}(2n+1)$. Consider the $\mathbb{Z}_2$-grading is defined by the model automorphism of type $(1,1,0,0,\dots,0;1)$. Its zero subalgebra $\mathfrak{g}_0$ is the direct sum of the subalgebra $\mathfrak{so}(2n-1) =: \mathfrak{g}_{t_2t_2}$ embedded in the lower right corner and an arbitrary 1-dimensional subalgebra $\mathfrak{s} =: \mathfrak{g}_{t_1t_1} \subset \mathfrak{h}$ such that $\mathfrak{s}$ is not contained in $\mathfrak{g}_{t_2t_2} \cap \mathfrak{h}$. It remains to put $\mathfrak{g}_{t_1t_2} := \mathfrak{g}_1$ as in Example 12.6. The situation here is similar to the preceding example. In order for the bi-Lie structures corresponding to subalgebras $\mathfrak{s}, \mathfrak{s}'$ to be isomorphic an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{g}_{t_2t_2}) = \mathfrak{g}_{t_2t_2}$ and $\varphi(\mathfrak{s}) = \mathfrak{s}'$ should exist. However such an automorphism should preserve the Cartan subalgebra $\mathfrak{h}$ and should belong to a discrete group.

This example is also a good illustration of the difference between the notions of the central and basic subalgebras of the corresponding bi-Lie structure. The basic subalgebra is equal to $\mathfrak{g}_0 = \mathfrak{s} + \mathfrak{g}_{t_2t_2}$ and the central one to $\mathfrak{s} + \mathfrak{g}_{t_2t_2} \cap \mathfrak{g}^{\mathfrak{s}}$, where $\mathfrak{g}^{\mathfrak{s}}$ is the centralizer of the subspace $\mathfrak{s}$.

Similar examples exist for the Lie algebras $\mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7$ (cf. [GOV94, Table 6]).

The following simple observation, the proof of which follows from the definition of an admissible pairoid quasigrading of Class I, produces new regular semisimple bi-Lie structures of Class I from known ones.

Lemma 12.20. *Let $X := \{t_1,\dots,t_n\}$ and let $\mathfrak{g} = \bigoplus_{t_i,t_j\in X} \mathfrak{g}_{t_it_j}$ be an admissible pairoid quasigrading on $(\mathfrak{g},[\,,])$ with the base X. Then for any surjective map $\mu : X \to Y := \{u_1,\dots,u_m\}$ the formula $\mathfrak{g} = \bigoplus_{u_i,u_j\in Y} \mathfrak{g}^{\mu}_{u_iu_j}$, where $\mathfrak{g}^{\mu}_{u_iu_j} := \bigoplus_{x\in\mu^{-1}(u_i),y\in\mu^{-1}(u_j)} \mathfrak{g}_{xy}$, gives an admissible pairoid quasigrading on $(\mathfrak{g},[\,,])$ with the base Y.*

In particular, if $\mu_i : X \to Y := \{u_1,u_2\}, i = 1,\dots,n$, is given by $\mu_i(t_i) := u_1, \mu_i(t_j) := u_2$ for any $j \neq i$, we get quasigradings $\mathfrak{g} = \mathfrak{g}^{\mu_i}_{u_1u_1} \oplus \mathfrak{g}^{\mu_i}_{u_2u_2} \oplus \mathfrak{g}^{\mu_i}_{u_1u_2}$.

The second part of this result gives a hope that admissible pairoid quasigradings can be completely classified since they are "woven" from quasigradings with two times, the structure of which is understood (cf. Example 12.6).

Conjecture 12.21. Any admissible pairoid quasigrading on a simple Lie algebra is equivalent to one of that from Examples 12.13–12.18 or to their "coarsenings" by means of procedure from Lemma 12.20.

Remark 12.22. All the WNOs built in Theorem 12.8 by admissible quasigradings with the base $\{t_1,\dots,t_n\}$ linearly depend on these parameters. Hence all the examples above are in fact examples of families of Lie brackets parametrized by the linear space $\mathbb{C}^n$ (cf. the result of Kantor and Persits discussed in the Introduction).

Remark 12.23. The WNO of the bi-Lie structure from Example 2.3 in the case when the matrix A is diagonal with a simple spectrum is also related to a specific pairoid quasigrading on $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ which, however, is not toral. Let $e_{ij} := E_{ij} - E_{ji}, i < j$, where E_{ij} is the "matrix unit"; put $X = \{t_1, \dots, t_n\}$ and $\mathfrak{g} = \bigoplus_{i<j} \mathfrak{g}_{t_i t_j}$, where $\mathfrak{g}_{t_i t_j} := \mathbb{C} e_{ij}$. Now one can proceed as in Theorem 12.8 for constructing the WNO (and the formulae from the proof give the primitive).

13. Bi-Lie structures of Class II

In this section we will build a theory related to pair diagrams of Class II (see Definitions 10.6, 11.2) similar to that of the previous section.

We start from auxiliary results. Let $(\mathfrak{g}, [\,,])$ be a simple Lie algebra (cf. Remark 10.9), $\mathfrak{g}_0 \subset \mathfrak{g}$ a Lie subalgebra of maximal rank reductive in $\mathfrak{g}$. In particular, there exists a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$. In what follows we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Let $R = R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*_{\mathbb{R}}$ be the corresponding reduced irreducible root system. Recall [Bou68, VI, §1] that a set $R_0 \subset R$ is *closed* if for any $\alpha, \beta \in R_0$ such that $\gamma := \alpha + \beta \in R$ the root γ belongs to R_0. The set R_0 is *symmetric* if $R_0 = -R_0$. An empty set is closed symmetric by definition.

In terms of the root system R any Lie subalgebra $\mathfrak{g}_0 \supset \mathfrak{h}$ of maximal rank reductive in $\mathfrak{g}$ can be described as follows. If $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ is the corresponding root decomposition, then $\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in R_0} \mathfrak{g}_\alpha$, where $R_0 \subset R$ is some closed symmetric root set [Bou75, VIII, §3].

Definition 13.1. We say that a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}, \mathfrak{g}_0 \supset \mathfrak{h}$, is *admissible* if it is a subalgebra reductive in $\mathfrak{g}$ and it is not the fixed point subalgebra of an inner automorphism of order 2. Given a pair of subalgebras reductive in $\mathfrak{g}$ such that $\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2 = \mathfrak{g}_0$, we say that $(\mathfrak{g}_0^1, \mathfrak{g}_0^2)$ is an *admissible pair of subalgebras* (trivial cases $\mathfrak{g}_0^1 = \{0\}$ or $\mathfrak{g}_0^2 = \{0\}$ are admitted). The closed symmetric root set R_0 corresponding to an admissible subalgebra $\mathfrak{g}_0$ is called *admissible* too. We put $R^i := \{\alpha \in R_0 \mid \mathfrak{g}_\alpha \subset \mathfrak{g}_0^i\}, i = 1, 2$.

Remark 13.2. The fixed point subalgebras of an inner automorphism of order 2 of simple Lie algebras and their root sets are well known [Hel00, Chapter X].

Lemma 13.3. *Fix a set $X := \{t_1, t_2\}, t_1 \neq t_2, t_i \in \mathbb{C}$.*

Let $(\mathcal{T}, U), \mathcal{T} = \{T_\alpha\}_{\alpha \in R}$ be an augmented pair diagram (see Definition 10.6), where $T_{\mathcal{T}} = X$, $\mathcal{T}$ is a pair diagram of Class II, and $U \in$ End$(\mathfrak{h})$ *(we transport the root system R to $\mathfrak{h}$ by means of the Killing form, see the discussion before Definition 10.4). Write $\mathfrak{h}_{t_i}$ for the eigenspace of the operator U correspond-*

ing to the eigenvalue t_i. *Put* $\mathfrak{g}_0^i := \mathfrak{h}_{t_i} \oplus \bigoplus_{\alpha, T_\alpha = \{t_i t_i\}} \mathfrak{g}_\alpha, i = 1, 2$. *Then* $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ *is an admissible pair of subalgebras.*

Vice versa, given an admissible pair of subalgebras $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ *with an admissible root set* $R_0 = R^1 \cup R^2$, *one obtains an augmented pair diagram* $(\mathcal{T}, U)$ *with* $T_{\mathcal{T}} = X$ *and a pair diagram of Class II by putting*

$$T_\alpha := \begin{cases} \{t_i t_i\} & \text{if} \quad \alpha \in R^i \\ \{t_1 t_2\} & \text{if} \quad \alpha \in R \setminus R_0 \end{cases}$$

and $U|_{\mathfrak{g}_0^i \cap \mathfrak{h}} = t_i \operatorname{Id}_{\mathfrak{g}_0^i \cap \mathfrak{h}}, i = 1, 2$.

The built correspondence is one-to-one up to the transposition $(t_1 t_2)$.

For the proof we only have to mention that the definition of a pair diagram of Class II excludes the case when $\mathfrak{g}_0$ is the fixed point subalgebra of an inner automorphism of order 2. Indeed, assuming the contrary, we have $(R \setminus R_0) + (R \setminus R_0) \subset R_0$ and there does not exist any triangle (see Definition 10.6) $\alpha, \beta, \gamma \in R \setminus R_0$, i.e. such that $T_\alpha = T_\beta = T_\gamma = \{t_1 t_2\}$. It is also easy to see that the case of the fixed point subalgebra of an automorphism of order 2 is the only one which should be excluded. The rest of the proof follows from the definitions. ■

The following theorem says that in the particular case of the bi-Lie structures of Class II the necessary conditions obtained in Section 10 are in fact sufficient for an operator to be a WNO. This theorem takes into account the information about the restriction $W|_{\mathfrak{h}}$ of this operator to the Cartan subalgebra $\mathfrak{h}$ and the symmetric part of the operator $W|_{\mathfrak{h}^\perp}$ as well as the information about the antisymmetric part of this operator described in Corollary 10.15.

Theorem 13.4. *Let* $(\mathfrak{g}, [\,,])$ *be a simple Lie algebra.*

(i) *Let an ordered pair* (t_1, t_2) *of distinct complex numbers be given and an admissible pair of Lie subalgebras* $(\mathfrak{g}_0^1, \mathfrak{g}_0^2)$. *Let* $\mathfrak{g}_0 = \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2$ *and let*

$$\mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}_0)} \mathfrak{g}_i \tag{13.4.1}$$

be the toral irreducible $\Gamma(\mathfrak{g}_0)$*-grading corresponding to the reductive Lie subalgebra* $\mathfrak{g}_0$ *(see Definition 15.3). Assume that an operator* $W \in \operatorname{End}(\mathfrak{g})$, *which is scalar on the subspaces* $\mathfrak{g}_i, i \neq 0$, *and* $\mathfrak{g}_0^j, j = 1, 2$, *satisfies the following conditions:*

(a) $W|_{\mathfrak{g}_0^j} = t_j \operatorname{Id}_{\mathfrak{g}_0^j}, j = 1, 2$;

(b) *the symmetric part* W_{s} *of the operator* $W|_{\mathfrak{g}_0^\perp}$ *is equal to* $((t_1+t_2)/2) \operatorname{Id}_{\mathfrak{g}_0^\perp}$; *here* $\mathfrak{g}_0^\perp = \bigoplus_{i \in \Gamma(\mathfrak{g}_0), i \neq 0} \mathfrak{g}_i$;

(c) *The extension by zero of the antisymmetric part* W_{a} *of the operator* $W|_{\mathfrak{g}_0^\perp}$, *which we denote by* $\overline{W}_{\mathrm{a}}$, *obeys the* $((t_1 - t_2)/2)$*-triangle rule with respect to the grading (see Definition 10.13).*

Then the principal projection $\operatorname{pr}(W)$ *(see Theorem 9.4) of the operator* W *also satisfies Conditions (a), (b), (c), and the triple* $(\mathfrak{g}, [\,,], [\,,]_W) = (\mathfrak{g}, [\,,], [\,,]_{\operatorname{pr}(W)})$ *is a semisimple bi-Lie structure of Class II such that* (a') *$\{t_1,t_2\}$ is its set of times, the subalgebras $\mathfrak{g}_0^i$ coincide with the subalgebras $\mathfrak{g}_0^{t_i}$ from Theorem 10.7, in particular, its basic subalgebra is $\mathfrak{g}_0$;*

(b') *its central subalgebra $\mathfrak{z}$ contains $\mathfrak{h}$ and, moreover, $\mathfrak{z}=\mathfrak{z}^{t_1}\oplus\mathfrak{z}^{t_2}$ ($\mathfrak{z}^{t_i}$ being the centre of the exceptional Lie algebra $(\mathfrak{g},[\,,]'-t_i[\,,])$), $\mathfrak{z}^{t_i}=\mathfrak{g}_0^i\cap\mathfrak{h}+\bigoplus_{\alpha\in\widetilde{R}^i}\mathfrak{g}_\alpha$, where $\widetilde{R}^1:=\{\alpha\in R^1\mid\alpha(\mathfrak{g}_0^2\cap\mathfrak{h})=0\}$, $\widetilde{R}^2:=\{\alpha\in R^2\mid\alpha(\mathfrak{g}_0^1\cap\mathfrak{h})=0\}$; here R^i is the closed symmetric root set corresponding to the subalgebra $\mathfrak{g}_0^i$.*

(ii) *Any semisimple bi-Lie structure of Class II is of the form above.*

Proof. (i) We will first prove that the operator W satisfies the basic equality $T_W(x,y)=[x,y]_P$, where $P\in\mathrm{End}(\mathfrak{g})$ is a symmetric operator preserving the root decomposition (9.3.1), $P\mathfrak{h}\subset\mathfrak{h}, P|_{\mathfrak{g}_\alpha}=\pi_\alpha\,\mathrm{Id}_{\mathfrak{g}_\alpha}$, where π_α are some complex numbers, $\pi_\alpha=\pi_{-\alpha}$.

Let $t_{1,\alpha}=t_{2,\alpha}=t_i$ if $\alpha\in R^i$ and $t_{1,\alpha}=t_1, t_{2,\alpha}=t_2$ if $\alpha\notin R^1\cup R^2$. Let λ_α be the eigenvalue of W corresponding to the eigenspace $\mathfrak{g}_\alpha$ and let $\sigma_\alpha:=(1/2)(\lambda_\alpha+\lambda_{-\alpha}), \kappa_\alpha:=(1/2)(\lambda_\alpha-\lambda_{-\alpha})$ be the corresponding eigenvalues of $W_{\mathrm{s}}, W_{\mathrm{a}}$, in particular, $\sigma_\alpha=(t_{1,\alpha}+t_{2,\alpha})/2$ for any α by condition (b), $\kappa_\alpha=0$ for $\alpha\in R^1\cup R^2$ by condition (c).

Put $P|_{\mathfrak{h}}=0$, $\pi_\alpha:=((t_{1,\alpha}-t_{2,\alpha})/2)^2-\kappa_\alpha^2)/2$ (the last choice is suggested by Theorem 10.3(ii)). One checks the following formulae:

$$\lambda_\alpha+\lambda_{-\alpha}=t_{1,\alpha}+t_{2,\alpha}, \lambda_\alpha\lambda_{-\alpha}=\sigma_\alpha^2-\kappa_\alpha^2=t_{1,\alpha}t_{2,\alpha}+2\pi_\alpha.$$

We claim that

$$(\lambda_\alpha-\lambda_\gamma)(\lambda_\beta-\lambda_\gamma)=\pi_\alpha+\pi_\beta-\pi_\gamma$$

whenever $\alpha+\beta=\gamma$ for some $\alpha,\beta,\gamma\in R$. This can be proven by direct inspection taking into account conditions (a), (b) and considering the following cases:

- $\alpha,\beta,\gamma\in R^i\Rightarrow\sigma_\alpha=\sigma_\beta=\sigma_\gamma=t_i,\kappa_\alpha=\kappa_\beta=\kappa_\gamma=0,\pi_\alpha=\pi_\beta=\pi_\gamma=0$;
- $\alpha\in R^i,\beta,\gamma\notin R^1\cup R^2\Rightarrow\sigma_\alpha=t_i,\sigma_\beta=\sigma_\gamma=(t_1+t_2)/2,\kappa_\alpha=0,\kappa_\beta=\kappa_\gamma,\pi_\alpha=0,\pi_\beta=\pi_\gamma=((t_1-t_2)/2)^2-\kappa_\beta^2)/2$;
- $\alpha,\beta\notin R^1\cup R^2,\gamma\in R^i,\Rightarrow\sigma_\alpha=\sigma_\beta=(t_1+t_2)/2,\sigma_\gamma=t_i,\kappa_\alpha=-\kappa_\beta,\kappa_\gamma=0,\pi_\alpha=\pi_\beta=((t_1-t_2)/2)^2-\kappa_\alpha^2)/2,\pi_\gamma=0$;
- $\alpha,\beta,\gamma\notin R^1\cup R^2,\Rightarrow\sigma_\alpha=\sigma_\beta=\sigma_\gamma=(t_1+t_2)/2,\kappa_\alpha+\kappa_\beta-\kappa_\gamma=\pm(t_1-t_2)/2,\pi_\delta=((t_1-t_2)/2)^2-\kappa_\delta^2)/2$, where $\delta=\alpha,\beta,\gamma$.

Now the proof of the main equality $T_W(x,y)=[x,y]_P$ is literally the same as in the proof of Theorem 12.8.

Hence we have proven that $(\mathfrak{g},[\,,],[\,,]_W)$ is a semisimple bi-Lie structure. Now let us show that $\mathrm{pr}(W)$ satisfies conditions (a), (b), (c). Indeed, since the projecting does not affect $W|_{\mathfrak{h}}$ and W_{s}, conditions (a), (b) follow from the assumptions. The antisymmetric parts of $W|_{\mathfrak{h}^\perp}$ and $\mathrm{pr}(W)|_{\mathfrak{h}^\perp}$ differ by an operator of the form $\operatorname{ad} H, H \in \mathfrak{h}$, due to Theorem 9.4, hence obey the same $((t_1 - t_2)/2)$-triangle rule by Lemma 13.5, below. Finally, the condition $\kappa'_\alpha = 0, \alpha \in R^1 \cup R^2$, where κ'_α is the eigenvalue of the antisymmetric part of $\operatorname{pr}(W)|_{\mathfrak{h}^\perp}$ corresponding to $\mathfrak{g}_\alpha$, is a consequence of Theorem 10.7(iv) and the fact that $[\,,]_W = [\,,]_{\operatorname{pr}(W)}$.

It is clear that $(\mathfrak{g}, [\,,], [\,,]_W)$ is of Class II. Property (a') follows from Theorem 10.3. Property (b') is a consequence of item (iv) of that theorem.

(ii) follows from Theorems 10.3, 10.7 and Corollary 10.15. ■

Later we will use the theorem above to construct a series of examples of bi-Lie structures of Class II. In order to formulate our next theorem (13.8), dealing with the uniqueness questions, we need two more lemmata.

Lemma 13.5. *Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$ be the corresponding root decomposition. Assume that $R_0 \subset R$ is an admissible root set (see Definition 13.1) and $S : \mathfrak{h}^\perp \to \mathfrak{h}^\perp$ is an antisymmetric operator preserving the root spaces $\mathfrak{g}_\alpha$ and obeying the a-triangle rule with respect to R_0 for some $a \in \mathbb{C}$ (see Definition 10.11). Then*

(i) *given any basis $B = \{\alpha_1, \ldots, \alpha_n\}$ of the root system R, the operator S can be reconstructed from a, its system of labels $\mathcal{L}_S$, and its eigenvalues $\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n}$ corresponding to the eigenspaces $\mathfrak{g}_{\alpha_1}, \ldots, \mathfrak{g}_{\alpha_n}$;*

(ii) *an antisymmetric operator $S' : \mathfrak{h}^\perp \to \mathfrak{h}^\perp$ preserving the root spaces $\mathfrak{g}_\alpha$ obeys the a-triangle rule with respect to the same root set with $\mathcal{L}_{S'} = \mathcal{L}_S$ if and only if $S' = S + (\operatorname{ad} H)|_{\mathfrak{h}^\perp}$ for some $H \in \mathfrak{h}$ such that $\alpha(H) = 0$ for any $\alpha \in R_0$;*

(iii) *in the case when S is pricipal (i.e. if $S = L|_{\mathfrak{h}^\perp}$ for some principal operator $L \in \operatorname{End}(\mathfrak{g})$, see Definition 7.3), any other principal antisymmetric operator $S' \in \operatorname{End}(\mathfrak{h}^\perp)$ preserving the root spaces $\mathfrak{g}_\alpha$ and obeying the a-triangle rule with respect to the same root set with $\mathcal{L}_{S'} = \mathcal{L}_S$ coincides with S; moreover, S can be uniquely reconstructed from a and its system of labels $\mathcal{L}_S$.*

Proof. (i) Proceeding by induction assume that κ_α, the eigenvalue of S corresponding to the eigenspace $\mathfrak{g}_\alpha$, is already reconstructed for all positive roots of height $\operatorname{ht}(\alpha) = k$. Since any positive root α of height $k+1$ can be decomposed as a sum $\alpha = \beta + \gamma$, where β, γ are positive roots with $\operatorname{ht}(\beta), \operatorname{ht}(\gamma) \le k$, we can put $\kappa_\alpha = \kappa_\beta + \kappa_\gamma$ or $\kappa_\alpha = \kappa_\beta + \kappa_\gamma \pm a$ depending on the label of the triangle $\alpha, -\beta, -\gamma$.

(ii) Obviously, for any $H \in \mathfrak{h}$ the operator $\operatorname{ad} H|_{\mathfrak{h}^\perp}$ is diagonal: $\operatorname{ad} H|_{\mathfrak{g}_\alpha} = \alpha(H) \operatorname{Id}_{\mathfrak{g}_\alpha}$. Since for any triangle α, β, γ we have $\alpha(H) + \beta(H) + \gamma(H) = 0$ (i.e. $\operatorname{ad} H$ satisfies the 0-triangle rule), adding $\operatorname{ad} H$ to S will not affect the system of labels $\mathcal{L}_S$. On the other hand, the condition $\alpha(H) = 0, \alpha \in R_0$, guarantees that $\operatorname{ad} H|_{\mathfrak{g}_\alpha} = 0$ and $(S + \operatorname{ad} H)|_{\mathfrak{g}_\alpha} = 0$ for any $\alpha \in R_0$. The other implication will be proven below.

We will first show that for the operator $S := (\operatorname{ad} H)|_{\mathfrak{h}^\perp}$ the vector $(\kappa_{\alpha_1}, \ldots, \kappa_{\alpha_n})$ runs through all $(z_1, \ldots, z_n) \in \mathbb{C}^n$ as H runs through $\mathfrak{h}$. Indeed, we have

$\kappa_{\alpha_i} = \alpha_i(H)$ and in order to find $H \in \mathfrak{h}$ such that $\alpha_i(H) = z_i, i = 1, \ldots, n$, we have to solve a system of linear equations with the matrix $\alpha_i(H_{\alpha_j})$ the determinant of which is proportional to that of the Cartan matrix of the base B, hence is nonzero.

We can now prove the second implication. The operator $S'' := S' - S$ obeys the 0-triangle rule. Let $B = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of the root system R and let κ''_α be the corresponding eigenvalues of S''. The considerations above show that there exists $H \in \mathfrak{h}$ such that $(\operatorname{ad} H)|_{\mathfrak{g}_{\alpha_i}} = \kappa''_{\alpha_i}, i = 1, \ldots, n$. However by (i) the operators $S'', \operatorname{ad} H$ coincide. Moreover $\alpha(H) = 0$ for any $\alpha \in R_0$ since $S''|_{\mathfrak{g}_\alpha} = 0$.

(iii) By (ii) we have $S' = S + (\operatorname{ad} H)|_{\mathfrak{h}^\perp}$ for some $H \in \mathfrak{h}$. However, both S, S' are principal and by definition $H = 0$.

Now let $S = L|_{\mathfrak{h}^\perp}$ for some principal operator $L \in \operatorname{End}(\mathfrak{g})$. Starting the induction process of the proof of (i) from arbitrary vector $(c_1, \ldots, c_n)$ we will obtain one of the operators $S + (\operatorname{ad} H)|_{\mathfrak{h}^\perp}, H \in \mathfrak{h}$. To get S one has to project $L + \operatorname{ad} H$ onto $\tilde{\mathfrak{g}}^\perp$ along $\tilde{\mathfrak{g}} = \operatorname{ad} \mathfrak{g} \subset \operatorname{End}(\mathfrak{g})$ and to restrict the result to $\mathfrak{h}^\perp$ (cf. Theorem 9.4). ■

Let $\varphi \in \operatorname{Aut}(\mathfrak{g})$ and let $\mathfrak{g}_0 \subset \mathfrak{g}$ be an admissible subalgebra (see Definition 13.1). Put $\mathfrak{g}'_0 = \varphi(\mathfrak{g}_0)$. If $\mathfrak{h} \subset \mathfrak{g}_0$ is a Cartan subalgebra, $Q \subset \mathfrak{h}^*_{\mathbb{R}}$ the corresponding root lattice, and $Q_0 \subset Q$ the sublattice related with the subalgebra $\mathfrak{g}_0$, put also $\mathfrak{h}' := \varphi(\mathfrak{h}), Q' := \tilde{\varphi}(Q)$ and $Q'_0 := \varphi(Q_0)$; here $\tilde{\varphi} := ((\varphi|_{\mathfrak{h}})^*)^T = ((\varphi|_{\mathfrak{h}})^{-1})^T$ is the transpose of the conjugate operator with respect to the Killing form to $\varphi|_{\mathfrak{h}}$. Clearly $\mathfrak{h}' \subset \mathfrak{g}'_0$ is a Cartan subalgebra, $Q' \subset (\mathfrak{h}'_{\mathbb{R}})^*$ is the corresponding root lattice, and Q'_0 is the sublattice related to the subalgebra $\mathfrak{g}'_0$.

It is easy to see that $\tilde{\varphi}$ induces an isomorphism of groups $\bar{\varphi} : \Gamma \to \Gamma'$, where $\Gamma := \Gamma(\mathfrak{h}, \mathfrak{g}_0) = Q/Q_0$ and $\Gamma' := \Gamma(\mathfrak{h}', \mathfrak{g}'_0) = Q'/Q'_0$ (see Appendix 15) and the operator φ preserves the corresponding toral irreducible gradings $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i = \bigoplus_{i \in \Gamma'} \mathfrak{g}'_i$, or, more precisely, $\varphi(\mathfrak{g}_i) = \mathfrak{g}'_{\bar{\varphi}(i)}$.

If $i, j, k \in \Gamma$ is a triangle, the triple $\bar{\varphi}(i), \bar{\varphi}(j), \bar{\varphi}(k)$ is obviously a triangle again (see Definition 10.13). Moreover, if $\mathcal{L}_S$ is the system of labels of an operator $S \in \operatorname{End}(\mathfrak{g})$ diagonal with respect to the grading $\mathfrak{g} = \bigoplus_{i \in \Gamma} \mathfrak{g}_i$, we can endow the triangle $\bar{\varphi}(i), \bar{\varphi}(j), \bar{\varphi}(k)$ with the same label as the label of i, j, k from the family $\mathcal{L}_S$ and obtain a new system of labels, which will be denoted $\bar{\varphi}\mathcal{L}_S$. Also, by $-\mathcal{L}_S$ we will denote the system of labels *opposite* to $\mathcal{L}_S$ (i.e. with interchanged pluses and minuses).

Lemma 13.6. *Let $\mathfrak{g}_0, \mathfrak{g}'_0 \subset \mathfrak{g}$ be admissible subalgebras such that $\varphi(\mathfrak{g}_0) = \mathfrak{g}'_0$ for some $\varphi \in \operatorname{Aut}(\mathfrak{g})$. Let $S, S' \in \operatorname{End}(\mathfrak{g})$ be principal (see Definition 7.3) operators scalar on the components of the toral irreducible gradings $\mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}_0)} \mathfrak{g}_i, \mathfrak{g} = \bigoplus_{i \in \Gamma(\mathfrak{g}'_0)} \mathfrak{g}'_i$ and obeying the a, a'-triangle rule with respect to these gradings respectively, where $a \neq 0, a' \neq 0$.*

Then the following conditions are equivalent

(i) *$S' \circ \varphi := \varphi \circ S$;*

(ii) *either $a = a'$ and $\mathcal{L}_{S'} = \bar{\varphi}\mathcal{L}_S$ or $a = -a'$ and $\mathcal{L}_{S'} = -\bar{\varphi}\mathcal{L}_S$.*

Proof. Assume $S' \circ \varphi := \varphi \circ S$. Let κ_i, κ_i' be the eigenvalues of the corresponding operators related to the eigenspaces $\mathfrak{g}_i$ and $\mathfrak{g}_i'$. Since $\varphi(\mathfrak{g}_i) = \mathfrak{g}'_{\bar\varphi(i)}$, then $\kappa_i = \kappa'_{\bar\varphi(i)}$ and, given any triangle i, j, k, we have $\kappa_i + \kappa_j + \kappa_k = \kappa_{\bar\varphi(i)} + \kappa_{\bar\varphi(j)} + \kappa_{\bar\varphi(k)}$. Thus either $a = a'$ and the labels of the corresponding triangles coincide, or $a = -a'$ and they are opposite.

Conversely, let condition (ii) hold. Put $S'' := \varphi \circ S \circ \varphi^{-1}$ and let κ_i'' be the corresponding eigenvalues. Then $\kappa_i = \kappa''_{\bar\varphi(i)}$ and in both cases S'' obeys the a'-triangle rule with the same labels as S'. Now we have to choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ and observe that the operators S', S'' preserve the root decomposition of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}' := \varphi(\mathfrak{h})$ and their restrictions $S'|_{(\mathfrak{h}')^\perp}, S''|_{(\mathfrak{h}')^\perp}$ obey the a'-triangle rule with respect to the admissible root set R_0' corresponding to the subalgebra $\mathfrak{g}_0'$ (see Definition 10.11).

On the other hand, it is easy to see that S'' is principal (indeed,

$$\operatorname{Tr}(\varphi \circ S \circ \varphi^{-1} \circ \operatorname{ad} x) = \operatorname{Tr}(S \circ \varphi^{-1} \circ \operatorname{ad} x \circ \varphi) = \operatorname{Tr}(S \circ \operatorname{ad}(\varphi^{-1} x)) = 0$$

since $\operatorname{Tr}(S \circ \operatorname{ad} x) = 0$ for any $x \in \mathfrak{g}$). By Lemma 13.5(iii) we conclude that $S' = S''$ (cf. Remark 10.14). ∎

Remark 13.7. Using Lemma 13.5 one can reformulate Theorem 13.4 by saying that the corresponding bi-Lie structure is built from a set of distinct complex numbers $\{t_1, t_2\}$, an admissible pair of subalgebras $\mathfrak{g}_0^1, \mathfrak{g}_0^2$ and a system of labels $\mathcal{L}$ corresponding to the antisymmetric part of the operator $W|_{\mathfrak{h}^\perp}$.

Theorem 13.8. *Two bi-Lie structures,* $(\mathfrak{g}, [\,,], [\,,]_W)$ *and* $(\mathfrak{g}, [\,,], [\,,]_{W'})$*, each of which is constructed as in Theorem 13.4 by data* $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathcal{L})$ *and* $(t_1', t_2', (\mathfrak{g}_0^1)', (\mathfrak{g}_0^1)', \mathcal{L}')$*, where* $\mathcal{L}, \mathcal{L}'$ *denote the systems of labels related to the antisymmetric parts of the corresponding operators, are strongly isomorphic (see Definition 2.6) if and only if either Conditions (i), (ii) are satisfied or Conditions (iii), (iv), which are as follows:*

(i) *there exists* $\varphi \in \operatorname{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^1)', \varphi(\mathfrak{g}_0^2) = (\mathfrak{g}_0^2)'$ *and* $\bar\varphi\mathcal{L} = \mathcal{L}'$*, see notations introduced before Lemma 13.6;*

(ii) $t_1 = t_1', t_2 = t_2'$*;*

(iii) *there exists* $\varphi \in \operatorname{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^2)', \varphi(\mathfrak{g}_0^2) = (\mathfrak{g}_0^1)'$ *and* $\bar\varphi\mathcal{L} = -\mathcal{L}'$*;*

(iv) $t_1 = t_2', t_2 = t_1'$.

The bi-Lie structures $(\mathfrak{g}, [\,,], [\,,]_W), (\mathfrak{g}, [\,,], [\,,]_{W'})$ *are isomorphic if and only if either Condition (i) or (iii) is satisfied.*

Proof. The "strong" case follows from the construction of the operators W, W', Lemma 13.6, and Theorem 7.12. To manage the general case we have only to prove that the *principal* WNOs W and W' built by data $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathcal{L})$ and $(t_1', t_2', (\mathfrak{g}_0^1)', (\mathfrak{g}_0^1)', \mathcal{L}')$ are related by the formula $W = \lambda W' + \mu \operatorname{Id}_{\mathfrak{g}}$ for some $\lambda, \mu \in \mathbb{C}$

if and only if either $\mathfrak{g}_0^1 = (\mathfrak{g}_0^1)', \mathfrak{g}_0^2 = (\mathfrak{g}_0^2)'$ and $\mathcal{L} = \mathcal{L}'$, or $\mathfrak{g}_0^1 = (\mathfrak{g}_0^2)', \mathfrak{g}_0^2 = (\mathfrak{g}_0^1)'$ and $\mathcal{L} = -\mathcal{L}'$.

Indeed, if $W = \lambda W' + \mu \operatorname{Id}_{\mathfrak{g}}$, the operators W and W' should have the same eigenspaces $\mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathfrak{g}_j, j \in \Gamma, j \neq 0$. In particular, either (1) $t_i = \lambda t_i' + \mu$ and $\mathfrak{g}_0^i = (\mathfrak{g}_0^i)', i = 1,2$; or (2) $t_i = \lambda t_{i'}' + \mu$ and $\mathfrak{g}_0^i = (\mathfrak{g}_0^{i'})'$, where $i = 1,2, i' = 2,1$. Since $\operatorname{Id}_{\mathfrak{g}}$ does not affect the antisymmetric part of an operator, the systems of labels of the corresponding antisymmetric operators $\overline{W}_{\mathrm{a}}, \overline{W}_{\mathrm{a}}'$ (see Theorem 13.4) should coincide in Case (1) or be opposite in Case (2).

Conversely, assume principal WNOs W, W' are built by data $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathcal{L})$ and $(t_1', t_2', \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathcal{L})$. Since $t_1' \neq t_2'$, the system of equations $t_i = \lambda t_i' + \mu, i = 1,2$, has a unique solution λ_0, μ_0 with $\lambda_0 \neq 0$. We have decompositions $W = W|_{\mathfrak{g}_0} + W_{\mathrm{s}} + W_{\mathrm{a}}, W' = W'|_{\mathfrak{g}_0} + W_{\mathrm{s}}' + W_{\mathrm{a}}'$ and the obvious equalities $W|_{\mathfrak{g}_0} = \lambda_0 W'|_{\mathfrak{g}_0} + \mu_0 \operatorname{Id}_{\mathfrak{g}_0}, W_{\mathrm{s}} = \lambda_0 W_{\mathrm{s}}' + \mu_0 \operatorname{Id}_{\mathfrak{g}_0^\perp}$. Thus it remains only to show that $W_{\mathrm{a}} = \lambda_0 W_{\mathrm{a}}'$.

To this end consider the operator $W'' \in \operatorname{End}(\mathfrak{h}^\perp)$ which is the extension by zero to $\mathfrak{h}^\perp \setminus \mathfrak{g}_0^\perp$ of the operator $(1/\lambda_0) W_{\mathrm{a}} \in \operatorname{End}(\mathfrak{g}_0^\perp)$. Then W'' is the restriction to $\mathfrak{h}^\perp$ of the principal operator $(1/\lambda_0)\overline{W}_{\mathrm{a}}$, obeys the $((t_1'-t_2')/2)$-rule with respect to the same admissible root set with the same labels as W', hence $W'' = W_{\mathrm{a}}'$ on $\mathfrak{g}_0^\perp$ by Lemma 13.5(iii).
The case when W, W' are built by data $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathcal{L})$ and $(t_1', t_2', \mathfrak{g}_0^2, \mathfrak{g}_0^1, -\mathcal{L})$ is similar. ■

Now we will use Theorem 13.4 to construct a series of examples of bi-Lie structures of Class II.

Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a Levi subalgebra. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ and a basis B of the root system $R(\mathfrak{g}, \mathfrak{h})$ such that the corresponding root system R_0 of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset $B_0 \subset B$ (see Appendix 15). Put $R^\pm$ for the set of positive (negative) roots with respect to B. Then $\mathfrak{g}_0^\perp = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^-$, where $\mathfrak{g}_0^\pm := \sum_{\alpha \in R^\pm \setminus R_0} \mathfrak{g}_\alpha$. The subspace $\mathfrak{p}^\pm(\mathfrak{g}_0, B) := \mathfrak{g}_0 \oplus \mathfrak{g}_0^\pm$ is a parabolic subalgebra.

In the following theorem we generalize Example 4.9.

Theorem 13.9. *Let $\mathfrak{g}_0$ be an admissible subalgebra which is a Levi subalgebra and let $(\mathfrak{g}_0^1, \mathfrak{g}_0^2), \mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2 = \mathfrak{g}_0$, be an admissible pair of subalgebras. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$ and a basis B of the root system $R(\mathfrak{g}, \mathfrak{h})$ such that the corresponding root system R_0 of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset $B_0 \subset B$. Let $\mathfrak{g} = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_0^-$ be the corresponding decomposition and let $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$. Define an operator $W \in \operatorname{End}(\mathfrak{g})$ by:*

- $W|_{\mathfrak{g}_0^j} = t_j \operatorname{Id}_{\mathfrak{g}_0^j}, j = 1,2;$
- $W|_{\mathfrak{g}_0^+} = t_1 \operatorname{Id}_{\mathfrak{g}_0^+}, W|_{\mathfrak{g}_0^-} = t_2 \operatorname{Id}_{\mathfrak{g}_0^-}.$

Then the operator W satisfies the assumptions of Theorem 13.4(i) and the triple $(\mathfrak{g}, [\,,], [\,,]_W)$ is a semisimple bi-Lie structure of Class II satisfying conditions (i) (a'), (b') of this theorem.

Moreover, two bi-Li structures $(\mathfrak{g}, [\,,], [\,,]_W)$ *and* $(\mathfrak{g}, [\,,], [\,,]_{W'})$ *constructed from data* $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathfrak{h}, B)$ *and* $(t_1', t_2', (\mathfrak{g}_0^1)', (\mathfrak{g}_0^2)', \mathfrak{h}', B')$*, respectively, are isomorphic if and only if either there exists* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^j) = (\mathfrak{g}_0^j)', j = 1, 2$*, and* $\varphi(\mathfrak{p}^+(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^+((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$ *or there exists* $\varphi \in \mathrm{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^2)', \varphi(\mathfrak{g}_0^2) = (\mathfrak{g}_0^1)'$*, and* $\varphi(\mathfrak{p}^+(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^-((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$*.*

Proof. First note that the subspaces $\mathfrak{g}_0^\pm$ are invariant with respect to the action of the subalgebra $\mathfrak{g}_0$ on $\mathfrak{g}_0^\perp = \mathfrak{g}_0^+ \oplus \mathfrak{g}_0^-$, i.e. $\mathfrak{g}_0^\pm$ are direct sums of the components of the toral irreducible $\Gamma(\mathfrak{g}_0)$-grading. The components of this grading and the subalgebras $\mathfrak{g}_0^j$ are eigenspaces of the operator W and, moreover, it obviously satisfies conditions (a), (b) of Theorem 13.4(i).

Thus we have only to show that the operator $\overline{W}_{\mathrm{a}}$ obeys the $((t_1 - t_2)/2)$-triangle rule with respect to the grading.

To prove the last fact observe that the triangles $i, j, k \in \overline{\Gamma(\mathfrak{g}_0)}$ are of two kinds:

- Two of three quasiroots (see Definition 10.13), say i, j, are such that $\mathfrak{g}_i, \mathfrak{g}_j \subset \mathfrak{g}_0^+$, for the third one, k, we have $\mathfrak{g}_k \subset \mathfrak{g}_0^-$. Then $\kappa_i = \kappa_j = (t_1 - t_2)/2 = -\kappa_k$, the label is $+$.
- Two of three quasiroots, say i, j, are such that $\mathfrak{g}_i, \mathfrak{g}_j \subset \mathfrak{g}_0^-$, for the third one, k, we have $\mathfrak{g}_k \subset \mathfrak{g}_0^+$. Then $\kappa_i = \kappa_j = -(t_1 - t_2)/2 = -\kappa_k$, the label is $-$.

The last statement of the theorem follows from Theorem 13.8. ∎

Remark 13.10. It is easy to see that the eigenspaces of the operator W built in Theorem 13.9 are subalgebras in $(\mathfrak{g}, [\,,])$. Hence the operator W is a Nijenhuis operator, i.e. its torsion T_W vanishes (cf. Example 4.9).

Another series of examples (generalizing Example 4.10, cf. [GS02]) will be related to inner automorphisms of finite order (see Appendix 14) and to more general admissible subalgebras.

Remark 13.11. Any inner automorphism σ of order m induces a $\mathbb{Z}_m$-grading $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \cdots \oplus \mathfrak{g}_{\overline{m-1}}$, where $\mathfrak{g}_{\bar{j}}$ is the eigenspace of σ corresponding to the eigenvalue $e^{(2i\pi/m)j(\mathrm{mod}\, m)}$ (in particular $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0$). This grading is toral with respect to $\mathfrak{h}$. On the other hand, this grading is a coarsening of the toral irreducible $\Gamma(\mathfrak{g}_0)$-grading (13.4.1) and there exists an additive epimorphism $\psi : \overline{\Gamma(\mathfrak{g}_0)} \to \overline{\mathbb{Z}_m}$ such that $\mathfrak{g}_{\bar{j}} = \bigoplus_{i \in \psi^{-1}(\bar{j})} \mathfrak{g}_i, j = 1, \ldots, m-1$ (recall that we denote by $\overline{\Gamma}$ the set of quasiroots of a Γ-grading, see Definition 10.13).

Theorem 13.12. (i) *Let* $\mathfrak{g}_0$ *be the fixed point subalgebra of an inner automorphism of order* $m > 2$ *of type* $(s_0, \ldots, s_n; 1)$ *(see Appendix 14; in particular* $\mathfrak{g}_0$ *is admissible) and let* $(\mathfrak{g}_0^1, \mathfrak{g}_0^2)$ *be an admissible pair of subalgebras such that* $\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2 = \mathfrak{g}_0$*. Let* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \cdots \oplus \mathfrak{g}_{\overline{m-1}}$ *be the corresponding grading. Given* $t_1, t_2 \in \mathbb{C}, t_1 \neq t_2$*, define an operator* $W \in \mathrm{End}(\mathfrak{g})$ *by* *(a)* $W|_{\mathfrak{g}_0^j} = t_i \operatorname{Id}_{\mathfrak{g}_0^i}, i = 1, 2$*;*

(b) $W|_{\mathfrak{g}_{\bar{j}}} = (((m-j)t_1 + jt_2)/m) \operatorname{Id}_{\mathfrak{g}_{\bar{j}}}$ *if* $j > 0$.

Then the operator W *satisfies the assumptions of Theorem 13.4(i); consequently,* $(\mathfrak{g}, [\,,], [\,,]_W)$ *is a semisimple bi-Lie structure of Class II satisfying conditions (a'), (b') of this theorem.*

(ii) *If* $\mathfrak{g}_0$ *is a Levi subalgebra (i.e.* $s_0 \neq 0$*), the bi-Lie structure* $(\mathfrak{g}, [\,,], [\,,]_W)$ *is isomorphic to that constructed from data* $(t_1, t_2, \mathfrak{g}_0^1, \mathfrak{g}_0^2, \mathfrak{h}, B)$ *in Theorem 13.9.*

(iii) *Two bi-Lie structures constructed from data* $(t_1, t_2, \mathfrak{h}, B, (s_0, \ldots, s_n), \mathfrak{g}_0^1, \mathfrak{g}_0^2)$ *and* $(t_1', t_2', \mathfrak{h}', B', (s_0', \ldots, s_n'), (\mathfrak{g}_0^1)', (\mathfrak{g}_0^2)')$ *are isomorphic if and only if either there exists* $\varphi \in \operatorname{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^j) = (\mathfrak{g}_0^j)', j = 1, 2$*, and* $\varphi(\mathfrak{p}^+(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^+((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$ *or there exists* $\varphi \in \operatorname{Aut}(\mathfrak{g})$ *such that* $\varphi(\mathfrak{g}_0^1) = (\mathfrak{g}_0^2)', \varphi(\mathfrak{g}_0^2) = (\mathfrak{g}_0^1)'$*, and* $\varphi(\mathfrak{p}^+(\mathfrak{g}_0^1 \oplus \mathfrak{g}_0^2, B)) = \mathfrak{p}^-((\mathfrak{g}_0^1)' \oplus (\mathfrak{g}_0^2)', B')$.

Remark 13.13. The formulae (a), (b) appeared in [GS02, Example 3].

Proof. (i) Clearly it is enough to consider a model automorphism of type $(s_0, \ldots, s_n; 1)$. Since the $\mathbb{Z}_m$-grading is a coarsening of the toral irreducible $\Gamma(\mathfrak{g}_0)$-grading, the operator W is scalar on the components of the latter grading. By Theorem 9.2(iv) we have $W_{\mathrm{s}}|_{\mathfrak{g}_{\bar{j}}} = ((t_1 + t_2)/2) \operatorname{Id}_{\mathfrak{g}_{\bar{j}}}$ for any $j > 0$ and Conditions (a), (b) of Theorem 13.4(i) are satisfied.

To check Condition (c) due to Remark 13.11 it is enough to check the $((t_1 - t_2)/2)$-triangle rule with respect to the $\mathbb{Z}_m$-grading. Observe that $W_{\mathrm{a}}|_{\mathfrak{g}_{\bar{j}}} = \kappa_j \operatorname{Id}_{\mathfrak{g}_{\bar{j}}}$, where $\kappa_j = ((-j + m/2)(t_1 - t_2)/m)$, and that triangles $\bar{i}, \bar{j}, \bar{k} \in \overline{\mathbb{Z}_m}$, are of two kinds:

- $i + j + k = m$ (then $\kappa_i + \kappa_j + \kappa_k = (t_1 - t_2)/2$, the label is $+$);
- $i + j + k = 2m$ (then $\kappa_i + \kappa_j + \kappa_k = -(t_1 - t_2)/2$, the label is $-$).

(ii) This follows from Theorem 13.8 and from the fact that the corresponding operators have identical systems of labels with respect to the toral irreducible $\Gamma(\mathfrak{g}_0)$-grading.

To prove this fact notice that, if $\bar{i}, \bar{j}, \bar{k}$ is a triangle, then there exist roots α, β, γ such that $\mathfrak{g}_\alpha \subset \mathfrak{g}_{\bar{i}}, \mathfrak{g}_\beta \subset \mathfrak{g}_{\bar{j}}, \mathfrak{g}_\gamma \subset \mathfrak{g}_{\bar{k}}$ and $\alpha + \beta + \gamma = 0$. We will show that in the case $i + j + k = m$ two of these roots are positive (and the third negative) and vice versa in the case $i + j + k = 2m$ (cf. the proof of Theorem 13.9).

Note that any root α can be uniquely represented as $\alpha = \sum_{l=0}^n k_l^\alpha \alpha_l$, where $0 \leq k_l^\alpha \leq a_l$ ($k_0^\alpha = 0$ if $\alpha > 0$ and $k_0^\alpha = 1$ if $\alpha < 0$). Moreover, for any $p \in \{1, \ldots, m-1\}$ the subspace $\mathfrak{g}_{\bar{p}}$ consists of those root spaces $\mathfrak{g}_\alpha$ for which $\sum_{l=0}^n k_l^\alpha s_l = p$ [GOV94, §3.7].

Let $i + j + k = 2m$ and assume that two of the three roots, say α, β, are positive. Then $\alpha = \sum_{l=1}^n k_l^\alpha \alpha_l, \beta = \sum_{l=1}^n k_l^\beta \alpha_l$ and $\alpha + \beta = \sum_{l=1}^n (k_l^\alpha + k_l^\beta) \alpha_l$, in particular $k_l^\alpha+k_l^\beta \le a_l$ for any l. On the other hand, $i+j=\sum_{l=1}^n(k_l^\alpha+k_l^\beta)s_l$ has to be greater than $m=s_0+\sum_{l=1}^n a_l s_l$, whence $k_l^\alpha+k_l^\beta > a_l$ for some $l\in\{1,\dots,n\}$, a contradiction.

Now let $i+j+k=m$ and assume that two of the three roots, say α,β, are negative. Then $\alpha=\alpha_0+\sum_{l=1}^n k_l^\alpha\alpha_l, \beta=\alpha_0+\sum_{l=1}^n k_l^\beta\alpha_l$ and $\alpha+\beta=\alpha_0+\sum_{l=1}^n(k_l^\alpha+k_l^\beta-a_l)\alpha_l$, in particular $k_l^\alpha+k_l^\beta\ge a_l$ for any l. On the other hand, $i+j=2s_0+\sum_{l=1}^n(k_l^\alpha+k_l^\beta)s_l$ has to be less than $m=s_0+\sum_{l=1}^n a_l s_l$, whence $k_l^\alpha+k_l^\beta<a_l$ for some $l\in\{1,\dots,n\}$, a contradiction.

(iii) The proof of the previous item shows that the system of labels of the antisymmetric part of the operator W depends only on the subalgebra $\mathfrak{g}_0$ and the decomposition of $\mathfrak{g}_0^\perp$ into subspaces corresponding to positive and negative roots. Now we can use Theorem 13.8. ■

We continue by presenting an example of a bi-Lie structure of Class II which is not isomorphic to any of the examples appeared in Theorems 13.9, 13.12.

Example 13.14. Let $\mathfrak{g}=\mathfrak{e}_8$. We will use notations from [Bou68, Table VII]. Let $R_0=\mathrm{Span}_{\mathbb{Z}}\{\alpha_1,\alpha_2,\alpha_3,\alpha_5,\alpha_6,\alpha_8,\delta,\zeta\}\cap R$; here $\delta=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+2\alpha_7+\alpha_8, \zeta:=\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6$. Then the corresponding regular reductive Lie subalgebra $\mathfrak{g}_0$ is of Class (3) (see Appendix 15) since $Q_0:=\mathrm{Span}_{\mathbb{Z}}\{\alpha_1,\alpha_2,\alpha_3,3\alpha_4,\alpha_5,\alpha_6,2\alpha_7,\alpha_8\}$ and $\Gamma(\mathfrak{g}_0)=Q/Q_0\cong\mathbb{Z}_3\times\mathbb{Z}_2$. Thus the subalgebra $\mathfrak{g}_0$ is neither a Levi subalgebra nor a fixed point subalgebra of any single inner automorphism of finite order (i.e. $\mathfrak{g}_0$ is not of Classes (1), (2)).

The eigenvalues of the antisymmetric part W_{a} of the operator $W|_{\mathfrak{g}_0^\perp}$ are presented in the following table (we set $a:=(t_1-t_2)/2$ and denote by $\kappa_{\overline{m\alpha_4+n\alpha_7}}$ the eigenvalue of W_{a} corresponding to all the roots containing the combination $m\alpha_4+n\alpha_7 \bmod \mathfrak{g}_0$).

$\kappa_{\overline{\alpha_4}}$	$a/3$	$\kappa_{\overline{\alpha_7}}$	0	$\kappa_{\overline{\alpha_4+\alpha_7}}$	$-2a/3$
$\kappa_{\overline{2\alpha_4+\alpha_7}}$	$2a/3$	$\kappa_{\overline{2\alpha_4}}$	$-a/3$		

Note that the combinations of roots $\alpha_4,2\alpha_4,\alpha_7,\alpha_4+\alpha_7,2\alpha_4+\alpha_7$ correspond respectively to the elements $(1,0),(2,0),(0,1),(1,1),(2,1)$ of the group $\mathbb{Z}_3\times\mathbb{Z}_2$. In particular, it is easy to see that the operator W_{a} so defined is antisymmetric and obeys the a-triangle rule with respect to the grading in the sense of Definition 10.13.

To complete the construction we need to: 1) split $\mathfrak{g}_0$ to an admissible pair $(\mathfrak{g}_0^1,\mathfrak{g}_0^2)$ (note that the subalgebra $[\mathfrak{g}_0,\mathfrak{g}_0]$ is a sum of several simple subalgebras, so such splitting can be done in several ways; the simplest one is $\mathfrak{g}_0^1:=\mathfrak{g}_0,\mathfrak{g}_0^2:=\{0\}$); 2) put $W|_{\mathfrak{g}_0^i}=t_i\,\mathrm{Id}_{\mathfrak{g}_0^i}$ for $i=1,2$ and $W_{\mathrm{s}}|_{\mathfrak{g}_\alpha}=((t_1+t_2)/2)\,\mathrm{Id}_{\mathfrak{g}_\alpha}$ for $\alpha\notin R_0$. By Theorem 13.4 we get a family of bi-Lie structures of Class II with the set of times $\{t_1,t_2\}$ and the basic subalgebra $\mathfrak{g}_0$.

The proofs of Theorems 13.9, 13.12 show that the corresponding systems of labels up to a sign satisfy the following simple property:

- any triangle with nonzero label and two positive (negative) roots has the label $+$ $(-)$.

Using this observation we can now formulate a conjecture (unfortunately the author does not know whether the bi-Lie structures from Example 13.14 fit this conjecture or contradict it).

Conjecture 13.15. For any bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ of Class II there exists a Cartan subalgebra $\mathfrak{h}$ and a basis of the root system $R = R(\mathfrak{g}, \mathfrak{h})$ such that the system of labels of the antisymmetric part of the corresponding principal WNO restricted to $\mathfrak{h}^\perp$ has the property mentioned. In particular,

- any bi-Lie structure of Class II where the basic subalgebra $\mathfrak{g}_0$ is a Levi subalgebra (i.e. a regular reductive subalgebra of Class (1), see Appendix 15) is isomorphic to one of those built in Theorem 13.9;
- any bi-Lie structure of Class II where the basic subalgebra $\mathfrak{g}_0$ is a regular reductive subalgebra of Class (2) (see Appendix 15) is isomorphic to one of those built in Theorem 13.12.

We conclude this section by proving this conjecture in a particular case $(\mathfrak{g}, [\,,]) = \mathfrak{a}_n$ thus obtaining a complete classification of bi-Lie structures of Class II in this case.

Theorem 13.16. *Let $(\mathfrak{g}, [\,,]) = \mathfrak{a}_n$. Then any semisimple regular bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$ of Class II is isomorphic to one of those built in Theorem 13.9.*

Proof. Let $\{t_1, t_2\}$ be the set of times of the bi-Lie structure $(\mathfrak{g}, [\,,], [\,,]')$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a basis of roots of the corresponding root system $R(\mathfrak{g}, \mathfrak{h})$ and let $R_0 \subset R$ be the root set corresponding to the basic subalgebra $\mathfrak{g}_0$. By Corollary 10.12 the corresponding principal WNO obeys the a-triangle rule with respect to R_0, where $a = (t_1 - t_2)/2$. We will first prove that there exists a WNO W of this bi-Lie structure such that all the eigenvalues of $W_{\mathrm{a}}|_{\mathfrak{h}^\perp}$ are equal to zero or $\pm a$.

Consider the triangle $\{\beta_{i-1}, \alpha_i, -\beta_i\}, i = 2, \ldots, n$, where we write $\beta_i := \alpha_1 + \cdots + \alpha_i$, and put $d_i = 0, \pm 1$ for the corresponding label. Consider the vector $(d_1 a, d_2 a, \ldots, d_n a)$, where by definition $d_1 = 0$ if $\alpha_1 \in R_0$ and $d_1 = 1$ otherwise, and start the induction process of the proof of Lemma 13.5 from this vector. We claim that the eigenvalues κ_α of the resulting antisymmetric operator will satisfy the required condition.

Indeed, let $\alpha := \alpha_k + \alpha_{k+1} + \cdots + \alpha_{k+m} = \beta_{k+m} - \beta_{k-1}$ for some $1 \le k \le n-1, m \ge 0$. Then it belongs to the triangle $\{\alpha, \beta_{k-1}, -\beta_{k+m}\}$ with the label $d = 0, \pm 1$ and

$$\kappa_\alpha = -\kappa_{\beta_{k-1}} + \kappa_{\beta_{k+m}} + da. \tag{13.16.1}$$

On the other hand, the induction process gives $\kappa_{\beta_1} = \kappa_{\alpha_1} = d_1 a, \kappa_{\beta_2} = \kappa_{\beta_1} + \kappa_{\alpha_2} - d_2 a = d_1 a + d_2 a - d_2 a = d_1 a, \ldots, \kappa_{\beta_{k-1}} = \kappa_{\beta_{k-2}} + \kappa_{\alpha_{k-1}} - d_{k-1} a = d_1 a + d_{k-1} a - d_{k-1} a = d_1 a, \ldots, \kappa_{\beta_{k+m}} = d_1 a$. Hence by (13.16.1) we get $\kappa_\alpha = da$.

In the next step of the proof consider the operator just built and notice that the set $Y := \{\alpha \in R \mid \kappa_\alpha = a\}$ is closed: if $\alpha, \beta \in Y$ and $\gamma := \alpha + \beta \in R$, then $\kappa_\gamma = \kappa_\alpha + \kappa_\beta - d'a = a$; here d' is the corresponding label, which necessarily equals $+1$. Now note that $Y \cap (-Y) = \emptyset$ and use [Bou68, VI, §1, Prop. 22] to deduce that there exists a basis of R such that Y is contained in the set of positive roots R^+ with respect to this basis.

Finally, observing that any triangle with a nonzero label and with two positive roots by construction has label $+1$, we prove Conjecture 13.15.

The rest of the proof follows from Theorems 13.4, 13.8 and Remark 15.4. ■

14. Appendix: Inner automorphisms of finite order of simple Lie algebras

Recall [Hel00, Ch. X] that any such automorphism of a simple Lie algebra $\mathfrak{g}$ is conjugate to a model one which can be described as follows. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let R stand for the corresponding root system. Pick a basis $B = \{\alpha_1, \ldots, \alpha_n\}$ of R and let $a_0 = 1, a_1, \ldots, a_n$ be the labels of the roots in the Dynkin diagram $\widetilde{\Pi}$ of the extended system $\{\alpha_0, \ldots, \alpha_n\}$ (here α_0 denotes the lowest root and the *labels* are defined by $-\alpha_0 = \sum_{l=1}^n a_l\alpha_l$). Let $s_0, \ldots, s_n$ be nonnegative integers without nontrivial common factor. Then there exists a system of canonical generators $X_0, \ldots, X_n, X_k \in \mathfrak{g}_{\alpha_k}$, of the Lie algebra $\mathfrak{g}$ such that the formula $\sigma(X_k) = e^{s_k(2i\pi/m)}X_k$, where $m = \sum_{k=0}^n a_k s_k$, defines uniquely an inner automorphism of m-th order, called a *model automorphism of type* $(s_0, \ldots, s_n; 1)$ (the number 1 after the semicolon means that the corresponding automorphism is inner unlike those related to automorphisms of the Dynkin diagram of order 2 and 3). Any conjugate automorphism is said to be an *automorphism of type* $(s_0, \ldots, s_n; 1)$.

The fixed point subalgebra $\mathfrak{g}_0$ of the automorphism σ contains $\mathfrak{h}$ and is a direct sum of its $(n-p)$-dimensional centre and the semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ whose Dynkin diagram π is the subdiagram of $\widetilde{\Pi}$ consisting of the vertices $i_1, \ldots, i_p$ for which $s_{i_1} = \cdots = s_{i_p} = 0$. In particular, if $s_0 \neq 0$, the subalgebra $\mathfrak{g}_0$ is a Levi subalgebra, and if $s_0 = 0$, the subalgebra $\mathfrak{g}_0$ is a regular subalgebra of Class (2) (see Appendix 15).

15. Appendix: Toral gradings and regular reductive Lie subalgebras

In this appendix we discuss toral gradings of semisimple Lie algebras $\mathfrak{g}$ corresponding to regular reductive subalgebras $\mathfrak{g}_0$. In particular, we recall [Ost] that the family of irreducible components of the natural representation of $\mathfrak{g}_0$ in $\mathfrak{g}$ forms a group grading. We also outline the classification of regular reductive subalgebras [Dyn52], [DO02].

Definition 15.1. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$ with respect to some Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We say that a Γ-grading $\mathfrak{g} = \bigoplus_{i\in\Gamma} \mathfrak{g}_i$, where Γ is an abelian group, is *toral* with respect to $\mathfrak{h}$ if the subspaces $\mathfrak{g}_i$ are spanned by the subspaces $\mathfrak{h}, \mathfrak{g}_\alpha$ and, moreover, $\mathfrak{g}_0 \supset \mathfrak{h}$; here 0 is the neutral element of Γ.

Let $\mathfrak{g}_0$ be a subalgebra reductive in $\mathfrak{g}$ such that $\mathfrak{g}_0 \supset \mathfrak{h}$. In particular the representation $x \mapsto \operatorname{ad}_{\mathfrak{g}} x$ of the Lie algebra $\mathfrak{g}_0$ in $\mathfrak{g}_0^\perp$ (here $\mathfrak{g}_0^\perp$ is the orthogonal complement to $\mathfrak{g}_0$ in $\mathfrak{g}$ with respect to the Killing form) is semisimple.

Let $R_0 := \{\alpha \in R \mid \mathfrak{g}_\alpha \subset [\mathfrak{g}_0, \mathfrak{g}_0]\} \subset R \subset \mathfrak{h}^*_{\mathbb{R}}$ be the root system of the subalgebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ considered as a subsystem of R and let $Q_0 \subset Q \subset \mathfrak{h}^*_{\mathbb{R}}$ be the corresponding root lattices of $[\mathfrak{g}_0, \mathfrak{g}_0]$ and $\mathfrak{g}$. Given $\alpha \in Q$, put $\mathfrak{g}_{\alpha+Q_0} := \bigoplus_{\beta \in (\alpha+Q_0)\cap R} \mathfrak{g}_\beta$.

Theorem 15.2. *[Ost, Section 6.2] The decomposition $\mathfrak{g} = \bigoplus_{\alpha+Q_0 \in Q/Q_0} \mathfrak{g}_{\alpha+Q_0}$ is a toral Q/Q_0-grading. Moreover, the subspaces $\mathfrak{g}_{\alpha+Q_0}, \alpha \notin Q_0$, are the irreducible components of the representation of $\mathfrak{g}_0$ in $\mathfrak{g}_0^\perp$.*

Let $\mathfrak{h}'$ be another Cartan subalgebra such that $\mathfrak{h}' \subset \mathfrak{g}_0$ and let Q', Q'_0 stand for the corresponding root lattices. Then there exists $\varphi \in \mathrm{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{h}) = \mathfrak{h}'$. It is easy to see that $(\varphi|_{\mathfrak{h}})^T(Q') = Q$ and $(\varphi|_{\mathfrak{h}})^T(Q'_0) = Q_0$; here $(\varphi|_{\mathfrak{h}})^T$ is the transpose of the operator $\varphi|_{\mathfrak{h}}$. In particular, the groups $\Gamma(\mathfrak{h}, \mathfrak{g}_0) := Q/Q_0$ and $\Gamma(\mathfrak{h}', \mathfrak{g}_0) := Q'/Q'_0$ are isomorphic. The theorem just cited shows that the grading mentioned therein does not depend on the choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$.

Definition 15.3. Let $\mathfrak{g}_0 \subset \mathfrak{g}$ be a subalgebra of maximal rank reductive in $\mathfrak{g}$ (call such a subalgebra *regular*) and let $\Gamma(\mathfrak{g}_0)$ be an abstract group isomorphic to one of the groups $\Gamma(\mathfrak{h}, \mathfrak{g}_0)$, where $\mathfrak{h}$ is a Cartan subalgebra such that $\mathfrak{h} \subset \mathfrak{g}_0$. The grading constructed above will be called the *irreducible toral* $\Gamma(\mathfrak{g}_0)$*-grading* corresponding to the subalgebra $\mathfrak{g}_0$.

Regular subalgebras can be divided into the following three classes depending on the structure of the group $\Gamma(\mathfrak{g}_0)$ (cf. [DO02]):

(1) $\Gamma(\mathfrak{g}_0)$ is free;

(2) the torsion component $\mathrm{Tor}(\Gamma(\mathfrak{g}_0))$ of $\Gamma(\mathfrak{g}_0)$ is cyclic;

(3) $\mathrm{Tor}(\Gamma(\mathfrak{g}_0))$ is not cyclic.

In the first case the subalgebra $\mathfrak{g}_0$ is the so-called Levi subalgebra, which can be also characterized by the following equivalent conditions ([GOV94, Th. 1.3, Ch. 6]):

- $\mathfrak{g}_0$ is a centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{c})$ of some subspace $\mathfrak{c} \subset \mathfrak{h}$, where $\mathfrak{h}$ is some Cartan subalgebra.
- $\mathfrak{g}_0$ contains a Cartan subalgebra $\mathfrak{h}$ and the corresponding system $R_0 \subset R$ of roots of $\mathfrak{g}_0$ is generated over $\mathbb{Z}$ by some subset B_0 of some system $B(\mathfrak{g})$ of simple roots of $\mathfrak{g}$.

To describe the other two cases recall [Dyn52, GOV94] that any regular subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ can be obtained as the last element in a nested sequence of subalgebras $\mathfrak{g} = \mathfrak{g}^0 \supset \mathfrak{g}^1 \supset \cdots \supset \mathfrak{g}^k = \mathfrak{g}_0$, where each $\mathfrak{g}^j, j = 1, \ldots, k-1$, is obtained from $\mathfrak{g}^{j-1}$ by an elementary transformation and $\mathfrak{g}_0$ is a Levi subalgebra in $\mathfrak{g}^{k-1}$. We say that a semisimple subalgebra $\mathfrak{k} \subset \mathfrak{g}$ of a semisimple Lie algebra

$\mathfrak{g} = \bigoplus_{j\in J} \mathfrak{g}_j$, where $\mathfrak{g}_j$ are the simple components, is obtained by an *elementary transformation* from $\mathfrak{g}$ if $\mathfrak{k} = \bigoplus_{j\neq j'} \mathfrak{g}_j \oplus \tilde{\mathfrak{g}}_{j'}$ for some $j' \in J$. Here $\tilde{\mathfrak{g}}_{j'}$ is a subalgebra in $\mathfrak{g}_{j'}$ such that its system of simple roots $B(\tilde{\mathfrak{g}}_{j'})$ is obtained by eliminating some root $\alpha_i, i > 0$, from the system $B(\mathfrak{g}_{j'})\cup\{\alpha_0\}$ of simple roots of $\mathfrak{g}_{j'}$ extended by the lowest root α_0. In other words, $\tilde{\mathfrak{g}}_{j'}$ is the fixed point subalgebra of an (inner) automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g}_{j'})$ of order a_i of type $(0,\ldots,0,1,0,\ldots,0;1)$ (the first unit is on the i-th place and $i > 0$) in terminology of [Hel00, Ch. X] (see also Appendix 14); here a_i is the label of the root α_i.

The equivalent characterization of regular subalgebras of Class (2) (respectively (3)) is, [DO02], that an elementary transformation in the sequence $\mathfrak{g} \supset \cdots \supset \mathfrak{g}_0$ is made only once (respectively more than once).

Remark 15.4. The structure of the extended Dynkin diagram of the Lie algebra $\mathfrak{g} = \mathfrak{a}_n$ implies that any regular subalgebra in $\mathfrak{g}$ is a Levi subalgebra.

16. Appendix: Bi-Lie structures of Class I on compact real forms of complex semisimple Lie algebras

Theorem 16.1. *Let $(\mathfrak{g}, [\,,])$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$ the corresponding root grading. Let $\mathfrak{g} = \bigoplus_{t_i,t_j\in X} \mathfrak{g}_{t_it_j}$ be an admissible pairoid quasigrading (see Definition 12.4) with the base $X = \{t_1,\ldots,t_n\}$, where $t_i \in \mathbb{R}$, such that the subspaces $\mathfrak{g}_{t_it_i} \cap \mathfrak{h}$ are the complexifications of subspaces in $\mathfrak{h}_\mathbb{R}$.*

For any $\alpha \in R$ choose $E_\alpha \in \mathfrak{g}_\alpha$ such that $B_\mathfrak{g}(E_\alpha, E_{-\alpha}) = 1$ and put $H_\alpha := [E_\alpha, E_{-\alpha}]$ as usual. Let

$$\mathfrak{u} = \textstyle\sum_{\alpha\in R} \mathbb{R}(iH_\alpha) + \sum_{\alpha\in R} \mathbb{R}(E_\alpha - E_{-\alpha}) + \sum_{\alpha\in R} \mathbb{R}(i(E_\alpha + E_{-\alpha})) \subset \mathfrak{g}$$

be the compact real form of $\mathfrak{g}$ related to the root decomposition and the choice of E_α (see [Hel00, Th. 6.3, Ch. III]).

Then the WNO W built by the quasigrading (see Theorem 12.8) restricts to $\mathfrak{u}$ and its restriction is a real WNO and induces a real bi-Lie structure on $\mathfrak{u}$.

Proof. The fact that W preserves $\mathfrak{u}$ follows from the assumptions and from the construction of W (recall that $W|_{\mathfrak{h}^\perp}$ is symmetric: $W|_{\mathfrak{g}_\alpha} = W|_{\mathfrak{g}_{-\alpha}}$). The principal primitive P of W is symmetric too (see Section 7 or check this directly using the formulae from the proof of Theorem 12.8). Thus the "main identity" $T_W(\,,) = [\,,]_P$ implies $T_{N|_\mathfrak{u}}(\,,) = [\,,]_{P|_\mathfrak{u}}$. ∎

Example 16.2. From Example 12.17 we get the following example of WNO on $\mathfrak{su}(n+1)$: $WX := (1/2)(L_A + R_A)X - \mathrm{Tr}((1/2)(L_A + R_A)X)B$, where $A = \mathrm{diag}(t_1, t_2,\ldots,t_{n+1}), B = \mathrm{diag}(0,0,\ldots,0,1)$ and $t_i \in \mathbb{R}$. Explicitly, if $X = ||x_{ij}|| \in \mathfrak{su}(n+1)$, then $WX = ||y_{ij}||$, where $y_{ij} = x_{ij}(t_i + t_j)/2$ for $i \neq j$, $y_{ii} = x_{ii}t_i$ for $i = 1,\ldots,n$, and $y_{n+1,n+1} = -\sum_{j=1}^n x_{jj}t_j$.

An analogous example can be obtained from Example 12.18 taking the parameter a to be real.

Example 16.3. Here we will only mention the existence of a nonstandard bi-Lie structure on $\mathfrak{so}(6,\mathbb{R})$ coming from the isomorphism $\mathfrak{so}(6,\mathbb{C}) \cong \mathfrak{sl}(4,\mathbb{C})$ and Examples 12.17, 12.18.

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Added in Proof, February 18, 2014. The present version replaces an earlier version published online in December 2013. The revision was necessitated by the discovery of an erroneous statement in Example 13.14 presenting a bi-Lie structure on the exceptional Lie algebra $\mathfrak{e}_7$: the regular reductive subalgebra $\mathfrak{g}_0$ used in this example is of Class (2), not (3), as was claimed. This mistake is corrected in the present version, where in Example 13.14 a bi-Lie structure on the exceptional Lie algebra $\mathfrak{e}_8$ is built with the corresponding regular reductive subalgebra $\mathfrak{g}_0$ of Class (3). In particular this guarantees that the example is new, as was stated in the Introduction, since in all the examples known in the literature, $\mathfrak{g}_0$ is of Class (1) or (2).

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