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The Conjugate Loci and Cut Loci on Simply-Connected Lorentzian Symmetric Spaces*

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Abstract. In this paper we study conjugate loci and cut loci of Lorentzian symmetric spaces. We prove that if M_1 is a connected simply connected Lorentzian symmetric space of the form $\mathbb{R} \times M$, $D \times M$, and $C \times M$, where M is a connected simply connected compact Riemannian symmetric space, D is the universal covering of the de Sitter space-time with dimension ≥ 3, and C is a Cahen-Wallach manifold, then for any given point $x \in M_1$, all future (past) nonspacelike cut loci and the locus of first future (past) nonspacelike conjugate loci coincide.
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1. Introduction

In this paper we study the relationship between conjugate loci and cut loci of Lorentzian symmetric spaces. Let $\gamma : [0, a) \to M$ be a future directed, future inextendible timelike geodesic in a space-time M and $t_0 = \sup\{t : d(\gamma(0), \gamma(t)) = L_g(\gamma|[0, t])\}$. If $0 < t_0 < a$, then $\gamma(t_0)$ is called the future timelike cut point of $\gamma(0)$ along γ. Similarly, let $\gamma : [0, a) \to M$ be a future directed, future inextendible lightlike geodesic and $t_0 = \sup\{t : d(\gamma(0), \gamma(t)) = 0\}$. If $0 < t_0 < a$, then $\gamma(t_0)$ is called the future lightlike cut point of $\gamma(0)$ along γ. Let (M, g) be a globally hyperbolic Lorentzian manifold. It is proved (see [3]) that if $q = c(t)$ is the future cut point of $p = c(0)$ along the timelike geodesic c from p to q, then at least one of the following assertions holds:

(1) q is the first future conjugate point of p along c;

(2) There exists at least two future directed maximal timelike geodesic segments from p to q.

A similar conclusion also holds for the future cut point along a lightlike geodesic.

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Globally hyperbolic Lorentzian symmetric manifolds have been classified; see [1] and [5]. It is shown that there are three types of connected and simply-connected Lorentzian symmetric spaces: $\mathbb{R}\times M$, $D\times M$ and $C\times M$, where M is a connected and simply-connected Riemannian symmetric space, D is the universal cover of the de Sitter or anti de Sitter space-time, and C is the Cahen-Wallach manifold. Moreover, it is proved in [2] and [3] that the above globally Lorentzian symmetric spaces are hyperbolic except when D is the universal cover of anti de Sitter space-time. For information on conjugate locus and cut locus, we refer to [3, 4, 6, 7, 14, 15, 16] and [17].

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Suppose that M_1 is a connected simply connected Lorentzian symmetric space of the form $\mathbb{R}\times M$, $D\times M$, and $C\times M$, where M is a connected simply connected compact Riemannian symmetric space, D is the universal cover of the de Sitter space-time with dimension ≥ 3, and C is a Cahen-Wallach manifold. Then for any given point $x\in M_1$, all future (past) nonspacelike cut locus and the locus of first future (past) nonspacelike conjugate locus coincide.*

The arrangement of this article is as the following. In Section 2, we present some preliminaries on Lorenztian manifolds and symmetric spaces. Section 3 is devoted to the study of the de Sitter space-time. In Section 4, we prove the main theorem for Cahen-Wallach manifolds. Finally, in Section 5, we consider the general cases and complete the proof of the main theorem.

2. Preliminaries

2.1. Lorentzian geometry.

Let (M,g) be a Lorentzian manifold of dimension $n\geq 2$. Given $p,q\in M$ with $p\leq q$, let $\Omega_{p,q}$ denote the path space of all future directed nonspacelike curves $\gamma:[0,1]\to M$ with $\gamma(0)=p$ and $\gamma(1)=q$. The Lorentzian arc length function L_g is defined as in [8] and we have the Lorentzian distance function $d(p,q)$ defined as the following: if $q\in J^+(p)$ (here $J^+(p)$ denotes the causal future of p), then we define $d(p,q)=\sup\{L_g(\gamma):\gamma\in\Omega_{p,q}\}$; Otherwise we set $d(p,q)=0$.

Given a plane $\sigma=$span$\{u,v\}$, we denote by $K(\sigma)$ the sectional curvature if σ is nondegenerate. When σ is degenerate with v lightlike, we denote by $K_v(\sigma)=\frac{g(R(u,v)v,u)}{g(u,u)}$ the lightlike sectional curvature associated to v (see [3], [6] and [9]).

As a special case, for a symmetric space, denote $\mathcal{R}_v:u\to R(u,v)v$. Then we have ([11])

$$\mathcal{R}_v(u)=-(\mathrm{ad}v)^2(u).$$

In [6], the following assertion is proved:

Lemma 2.1. *Let (M,g) be a Lorentz symmetric manifold, $p\in M$ and $v\in T_pM$.*

a. *If v is timelike, then p has no conjugate points along γ_v if and only if*

$k(\sigma) \geq 0$ for any timelike plane $\sigma \subseteq T_pM$ containing v.

b. *If v is lightlike, then p has no conjugate points along γ_v if and only if $k(\sigma) \leq 0$ for any degenerate plane $\sigma \subseteq T_pM$ containing v.*

On the other hand, from [13] we have the following

Lemma 2.2. *Let M be a Lorentzian symmetric manifold, and $\gamma_X : I \to M, X \in T_pM$ be a geodesic in M. Then the conjugate points of $\gamma_X(0)$ along γ_X are $\gamma_X(\frac{m\pi}{\sqrt{\lambda}})$, where $m \in \mathbb{Z} - \{0\}$, $\frac{m\pi}{\sqrt{\lambda}} \in I$, and λ is a real positive eigenvalue of $\mathcal{R}_X = -(\mathrm{ad}X)^2$.*

2.2. Orthogonal symmetric Lie algebra of compact type $(\mathfrak{u}, \mathfrak{k})$. Let $(\mathfrak{u}, \mathfrak{k})$ be an orthogonal symmetric Lie algebra of compact type with a canonical decompostion $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, and $\mathfrak{h}_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. If we denote the restricted root system by Σ , and the set of positive roots by Σ^+, then we have

$$\begin{aligned} \mathfrak{k} &= \mathfrak{k}_0 + \sum_{\gamma \in \Sigma^+} \mathfrak{k}_\gamma, \\ \mathfrak{p} &= \mathfrak{h}_\mathfrak{p} + \sum_{\gamma \in \Sigma^+} \mathfrak{p}_\gamma, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{k}_\gamma &= \mathfrak{k} \cap (\mathfrak{u}_\gamma + \mathfrak{u}_{-\gamma}), \\ \mathfrak{p}_\gamma &= \mathfrak{p} \cap (\mathfrak{u}_\gamma + \mathfrak{u}_{-\gamma}), \\ \mathfrak{u}_\gamma &= \{X \in \mathfrak{g}^\mathbb{C} | [H, X] = \gamma(H)X, \ \forall H \in \mathfrak{h}_\mathfrak{p}\}. \end{aligned}$$

In the following sections, we shall denote by exp and Exp the exponential mappings of Lie groups and manifolds, respectively.

3. The de Sitter space-time of dimension $n \geq 3$

The de Sitter space-time is the Lorentz symmetric manifold $(\mathfrak{so}(1, n+1), \mathfrak{so}(1, n))$. Denoting the pair simply as $(\mathfrak{g}, \mathfrak{h})$, we have the decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q},$$

where $\mathfrak{q}$ consists of the vectors of the form

$$\begin{pmatrix} 0 & 0 & y \\ 0 & 0 & \alpha \\ y & -\alpha^T & 0 \end{pmatrix},$$

with $y \in \mathbb{R}$ and $\alpha \in \mathbb{R}^n$; see [12] and [10]. Let $B_d = -\frac{B}{2n}$ be the Lorentzian metric, where B is the Killing form of $\mathfrak{g}$.

On the other hand, the de Sitter space-time can also be represented as $\mathrm{SO}_0(1, n+1)/\mathrm{SO}_0(1, n)$ (see [12], page 27), where $\mathrm{SO}_0(1, n+1)$ can be realized in the following way. Denote by $M(p\times q,\mathbb{R})$ the $p\times q$ matrices with entries in $\mathbb{R}$. Then $\mathrm{SO}_0(1,n+1)$ consists of the matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A\in\mathbb{R}$, $B\in M(1\times(n+1),\mathbb{R})$, $C\in M((n+1)\times 1,\mathbb{R})$, $D\in M((n+1)\times(n+1),\mathbb{R})$, with $A\geq 1$, $\det D\geq 1$, and the following conditions being satisfied:

$$\begin{aligned} -A^TB+C^TD &= 0, \\ -A^TA+C^TC &= -1, \\ -B^TB+D^TD &= I_{n+1}. \end{aligned}$$

The subgroup $SO_0(1,n)$ consists of the elements of the form

$$\begin{pmatrix} a & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a\in\mathbb{R}$, $B\in M(1\times n,\mathbb{R})$, $C\in M(n\times 1,\mathbb{R})$, $D\in M(n\times n,\mathbb{R})$, with $a\geq 1$, $\det D\geq 1$ and the following conditions being satisfied:

$$\begin{aligned} -aB+C^TD &= 0, \\ -a^2+C^TC &= -1, \\ -B^TB+D^TD &= I_n. \end{aligned}$$

Since $\mathrm{SO}_0(1,n)$ acts transitively on the set $\{X\in\mathfrak{q}:B_d(X,X)=-1,y>0\}$ and the set $\{Y\in\mathfrak{q}:B_d(Y,Y)=0,y=1\}$, it follows from Lemma 2.1 that o has no conjugate point along any nonspacelike geodesic. Now we consider the future nonspacelike cut locus of o. Given a unit future timelike vector

$$X_0=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

denote by $\mathrm{Exp}t_0X_0$ the future cut point $(t_0>0)$ of o along the geodesic $\mathrm{Exp}tX_0$. Since $SO_0(1,n)$ acts transitively on the set $\{X\in\mathfrak{q}:B_d(X,X)=-1,y>0\}$, one can find $h\in SO_0(1,n)$ such that $\mathrm{Ad}(h)X_0\neq X_0$ and $\mathrm{Exp}t_0X_0=\mathrm{Exp}t_0\mathrm{Ad}(h)X_0$. Then we have $\exp(-t_0X_0)\cdot h^{-1}\cdot\exp t_0X_0\in SO_0(1,n)$. If we write

$$h^{-1}=\begin{pmatrix} a & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $a\geq 1$ and $\det D\geq 1$, then h is equal to

$$\begin{pmatrix} a & -C^T & 0 \\ -B^T & D^T & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, $\exp t_0 X$ is equal to

$$\begin{pmatrix} \cosh t_0 & 0 & \sinh t_0 \\ 0 & I_n & 0 \\ \sinh t_0 & 0 & \cosh t_0 \end{pmatrix}.$$

Thus $\exp(-t_0X_0)\cdot h^{-1}\cdot \exp t_0X_0$ is equal to

$$\begin{pmatrix} a\cosh^2 t_0 - \sinh^2 t_0 & \cosh t_0 B & (a-1)\sinh t_0\cosh t_0 \\ \cosh t_0 C & D & \sinh t_0 C \\ (1-a)\sinh t_0\cosh t_0 & -\sinh t_0 B & \cosh^2 t_0 - a\sinh^2 t_0 \end{pmatrix}.$$

Suppose $a=1$. Then it follows from the fact $\exp(-t_0X_0)\cdot h^{-1}\cdot \exp t_0X_0 \in SO_0(1,n)$ that $B=0$ and $C=0$, which implies that $h^{-1}\in \mathrm{SO}(n)\hookrightarrow \mathrm{SO}_0(1,n)$. Then we have $\mathrm{Ad}(h)X_0 = X_0$, contradicting to the fact that $\mathrm{Ad}(h)X_0\neq X_0$.

Suppose $a>1$. Then we have $\cosh^2 t_0 - a\sinh^2 t_0 = 1$. This implies that $t_0=0$, contradicting to the fact that $t_0>0$.

From the above arguments we conclude that there does not exist any future cut point of o along $\mathrm{Exp}tX_0$. In other words, o has no future timelike cut point.

Next we consider a future lightlike vector

$$Y_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \alpha^T \\ 1 & -\alpha & 0 \end{pmatrix},$$

where $\alpha=(1,0,\dots,0)^T$. Let $\mathrm{Exp}s_0Y_0, s_0>0$, be the future cut point of o along $\mathrm{Exp}sY_0$. When $s>s_0$, there exists a future timelike geodesic connecting o and $\mathrm{Exp}sY_0$ (see [3]). Now we prove that there does not exist any future cut point of o along $\mathrm{Exp}sY_0$.

In fact, given $s_*>s_0$, one can verify that

$$\exp s_*Y_0 = \begin{pmatrix} 1+\frac{s_*^2}{2} & -\frac{s_*^2}{2}\alpha & s_* \\ \frac{s_*^2}{2}\alpha^T & I_n - \frac{s_*^2}{2}\alpha^T\alpha & s_*\alpha^T \\ s_* & -s_*\alpha & 1 \end{pmatrix}.$$

Since $\mathrm{SO}_0(1,n)$ acts transitively on the set $\{X\in\mathfrak{q} : B_d(X,X)=0, y>0\}$, there exist $t_*>0$ and $h\in \mathrm{SO}_0(1,n)$, such that

$$\mathrm{Exp}s_*Y_0 = \mathrm{Exp}t_*\mathrm{Ad}hX_0.$$

Then we have

$$\exp(-t_*X_0)\cdot h^{-1}\cdot \exp s_*Y_0 \in \mathrm{SO}_0(1,n).$$

Now write h^{-1} as

$$\begin{pmatrix} a & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $a\geq 1$, and $\det D\geq 1$. Then a direct computation shows that the $(2,3)$-entry of $\exp(-t_*X)\cdot h^{-1}\cdot \exp s_*Y \in \mathrm{SO}_0(1,n)$ must be 0. Thus $C=-D\alpha^T$. On the other hand, we also have $a^2-C^TC=1$, $aB=C^TD$, and $-B^TB+D^TD=I_n$. Since $C=-D\alpha^T$, we have $a^2-\alpha D^TD\alpha^T=1$ and $B=-\frac{1}{a}\alpha D^TD$. Meanwhile, since $-B^TB+D^TD=I_n$ and $\alpha^T\alpha=1$, we also have $-a^2+1=-a^2$. This is a contradiction. This proves the above assertion.

Finally, since $\mathrm{SO}_0(1,n)$ acts transitively on the set $\{Y\in\mathfrak{q}:B_d(Y,Y)=0, y=1\}$, we conclude that for any future lightlike vector Y, there does not exist any future cut point of o along $\mathrm{Exp}tY$.

In summarizing, we have proved the following

Theorem 3.1. *For the de Sitter space-time of dimension $n\geq 3$, any future directed nonspacelike geodesics must be a future directed nonspacelike geodesic ray; Dually, any past directed nonspacelike geodesics must be a past directed nonspacelike geodesic ray.*

4. Cahen-Wallach manifolds

We first recall the definition of Cahen-Wallach manifolds (see [6], [2] and [5]).

Let $\mathfrak{g}=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}$, with $n\geq 1$. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a symmetric endomorphism of $\mathbb{R}^n$ with respect to the standard inner product $\langle,\rangle$, and define Lie brackets on $\mathfrak{g}$ as follows:

$$[(x,y,t,u),(x',y',t',u')]=(u'y-uy',uf(x')-u'f(x),\langle f(x'),y\rangle-\langle f(x),y'\rangle,0).$$

Denote $\mathfrak{h}=\{(x,0,0,0):x\in\mathbb{R}^n\}\subseteq\mathfrak{g}$. Let G be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$ and H the connected subgroup generated by $\mathfrak{h}$ (which is a closed subgroup of G). Then $\mathfrak{m}=\{(0,y,t,u)\}\subseteq\mathfrak{g}$ is a $\mathrm{Ad}(H)-$invariant complement subspace of $\mathfrak{h}$ in $\mathfrak{g}$. Let $q((0,y,t,u),(0,y',t',u'))=\langle y,y'\rangle-tu'-t'u$ be a Lorentzian inner product on $\mathfrak{m}$. Then q induces a G-invariant metric on G/H, denoted as Q_c. We call $(G/H,Q_c)$ a Cahen-Wallach manifold.

The following lemma is obvious.

Lemma 4.1. *The following equality holds:*

$$\mathrm{Ad}(\exp(x,0,0,0))(0,y',t',u')=(0,y'-u'f(x),t'-\langle f(x),y'\rangle+\frac{u'}{2}\langle f(x),f(x)\rangle,u').$$

We next consider some special cases and then deduce our conclusion of this section.

4.1. f is nondegenerate and has at least one positive eigenvalue. Suppose that the eigenvalues of f are as follows:

$$\lambda_1\geq\lambda_2\geq\ldots\lambda_l>0>\lambda_{l+1}\geq\cdots\geq\lambda_n,\quad l\geq 1.$$

We first give two lemmas.

Lemma 4.2. *Each future directed timelike unit vector $(0,y,t,u), u>0$, is $\mathrm{Ad}H$-conjugate to $(0,0,\frac{1}{2u},u)$.*

Proof. By Lemma 4.1, we have
$\mathrm{Ad}\exp(x,0,0,0)(0,y,t,u)=(0,y-uf(x),t-\langle f(x),y\rangle+\frac{u}{2}\langle f(x),f(x)\rangle,u)$. Since f is nondegenerate, for any $y\in\mathbb{R}^n$, there exists a unique $x\in\mathbb{R}^n$ such that $y-uf(x)=0$. Since $q((0,y,t,u),(0,y,t,u))=-1$, the lemma follows. ∎

A similar argument can be used to prove the following

Lemma 4.3. *Each future directed lightlike vector is* $\mathrm{Ad}H$*-conjugate to an element in the subset* $\{(0,0,0,a):a>0\}\bigcup\{(0,0,b,0):b>0\}$.

Now we consider the linear transformation $\mathrm{ad}^2(0,0,s_1,s_2)((0,y,t,u))$ on $\mathfrak{m}$ and $\mathfrak{h}$ (it can be easily checked that the map keeps both the subspaces invariant). The following equalities are easily verified:

$$\mathrm{ad}^2(0,0,s_1,s_2)((0,y,t,u))=-s_2^2(0,f(y),0,0);$$

$$\mathrm{ad}^2(0,0,s_1,s_2)((x,0,0,0))=-s_2^2(f(x),0,0,0).$$

Thus we have the direct sum decomposition of $\mathfrak{m}$:

$$\mathfrak{m}=\mathfrak{m}_{\lambda_1}\oplus\mathfrak{m}_{\lambda_2}\oplus\cdots\oplus\mathfrak{m}_{\lambda_n}\oplus\overline{\mathfrak{m}_0},$$

where $\mathfrak{m}_{\lambda_i}$ is the eigenspace of $\mathrm{ad}^2(0,0,s_1,s_2)$ with eigenvalue $-s_2^2\lambda_i$ and $\overline{\mathfrak{m}_0}$ is the subspace spanned by $\{(0,0,s,t)\}$. Obviously, $\mathfrak{m}_{\lambda_i}$ can also be viewed as the eigenspace of f with eigenvalue λ_i. Similarly, we have a direct sum decomposition of $\mathfrak{h}$:

$$\mathfrak{h}=\mathfrak{h}_{\lambda_1}\oplus\mathfrak{h}_{\lambda_2}\oplus\cdots\oplus\mathfrak{h}_{\lambda_n},$$

where the subspaces $\mathfrak{h}_{\lambda_i}$ can be defined similarly as before.

Denote $S_{\lambda_i}=(e_i,0,0,0)\in\mathfrak{h}_{\lambda_i}$, and $T_{\lambda_i}=(0,e_i,0,0)\in\mathfrak{m}_{\lambda_i}$, where e_i is the unit eigenvector of f with eigenvalue λ_i. Let $X=(0,0,\frac{1}{2u},u)$, where $u>0$.

Lemma 4.4. *If* $\lambda_i>0$*, then*

$$\mathrm{Ad}(\exp X)T_{\lambda_i}=\cos u\sqrt{\lambda_i}T_{\lambda_i}-\frac{1}{\sqrt{\lambda_i}}\sin u\sqrt{\lambda_i}S_{\lambda_i};$$

if $\lambda_i<0$*, then*

$$\mathrm{Ad}(\exp X)T_{\lambda_i}=\cosh u\sqrt{-\lambda_i}T_{\lambda_i}-\frac{1}{\sqrt{-\lambda_i}}\sinh u\sqrt{-\lambda_i}S_{\lambda_i}.$$

Proof. The lemma follows directly from the equalities: $[X,S_{\lambda_i}]=u\lambda_iT_{\lambda_i}$ and $[X,T_{\lambda_i}]=-uS_{\lambda_i}$. ∎

Now we can prove

Theorem 4.5. *If* Expt_0X *is the future cut point of* o *along the future direction of* γ_X*, then it must be the first future conjugate point.*

Proof. Suppose that $\mathrm{Exp}t_0X$ is not the first future conjugate point of o. Then there exists a future timelike unit vector $Z \neq X$ in $\mathfrak{m}$, such that $\mathrm{Exp}t_0X = \mathrm{Exp}t_0Z$. Then there exists $h \in H$ such that $\exp t_0X = \exp t_0Z \cdot h$, that is, $\mathrm{Ad}(\exp(-t_0X))Z = \mathrm{Ad}h^{-1}Z \in \mathfrak{m}$. Now one can write Z as:

$$Z = Z_0 + \sum_{i=1}^{l} a_i T_{\lambda_i} + \sum_{i=l+1}^{n} b_i T_{\lambda_i},$$

where $Z_0 \in \overline{\mathfrak{m}_0}$. Then we have

$$\begin{aligned}\mathrm{Ad}(\exp(-t_0X))Z &= Z_0 + \sum_{i=1}^{l} a_i(\cos t_0u\sqrt{\lambda_i}T_{\lambda_i} + \frac{1}{\sqrt{\lambda_i}}\sin t_0u\sqrt{\lambda_i}S_{\lambda_i}) \\ &+ \sum_{i=l+1}^{n} b_i(\cosh t_0u\sqrt{-\lambda_i}T_{\lambda_i} + \frac{1}{\sqrt{-\lambda_i}}\sinh t_0u\sqrt{-\lambda_i}S_{\lambda_i}) \in \mathfrak{m}.\end{aligned}$$

Since $t_0 \neq 0$ and $S_{\lambda_i} \in \mathfrak{h}$, we have $b_i = 0$, for any i. On the other hand, if some of the a_i is nonzero, then we have $\sin t_0u\sqrt{\lambda_i} = 0$. This implies that there exists $m \in \mathbb{N} - \{0\}$ such that $t_0 = \frac{m\pi}{u\sqrt{\lambda_i}}$. Then it follows from Lemma 2.2 and the statement after Lemma 4.3 that $\mathrm{Exp}t_0X$ must be the conjugate point of o, contradicting to the assumption that $\mathrm{Exp}t_0X$ is not the future conjugate point of o. Therefore we have $a_i = 0$, for any i. Then we have $Z = Z_0 \in \overline{\mathfrak{m}_0}$, and $[X, Z] = 0$. Consequently we have $t_0(X - Z) \in \mathfrak{h}$, that is, $X = Z$, contradicting to the condition $X \neq Z$. ■

Combining the above theorem with Lemma 4.2, we have the following

Corollary 4.6. *If f is nondegenerate and has at least one positive eigenvalue, then the future timelike cut locus coincides with the first future timelike conjugate locus.*

Similarly, we can prove the following

Theorem 4.7. *If f is nondegenerate and has at least one positive eigenvalue, then the future lightlike cut locus coincides with the first future lightlike conjugate locus except for the direction $(0, 0, 1, 0)$.*

Remark 4.8. The geodesic along the lightlike vector $(0, 0, 1, 0)$ is the unique geodesic ray described in Chapter 8 of [3].

4.2. f is degenerate and has at least one positive eigenvalue. In this case, we can suppose that the eigenvalues of f are as follows:

$$\lambda_1 \geq \cdots \lambda_p > 0 = \lambda_{p+1} = \cdots \lambda_q > \lambda_{q+1} \geq \cdots \geq \lambda_n, \quad p \geq 1, q > p.$$

Since $\mathbb{R}^n = \ker f \oplus \mathrm{Im}f$ and $\ker f \neq 0$, we need only consider the vector $(0, y, t, u)$, where y has no pre-image in $\mathbb{R}^n$. Writing y as $y_1 + y_2$, with $y_1 \in \mathrm{Im}f$ and $y_2 \in \ker f$, we have the following

Lemma 4.9. *For each future timelike unit vector* $(0,y,t,u)$ *with* $u>0$, $y=y_1+y_2$, $y_1\in \mathrm{Im}f$ *and* $0\neq y_2\in\ker f$, *there exists* $s\in\mathbb{R}$ *such that* $(0,y,t,u)$ *is* $\mathrm{Ad}H$*-conjugate to* $(0,y_2,s,u)$.

Proof. From Lemma 4.1, we have:

$$\mathrm{Ad}\exp(x,0,0,0)(0,y,t,u)=(0,y-uf(x),t-\langle f(x),y\rangle+\frac{u}{2}\langle f(x),f(x)\rangle,u).$$

For $y_1\in\mathrm{Im}f$, we can find $x\in\mathbb{R}^n$ such that $y_1=uf(x)$. Let $s=t-\langle f(x),y_1\rangle+\frac{u}{2}\langle f(x),f(x)\rangle$. Then we have $y_2=y-y_1\in\ker f$, and the lemma follows. ■

From lemma 4.2 and lemma 4.5, we have the following

Corollary 4.10. *If* f *is degenerate, then the future timelike unit vector* $(0,y,t,u)$ *is* $\mathrm{Ad}H$*-conjugate to* $(0,y_2,s,u)$ *or* $(0,0,\frac{1}{2u},u)$.

For the future lightlike vector $(0,y,t,u)$, we can prove the following corollary using a similar argument.

Corollary 4.11. *If* f *is degenerate, then the future lightlike unit vector* $(0,y,t,u)$ *is* $\mathrm{Ad}H$*-conjugate to* $(0,y_2,s,u)$, $(0,0,a,0)$ *or* $(0,0,0,b)$, *where* a *and* b *are positive, and* $\langle y_2,y_2\rangle-2su=0$.

Now we consider the property of future nonspacelike vectors $X=(0,y_2,s,u)$ with $y_2\in\ker f$. It is easy to check that

$$\mathrm{ad}^2X((0,x,0,0))=-u^2(0,f(x),0,0),$$

that

$$\mathrm{ad}^2X((x,0,0,0))=-u^2(f(x),0,0,0),$$

and that

$$\mathrm{ad}^2X(0,0,a,b)=0.$$

Thus we have the direct sum decompositions:

$$\mathfrak{m}=\mathfrak{m}_{\lambda_1}\oplus\mathfrak{m}_{\lambda_2}\oplus\cdots\oplus\mathfrak{m}_{\lambda_n}\oplus\overline{\mathfrak{m}_0},$$

and

$$\mathfrak{h}=\mathfrak{h}_{\lambda_1}\oplus\mathfrak{h}_{\lambda_2}\oplus\cdots\oplus\mathfrak{h}_{\lambda_n}.$$

Denote $S_{\lambda_i}=(e_i,0,0,0)\in\mathfrak{h}_{\lambda_i}$, and $T_{\lambda_i}=(0,e_i,0,0)\in\mathfrak{m}_{\lambda_i}$. Similarly as in the case that f is nondegenerate, we have the following

Lemma 4.12. *If* $\lambda_i>0$, *then*

$$\mathrm{Ad}(\exp X)T_{\lambda_i}=\cos u\sqrt{\lambda_i}T_{\lambda_i}-\frac{1}{\sqrt{\lambda_i}}\sin u\sqrt{\lambda_i}S_{\lambda_i};$$

if $\lambda_i<0$, *then*

$$\mathrm{Ad}(\exp X)T_{\lambda_i}=\cosh u\sqrt{-\lambda_i}T_{\lambda_i}-\frac{1}{\sqrt{-\lambda_i}}\sinh u\sqrt{-\lambda_i}S_{\lambda_i};$$

if $\lambda_i = 0$*, then*

$$\mathrm{Ad}(\exp X)T_{\lambda_i} = T_{\lambda_i} - uS_{\lambda_i},$$

and

$$\mathrm{Ad}(\exp X)(0,0,a,b) = (by_2, 0, a, b).$$

Theorem 4.13. *If f is degenerate, and* $\mathrm{Expt}_0 X$ *is the future cut point of o along the future direction of γ_X, then it must be the first future conjugate point.*

Proof. By Corollary 4.2, we need only consider the case that X is the future timelike unit vectors, namely, $X = (0, y_2, s, u)$, $u > 0$, $0 \neq y_2 \in \ker f$, or $X = (0, 0, \frac{1}{2u}, u)$, $u > 0$. Suppose conversely that $\mathrm{Expt}_0 X$ is not the first future conjugate point of o. Then there exists a future timelike unit vector $Z \neq X$ in $\mathfrak{m}$, such that $\mathrm{Expt}_0 X = \mathrm{Expt}_0 Z$. Hence there exists $h \in H$ such that $\exp t_0 X = \exp t_0 Z \cdot h$, that is, $\mathrm{Ad}(\exp(-t_0 X))Z = \mathrm{Ad}h^{-1}Z \in \mathfrak{m}$. Now write Z as:

$$Z = Z_0 + \sum_{i=1}^{p} a_i T_{\lambda_i} + \sum_{i=p+1}^{q} b_i T_{\lambda_i} + \sum_{i=q+1}^{n} c_i T_{\lambda_i},$$

where $Z_0 \in \overline{\mathfrak{m}_0}$. Suppose $X = (0, y_2, s, u)$ and $Z_0 = (0, 0, a, b)$. Then we have

$$\begin{aligned}\mathrm{Ad}(\exp(-t_0 X))Z &= \sum_{i=1}^{p} a_i(\cos t_0 u\sqrt{\lambda_i}T_{\lambda_i} + \frac{1}{\sqrt{\lambda_i}}\sin t_0 u\sqrt{\lambda_i}S_{\lambda_i}) \\ &\quad + (-t_0 b y_2, 0, a, b) + \sum_{i=p+1}^{q} b_i(T_{\lambda_i} + t_0 u S_{\lambda_i}) \\ &\quad + \sum_{i=q+1}^{n} c_i(\cosh t_0 u\sqrt{-\lambda_i}T_{\lambda_i} + \frac{1}{\sqrt{-\lambda_i}}\sinh t_0 u\sqrt{-\lambda_i}S_{\lambda_i}) \in \mathfrak{m}.\end{aligned}$$

Using the fact that $y_2 \in \ker f$, one easily proves that $a_i = c_i = 0$ by a similar argument as in the proof of Theorem 4.1. On the other hand, the above equality implies that $by_2 = u\sum_{i=p+1}^{q} b_i e_i$. Thus $Z = (0, \frac{b}{u}y_2, a, b)$. From this one easily deduces that $[X, Z] = 0$. Then we have $X - Z \in \mathfrak{h}$, that is, $X = Z$, which is a contradiction.

Finally, the case $X = (0, 0, \frac{1}{2u}, u)$ can be proved by a similar argument as in the proof of Theorem 4.1. This completes the proof of the theorem. ■

From the above lemmas and the theorem one easily deduces the following

Corollary 4.14. *Let M be a Cahen-wallach manifold and f the corresponding map. If f is degenerate and has at least one positive eigenvalue, then the future timelike cut locus coincides with the first future timelike conjugate locus.*

Theorem 4.15. *Let M be a Cahen-wallach manifold and f the corresponding map. If f is degenerate and has at least one positive eigenvalue, then the future lightlike cut locus coincides with the first future lightlike conjugate locus except for the direction* $(0,0,1,0)$.

Remark 4.16. The same reasoning shows that the geodesic along the lightlike vector $(0, 0, 1, 0)$ is the unique geodesic ray.

4.3. The conclusion. In the above two subsections we have considered the cases that f has at least one positive eigenvalue. If f has no positive eigenvalue, then neither cut points nor conjugate points of o exist. In conclusion, we have the following

Theorem 4.17. *On any Cahen-Wallach manifold, the future nonspacelike cut locus coincides with the first future nonspacelike conjugate locus.*

5. $\mathbb{R}\times M$, $D\times M$, and $C\times M$

Let M be a connected simply connected compact Riemannian symmetric space $(U/K, \underline{Q})$ associated with $(\mathfrak{u}, \mathfrak{k})$. Let $(G/H, Q)$ be one of $(\mathbb{R}, -\mathrm{d}t^2)$, (D, B_d) and (C, Q_c). Then the spaces $\mathbb{R}\times M$, $D\times M$, and $C\times M$ can be written as a uniform form $(G/H\times U/K, Q+\underline{Q})$. As in the previous section, we first study some special cases and then deduce our conclusion.

5.1. $G/H = (\mathbb{R}, -dt^2), (D, B_d)$ **or** (C, Q_c) **and** f **has no positive eigenvalues.** In this case, it is easily seen that the timelike unit vector has the form

$$\widehat{X} = xX + \sqrt{x^2-1}\underline{X}, \quad x \geq 1,$$

here $Q(X, X) = -1$, and $\underline{Q}(\underline{X}, \underline{X}) = 1$.

Let $\underline{X}$ be $\mathrm{Ad}K$-conjugate to $\underline{X}' \in \mathfrak{h}_{\mathfrak{p}}$. Consider the tangent conjugate point of $\widehat{X}' = xX + \sqrt{x^2-1}\underline{X}'$, $x \geq 1$. First we have

Lemma 5.1. *If* $\widehat{X}' = xX + \sqrt{x^2-1}\underline{X}', x > 1$*, is a future timelike unit vector, then* $t_0\widehat{X}'$ *is the tangent conjugate point of* $o = \{H\times K\}$ *along* $\gamma_{\widehat{X}'}$ *if and only if there exists* $\alpha \in \Sigma^+$ *and* $m \in \mathbb{Z}-\{0\}$ *such that* $t_0 = \frac{m\pi}{\sqrt{1-x^2\alpha(\underline{X}')}}$.

Proof. If $t_0\widehat{X}'$ is the tangent conjugate point of $o = \{H\times K\}$ along $\gamma_{\widehat{X}'}$, then there exists a non-zero Jacobi field $\mathcal{J}(t)$ such that $\mathcal{J}(0) = \mathcal{J}(t_0) = 0$. Since $\mathrm{ad}^2\widehat{X}' = x^2\mathrm{ad}^2X + (x^2-1)\mathrm{ad}^2\underline{X}'$, and $R(\ ,\widehat{X}')\widehat{X}' = -\mathrm{ad}^2\widehat{X}'$ (see [11]), we deduce from Section 3 and Section 4 that the eigenspaces of $R(\ ,\widehat{X}')\widehat{X}'$ with positive eigenvalues should be subspaces of $\mathfrak{p}$. Then the subspace $\mathfrak{p}$ has a decomposition

$$\mathfrak{p} = \mathfrak{h}_{\mathfrak{p}} + \sum_{\gamma\in\Sigma^+}\mathfrak{p}_\gamma.$$

Select a basis $\{E_i\}\cup\{F_j\}$ of the tangent space of $G/H\times U/K$ at the origin such that $\{E_i\}$ are unit eigenvectors of $R(\ ,\widehat{X}')\widehat{X}'$ in the decomposition of $\mathfrak{p}$ and $\{F_j\}$ are unit eigenvectors of $R(\ ,\widehat{X}')\widehat{X}'$ on the tangent space of $\{H\}$. Let $\{E_i(t)\}$ and $\{F_j(t)\}$ be the parallel translations of $\{E_i\}$ and $\{F_j\}$ along the geodesic $\gamma_{\widehat{X}'}$, respectively. Then one can write $\mathcal{J}(t)$ as

$$\mathcal{J}(t) = \sum_i h_i(t)E_i(t) + \sum_j g_i(t)F_i(t).$$

By the Jacobi equation we have

$$\mathcal{J}''(t) + R(\mathcal{J}(t), \widehat{X}')\widehat{X}' = 0.$$

From this one easily deduces that only $h_i(t)$ may have non-zero solutions

$$C\alpha(\underline{X}')\sin(t\sqrt{1-x^2}),$$

where $\alpha(\underline{X}') \neq 0$. From this the "only if" part of the lemma follows.

Conversely, if $t_0 = \frac{m\pi}{\sqrt{1-x^2}\alpha(\underline{X}')}$, with $m \in \mathbb{Z} - \{0\}$, then we can directly construct a non-zero Jacobi field $\mathcal{J}$ by $\mathcal{J}(t) = \alpha(\underline{X}')\sin(t\cdot\sqrt{1-x^2})E_1(t)$, where $E_1(t)$ is defined as above. Then we have $\mathcal{J}(0) = \mathcal{J}(t_0) = 0$. This completes the proof of the lemma. ∎

Corollary 5.2. *Let $\widehat{X} = xX + \sqrt{x^2-1}\underline{X}, x > 1$, be a future timelike unit vector and $\underline{X}$ be* AdK*-conjugate to $\underline{X}' \in \mathfrak{h}_\mathfrak{p}$. Then the first future tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}}$ is $t_0\widehat{X}$, where*

$$t_0 = \min_{\alpha\in\Sigma^+} \frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}.$$

Remark 5.3. From section 3 and section 4, we know that if $x \neq 1$, then the first future tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}}$ is $t_0\widehat{X}$; If $x = 1$, then there doesn't exist any conjugate point along $\gamma_{\widehat{X}}$.

Now we can prove the following

Theorem 5.4. *Let $\widehat{X} = xX + \sqrt{x^2-1}\underline{X}, x > 1$, be a future timelike unit vector and $t_0\widehat{X}$ be the future tangent cut point of o along the geodesic $\gamma_{\widehat{X}}$, then it must be the first future tangent conjugate point.*

Proof. We only need to prove the equality $t_0 = \min_{\alpha\in\Sigma^+}\frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}$. Suppose conversely that $t_0 < \min_{\alpha\in\Sigma^+}\frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}$. Then there exists a future timelike unit vector Y and unit vecor $\underline{Y}$ with $\widehat{Y} = yY + \sqrt{y^2-1}\underline{Y} \neq \widehat{X}$ such that $\mathrm{Exp}t_0\widehat{X} = \mathrm{Exp}t_0\widehat{Y}$, i.e.,

$$(\exp t_0 xX \cdot H,\ \exp t_0\sqrt{x^2-1}\underline{X}\cdot K) = (\exp t_0 yY \cdot H,\ \exp t_0\sqrt{y^2-1}\underline{Y}\cdot K).$$

Then from Sections 3 and 4 we have $x = y$. Since $t_0 < \min_{\alpha\in\Sigma^+}\frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}$, i.e., $t_0\sqrt{x^2-1} < \min_{\alpha\in\Sigma^+}\frac{\pi}{|\alpha(\underline{X}')|}$, by [4] or [14], we have $\underline{X} = \underline{Y}$, which is a contradiction to the fact $\widehat{X} \neq \widehat{Y}$. ∎

Remark 5.5. If x=1, then from Sections 3 and 4, we know that neither cut point nor conjugate point of o exists along $\gamma_{\widehat{X}}$.

Finally, we consider future lightlike vectors. We only need to consider vectors of the form $\widehat{X} = X + \underline{X}$ with $Q(X,X) = -1$ and $\underline{Q}(\underline{X},\underline{X}) = 1$. Similarly as for future timelike vectors, we have the following

Lemma 5.6. *$t_0\widehat{X}'$ is the tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}'}$ if and only if there exists $\alpha \in \Sigma^+$ and $m \in \mathbb{Z} - \{0\}$ such that $t_0 = \frac{m\pi}{\alpha(\underline{X}')}$.*

Corollary 5.7. *Let $\widehat{X} = X + \underline{X}$ be a future lightlike vector and $\underline{X}$ be $\mathrm{Ad}K$-conjugate to $\underline{X}' \in \mathfrak{h}_\mathfrak{p}$. Then the first future tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}}$ is $t_0\widehat{X}$, where*

$$t_0 = \min_{\alpha\in\Sigma^+} \frac{\pi}{|\alpha(\underline{X}')|}.$$

Theorem 5.8. *Let $\widehat{X} = X + \underline{X}$ be a future lightlike unit vector and $t_0\widehat{X}$ be the future tangent cut point of o along the geodesic $\gamma_{\widehat{X}}$, then it must be the first future tangent conjugate point.*

5.2. $G/H = (C, Q_c)$ and f has at least one positive eigenvalue. We keep the notations as in the previous subsection. Suppose $X = (0, y, t, u), u > 0$. Write the positive eigenvalues of f as

$$\lambda_1 \geq \cdots \geq \lambda_p > 0, \quad p \geq 1.$$

To deduce our conclusion for this case, we only need to modify some assertions of Section 5.1.

Lemma 5.9. *$t_0\widehat{X}'$ is the tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}'}$ if and only if there exists $\alpha \in \Sigma^+$ and $m \in \mathbb{Z} - \{0\}$ such that $t_0 = \frac{m\pi}{\sqrt{1-x^2}\alpha(\underline{X}')}$, or $t_0 = \frac{m\pi}{xu\sqrt{\lambda_i}}$, $1 \leq i \leq p$.*

Proof. Consider the eigenvalues λ_i of f. Denote by $T_{\lambda_i}(t)$, $A(t)$ and $B(t)$ the parallel translations of T_{λ_i}, $A = (0, 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, and $B = (0, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ along $\gamma_{\widehat{X}'}$, respectively. Let $E_j(t)$ be as in subsection 5.1. Then we can write $\mathcal{J}(t)$ as

$$\mathcal{J}(t) = \sum_{i=1}^{n} f_i(t)T_{\lambda_i}(t) + a(t)A(t) + b(t)B(t) + \sum_j g_j(t)E_j(t).$$

Therefore the lemma follows. ■

Corollary 5.10. *Let $\widehat{X} = xX + \sqrt{x^2-1}\underline{X}, x \geq 1$, be a future timelike unit vector and $\underline{X}$ be $\mathrm{Ad}K$-conjugate to $\underline{X}' \in \mathfrak{h}_\mathfrak{p}$. Then the first future tangent conjugate point of $o = \{H \times K\}$ along $\gamma_{\widehat{X}}$ is $t_0\widehat{X}$, where*

$$t_0 = \min\{\min_{\alpha\in\Sigma^+} \frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}, \frac{\pi}{xu\sqrt{\lambda_1}}\}.$$

Now we prove

Theorem 5.11. *Let $\widehat{X} = xX + \sqrt{x^2-1}\underline{X}\, x \geq 1$, be a future timelike unit vector and $t_0\widehat{X}$ be the future tangent cut point of o along the geodesic $\gamma_{\widehat{X}}$. Then $t_0\widehat{X}$ must be the first future tangent conjugate point.*

Proof. It suffices to prove that

$$t_0 = \min\{\min_{\alpha\in\Sigma^+} \frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|},\ \frac{\pi}{xu\sqrt{\lambda_1}}\}.$$

Suppose conversely that

$$t_0 < \min\{\min_{\alpha\in\Sigma^+} \frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|},\ \frac{\pi}{xu\sqrt{\lambda_1}}\}.$$

Then there exists another future timelike unit vector $\widehat{Z} = zZ + \sqrt{z^2-1}\underline{Z} \neq \widehat{X}$ such that

$$(\exp t_0 xX\cdot H,\ \exp t_0\sqrt{x^2-1}\underline{X}\cdot K) = (\exp t_0 zZ\cdot H,\ \exp t_0\sqrt{z^2-1}\underline{Z}\cdot K).$$

Then we have $x = z$. Since $t_0 < \frac{\pi}{xu\sqrt{\lambda_1}}$, by Theorems 4.4 and 4.7, we have $X = Z$. Similarly, if $t_0 < \min_{\alpha\in\Sigma^+}\frac{\pi}{\sqrt{x^2-1}|\alpha(\underline{X}')|}$, then by the results of Critendden abd Sakai (see [4, 14, 15]), we have $\underline{X} = \underline{Z}$, which is a contradiction to the fact $\widehat{X} \neq \widehat{Z}$.

In the case of $x = 1$, the assertion of the theorem follows directly from Theorems 4.4 and 4.7. ∎

For future lightlike vector X, we have the following

Lemma 5.12. *$t_0\widehat{X}'$ is the tangent conjugate point of $o = \{H\times K\}$ along $\gamma_{\widehat{X}'}$ if and only if there exists $\alpha\in\Sigma^+$ and $m\in\mathbb{Z}-\{0\}$ such that $t_0 = \frac{m\pi}{\alpha(\underline{X}')}$, or $t_0 = \frac{m\pi}{u\sqrt{\lambda_i}}$, $1\le i\le p$.*

Corollary 5.13. *Let $\widehat{X} = X + \underline{X}$ be a future lightlike vector and $\underline{X}$ be* AdK*-conjugate to $\underline{X}'\in\mathfrak{h}_\mathfrak{p}$. Then the first future tangent conjugate point of $o = \{H\times K\}$ along $\gamma_{\widehat{X}}$ is $t_0\widehat{X}$, where $t_0 = \min\{\min_{\alpha\in\Sigma^+}\frac{\pi}{|\alpha(\underline{X}')|},\ \frac{\pi}{u\sqrt{\lambda_1}}\}$.*

Finally, we have

Theorem 5.14. *Let $\widehat{X} = X + \underline{X}$ be a future lightlike unit vector and $t_0\widehat{X}$ be the future tangent cut point of o along the geodesic $\gamma_{\widehat{X}}$. Then $t_0\widehat{X}$ must be the first future tangent conjugate point.*

The main theorem (Theorem 1.1) of this paper now follows from Theorems 5.4, 5.8, 5.11 and 5.14 of this section.

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