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Lifting Automorphisms of Quotients of Adjoint Representations

Gerald W. Schwarz

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Abstract. Let $\mathfrak{g}_i$ be a simple complex Lie algebra, $1 \leq i \leq d$, and let $G = G_1 \times \cdots \times G_d$ be the corresponding adjoint group. Consider the G-module $V = \oplus r_i \mathfrak{g}_i$ where $r_i \in \mathbb{N}$ for all i. We say that V is *large* if all $r_i \geq 2$ and $r_i \geq 3$ if G_i has rank 1. In Quotients, automorphisms and differential operators, http://arxiv.org/abs/1201.6369 (2012), we showed that when V is large any algebraic automorphism ψ of the quotient $Z := V/\!\!/G$ lifts to an algebraic mapping $\Psi\colon V \to V$ which sends the fiber over z to the fiber over $\psi(z)$, $z \in Z$. (Most cases were already handled in J. Kuttler, Lifting automorphisms of generalized adjoint quotients, Transformation Groups **16** (2011), 1115–1135.) We also showed that one can choose a biholomorphic lift Ψ such that $\Psi(gv) = \sigma(g)\Psi(v)$, $g \in G$, $v \in V$, where σ is an automorphism of G. This leaves open the following questions: Can one lift holomorphic automorphisms of Z? Which automorphisms lift if V is not large? We answer the first question in the affirmative and also answer the second question. Part of the proof involves establishing the following result for V large. Any algebraic differential operator of order k on Z lifts to a G-invariant algebraic differential operator of order k on V. We also consider the analogues of the questions above for actions of compact Lie groups.
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1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let G be a complex reductive group and X a smooth affine G-variety. We denote the algebra of polynomial functions on X by $\mathcal{O}(X)$. For the following, we refer to [Kra84], [Lun73] and [VP89]. By Hilbert, the algebra $\mathcal{O}(X)^G$ is finitely generated, so that we have a quotient variety $Z := X/\!\!/G$ with coordinate ring $\mathcal{O}(Z) = \mathcal{O}(X)^G$. Let $\pi\colon X \to Z$ denote the morphism dual to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$. Then π sets up a bijection between the points of Z and the closed orbits in X. If Gx is a closed orbit, then the isotropy group $H = G_x$ is reductive. The *slice representation of H at x* is its action on N_x where N_x is an H-complement to $T_x(Gx)$ in $T_x(X)$. Let $Z_{(H)}$ denote the points of Z such that the isotropy groups of the

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corresponding closed orbits are in the conjugacy class (H) of H. The $Z_{(H)}$ give a finite stratification of Z by locally closed smooth subvarieties. In particular, there is a unique open stratum $Z_{(H)}$, the *principal stratum*, which we also denote by Z_{pr}. We call H a *principal isotropy group* and any associated closed orbit a *principal orbit* of G.

As shorthand for saying that X has finite principal isotropy groups we say that X has FPIG. If X has FPIG, then there is an open set of closed orbits and a closed orbit is principal if and only if the slice representation of its isotropy group is trivial. Set $X_{\text{pr}} := \pi^{-1}(Z_{\text{pr}})$. We say that X is *k-principal* if X has FPIG and $\operatorname{codim} X \setminus X_{\text{pr}} \geq k$.

For Y an affine variety let $\operatorname{Aut}(Y)$ (resp. $\operatorname{Aut}_h(Y)$) denote the automorphisms (resp. biholomorphisms) of Y. If $y \in Y$, we denote by $\operatorname{Aut}_h(Y, y)$ the germs of biholomorphisms of Y which fix y.

Our main interest is in the following case. Let $V = \oplus_{i=1}^{d} r_i\mathfrak{g}_i$ where the $\mathfrak{g}_i$ are simple complex Lie algebras and $r_i\mathfrak{g}_i$ denotes the direct sum of r_i copies of $\mathfrak{g}_i$, $r_i \in \mathbb{N}$. Let G_i denote the adjoint group of $\mathfrak{g}_i$ and let G denote the product of the G_i. We denote by $0 \in Z := V/\!\!/G$ the image of $0 \in V$. Let $\psi \in \operatorname{Aut}_h(Z)$. We say that $\Psi \in \operatorname{Aut}_h(V)$ is a *lift of* ψ if $\pi \circ \Psi = \psi \circ \pi$. Equivalently, Ψ maps the fiber $\pi^{-1}(z)$ to the fiber $\pi^{-1}(\psi(z))$, $z \in Z$. Let σ be an automorphism of G. We say that Ψ is *σ-equivariant* if $\Psi(gv) = \sigma(g)\Psi(v)$ for all $v \in V$, $g \in G$. We say that V is *large* if $r_i \geq 2$ for all i and $r_i \geq 3$ if $\mathfrak{g}_i \simeq \mathfrak{sl}_2$.

Lemma 1.1. *The following are equivalent:*

1. *V is large.*
2. *Z has no codimension one strata.*
3. *V is 2-principal.*
4. *Any $\psi \in \operatorname{Aut}_h(Z)$ preserves the stratification of Z.*
5. *No $(r_i\mathfrak{g}_i)/\!\!/G_i$ is smooth.*

Proof. By [Sch13, Proposition 3.1], (1) and (2) are equivalent. Since G is semisimple, [Sch80, Corollary 7.4] shows that (2) implies (3). Since V is an orthogonal representation, [Sch13, Theorem 1.2] shows that (3) implies (4), which in turn clearly implies (5). Since some $(r_i\mathfrak{g}_i)/\!\!/G_i$ is smooth when V is not large, (5) implies (1) ∎

The lifting problem for V has been investigated in [Kut11] and [Sch12]. In [Sch12] we showed that V has the following lifting property.

Theorem 1.2. *Let V be large. Then for any $\psi \in \operatorname{Aut}(Z)$ there is a morphism $\Psi\colon V \to V$ such that $\pi \circ \Psi = \psi \circ \pi$.*

Most cases of the theorem were first established by Kuttler [Kut11]. The lift given in the theorem is not necessarily an automorphism of V nor is it necessarily σ-equivariant for some σ. From [Sch12] we get the following

Corollary 1.3. *Let V and ψ be as above. Then there is an automorphism σ of G and a σ-equivariant lift $\Psi \in \mathrm{Aut}_h(V)$. Hence ψ sends $Z_{(H)}$ to $Z_{(\sigma(H))}$ for every stratum $Z_{(H)}$ of Z.*

The action of ψ on the strata is obtained by a different method in [Kut11].

The above results lead to the following

Problems 1.4. 1. Can we lift elements of $\mathrm{Aut}_h(Z)$ to σ-equivariant elements of $\mathrm{Aut}_h(V)$?

2. Which automorphisms lift if V is not large?

We find necessary and sufficient conditions for a ψ in $\mathrm{Aut}_h(Z)$ to have a holomorphic σ-equivariant lift. The extra conditions that one needs have to do with the codimension one strata of Z.

Suppose that some r_i is 1. Then $Z_i := \mathfrak{g}_i/\!\!/G_i \simeq \mathfrak{t}_i/\mathcal{W}_i$ where $\mathfrak{t}_i$ is the Lie algebra of a maximal torus of G_i and $\mathcal{W}_i$ is the Weyl group. Now the closures of the codimension one strata of Z_i are the images of the reflection hyperplanes in $\mathfrak{t}_i$. If $\mathfrak{g}_i$ has two root lengths, then we get two strata of codimension one in Z, corresponding to the reflection hyperplanes of the short roots and the long roots. Let $D_{i,s}$ and $D_{i,\ell}$ be the corresponding divisors of Z. It $\mathfrak{g}_i$ is simply laced, then $D_{i,s} = \emptyset$. Let D_s and D_ℓ be the union of the $D_{i,s}$ and $D_{i,\ell}$ for $r_i = 1$.

There is one other way of getting a codimension one stratum, which occurs if $r_i = 2$ and $\mathfrak{g}_i \simeq \mathfrak{sl}_2$. Then $(2\mathfrak{sl}_2, \mathrm{PSL}_2) \simeq (2\mathbb{C}^3, \mathrm{SO}_3)$. Thus the quotient is $\mathbb{C}^3$ and the closure of the codimension one stratum is the zero set D_i of a non degenerate quadratic form $xy - z^2$. The corresponding isotropy class is (SO_2). Let D_0 denote the union of the D_i for $r_i = 2$ and $\mathfrak{g}_i \simeq \mathfrak{sl}_2$. Then the closure of the codimension one strata of Z is $D := D_0 \cup D_s \cup D_\ell$.

Here is our main theorem.

Theorem 1.5. *Let V, G, D and D_s be as above. Let $\psi \in \mathrm{Aut}_h(Z)$. Then ψ has a σ-equivariant biholomorphic lift for some σ if and only if ψ preserves D and D_s.*

If $\Psi\colon V \to V$ is a σ-equivariant biholomorphic lift of ψ, then ψ sends $Z_{(H)}$ to $Z_{\sigma(H)}$ for all (H). Thus ψ permutes the strata, in particular the codimension one strata. Moreover, if (H) corresponds to the short roots of some $\mathfrak{g}_i$ (so $r_i = 1$), then $(\sigma(H))$ corresponds to the short roots of some $\mathfrak{g}_j$ with $r_j = 1$. Hence ψ has to stabilize D_s and we see that the conditions of the theorem are necessary. For sufficiency we proceed in several steps. The first step, taken in §2, is to show that if ψ preserves D, then ψ *preserves the stratification*, i.e., it permutes the strata. If ψ also preserves D_s, then we say that ψ is *strongly stratification preserving*.

Next we need to use an action of $\mathbb{C}^*$ on Z. For $z = \pi(v)$, $t \cdot z = \pi(tv)$, $t \in \mathbb{C}^*$. One can also see the action as follows. Let $p_1, \dots, p_r$ be homogeneous generators of $\mathcal{O}(V)^G$ and let $p = (p_1, \dots, p_r)\colon V \to \mathbb{C}^r$. Let Y denote the image of p. Then we can identify Y with $Z = V/\!\!/G$. If e_i is the degree of p_i, then we have a $\mathbb{C}^*$-action on Y where $t \in \mathbb{C}^*$ sends $(y_1, \dots, y_r) \in Y$ to $(t^{e_1}y_1, \dots, t^{e_r}y_r)$. The isomorphism $Y \simeq Z$ identifies the two $\mathbb{C}^*$-actions.

We say that $\psi \in \operatorname{Aut}(Z)$ is *quasilinear* if $\psi(t \cdot z) = t \cdot \psi(z)$ for all $z \in Z$ and $t \in \mathbb{C}^*$. We write $\psi \in \operatorname{Aut}_{q\ell}(Z)$. A main idea, as in [Sch12], is to deform a general ψ to one that is quasilinear. Assume that $\psi(0) = 0$. For $t \in \mathbb{C}^*$, let $\psi_t(z)$ denote $t^{-1} \cdot \psi(t \cdot z)$. We say that $\psi \in \operatorname{Aut}_h(Z, 0)$ is *deformable* if $\psi_0(z) := \lim_{t \to 0} \psi_t(z)$ exists for all z sufficiently close to 0. Let $S = \oplus S_k$ denote the graded ring $\mathcal{O}(Z)$, where the grading comes from the $\mathbb{C}^*$-action. Let $\mathcal{H}(Z)$ denote the holomorphic functions on Z and $\mathcal{H}(Z, 0)$ the germs of holomorphic functions at $0 \in Z$. Then ψ is deformable if and only if $\psi^* S_k \subset S_k \cdot \mathcal{H}(Z, 0)$ for all k. Equivalently, $\psi^* y_i \in S_{e_i} \cdot \mathcal{H}(Z, 0)$ for all i. This is a vanishing condition on some of the partial derivatives of ψ at 0. If ψ is deformable, then ψ_0^* sends y_i to the term homogeneous of degree e_i of the Taylor series of $\psi^* y_i$ at 0, considered as an element of $\mathcal{O}(Z)$. Clearly, if ψ is deformable, the family ψ_t (including $t = 0$) is holomorphic in z and t. (If $\psi = \psi(x, z)$ also depends holomorphically on parameters $x \in X$, then the family $\psi_t(x, z)$ is holomorphic in t, x and z.) Each ψ_t preserves the germ of Z at 0 and ψ_0 preserves Z. If ψ and ψ^{-1} are deformable, then $(\psi^{-1})_0$ is an inverse to ψ_0 so that $\psi_0 \in \operatorname{Aut}_{q\ell}(Z)$. If ψ is also (strongly) stratification preserving, then so is ψ_0 since the strata are invariant under the action of $\mathbb{C}^*$. We say that V is *good* if every stratification preserving $\psi \in \operatorname{Aut}_h(Z, 0)$ is deformable. We say that V has the *lifting property* if every strongly stratification preserving $\psi \in \operatorname{Aut}_{q\ell}(Z)$ has a σ-equivariant lift $\Psi \in \operatorname{GL}(V)$ for some σ. Then $\Psi \circ g \circ \Psi^{-1} = \sigma(g)$, so that $\Psi \in N_{\operatorname{GL}(V)}(G)$, the normalizer of G in $\operatorname{GL}(V)$.

Theorem 1.6. *Let (V, G) be as above. Then*

1. *V is good.*
2. *V has the lifting property*

We deduce our main theorem from the one above using a result on lifting of isotopies as in [Sch80]. Our proof of Theorem 1.6 breaks up into three parts. Recall that $V = \oplus_{i=1}^d r_i \mathfrak{g}_i$. Let V_1 be the sum of the submodules $r_i \mathfrak{g}_i \simeq 2\mathfrak{sl}_2$, let V_2 be the sum of the submodules $r_i \mathfrak{g}_i$ with $r_i = 1$ and let V_3 be the sum of the $r_i \mathfrak{g}_i$ which are large. Let $\widetilde{G}_i$ be the image of G in $\operatorname{GL}(V_i)$, $1 \leq i \leq 3$. We show that Theorem 1.6 holds for each $(V_i, \widetilde{G}_i)$. The first two cases are handled in §3 and §4. The third case is a bit more difficult.

Let W be an H-module where H is reductive. Set $Y := W/\!\!/H$ and let $\pi\colon W \to Y$ be the quotient morphism. Let $\mathcal{D}^k(W)$ (resp. $\mathcal{D}^k(Y)$) denote the algebraic differential operators on W (resp. Y) of order at most k (see [Sch95, §3]). Then we have a morphism $\pi_*\colon \mathcal{D}^k(W)^H \to \mathcal{D}^k(Y)$ where $\pi_*(P)(f) = P(\pi^*(f))$ for $P \in \mathcal{D}^k(W)^H$ and $f \in \mathcal{O}(Y) \simeq \mathcal{O}(W)^H$.

Definition 1.7. We say that the H-module W is *admissible* if

1. W is 2-principal.

2. $\pi_*\colon \mathcal{D}^k(W)^H \to \mathcal{D}^k(Y)$ is surjective for all k.

From [Sch12, Theorem 2.2] we have the following result:

Theorem 1.8. *If W is admissible, then W is good.*

([Sch12, Theorem 2.2] is actually stated for $\operatorname{Aut}_h(Y)$, but the proof for $\operatorname{Aut}_h(Y,0)$ is the same.) In sections §5 and §6 we show that $(V_3,\widetilde{G}_3)$ is admissible, hence good. From [Sch12, 1.9, 1.16] we see that V_3 has the lifting property. Thus each $(V_i,\widetilde{G}_i)$ satisfies the conclusions of Theorem 1.6. In §7 we combine these facts to prove Theorems 1.5 and 1.6. In §8 we consider the analogues of our results for compact Lie groups.

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2. Preserving the stratification

Let H be reductive and W an H-module. Let $Y = W/\!\!/H$. For (L) and (M) conjugacy classes of subgroups of H we write $(M) \geq (L)$ if L is conjugate to a subgroup of M. From [Sch80, Lemma 5.5] one has

Lemma 2.1. *Let $S = Y_{(L)}$ be a stratum of Y. Then S is irreducible and*

$$\overline{S} = \bigcup_{(M)\geq(L)} Y_{(M)}$$

Let $\mathfrak{X}(Y)$ denote the strata preserving vector fields on Y. These are the derivations of $\mathcal{O}(Y)$ which are tangent to S along S for every stratum S of Y. Equivalently, they are the derivations preserving the ideals vanishing on the closures of the strata of Y. Let $\mathfrak{X}_h(Y)$ denote the holomorphic strata preserving vector fields on Y and let $\mathfrak{X}_h(Y,0)$ denote the corresponding germs at 0. Let $\mathfrak{X}(W)$ denote the vector fields on W. Then it is easy to see that any element of $\mathfrak{X}(W)^H$ preserves the pull-backs of the ideals vanishing on the closures of the strata of Z [Sch80, Corollary 1.3]. Thus we have a natural morphism $\pi_*\colon \mathfrak{X}(W)^H \to \mathfrak{X}(Y)$. We also have $\pi_*\colon \mathfrak{X}_h(W)^H \to \mathfrak{X}_h(Y)$ and similarly for $\mathfrak{X}_h(W,0)^H$ and $\mathfrak{X}_h(Y,0)$. Now we assume that W is orthogonal.

Proposition 2.2. *Let H be reductive and W an orthogonal H-module.*

1. *Let A be a derivation of $\mathcal{O}(Y)$ which preserves the codimension one strata of Y. Then A preserves all the strata of Y. The same result holds for holomorphic derivations of $\mathcal{H}(Y)$ and germs of holomorphic derivations at $0 \in Y$ [Sch80, 3.5, 5.8, 6.1].*

2. *Let $A \in \mathfrak{X}(Y)$ (resp. $\mathfrak{X}_h(Y)$, resp. $\mathfrak{X}_h(Y,0)$). Then there is a $B \in \mathfrak{X}(W)^H$ (resp. $\mathfrak{X}_h(W)^H$, resp. $\mathfrak{X}_h(W,0)^H$) such that $\pi_*B = A$ [Sch80, 3.7, 6.7, 6.9].*

Corollary 2.3. *Let $\psi \in \mathrm{Aut}(Y)$ such that ψ preserves the union of the codimension one strata. Then ψ preserves the stratification of Y.*

Proof. We follow the argument of [Sch13]. Let $A \in \mathfrak{X}(Y)$ and define $\psi^*(A) = \psi^* \circ A \circ (\psi^*)^{-1}$. This is a derivation of $\mathcal{O}(Y)$ which, by Proposition 2.2, again lies in $\mathfrak{X}(Y)$. Thus ψ^* preserves $\mathfrak{X}(Y)$. One computes that, in terms of tangent vectors, $(\psi^* A)(y) = d(\psi^{-1})_{\psi(y)} A(\psi(y))$, $y \in Y$. Let $s \in S$ where S is a stratum of Y. Then the evaluation at s of all elements of $\pi_* \mathfrak{X}(W)^H = \mathfrak{X}(Y)$ is precisely $T_s S$. This is established in [Sch13, Proposition 2.2]. Now the dimension of $\mathfrak{X}(Y) = \psi^* \mathfrak{X}(Y)$ evaluated at s is the same as the dimension of $\mathfrak{X}(Y)$ evaluated at $\psi(s)$. Hence the stratum S' containing $\psi(s)$ has the same dimension as S. Since there are finitely many strata and the strata are irreducible, we see that $\psi(S) = S'$. Hence ψ permutes the strata of Y. ■

Remark 2.4. The same argument works to show that if $\psi \in \mathrm{Aut}_h(Y)$ or $\mathrm{Aut}_h(Y, 0)$ and ψ preserves the union of the codimension one strata, then it preserves the stratification.

3. Copies of $2\mathfrak{sl}_2$

Let $(W_i, H_i) = (2\mathfrak{sl}_2, \mathrm{PSL}_2) \simeq (2\mathbb{C}^3, \mathrm{SO}_3)$, $i = 1, \dots, n$. Let $W = \oplus W_i$ have the product action of $H = \prod_i H_i$ with quotient Y. Let Y_i denote $W_i /\!\!/ H_i$. Then $Y = Y_1 \times \cdots \times Y_n$ where each Y_i is isomorphic to $\mathbb{C}^3$, and the non principal strata of Y_i are the nonzero points in a cone $\mathrm{Var}(xy - z^2)$ (denoted S_i) and the origin of $\mathbb{C}^3$. Let S_i' denote the product of S_i and the principal strata of the other Y_j. The S_i' are the codimension one strata of Y.

Proposition 3.1. *The H-module W is good and has the lifting property.*

Proof. Let $\psi \in \mathrm{Aut}_h(Y)$ preserve the codimension one strata. Then ψ preserves the stratification. The point 0 is the lowest dimensional stratum, hence it is preserved. Since each $\mathcal{O}(W_i)^{H_i}$ is generated by quadratic invariants, ψ_0 exists and is just $d\psi(0)$, considered as a mapping of $Y \simeq T_0(Y)$ to itself. The same argument works in case that $\psi \in \mathrm{Aut}_h(Y, 0)$. Hence W is good.

Now let ψ be quasilinear and strata preserving. Then ψ permutes the S_i' and modulo a permutation of the W_i (which induces an automorphism σ of H), we may reduce to the case that ψ preserves each S_i'. Now $\overline{S_i'}$ is the product of $\overline{S_i}$ with the other Y_j. The singular points of this space are the product of $0 \in Y_i$ with the product of the other Y_j. By taking intersections we see that ψ preserves the product of Y_i with the origin of the other Y_j. Thus $\psi = \mathrm{diag}(A_1, \dots, A_n)$ where $A_i \in \mathrm{GL}(3)$ preserves the cone $\overline{S_i}$, $i = 1, \dots, n$. But then A_i is in $\mathbb{C}^*$ times the orthogonal group of $xy - z^2$. This is the image of the group GL_2 which acts on $2\mathfrak{sl}_2 \simeq \mathfrak{sl}_2 \otimes \mathbb{C}^2$ via its action on $\mathbb{C}^2$, commuting with the action of PSL_2. Hence ψ lifts and W has the lifting property. ■

4. The adjoint representation

We now consider the case where $(W, H) = (\mathfrak{h}, H)$ and H is semisimple. We have $\mathfrak{h} = \oplus_{i=1}^{n} \mathfrak{h}_i$ where $\mathfrak{h}_i$ is simple. Let H_i denote the corresponding adjoint group so that $H = \prod_i H_i$. Let $\mathfrak{t}_i$ be a maximal toral subalgebra of $\mathfrak{h}_i$ and let $\mathcal{W}_i$ be the corresponding Weyl group. Set $\mathfrak{t} = \oplus_i \mathfrak{t}_i$ and $\mathcal{W} = \prod_i \mathcal{W}_i$. Let Y_i denote $\mathfrak{h}_i /\!\!/ H_i \simeq \mathfrak{t}_i/\mathcal{W}_i$. Define D_s and D_ℓ as in the introduction.

Let $\psi \in \mathrm{Aut}_h(Y, 0)$ preserve D_s and D_ℓ. First we only use the information that ψ preserves $D_s \cup D_\ell$. Let $\tilde{\pi}$ denote the quotient mapping $\mathfrak{t} \to Y = Y_1 \times \cdots \times Y_n$.

Proposition 4.1. *Let ψ be as above. Then*

1. *ψ is stratification preserving. In particular, $\psi(Y_{\mathrm{pr}}) \subset Y_{\mathrm{pr}}$.*
2. *ψ has a lift $\tau \in \mathrm{Aut}_h(\mathfrak{t}, 0)$ which is σ-equivariant for some $\sigma \in \mathrm{Aut}(\mathcal{W})$.*
3. *ψ is deformable and ψ_0 lifts to $\tau'(0) \in N_{\mathrm{GL}(\mathfrak{t})}(\mathcal{W})$.*
4. *W is good.*

Proof. Part (1) follows from Corollary 2.3. Then [Lya83], [KLM03, Theorem 5.4] or [Sch09, Theorem 3.1] show that ψ has a σ-equivariant lift τ to $\mathfrak{t}$, giving (2). Clearly $\tau(0) = 0$. Let $x \in \mathfrak{t}$ and set $y = \tilde{\pi}(x)$. Then $\lim_{t\to 0} \psi_t(y) = \tilde{\pi}(\lim_{t\to 0} t^{-1}\tau(tx)) = \tilde{\pi}(\tau'(0)x)$. Thus ψ is deformable and we have (3) and (4). ■

We are not done yet. If ψ is quasilinear, then we have a linear lift τ which is σ-equivariant. We have to lift τ to $\mathfrak{h}$. We have not used yet that ψ preserves D_s. Let $\Phi \subset \mathfrak{t}^*$ be the roots of $\mathfrak{h}$. For $\alpha \in \Phi$, let τ_α denote the corresponding reflection in $\mathfrak{t}$. We have the Killing form B_i on $\mathfrak{t}_i^*$. Since $\mathcal{W}_i$ acts irreducibly on $\mathfrak{t}_i^*$ for each i, any $\mathcal{W}$-invariant non degenerate bilinear form on $\mathfrak{t}^*$ has to be of the form $\oplus c_i B_i$ where $c_i \in \mathbb{C}^*$. Let $\tau \in N_{\mathrm{GL}(\mathfrak{t})}(\mathcal{W})$. Then τ permutes the factors $\mathfrak{t}_i$ so we have an action of τ on $\{1, \dots, n\}$ where $\tau(\mathfrak{t}_i) = \mathfrak{t}_{\tau(i)}$ Let $\tau^*\colon \mathfrak{t}^* \to \mathfrak{t}^*$ be composition with τ. We have the push-forward $\tau_*(B_i)$ where $\tau_*(B_i)(\xi, \eta) = B_i(\tau^*\xi, \tau^*\eta)$ for ξ, $\eta \in \mathfrak{t}_{\tau(i)}^*$. Set $\tau_* B = \oplus_i \tau_* B_i$ where $B = \oplus_i B_i$ is the Killing form. We will also denote B by $(\,,\,)$.

Lemma 4.2. *We can modify $\tau|_{\mathfrak{t}_i}$ by a scalar d_i, $1 \le i \le n$, such that $\tau_* B = B$.*

Proof. Since τ normalizes $\mathcal{W}$, $\tau_* B$ is invariant under $\mathcal{W}$, hence is of the form $\oplus c_i B_i$. Since τ acts on $\{1, \dots, n\}$ by a product of cycles, one easily sees that one can modify the restriction of τ to $\mathfrak{t}_i$ by a scalar d_i such that $\tau_* B_i = B_{\tau(i)}$, $1 \le i \le n$. Note that this modification lifts to $\mathrm{GL}(\mathfrak{h})^H \simeq (\mathbb{C}^*)^n$. ■

Lemma 4.3. *Assume that τ preserves the Killing form. Then for all $\alpha \in \Phi$, $\tau^*(\alpha) = c(\alpha)\varphi(\alpha)$ where $\varphi\colon \Phi \to \Phi$ and $c(\alpha) > 0$.*

Proof. The composition $\tau^{-1} \circ \tau_\alpha \circ \tau$ is a reflection in $\mathcal{W}$, hence it is τ_β for some $\beta \in \Phi$. Thus $\tau^*\alpha = c\beta$ for some $c \in \mathbb{C}^*$. Since τ preserves the Killing form,

$c^2(\beta,\beta)$ is positive, hence $c \in \mathbb{R}^*$. Perhaps changing β to $-\beta$, we can arrange that $c > 0$. Set $\varphi(\alpha) = \beta$ and $c(\alpha) = c$. ∎

Theorem 4.4. *Let $\tau \in N_{\mathrm{GL}(\mathfrak{t})}(\mathcal{W})$ be a lift of $\psi \in \mathrm{Aut}_{\mathrm{q}\ell}(Y)$ where ψ is strongly stratification preserving. Then τ lifts to $N_{\mathrm{GL}(\mathfrak{h})}(H)$. Hence W has the lifting property.*

Proof. We may assume that τ preserves the Killing form and induces a bijection φ of Φ where $\tau^*(\alpha) = c(\alpha)\varphi(\alpha)$ and $c(\alpha) > 0$, $\alpha \in \Phi$. Thus τ^* permutes the reflection hyperplanes in $\mathfrak{t}^*_{\mathbb{R}}$, the real span of the roots. Hence τ^* permutes the chambers and there is an element $w \in \mathcal{W}$ such that $w\tau^*$ preserves the Weyl chamber $C(\Delta)$ where Δ is a base of Φ. Hence we may assume that τ^* preserves $C(\Delta)$ and it follows that φ preserves Δ.

Let $i \in \{1,\dots,n\}$ and set $j = \tau(i)$. Let $\alpha,\ \beta \in \Delta_j$ where Δ_j is the base of $\Phi_j \subset \mathfrak{t}^*_j$. Let $\langle\beta,\alpha\rangle$ denote $2(\beta,\alpha)/(\alpha,\alpha)$. Since τ^* is an isometry we have

$$\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = \langle\tau^*(\beta),\tau^*(\alpha)\rangle\langle\tau^*(\alpha),\tau^*(\beta)\rangle = \langle\varphi(\beta),\varphi(\alpha)\rangle\langle\varphi(\alpha),\varphi(\beta)\rangle.$$

Thus $\varphi\colon \Delta_j \to \Delta_i$ preserves the product $\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle$. Since τ preserves the collection of reflection hyperplanes for the short roots, we have that $|\alpha| < |\beta|$ if and only if $|\varphi(\alpha)| < |\varphi(\beta)|$ where $|\alpha|$ is the norm of α, etc. Now the $\langle\beta,\alpha\rangle$ are negative (or zero) and one can see that the numbers $\langle\beta,\alpha\rangle$ are preserved by φ (see [Hum72, Ch. III, Table 1]). Hence the restriction of φ to Δ is a diagram automorphism. Now changing τ by a diagram automorphism, which lifts to $N_{\mathrm{GL}(\mathfrak{h})}(H)$, we can reduce to the case that φ is the identity on Δ. Thus τ^* preserves the $\mathfrak{t}^*_i$ and sends α to $c(\alpha)\alpha$, $\alpha \in \Delta_i$, where $c(\alpha) > 0$. Hence τ^* commutes with the simple reflections on $\mathfrak{t}^*$, so it centralizes $\mathcal{W}$. Since each $\mathcal{W}_i$ acts irreducibly on $\mathfrak{t}^*_i$, τ^* must be multiplication by a (positive) scalar on each $\mathfrak{t}^*_i$. Since τ^* is an isometry, we see that we have reduced to the case that τ^* is the identity. ∎

5. Modularity

We want to prove that (V,G) is admissible if it is large. We first need to recall some notions and results from [Sch94, Sch95]. Let H be a linear algebraic group and X an H-variety. Let $X_{(n)}$ or $(X,H)_{(n)}$ denote the set of points of X with isotropy group of dimension n. Then $X_{(n)}$ is constructible and we define the *modularity of* X, $\mod(X,H)$, by

$$\mod\ (X,H) = \sup_{n\geq 0}(\dim X_{(n)} - \dim H + n)$$

where, as usual, the dimension of the empty set is $-\infty$. If $X_{(n)}$ is irreducible, then $\dim X_{(n)} - \dim H + n$ is the transcendence degree of $\mathbb{C}(X_{(n)})^H$. We have that

$$\mod\ (X,H) = \dim X - \dim H + \sup_{n\geq 0}(n - \operatorname{codim} X_{(n)}).$$

Now suppose that $X_{(0)} \neq \emptyset$. We say that X is *k-modular* if

$$\text{mod}\ (X \setminus X_{(0)}, H) \le \dim X - \dim H - k\,.$$

Equivalently, $\operatorname{codim} X_{(n)} \ge n+k$ for all $n \ge 1$. Let us assume from now on that H is reductive and that X is an affine H-variety with FPIG. Then $X_{(0)} \ne \emptyset$ and $\dim X - \dim H = \dim X/\!\!/H$ so that X is k-modular if and only if $\text{mod}\ (X \setminus X_{(0)}, H) \le \dim X/\!\!/H - k$.

There are geometric interpretations of X being 0-modular. From now on assume that X is smooth. Let $A_1, \dots, A_r$ be a basis of the Lie algebra of H. We may consider the A_i as vector fields $\tilde{A}_i$ on X, hence as functions f_{A_i} on T^*X. Then X is 0-modular if and only if the f_{A_i} are a regular sequence in $\mathcal{O}(T^*X)$, i.e., their set of common zeroes has codimension r. Another interpretation is the following. Let $\mu\colon T^*X \to \mathfrak{h}^*$ be the moment mapping which sends $\xi \in T^*_xX$ to the element of $\mathfrak{h}^*$ whose value on $A \in \mathfrak{h}$ is $\tilde{A}(x)(\xi)$. Then X is 0-modular if and only if all the fibers of μ have codimension r [Sch95, Remark 8.6].

Remark 5.1. Note that if X is k-modular for any $k \ge 0$, then $\text{mod}\ (X, H) = \dim X - \dim H$.

Suppose that X is 2-principal. Then we can find invariant functions h_1 and h_2 which vanish on $X \setminus X_{\text{pr}}$ and are a regular sequence in $\mathcal{O}(X)$. If X is also 2-modular, then h_1, h_2, $f_{A_1}, \dots, f_{A_r}$ is a regular sequence in $\mathcal{O}(T^*X)$ [Sch95, Lemma 9.7]. Then we can apply [Sch95, Proposition 8.15]:

Proposition 5.2. *Suppose that there are h_1, h_2 as above such that h_1, h_2 and the f_{A_i} are a regular sequence in $\mathcal{O}(T^*X)$. Then $\pi_*\colon \mathcal{D}^k(X)^H \to \mathcal{D}^k(X/\!\!/H)$ is surjective for every $k \ge 0$.*

Corollary 5.3. *Suppose that X is 2-principal and 2-modular. Then $\pi_*\colon \mathcal{D}^k(X)^H \to \mathcal{D}^k(X/\!\!/H)$ is surjective for every $k \ge 0$.*

In the next section we will show that our V of interest is 2-modular. We will need the following result. Recall that a group action is *almost effective* if the kernel of the action is finite.

Lemma 5.4. *Let H be connected reductive. Let T denote the connected center of H and let H_s denote $[H,H]$. Let $W := W' \oplus W''$ be a direct sum of H-modules where T acts trivially on W'. Assume that the action of T on W'' is almost effective. If (W', H_s) is 0-modular, then so is (W, H).*

Proof. By Vinberg [Vin86], for every $q \ge 0$, $\operatorname{codim}(W'', T)_{(q)} \ge q$. Let $w = (w', w'')$ where $w' \in W'$ and $w'' \in (W'', T)_{(q)}$. If $\dim H_w = p$, then $p \ge q$ and $\dim (H_s)_{w'} = p' - q$ for $p' \ge p$. Thus

$$\begin{aligned}
&\sup_{p \ge q}(p - \operatorname{codim}((W' \times (W'', T)_{(q)}, H)_{(p)})\\
\le &\sup_{p' \ge q}(p' - q - \operatorname{codim}((W', H_s)_{(p'-q)}) + q - \operatorname{codim}(W'', T)_{(q)}\\
\le &\sup_{p' \ge 0}(p' - \operatorname{codim}(W', H_s)_{(p')}) \le 0.
\end{aligned}$$

Since $q \le p$ is arbitrary, we see that (W, H) is 0-modular. ■

6. Large representations.

We assume that $W = \oplus_{i=1}^n r_i \mathfrak{h}_i$ is large. Recall that W is 2-principal. By Corollary 1.3 a quasilinear $\psi \in \mathrm{Aut}_{q\ell}(W/\!\!/H)$ has a biholomorphic σ-equivariant lift Ψ. Then $\Psi'(0)$ is also a lift, so that W has the lifting property. Thus, by Corollary 5.3, we only need to show that W is 2-modular. It is easy to see that W is 2-modular if and only if each $r_i\mathfrak{h}_i$ is 2-modular, so we may reduce to the case that H is simple. The case $H = \mathrm{PSL}_2$ follows from [Sch95, Theorem 11.9], so we only have to consider the case where the rank of H is at least two. Moreover, we need only consider the case of two copies of $\mathfrak{h}$ since increasing the number of copies can only increase the modularity [Sch95, Proposition 11.5].

Let $v \in W \setminus W_{(0)}$. Then $\dim H_v > 0$ and there is an injective homomorphism $\lambda : L \to H_v$ where L is a copy of the multiplicative group $\mathbb{C}^*$ or the additive group $\mathbb{C}^+$. First consider the case where $L = \mathbb{C}^*$. Acting by an element of H, we may assume that $\mathrm{Im}\,\lambda \subseteq T$, where T denotes a maximal torus of H. Identify L with its image in T. Since the action of T on W is diagonalizable and $L \subset T$, there are only finitely many possible W^L. Let C_L denote $C_H(L)$. Then $H \cdot W^L$ is constructible and the H-orbits there all intersect W^L in a union of C_L orbits. Hence $\mod (H \cdot W^L, H) \le \mod (W^L, C_L)$.

Now suppose that $L = \mathbb{C}^+$. By the Jacobson-Morozov theorem, λ extends to an injective homomorphism, also called λ, from SL_2 or PSL_2 to H. We identify L with its image in H and we let S denote $\lambda(\mathrm{SL}_2)$ or $\lambda(\mathrm{PSL}_2)$ as the case may be. Up to conjugation in H, there are only finitely many possible λ. Let C_L denote $C_H(S) \times T''$ where T'' is the maximal torus of S. Then again we have the estimate that $\mod (H \cdot W^L, H) \le \mod (W^L, C_L)$.

Since the orbit of every point in $W \setminus W_{(0)}$ intersects one of our finite collection of sets W^L, and since $\dim W/\!\!/H = \dim H$, our discussion above gives the following.

Lemma 6.1. *Suppose that* $\mod (W^L, C_L) \le \dim H - 2$ *for all* L *of dimension one. Then* W *is* 2*-modular.*

The next two propositions show that we always have $\mod (W^L, C_L) \le \dim H - 2$, hence they complete our proof that W is admissible, hence good.

Proposition 6.2. *Suppose that* $L = \mathbb{C}^*$*. Then* $\mod (W^L, C_L) \le \dim H - 2$.

Proof. As above, we may assume that L is contained in a maximal torus T of H. The action of C_L on W^L is twice the adjoint representation of C_L. A maximal torus of C_L is T, and the connected center of C_L is a subtorus T' of T containing L. Then C_L/T' is semisimple, so its action on W^L is a trivial representation plus the sum of twice the adjoint representation of each simple component M_i of C_L/T'. By induction on rank, $2\mathfrak{m}_i$ is 2-modular if M_i has rank at least two. If

$\mathfrak{m}_i \simeq \mathfrak{sl}_2$, then $2\mathfrak{m}_i$ is 1-modular. It then follows from Remark 5.1 that

$$\operatorname{mod}(W^L, C_L) = \dim W^L - \dim C_L + \dim T' = \dim \mathfrak{h}^L + \dim T'.$$

Let $\alpha_1, \dots, \alpha_k$ be the positive roots of $\mathfrak{h}$ which are nontrivial when restricted to T'. As T'-module,

$$\mathfrak{h} = \theta_m + \bigoplus_{i=1}^{k} (\mathfrak{h}_{\alpha_i} + \mathfrak{h}_{-\alpha_i})$$

where $m = \dim \mathfrak{h}^L$ and θ_m denotes the trivial module of dimension m. We need to show that $\dim \mathfrak{h} - 2 - (m + \dim T')$ is nonnegative, i.e., that $-2 - \dim T' + 2k \geq 0$. Since the action of T' on $\mathfrak{h}$ is effective, $k \geq \dim T'$. Thus, if $\dim T' \geq 2$, we obtain the desired estimate. The only problem is the case that $T' = L$ has dimension one. Suppose that only one positive root α of H acts nontrivially on L. We may assume that α is simple. Then, since H has rank at least two, there is a simple root β (acting trivially on L) such that $\alpha + \beta$ is a root. Then $\alpha + \beta$ acts nontrivially on L and we have a contradiction. Hence $k \geq 2$ and $\operatorname{mod}(W^L, C_L) \leq \dim H - 2$. ■

Proposition 6.3. *Suppose that* $L = \mathbb{C}^+$. *Then* $\operatorname{mod}(W^L, C_L) \leq \dim W/\!\!/H - 2$.

Proof. Let $L \subset S \subset H$ be as discussed above. As S-module, $\mathfrak{h} = \sum_i m_i R_i$ where R_i denotes the space of binary forms of degree i and $m_i \geq 0$ is the multiplicity of R_i. The fixed points of L in each R_i have dimension one, so that W^L has dimension $2\sum_i m_i$. The Lie algebra of C_L has dimension $m_0 + 1$ and the action of $C_H(S)$ on $2\mathfrak{h}^S$ is twice its adjoint representation. Write $C_H(S)^0 = C_s T'$ where C_s is semisimple and T' is the connected center. As we saw above, if C_s is nontrivial, its action on $2\mathfrak{h}^S$ is 1-modular. Recall that T'' is the maximal torus of S. Now T' acts on $m_i R_i$ via a homomorphism to GL_{m_i}, hence T' acts almost effectively on $\oplus_{i\geq 1} m_i R_i^L$. Since T' commutes with S, it acts trivially on $\mathfrak{s} \subset m_2 R_2$. However, T'' acts nontrivially on $\mathfrak{s}^L$. Thus $T' \times T''$ acts almost effectively on W^L. Using Lemma 5.4, we get that $\operatorname{mod}(W^L, C_L)$ is $\dim W^L - \dim C_H(S) - 1$. Now $\dim W^L$ is $2(m_0 + \sum_{i\geq 1} m_i)$ and $\dim C_H(S) = m_0$. We need that $\dim W^L - m_0 - 1 \leq \dim \mathfrak{h} - 2$. Hence we need

$$2(m_0 + \sum_{i\geq 1} m_i) - m_0 - 1 \leq \sum_{i\geq 0} (i+1) m_i - 2, \text{ i.e., } 0 \leq -1 + \sum_{i\geq 2} (i-1) m_i.$$

Since $\mathfrak{s} \simeq R_2 \subset \mathfrak{h}$, we have that $m_2 \geq 1$, giving us the desired inequality. ■

7. Proofs of the main theorems

Lemma 7.1. *Let H be a simple adjoint group of rank at least two. Let S be a codimension one stratum of $Y := \mathfrak{h}/\!\!/H$. Then $\overline{S}$ contains a codimension two stratum of Y.*

Proof. Let $\mathfrak{t}$ be a maximal toral subalgebra of $\mathfrak{h}$. Then there is a simple root α such that $\overline{S}$ is the image in Y of the corresponding root hyperplane $\mathfrak{t}_\alpha$. Let β be another simple root and let S' denote the corresponding stratum of Y. Then $\overline{S} \cap \overline{S'}$ is the image of $\mathfrak{t}_\alpha \cap \mathfrak{t}_\beta$ and contains a stratum of codimension two. ■

Let $V = \oplus_i r_i \mathfrak{g}_i$ and G be as in Theorem 1.6. We have decomposed V as $V_1 \oplus V_2 \oplus V_3$ and G as $\widetilde{G}_1 \times \widetilde{G}_2 \times \widetilde{G}_3$ where $(V_1, \widetilde{G}_1)$ is a direct sum of representations isomorphic to $(2\mathfrak{sl}_2, \mathrm{PSL}_2)$, $(V_2, \widetilde{G}_2)$ is the adjoint representation of a semisimple group and $(V_3, \widetilde{G}_3)$ is large. Then $Z = Z_1 \times Z_2 \times Z_3$ where $Z = V/\!\!/G$ and $Z_i = V_i/\!\!/\widetilde{G}_i$, $1 \le i \le 3$. We have shown that each $(V_i, \widetilde{G}_i)$ is good and has the lifting property. Now we have to show the same for (V, G).

For S_i a stratum of Z_i, let S_i' denote the product of S_i and the principal strata of the other two factors of Z, $1 \le i \le 3$. Let $\psi \in \mathrm{Aut}_h(Z, 0)$ be stratification preserving and write $\psi = (\psi_1, \psi_2, \psi_3)$ where ψ_i is a germ sending Z to Z_i, $1 \le i \le 3$.

Lemma 7.2. *1. ψ preserves the strata of the form R_3' and the sets of the form $Z_1 \times Z_2 \times R_3$ where R_3 is a stratum of Z_3.*

2. *The analogues of (1) hold for strata R_1 of Z_1 and R_2 of Z_2.*
3. *$d\psi(0)$ has the form $\mathrm{diag}(A_1, A_2, A_3)$ where $A_i \colon T_0(Z_i) \to T_0(Z_i)$ is an isomorphism, $1 \le i \le 3$.*
4. *For z near 0, $\psi_3(z_1, z_2, z_3)$ is a strata preserving automorphism of Z_3, fixing 0, parameterized by $(z_1, z_2) \in Z_1 \times Z_2$.*
5. *$\psi_{3,0}(z) = \lim_{t\to 0} t^{-1} \cdot \psi_3(t \cdot z) = \lim_{t \to 0} t^{-1} \cdot \psi_3(0, 0, t \cdot z_3)$ exists. Thus $\psi_{3,0}(z) = \psi_{3,0}(0, 0, z_3)$ can be considered as an element of $\mathrm{Aut}_{q\ell}(Z_3)$.*
6. *The analogues of (4)–(5) hold for ψ_1 and ψ_2.*
7. *ψ is deformable and $\psi_0 = (\psi_{1,0}, \psi_{2,0}, \psi_{3,0})$ where $\psi_{i,0} \in \mathrm{Aut}_{q\ell}(Z_i)$ is strata preserving, $1 \le i \le 3$.*

Proof. Let R_3 be a stratum of Z_3. Then $\psi(R_3')$ is of the form $S_1 \times S_2 \times S_3$ where S_i is a stratum of Z_i, $1 \le i \le 3$. If S_1 is not $Z_{1,\mathrm{pr}}$, then $\psi(R_3')$ is in the closure of a codimension one stratum, hence so is R_3'. Since Z_3 has no codimension one strata, we get a contradiction. Thus $S_1 = Z_{1,\mathrm{pr}}$ and similarly $S_2 = Z_{2,\mathrm{pr}}$. Hence ψ permutes the strata of the form R_3' and all sets of the form $Z_1 \times Z_2 \times \overline{R_3}$. By Lemma 2.1 we see that ψ preserves sets of the form $Z_1 \times Z_2 \times R_3$, giving (1).

Let S_1 be a codimension one stratum of Z_1. Then S_1' is the codimension one stratum R of a copy of $Y := (2\mathfrak{sl}_2)/\!\!/\mathrm{PSL}_2$ times the principal stratum of the other factors of Z. Moreover, $\overline{R}$ is R union a point which has codimension three in $Y \simeq \mathbb{C}^3$. Now consider the case where we have a codimension one stratum S_2 of Z_2. Then S_2' is the codimension one stratum R of a copy of $Y := \mathfrak{l}/\!\!/L$, where L is simple, times the principal stratum of the other factors of Z. If ψ sends S_1' to S_2', then the corresponding space $\mathfrak{l}/\!\!/L$ has dimension at least three and contains no codimension two strata. Thus $\operatorname{rank} L > 1$ and we get a contradiction to Lemma 7.1. It follows that ψ permutes all strata of the form S_1' so ψ permutes the sets of the form $S_1 \times Z_2 \times Z_3$. The analogous results hold for strata S_2 of Z_2 and we have (2).

Clearly $d\psi_0$ has the form given in (3). Thus for (z_1, z_2) near 0, $\psi_3(z_1, z_2, z_3)$ is a germ of an automorphism of Z_3 which fixes 0. For this one uses the inverse function theorem for varieties, see, for example, [HS07, Lemma 14.15]. Here we don't need to worry about preserving strata since V_3 is large. Hence we have (4) and since V_3 is good, we have (5). In the case of ψ_1, we note that ψ respects the stratification by strata $S_1 \times Z_2 \times Z_3$ where S_1 is a stratum of Z_1. Thus $\psi_1(z_1, z_2, z_3)$, near 0, is a family of strata preserving automorphisms of Z_1, fixing $0 \in Z_1$, with parameters $(z_2, z_3) \in Z_2 \times Z_3$. The analogous result holds for ψ_2 and since V_1 and V_2 are good, we have (6). Part (7) is immediate. ∎

Proof of Theorem 1.6. It follows from Lemma 7.2 that V is good, and we also have that any $\psi \in \mathrm{Aut}_{q\ell}(Z)$ is a product of elements $\psi_i \in \mathrm{Aut}_{q\ell}(Z_i)$, $1 \le i \le 3$, where ψ_2 preserves the closures of the strata corresponding to short roots. Then it follows from Proposition 3.1, Theorem 4.4 and Corollary 1.3 that V has the lifting property. ∎

Proof of Theorem 1.5. By Theorem 1.6 we are in the following situation. We have $\psi \in \mathrm{Aut}_h(Z)$ and a deformation ψ_t with $\psi_1 = \psi$ and ψ_0 lifts to a σ-equivariant $\Psi_0 \in \mathrm{GL}(V)$. Let $\widetilde{\psi}_t = \psi_t \circ (\psi_0^{-1})$. Then $\widetilde{\psi}_t$ is an isotopy starting at the identity consisting of strata preserving automorphisms. Thus it is obtained by integrating a holomorphic (complex) time dependent strata preserving vector field $A(t, z)$. Since $\pi_* \colon \mathfrak{X}_h(V)^G \to \mathfrak{X}_h(Z)$ is surjective, we can lift A to a time dependent invariant holomorphic vector field $B(t, v)$. By [Sch12, Theorem 3.4] we can integrate B for $0 \le t \le 1$ to get G-equivariant biholomorphic lifts $\widetilde{\Psi}_t$ of $\widetilde{\psi}_t$. Then $\widetilde{\Psi}_1 \circ \Psi_0$ is a σ-equivariant biholomorphic lift of ψ. ∎

8. Compact Lie groups

Let K_i, $i = 1, \dots, d$, be simple compact adjoint Lie groups and consider the natural action of $K := \prod_i K_i$ on $W := \oplus r_i \mathfrak{k}_i$ where $r_i \in \mathbb{N}$, $1 \le i \le d$. Let $\pi \colon W \to W/K$ denote the quotient mapping. We can put a smooth structure on W/K (see [Sch75]), as follows. A function on W/K is smooth if it pulls back to a (necessarily K-invariant) smooth function on W. We can make this more concrete, as follows. Let $p_1, \dots, p_r$ be homogeneous generators of $\mathbb{R}[W]^K$ and let $p = (p_1, \dots, p_r) \colon W \to \mathbb{R}^r$. Now p is proper and separates the K-orbits in W. Let X denote $p(W) \subset \mathbb{R}^r$. Then X is a closed semialgebraic set and p induces a homeomorphism $\bar{p} \colon W/K \to X$. The main theorem of [Sch75] says that $p^* C^\infty(X) = C^\infty(W)^K$ where a function f on X is smooth if it extends to a smooth function in a neighborhood of X. Now we can define the notion of a smooth mapping $X \to X$ (or $W/K \to W/K$) in the obvious way. We also see an $\mathbb{R}^*$-action on X where $t \cdot (y_1, \dots, y_r) = (t^{e_1} y_1, \dots, t^{e_r} y_r)$ for $t \in \mathbb{R}^*$, $(y_1, \dots, y_r) \in X$ and e_j the degree of p_j, $j = 1, \dots, r$. We denote by $\mathrm{Aut}_{q\ell}(W/K)$ the quasilinear automorphisms of W/K, i.e., the automorphisms which commute with the $\mathbb{R}^*$-action.

We have the usual stratification of W/K determined by the conjugacy classes of isotropy groups of orbits. There is a unique open stratum, the principal stratum, which consists of the orbits with trivial slice representation. It is connected and dense in W/K. Let E_s be the union of the closures of the codimension one strata of W/K corresponding to the short roots. We call a diffeomorphism $\Psi\colon W \to W$ *σ-equivariant* if $\Psi \circ k = \sigma(k) \circ \Psi$, $k \in K$, where $\sigma \in \operatorname{Aut}(K)$.

Theorem 8.1. *Let $\psi\colon W/K \to W/K$ be a diffeomorphism. Then ψ lifts to a σ-equivariant diffeomorphism $\Psi\colon W \to W$ if and only if ψ preserves E_s*

Proof. Necessity of the condition on ψ is established as before. Now assume that ψ preserves E_s. By [Los01, Theorem 2.2], $\psi_0(x) = \lim_{t\to 0} t^{-1} \cdot \psi(t \cdot x)$ exists for every $x \in W/K$ where, of course, ψ_0 is quasilinear. By [Bie75, Theorem A], ψ_0 (and ψ) necessarily preserve the principal stratum and the union of the codimension one strata. We want to show that ψ_0 has a σ-equivariant lift to $\mathrm{GL}(W)$. Complexifying and applying Lemma 7.2 we see that it is enough to treat the following three cases:

1. W is a direct sum of representations isomorphic to $(2\mathbb{R}^3, \mathrm{SO}(3,\mathbb{R}))$.
2. $W = \mathfrak{k}$ where K is compact semisimple. Here we need that ψ_0 preserves E_s.
3. W is large (defined as before).

The proof of lifting in case (1) is as in Proposition 3.1. For case (3) one can use [Sch12, Proposition 8.5] which allows one to deduce the lifting property in (3) from Corollary 1.3. Note that (3) is not a trivial consequence of Corollary 1.3, since we have to show that the lift preserves $W \subset W \otimes_{\mathbb{R}} \mathbb{C}$. For case (2), a theorem of Strub [Str82] (see also [Sch09, Theorem 2.11]) shows that there is a lift of ψ_0 to $\tau \in N_{\mathrm{GL}(\mathfrak{t})}(\mathcal{W})$ where $\mathfrak{t}$ is the Lie algebra of a maximal torus of K and $\mathcal{W}$ is the Weyl group. Then the argument of Lemma 4.2 through Theorem 4.4 shows that τ lifts to an element of $N_{\mathrm{GL}(\mathfrak{k})}(K)$. Thus, if ψ preserves E_s, there is a smooth deformation ψ_t with $\psi_1 = \psi$ such that ψ_0 lifts to $\Psi_0 \in N_{\mathrm{GL}(W)}(K)$. By the isotopy lifting theorem of [Sch80] there is an equivariant diffeomorphism lifting $\psi \circ \psi_0^{-1}$, hence there is a σ-equivariant diffeomorphism Ψ lifting ψ. ■

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Gerald W. Schwarz
Department of Mathematics
Brandeis University
Waltham, MA 02454-9110
schwarz@brandeis.edu

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