

Sp-Irreducible Components in the Johnson Cokernels of the Mapping Class Groups of Surfaces, I

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Abstract. In “N. Enomoto and T. Satoh, New series in the Johnson cokernels of the mapping class groups of surfaces, to appear in Algebraic and Geometric Topology,” the second author and Takao Satoh introduced a new class in the Johnson cokernels for the mapping class groups of surfaces, and detected a series of Sp-irreducible components $[1^{4m+1}]$ ($m \geq 1$) in this class. In this paper, we detect another series $[\lambda]$ in this class for some hook type partitions λ .

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1. Introduction

Let $\Sigma_{g,1}$ be a compact oriented surface of genus g with one boundary component. The mapping class group $M_{g,1}$ is the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_{g,1}$ which fix the boundary component pointwise. Taking a base point $*$ on the boundary, the fundamental group $\pi_1(\Sigma_{g,1}, *)$ is isomorphic to the free group F_{2g} of rank $2g$. We fix a basis x_1, \dots, x_{2g} as shown in Figure 1. The integral first homology group $H := H_1(\Sigma_{g,1}, \mathbb{Z})$ is the free abelian group of rank $2g$. The homology classes e_1, \dots, e_{2g} of x_1, \dots, x_{2g} form a symplectic basis of H .

The mapping class group $M_{g,1}$ has a subgroup $\text{Torelli}_{g,1}$ which trivially acts on H . We have the following short exact sequence:

$$1 \rightarrow \text{Torelli}_{g,1} \rightarrow M_{g,1} \rightarrow \text{Sp}(2g, \mathbb{Z}) \rightarrow 1.$$

In the 1980s, D. Johnson established a method to study the structure of the mapping class group $M_{g,1}$ and its Torelli subgroup $\text{Torelli}_{g,1}$. That is an approximation of $M_{g,1}$ and $\text{Torelli}_{g,1}$ by some graded Lie algebras. For simplicity, we consider the lower central series $\text{Torelli}_{g,1} = \Gamma(1) \supset \Gamma(2) \supset \dots$ of $\text{Torelli}_{g,1}$. Here we inductively define $\Gamma(k+1) = [\text{Torelli}_{g,1}, \Gamma(k)]$ for $k \geq 1$. We

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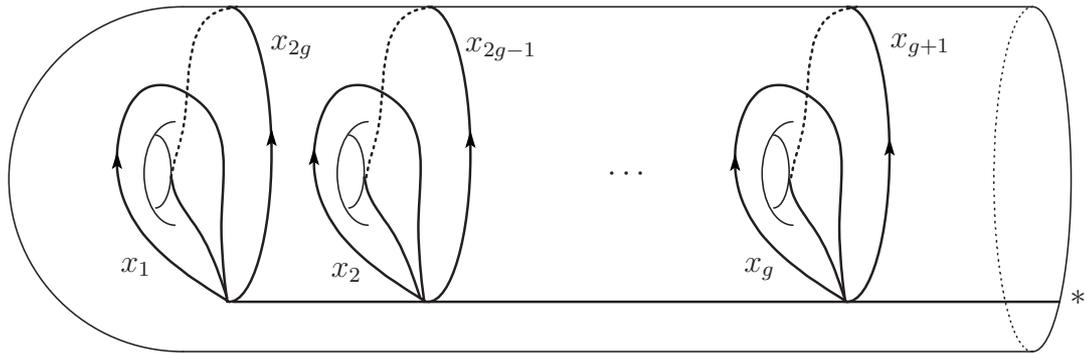


Figure 1: generators x_1, \dots, x_{2g} of $\pi_1(\Sigma_{g,1}, *)$

set $\text{gr}^k(\text{Torelli}_{g,1}) := \Gamma(k)/\Gamma(k + 1)$. These graded quotients become $\text{Sp}(2g, \mathbb{Z})$ -modules. Johnson introduced an $\text{Sp}(2g, \mathbb{Z})$ -homomorphism

$$\tau'_k : \text{gr}^k(\text{Torelli}_{g,1}) \rightarrow \text{Hom}_{\mathbb{Z}}(H, \mathcal{L}_{2g}(k + 1)) = H^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}(k + 1),$$

where $\mathcal{L}_{2g}(k + 1)$ is the degree $k + 1$ part of the free Lie algebra \mathcal{L}_{2g} generated by H . The righthand side can be regarded as the degree k part of the derivation algebra of \mathcal{L}_{2g} .

We remark that the Johnson filtration of $M_{g,1}$ is slightly different from the lower central series of $\text{Torelli}_{g,1}$ and τ'_k is the restriction of the Johnson homomorphism τ_k . But, by tensoring $- \otimes_{\mathbb{Z}} \mathbb{Q}$, the image of $\tau_k^{\mathbb{Q}}$ coincides with the image of $\tau'_k^{\mathbb{Q}}$ in $H_{\mathbb{Q}}^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1)$ proved by R. Hain [Ha]. Thus to study the structure of the Johnson images or cokernels over \mathbb{Q} , we may consider the settings above. For more details, see [ES2, Section 3].

S. Morita studied the structure of Johnson images and obtained some remarkable results. As an $\text{Sp}(2g, \mathbb{Q})$ -module, $H_{\mathbb{Q}}^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1)$ is isomorphic to $H_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1)$. Let us consider the left bracketing homomorphism $H_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \rightarrow \mathcal{L}_{2g}^{\mathbb{Q}}(k + 2)$ defined by $X \otimes Y \rightarrow [X, Y]$. We denote by $\mathfrak{h}_{g,1}(k)$ the kernel of this homomorphism. In [Mo1], Morita proved that the Johnson image $\text{Im}(\tau_k^{\mathbb{Q}}) = \text{Im}(\tau'_k{}^{\mathbb{Q}})$ is contained in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ for all $k \geq 1$. From now, we call $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\text{Im}(\tau_k^{\mathbb{Q}})$ the Johnson cokernel. Morita also constructed a homomorphism $\text{tr}_k : \mathfrak{h}_{g,1}^{\mathbb{Q}}(k) \rightarrow S^k H_{\mathbb{Q}}$ and proved $\text{tr}_k \circ \tau_k^{\mathbb{Q}} \equiv 0$ for any odd $k \geq 3$. By this, an Sp -irreducible component $S^k H_{\mathbb{Q}}$ is detected in the Johnson cokernel. This is called the Morita trace or the Morita obstruction for the surjectivity of the Johnson homomorphism.

The second author and T. Satoh studied the structure of the Johnson cokernels by using some representation theory and Satoh's results for the Johnson homomorphism for the automorphisms groups of the free groups. In [ES2], we detected an Sp -irreducible component $[1^k]$ for $k \equiv 1 \pmod{4}$ and $k \geq 5$. Moreover we introduced a new class in the Johnson cokernels. To explain this class, we consider the Sp -homomorphism

$$c_k : \mathfrak{h}_{g,1}^{\mathbb{Q}}(k) \subset H_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \cong H_{\mathbb{Q}}^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \subset H_{\mathbb{Q}}^* \otimes H_{\mathbb{Q}}^{\otimes k + 1} \xrightarrow{12\text{-cont}} H_{\mathbb{Q}}^{\otimes k} \twoheadrightarrow \mathcal{C}_{2g}^{\mathbb{Q}}(k).$$

Here 12-cont is the homomorphism $H_{\mathbb{Q}}^* \otimes H_{\mathbb{Q}}^{\otimes k + 1} \rightarrow H_{\mathbb{Q}}^{\otimes k}$ defined by

$$f \otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_{k+1}) \mapsto f(v_1)v_2 \otimes \cdots \otimes v_{k+1}.$$

The space $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ is the quotient with respect to the natural action of the cyclic group Cyc_k on $H_{\mathbb{Q}}^{\otimes k}$. This c_k is not surjective nor injective. But by using Satoh’s remarkable result that the space $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ coincides with the Johnson cokernels for the automorphism groups of the free groups, we proved that

$$\text{Im}(\tau_k^{\mathbb{Q}}) = \text{Im}(\tau_k^{\prime\mathbb{Q}}) \subset \text{Ker}(c_k) \subset \mathfrak{h}_{g,1}^{\mathbb{Q}}(k).$$

This is a new class in the Johnson cokernels for the mapping class group of surfaces. To detect Sp-irreducible components in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\text{Ker}(c_k)$, in [ES2], we described a kind of generators of the space of Sp-maximal vectors of $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$. This generating set is obtained by using the Brauer-Schur-Weyl duality, the Dynkin-Specht-Wever idempotent and Morita’s observation in [Mo2, Proposition 4.6]. (For more details, see Proposition 2.2.) By using the generating set mentioned as above, we proved that $c_k([1^k]) \neq 0$ for $k \equiv 1 \pmod{4}$ and $k \geq 5$ in [ES2].

In this note, using the same generators, we prove the following results for the multiplicities in the Johnson cokernels $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\text{Im}(\tau_k^{\mathbb{Q}})$ for the mapping class group of surfaces.

We only consider the stable range, namely assume $g \geq k + 2$.

Theorem 1.1. *Suppose $\lambda = (r + 1, 1^{k-r-1})$ for $r \geq 1$ and $k - r - 1 \geq 2$. Then the multiplicity of the Sp-irreducible module $[\lambda]$ in the k -th Johnson cokernel is larger than or equal to 1 if none of the following conditions are satisfied: (i) k is odd and r is even such that $k - r \equiv 3 \pmod{4}$, (ii) $r = 1$ and k is odd, (iii) $r = 1$ and $k - r \equiv 1 \pmod{4}$.*

For example, this theorem claims that the following Sp-irreducible components exist in the Johnson cokernels for the mapping class group of the surface $\Sigma_{g,1}$:

- $[2, 1^{k-2}]$ if $k \equiv 0 \pmod{4}$ and $k \geq 4$,
- $[3, 1^{k-3}]$ if $k \not\equiv 1 \pmod{4}$ and $k \geq 5$,
- $[4, 1^{k-4}]$ if $k \geq 6$,
- $[5, 1^{k-5}]$ if $k \not\equiv 3 \pmod{4}$ and $k \geq 7$,
- $[6, 1^{k-6}]$ if $k \geq 8$,

and so on.

Remark 1.2. In the case $k - r - 1 = 1$, $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ has no Sp-irreducible component $[k - 1, 1]$. In the case (ii), $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ has no Sp-irreducible component $[2, 1^{k-2}]$ (see, [ES1, Section 4.1 Table.1]). Thus $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\text{Ker}(c_k)$ has no such components. But in the cases (i) and (iii), the theorem above does not claim that the multiplicity of Sp-irreducible components $[r + 1, 1^{k-r-1}]$ is equal to 0 in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)/\text{Ker}(c_k)$. Indeed, for example, by a computer calculation, we can show that the multiplicity of $[5, 1^6]$ in $\mathfrak{h}_{g,1}/\text{Ker}(c_{11})$ is larger than or equal to 5.

We will prove the following theorem in [EE].

Theorem 1.3. *We have*

$$[\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) / \text{Ker}(c_k) : [k - 2]] = \delta_{k:\text{even}} \left\lfloor \frac{k}{3} \right\rfloor.$$

This paper is organized as follows. In section 2, we recall some notions and results from [ES2]. In section 3, we give a proof of Theorem 1.1. In the Appendix, we explain a combinatorial way to calculate multiplicities of Sp-irreducible components in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$.

2. Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$ and its Sp-maximal vectors

2.1. Lie algebra $\mathfrak{h}_{g,1}^{\mathbb{Q}}$. Let H be the free abelian group of rank $2g$ for an integer $g \geq 1$. We naturally regard $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$ as an $\text{Sp}(2g, \mathbb{Q})$ -module. We denote by $\mathcal{L}_{2g}^{\mathbb{Q}}$ the free Lie algebra generated by $H_{\mathbb{Q}}$, and by $\text{Der}(\mathcal{L}_{2g}^{\mathbb{Q}})$ its derivation algebra. We also regard their degree k parts $\mathcal{L}_{2g}^{\mathbb{Q}}(k)$ and $\text{Der}(\mathcal{L}_{2g}^{\mathbb{Q}})(k)$ as $\text{Sp}(2g, \mathbb{Q})$ -modules. As $\text{Sp}(2g, \mathbb{Q})$ -modules, $\text{Der}(\mathcal{L}_{2g}^{\mathbb{Q}})(k) = H_{\mathbb{Q}}^* \otimes_{\mathbb{Z}} \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1)$ is isomorphic to $H_{\mathbb{Q}} \otimes \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1)$. We consider an $\text{Sp}(2g, \mathbb{Q})$ -module homomorphism $[\cdot, \cdot] : H_{\mathbb{Q}} \otimes \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \rightarrow \mathcal{L}_{2g}^{\mathbb{Q}}(k + 2)$ defined by $X \otimes Y \mapsto [X, Y]$. We define $\mathfrak{h}_{g,1}(k)$ by $\text{Ker}(H_{\mathbb{Q}} \otimes \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \rightarrow \mathcal{L}_{2g}^{\mathbb{Q}}(k + 2))$. Then $\mathfrak{h}_{g,1}^{\mathbb{Q}} := \bigoplus_{k \geq 1} \mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ becomes a Lie subalgebra of $\text{Der}(\mathcal{L}_{2g}^{\mathbb{Q}})$. This is introduced by M. Kontsevich in [Ko1], [Ko2] as a Lie version of the formal symplectic geometry.

2.2. Quotient space $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$. In [Sa], T. Satoh considered the cyclic quotient space $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ for studying the structure of Johnson cokernels for the automorphism group of the free groups. The cyclic group Cyc_k of order k generated by σ_k acts on $H_{\mathbb{Q}}^{\otimes k}$ by $\sigma_k(v_1 \otimes \cdots \otimes v_{k-1} \otimes v_k) = v_k \otimes v_1 \otimes \cdots \otimes v_{k-1}$. The space $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ is the quotient of $H_{\mathbb{Q}}^{\otimes k}$ by this action of Cyc_k . We have an $\text{Sp}(2g, \mathbb{Q})$ -homomorphism $\text{Der}(\mathcal{L}_{2g}^{\mathbb{Q}})(k) = H_{\mathbb{Q}}^* \otimes \mathcal{L}_{2g}^{\mathbb{Q}} \hookrightarrow H_{\mathbb{Q}}^* \otimes H_{\mathbb{Q}}^{\otimes k+1} \rightarrow H_{\mathbb{Q}}^{\otimes k}$ defined by a contraction $f \otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_{k+1}) = f(v_1)v_2 \otimes \cdots \otimes v_{k+1}$. In [Sa], Satoh characterized $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ as the Johnson cokernels for the automorphism groups of the free groups. Thus we have the following diagram of Sp-modules:

$$\begin{array}{ccccc} H_{\mathbb{Q}}^* \otimes \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) & \longrightarrow & H_{\mathbb{Q}}^{\otimes k} & \xrightarrow{\pi_k} & \mathcal{C}_{2g}^{\mathbb{Q}}(k) \\ & & \uparrow \wr & & \\ \mathfrak{h}_{g,1}^{\mathbb{Q}}(k) = \text{Ker}([\cdot, \cdot]) & \hookrightarrow & H_{\mathbb{Q}} \otimes \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) & \xrightarrow{[\cdot, \cdot]} & \mathcal{L}_{2g}^{\mathbb{Q}}(k + 1) \end{array}$$

We denote by c_k the Sp-homomorphism $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) \rightarrow \mathcal{C}_{2g}^{\mathbb{Q}}(k)$. In [ES2], to study the structure of Johnson cokernels for the mapping class groups of surfaces, the second author and Satoh introduced this homomorphism c_k and proposed the following problem.

Problem 2.1. Describe the Sp -module structures of $\mathrm{Im}(c_k)$ and $\mathrm{Ker}(c_k)$.

2.3. A description of Sp -maximal vectors in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$. First, we briefly recall the classical representation theory of $\mathrm{Sp}(2g, \mathbb{Q})$. Let T be a maximal torus of $\mathrm{Sp}(2g, \mathbb{Q})$ consisting of all diagonal matrices in $\mathrm{Sp}(2g, \mathbb{Q})$. We define the one-dimensional representation ε_i of T by $\varepsilon_i(\mathrm{diag}(x_1, \dots, x_g, x_g^{-1}, \dots, x_1^{-1})) = x_i$ for $1 \leq i \leq g$. For a rational representation of $\mathrm{Sp}(2g, \mathbb{Q})$, we have a weight decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$. Here $P := \{\lambda\varepsilon_1 + \dots + \lambda_g\varepsilon_g \mid \lambda_i \in \mathbb{Z}, 1 \leq i \leq g\}$, where a subspace $V_\lambda = \{v \in V \mid tv = \lambda(t)v \text{ for all } t \in T\}$. We call V_λ the weight space of V with the weight λ .

Let U be the subgroup of $\mathrm{Sp}(2g, \mathbb{Q})$ consisting of all upper unitriangular matrices in $\mathrm{Sp}(2g, \mathbb{Q})$. Then we call a non-zero element of

$$V^U := \{v \in V \mid uv = v \text{ for all } u \in U\}$$

an Sp -maximal vector. We also have a weight decomposition $(V^U) = \bigoplus_{\lambda \in P} V_\lambda^U$, where $V_\lambda^U := V_\lambda \cap V^U$. If a rational representation V of $\mathrm{Sp}(2g, \mathbb{Q})$ is irreducible, there exists a unique weight λ such that $V^U = (V^U)_\lambda$. Moreover, such λ satisfies $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g$. We say this λ a highest weight of V . Thus an irreducible rational representation of $\mathrm{Sp}(2g, \mathbb{Q})$ is uniquely determined by its highest weight. The set of isomorphism classes of irreducible rational representations of $\mathrm{Sp}(2g, \mathbb{Q})$ is equal to

$$P^+ := \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq 0 \mid \lambda_i \in \mathbb{Z}, 1 \leq i \leq g\},$$

namely, the set of partitions such that $\ell(\lambda) \leq g$. Here $\ell(\lambda)$ is the length of λ . In this note, we denote by $[\lambda]$ the irreducible rational representation of $\mathrm{Sp}(2g, \mathbb{Q})$ with the highest weight λ .

Note that any rational representation V of $\mathrm{Sp}(2g, \mathbb{Q})$ is completely reducible, namely V is isomorphic to a direct sum of some irreducible representations. Then we have the multiplicity of an irreducible Sp -representation $[\lambda]$ in V is equal to $\dim(V_\lambda^U)$.

We describe the space of maximal vectors $(\mathfrak{h}_{g,1}^{\mathbb{Q}}(k))_\lambda^U$ in the following way. We use some idempotents.

Now we fix a basis $\{e_1, \dots, e_g, e_{g+1}, \dots, e_{2g}\}$ in $H_{\mathbb{Q}}$. Set $i' := 2g - i + 1$ for each integer $1 \leq i \leq 2g$. For the standard basis $\{e_i\}_{i=1}^{2g}$ of $H_{\mathbb{Q}}$, we see

$$\langle e_i, e_j \rangle = 0 = \langle e_{i'}, e_{j'} \rangle, \quad \langle e_i, e_{j'} \rangle = \delta_{ij} = -\langle e_{j'}, e_i \rangle, \quad (1 \leq i, j \leq g).$$

For each integer $1 \leq i \leq 2g$, we define

$$e_i^* = \begin{cases} e_{i'}, & (1 \leq i \leq g), \\ -e_{i'}, & (g+1 \leq i \leq 2g). \end{cases} \tag{1}$$

Then both of $\{e_i\}_{i=1}^{2g}$ and $\{e_i^*\}_{i=1}^{2g}$ are basis for $H_{\mathbb{Q}}$ such that one is dual to the other in the sense that $\langle e_i, e_j^* \rangle = \delta_{ij}$ for any i, j . Let ω be the element $\sum_{i=1}^{2g} e_i \otimes e_i^* \in H_{\mathbb{Q}}^{\otimes 2}$. We identify $H_{\mathbb{Q}}$ with $H_{\mathbb{Q}}^*$ by $v \mapsto \langle v, \bullet \rangle$.

For a partition λ , we denote by λ^T its transpose, and by λ_i^T the i -th component of λ^T . For a partition λ of $k + 2 - 2j$ such that $\ell(\lambda) \leq g$, we define

$$v_\lambda := \omega^j \otimes (e_1 \wedge \dots \wedge e_{\lambda_1^T}) \otimes (e_1 \wedge \dots \wedge e_{\lambda_2^T}) \otimes \dots \in H_{\mathbb{Q}}^{k+2}.$$

Here $v_1 \wedge v_2 \wedge \cdots \wedge v_r$ is the anti-symmetrizer $\sum_{\sigma \in \mathfrak{S}_r} \text{sgn}(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_r) \cdot \sigma \in H_{\mathbb{Q}}^{\otimes r}$. This element v_{λ} is an Sp-maximal vector in $H_{\mathbb{Q}}^{\otimes k+2}$ with weight λ . The space of Sp-maximal vectors $(H_{\mathbb{Q}}^{\otimes k+2})_{\lambda}^U$ generate v_{λ} and the action of $\mathbb{Q}\mathfrak{S}_{k+2}$ via the Brauer-Schur-Weyl duality.

Let us consider the right action of \mathfrak{S}_{k+2} on $H_{\mathbb{Q}}^{\otimes(k+2)}$. Let s_i be the permutation of i and $i + 1$ for $1 \leq i \leq k + 1$ and σ_{k+2} the cyclic permutation of order $k + 2$. We define an element θ by

$$\theta = (1 - s_2)(1 - s_3s_2) \cdots (1 - s_{k+1}s_k \cdots s_2)$$

in $\mathbb{Q}\mathfrak{S}_{k+2}$. We also consider an element

$$\zeta_{k+2} = 1 + \sigma_{k+2} + \sigma_{k+2}^2 + \cdots + \sigma_{k+2}^{k+1}$$

in $\mathbb{Q}\mathfrak{S}_{k+2}$.

Proposition 2.2. *The space of maximal vectors $(\mathfrak{h}_{g,1}^{\mathbb{Q}}(k))_{\lambda}^U$ is generated by the set of elements*

$$\{v_{\lambda} \cdot \sigma \theta \zeta_{k+2} \mid \sigma \in \mathfrak{S}_{k+2}\}.$$

In [ES2, Section 4 and 5], this proposition is proved using the results of J. Hu, Dynkin-Specht-Wever and S. Morita.

We remark that the multiplicity of $[\lambda]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ (namely $\dim(\mathfrak{h}_{g,1}^{\mathbb{Q}}(k))_{\lambda}^U$) is computable by a totally combinatorial way based on a result of V. M. Zhuravlev. For more details, see the Appendix. By [ES1, Proposition 4.1], we can also calculate the multiplicity $[\lambda]$ in $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$ by a combinatorial formula. But, since our homomorphism $c_k^{\mathbb{Q}}$ is not surjective and not injective in general even if restricting to each λ -isotypic component for $\lambda \in P^+$, these combinatorial way is not sufficient to answer to our Problem 2.1. In this paper, we will use Proposition 2.2 to obtain our main theorems.

3. Proof of Theorem 1.1

In this section, we will study whether the element $c_k(v_{\lambda} \theta \zeta_{k+2})$ is 0 or not for the hook shape partitions $\lambda = [r + 1, 1^{k-r-1}] \vdash k$. If not, we obtain $[\mathfrak{h}_{g,1}^{\mathbb{Q}}(k) / \text{Ker}(c_k) : [\lambda]] \geq 1$.

3.1. Notations. We use the following notations. We consider the element $\Lambda_{[a,b]}$ of the form

$$\sum_{\tau \in \mathfrak{S}_{k-r}} \text{sgn}(\tau) e_1^{\otimes a} \otimes \overbrace{e_{\tau(1)} \otimes \cdots \otimes e_{\tau(b)}}^b \otimes e_1^{\otimes r-a} \otimes \overbrace{e_{\tau(b+1)} \otimes \cdots \otimes e_{\tau(k-r)}}^{k-r-b} \in H_{\mathbb{Q}}^{\otimes k},$$

where any permutation $\tau \in \mathfrak{S}_{k-r}$ acts on the set $\{1, 2, \dots, k - r\}$.

We also use an (i, j) -expansion operator $D_{ij} : H_{\mathbb{Q}}^{\otimes k} \rightarrow H_{\mathbb{Q}}^{\otimes(k+2)}$ defined by

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_k) \cdot D_{ij} := \sum_{r=1}^{2g} v_1 \otimes \cdots \otimes v_{i-1} \otimes e_r \otimes v_i \otimes \cdots \otimes v_{j-2} \otimes e_r^* \otimes v_{j-1} \otimes \cdots \otimes v_k$$

for $1 \leq i < j \leq k + 2$. For example,

$$v_\lambda = (e_1 \wedge \cdots \wedge e_{\lambda_1^r}) \otimes (e_1 \wedge \cdots \wedge e_{\lambda_2^r}) \otimes \cdots) D_{12} D_{34} \cdots D_{2j-1,2j}$$
 for a partition $\lambda \vdash (k + 2 - 2j)$.

We denote by ${}_i C_j$ the binomial coefficient $\begin{bmatrix} i \\ j \end{bmatrix}$.

3.2. Expansion of $v_\lambda \cdot \theta$. We consider partitions $\lambda = (r + 1, 1^{k-r-1})$. Then $v_\lambda = \omega \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r} = ((e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r}) D_{12}$. To expand the elements $v_\lambda \cdot \theta$, we prove the following two lemmas.

Lemma 3.1. *For any $m \geq 1$, the element $(e_1 \wedge \cdots \wedge e_m) D_{12} (1 - s_2)(1 - s_3 s_2) \cdots (1 - s_{m+1} \cdots s_3 s_2)$ is expanded as follows;*

$$\begin{cases} \sum_{j=1}^{\frac{m}{2}+1} (-1)^{j-1} {}_{\frac{m}{2}} C_{j-1} (e_1 \wedge \cdots \wedge e_m) D_{1,2j} & \text{if } m \text{ is even,} \\ \sum_{j=1}^{\frac{m+1}{2}} (-1)^{j-1} {}_{\frac{m-1}{2}} C_{j-1} (e_1 \wedge \cdots \wedge e_m) (D_{1,2j} - D_{1,2j+1}) & \text{if } m \text{ is odd.} \end{cases}$$

Proof. We prove the claim by the induction on m . ■

Lemma 3.2. *For any $v \in H_{\mathbb{Q}}^{\otimes m+2}$ and any $r \geq 1$, we have*

$$(v \otimes e_1^{\otimes r}) (1 - s_{m+2} \cdots s_3 s_2) \cdots (1 - s_{m+r+1} \cdots s_3 s_2) = \sum_{a=0}^r (-1)^a {}_r C_a e_1^{\otimes a} \otimes v \otimes e_1^{\otimes r-a}.$$

Proof. We prove the claim by the induction on r . ■

By these lemmas, we have an expansion of the element $v_\lambda \cdot \theta$ for $\lambda = (r + 1, 1^{k-r-1})$.

Proposition 3.3. *Suppose $\lambda = (r + 1, 1^{k-r-1})$ for $r \geq 1$ and $k - r - 1 \geq 2$. If $k - r$ is even, the element $v_\lambda \cdot \theta$ is equal to*

$$\sum_{a=0}^r \sum_{j=1}^{\frac{k-r}{2}+1} (-1)^{a+j-1} {}_r C_a {}_{\frac{k-r}{2}} C_{j-1} e_1^{\otimes a} \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r-a} D_{1,2j+a}.$$

If $k - r$ is odd, the element $v_\lambda \cdot \theta$ is equal to

$$\sum_{a=0}^r \sum_{j=1}^{\frac{k-r+1}{2}} (-1)^{a+j-1} {}_r C_a {}_{\frac{k-r-1}{2}} C_{j-1} e_1^{\otimes a} \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r-a} (D_{1,2j+a} - D_{1,2j+a+1}).$$

3.3. A presentation of $c_k(e_1^{\otimes a} \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r-a} \cdot D_{1j} \zeta_{k+2})$. Note that $\langle e_{r'}, e_r \rangle e_{r'}^* = e_r$ for $1 \leq r \leq 2g$ and $\pi_k(v) = \pi_k(v \sigma_{k+2})$ for any $v \in H_{\mathbb{Q}}^{\otimes k}$. We assume $g \geq k + 2$. We prove the following three lemmas. In the proof of this subsection, for simplicity, sometimes we omit \otimes .

Lemma 3.4. For $2 \leq j \leq k - r + 2$, the element $c_k(((e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r})D_{1j}\zeta_{k+2})$ is equal to

- (i) $\pi_k((2g - 2)\Lambda_{[r,k-r]})$ if $j = 2$,
- (ii) $\pi_k(-3(-1)^j\Lambda_{[r,k-r]} - \Lambda_{[r-1,j-2]})$ if $3 \leq j \leq k - r + 1$,
- (iii) $\pi_k(2((-1)^{k-r-1} - 1)\Lambda_{[r,k-r]})$ if $j = k - r + 2$.

Proof. If $j = 2$, the element $c_k(((e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r})D_{1j}\zeta_{k+2})$ is equal to

$$\begin{aligned} & c_k \left(\sum_{s=1}^{2g} \begin{pmatrix} e_s e_s^*(e_1 \wedge \cdots \wedge e_{k-r}) e_1^{\otimes r} \\ + e_1 e_s e_s^*(e_1 \wedge \cdots \wedge e_{k-r}) e_1^{\otimes r-1} \\ + e_s^*(e_1 \wedge \cdots \wedge e_{k-r}) e_1^{\otimes r} e_s \end{pmatrix} \right) \\ &= \pi_k \left(\begin{pmatrix} (2g)\Lambda_{[r,k-r]} - e_1(e_1 \wedge \cdots \wedge e_{k-r}) e_1^{\otimes r-1} \\ - \sum_{\tau \in \mathfrak{S}_{k-r}} \text{sgn}(\tau) e_{\tau(2)} \cdots e_{\tau(k-r)} e_1^{\otimes r} e_{\tau(1)} \end{pmatrix} \right) \\ &= \pi_k((2g - 2)\Lambda_{[r,k-r]}). \end{aligned}$$

If $3 \leq j \leq k - r - 1$, the element $c_k(((e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r})D_{1j}\zeta_{k+2})$ is equal to

$$\begin{aligned} & c_k \left(\sum_{s=1}^{2g} \sum_{\tau \in \mathfrak{S}_{k-r}} \text{sgn}(\tau) \begin{pmatrix} e_s e_{\tau(1)} e_{\tau(2)} \cdots e_{\tau(j-2)} e_s^* e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r} \\ + e_1 e_s e_{\tau(1)} e_{\tau(2)} \cdots e_{\tau(j-2)} e_s^* e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r-1} \\ + e_s^* e_{\tau(j-1)} e_{\tau(j)} \cdots e_{\tau(k-r)} e_1^{\otimes r} e_s e_{\tau(1)} \cdots e_{\tau(j-2)} \\ + e_{\tau(j-2)} e_s^* e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r} e_s e_{\tau(1)} \cdots e_{\tau(j-3)} \end{pmatrix} \right) \\ &= \pi_k \sum_{\tau \in \mathfrak{S}_{k-r}} \text{sgn}(\tau) \begin{pmatrix} e_{\tau(2)} \cdots e_{\tau(j-2)} e_{\tau(1)} e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r} \\ - e_{\tau(1)} e_{\tau(2)} \cdots e_{\tau(j-2)} e_1 e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r-1} \\ - e_{\tau(j)} \cdots e_{\tau(k-r)} e_1^{\otimes r} e_{\tau(j-1)} e_{\tau(1)} \cdots e_{\tau(j-2)} \\ + e_{\tau(j-1)} \cdots e_{\tau(k-r)} e_1^{\otimes r} e_{\tau(j-2)} e_{\tau(1)} \cdots e_{\tau(j-3)} \end{pmatrix} \\ &= \pi_k(3(-1)^{j-3}\Lambda_{[r,k-r]} - \Lambda_{[r-1,j-2]}). \end{aligned}$$

If $j = k - r + 2$, similarly we can obtain the claim. ■

Lemma 3.5. For $a + 2 \leq j \leq k - r + a + 2$ and $1 \leq a \leq r - 1$, the element $c_k((e_1^{\otimes a} \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r-a})D_{1j}\zeta_{k+2})$ is equal to

- (i) $\pi_k(\Lambda_{[r,k-r]} + (-1)^{k-r}\Lambda_{[a,k-r-1]})$ if $j = a + 2$,
- (ii) $\pi_k(2(-1)^{j-a-1}\Lambda_{[r-a,1]})$ if $a + 3 \leq j \leq k - r + a + 1$,
- (iii) $\pi_k(-\Lambda_{[r,k-r]} + (-1)^{k-r-1}\Lambda_{[r-a,1]})$ if $j = k - r + a + 2$.

Proof. We can prove the claim similarly as Lemma 3.4. ■

Lemma 3.6. For $r + 2 \leq j \leq k + 2$, the element $c_k((e_1^{\otimes r} \otimes (e_1 \wedge \cdots \wedge e_{k-r}))D_{1j}\zeta_{k+2})$ is equal to

- (i) $\pi_k(2(1 - (-1)^{k-r-1})\Lambda_{[r,k-r]})$ if $j = r + 2$.

- (ii) $\pi_k(3(-1)^{k-j}\Lambda_{[r,k-r]} + \Lambda_{[r-1,j-2-r]})$ if $r + 3 \leq j \leq k + 1$,
- (iii) $\pi_k(-(2g - 2)\Lambda_{[r,k-r]})$ if $j = k + 2$,

Proof. We can prove the claim similarly as Lemma 3.4. ■

3.4. A presentation of the image of maximal vectors by c_k . We describe $c_k(v_\lambda\theta\zeta_{k+2})$ as a sum of $\pi_k(\Lambda_{[a,b]})$ in the following proposition.

Proposition 3.7. For $\lambda = (r + 1, 1^{k-r-1})$ with $r \geq 1, k - r - 1 \geq 2$, the elements $c_k(v_\lambda\theta\zeta_{k+2})$ are equal to the following elements in $\mathcal{C}_{2g}(k)$;

- (i) If both k and r are odd such that $k \equiv r \pmod{4}$,

$$\pi_k \left((4g)\Lambda_{[r,k-r]} + 2 \sum_{a=1}^{r-1} (-1)^a {}_r C_a \Lambda_{[r-a,1]} - 2 \sum_{j=2}^{\frac{k-r}{2}} (-1)^{j-1} {}_{\frac{k-r}{2}} C_{j-1} \Lambda_{[r-1,2j-2]} \right). \tag{2}$$

- (ii) If both k and r are odd such that $k - r \equiv 2 \pmod{4}$,

$$\pi_k \left(-2 \sum_{j=2}^{\frac{k-r}{2}} (-1)^{j-1} {}_{\frac{k-r}{2}} C_{j-1} \Lambda_{[r-1,2j-2]} \right). \tag{3}$$

- (iii) If both k and r are even such that $k \equiv r \pmod{4}$,

$$\pi_k \left(2 \sum_{a=1}^{r-1} (-1)^a {}_r C_a \Lambda_{[r-a,1]} \right). \tag{4}$$

- (iv) If both k and r are even such that $k - r \equiv 2 \pmod{4}$, $\pi_k((4g)\Lambda_{[r,k-r]})$.

- (v) If k is odd and r is even such that $k - r \equiv 1 \pmod{4}$,

$$\pi_k \left((4g + 4)\Lambda_{[r,k-r]} + 2 \sum_{a=1}^{r-1} (-1)^a {}_r C_a \Lambda_{[r-a,1]} \right). \tag{5}$$

- (vi) If k is odd and r is even such that $k - r \equiv 3 \pmod{4}$, 0 .

- (vii) If k is even and r is odd such that $k - r \equiv 1 \pmod{4}$,

$$\pi_k \left(\begin{array}{l} -2(\Lambda_{[r-1,k-r-1]} - \Lambda_{[r-1,1]}) + 2 \sum_{a=1}^{r-1} (-1)^a {}_r C_a \Lambda_{[r-a,1]} \\ -2 \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} {}_{\frac{k-r-1}{2}} C_{j-1} (\Lambda_{[r-1,2j-2]} - \Lambda_{[r-1,2j-1]}) \end{array} \right). \tag{6}$$

(viii) If k is even and r is odd such that $k - r \equiv 3 \pmod{4}$,

$$\pi_k \left(\begin{array}{l} (4g - 4)\Lambda_{[r,k-r]} + 2(\Lambda_{[r-1,k-r-1]} + \Lambda_{[r-1,1]}) \\ -2 \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} \frac{k-r-1}{2} C_{j-1} (\Lambda_{[r-1,2j-2]} - \Lambda_{[r-1,2j-1]}) \end{array} \right). \tag{7}$$

Proof. We consider the case that $k - r$ is odd. By Proposition 3.3, the elements $c_k(v_\lambda \theta \zeta_{k+2})$ are equal to

$$c_k \left(\begin{array}{l} \sum_{j=1}^{\frac{k-r+1}{2}} (-1)^{j-1} \frac{k-r-1}{2} C_{j-1} (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r} (D_{1,2j} - D_{1,2j+1}) \zeta_{k+2} \\ + \sum_{a=1}^{r-1} \sum_{j=1}^{\frac{k-r+1}{2}} (-1)^{a+j-1} {}_r C_a \frac{k-r-1}{2} C_{j-1} \\ \quad e_1^{\otimes a} \otimes (e_1 \wedge \cdots \wedge e_{k-r}) \otimes e_1^{\otimes r-a} (D_{1,2j+a} - D_{1,2j+a+1}) \zeta_{k+2} \\ + \sum_{j=1}^{\frac{k-r-1}{2}} (-1)^{r+j-1} \frac{k-r-1}{2} C_{j-1} e_1^{\otimes r} (e_1 \wedge \cdots \wedge e_{k-r}) (D_{1,2j+r} - D_{1,2j+r+1}) \zeta_{k+2} \end{array} \right).$$

By Lemmas 3.4, 3.5 and 3.6, these elements are equal to

$$\pi_k \left(\begin{array}{l} (2g - 2)\Lambda_{[r,k-r]} - 3\Lambda_{[r,k-r]} + \Lambda_{[r-1,1]} \\ + \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} \frac{k-r-1}{2} C_{j-1} (-6\Lambda_{[r,k-r]} - \Lambda_{[r-1,2j-2]} + \Lambda_{[r-1,2j-1]}) \\ + (-1)^{\frac{k-r-1}{2}} (-3\Lambda_{[r,k-r]} - \Lambda_{[r-1,k-r-1]}) \\ + \sum_{a=1}^{r-1} (-1)^a {}_r C_a \left(\begin{array}{l} \Lambda_{[r,k-r]} - \Lambda_{[a,k-r-1]} - 2\Lambda_{[r-a,1]} \\ -4 \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} \frac{k-r-1}{2} C_{j-1} \Lambda_{[r-a,1]} \\ + (-1)^{\frac{k-r-1}{2}} (-2\Lambda_{[r-a,1]} + \Lambda_{[r,k-r]} - \Lambda_{[r-a,1]}) \end{array} \right) \\ + (-1)^r \left(\begin{array}{l} -3\Lambda_{[r,k-r]} - \Lambda_{[r-1,1]} \\ + \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} \frac{k-r-1}{2} C_{j-1} (-6\Lambda_{[r,k-r]} + \Lambda_{[r-1,2j-2]} - \Lambda_{[r-1,2j-1]}) \\ + (-1)^{\frac{k-r-1}{2}} (-3\Lambda_{[r,k-r]} + \Lambda_{[r-1,k-r-1]} + (2g - 2)\Lambda_{[r,k-r]}) \end{array} \right) \end{array} \right).$$

Note that $\Lambda_{[a,k-r-1]} = (-1)^{k-r-1} \Lambda_{[r-a,1]} = \Lambda_{[r-a,1]}$.

If r is even, the elements above are equal to

$$(1 + (-1)^{\frac{k-r-1}{2}}) \left((2g + 2)\Lambda_{[r,k-r]} + \sum_{a=1}^{r-1} (-1)^a {}_r C_a \Lambda_{[r-a,1]} \right).$$

Thus we obtain the claims (v) and (vi).

If r is odd, the elements above are equal to

$$\begin{aligned} & (1 - (-1)^{\frac{k-r-1}{2}})(2g - 2)\Lambda_{[r,k-r]} + 2\Lambda_{[r-1,1]} - 2(-1)^{\frac{k-r-1}{2}}\Lambda_{[r-1,k-r-1]} \\ & - \sum_{j=2}^{\frac{k-r-1}{2}} (-1)^{j-1} C_{j-1}^{\frac{k-r-1}{2}} (\Lambda_{[r-1,2j-2]} - \Lambda_{[r-1,2j-1]}) \\ & + (1 + (-1)^{\frac{k-r-1}{2}}) \sum_{a=1}^{r-1} (-1)^a C_a \Lambda_{[r-a,1]}. \end{aligned}$$

Thus we obtain the claims (vii) and (viii).

For (i),(ii),(iii) and (iv), the proofs are similar as above. ■

3.5. Non-vanishing of $c_k(v_\lambda \theta \zeta_{k+2})$. We study whether $c_k(v_\lambda \theta \zeta_{k+2}) = 0$ or not.

Proposition 3.8. *Suppose $g \geq k + 2$. For $\lambda = [r + 1, 1^{k-r-1}]$ with $r > 0, k - r - 1 \geq 2$, in the following three cases, the elements $c_k(v_\lambda \theta \zeta_{k+2})$ is equal to 0 in $\mathcal{C}_{2g}^{\mathbb{Q}}(k)$;*

- k is odd and r is even such that $k - r \equiv 3 \pmod{4}$,
- $r = 1$ and k is odd,
- $r = 1$ and $k - r \equiv 1 \pmod{4}$,
- $(r, k) = (3, 8)$.

Otherwise, $c_k(v_\lambda \theta \zeta_{k+2})$ is not zero.

Proof. We fix a standard basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid 1 \leq i_1 \leq \cdots \leq i_k \leq 2g\}$ in $H_{\mathbb{Q}}^{\otimes k}$. To prove $\pi_k(v) \neq 0$ for $v \in H_{\mathbb{Q}}^{\otimes k}$, it is sufficient to find a basis element $e_{i_1} \otimes \cdots \otimes e_{i_k}$ such that the sum of coefficients of $(e_{i_1} \otimes \cdots \otimes e_{i_k}) \sigma_k^j$ ($0 \leq j \leq k - 1$) is not 0 in the expansion of v with respect to the standard basis above.

(i) Suppose both k and r are odd such that $k - r \equiv 0 \pmod{4}$. If $r = 1$,

$c_k(v_\lambda \theta \zeta_{k+2})$ is presented by $\pi_k \left((4g)\Lambda_{[1,k-1]} - 2 \sum_{j=1}^{\frac{k-1}{2}} (-1)^{j-1} C_{j-1}^{\frac{k-1}{2}} \Lambda_{[0,2j-2]} \right)$ by

(2). Since $\pi_k(\Lambda_{[0,k-1]}) = \pi_k(\Lambda_{[0,2j-2]})$, we have

$$c_k(v_\lambda \theta \zeta_{k+2}) = (4g + 4)\pi_k(e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1})).$$

But, for odd k , in the expansion of $e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1})$ with respect to the standard basis, the coefficients of $e_1 \otimes e_{i_1} \otimes \cdots \otimes e_{i_t} \otimes e_1 \otimes e_{i_{t+1}} \otimes \cdots \otimes e_{i_{k-2}}$ and $e_1 \otimes e_{i_{t+1}} \otimes \cdots \otimes e_{i_{k-2}} \otimes e_1 \otimes e_{i_1} \otimes \cdots \otimes e_{i_t}$ have different signatures. Then $\pi_k(e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1})) = 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) = 0$.

If $r \geq 3$, then $k - r \geq 4$ and $k \geq 7$ in this case. We consider the basis element $e_1^{\otimes r} \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_4 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. In (2), they only appear in $\Lambda_{[r,k-r]}$ as $e_1^{\otimes r} \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_4 \otimes \cdots \otimes e_{k-r}$ with the coefficient 1, and in $\Lambda_{[r-1,2j-2]}$ for $j = 2$ as $e_1^{\otimes r-1} \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_4 \otimes \cdots \otimes e_{k-r} \otimes e_1$ with the

coefficient $(-1)^{k-r-1} = -1$. Then the sum of coefficients of them in (2) is equal to $4g - 2 \cdot \frac{k-r}{2} = 4g - k + r \geq 4(k+2) - k + r = 3k + r + 8 > 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(ii) Suppose both k and r are odd such that $k - r \equiv 2 \pmod{4}$. If $r = 1$, we obtain $c_k(v_\lambda \theta \zeta_{k+2}) = 0$ similarly as above. If $r \geq 3$, similarly as above, the sum of coefficients $e_1^{\otimes r} \otimes e_2 \otimes e_3 \otimes e_1 \otimes e_4 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations in (3) is equal to $-(k - r) < 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(iii) Suppose both k and r are even such that $k - r \equiv 0 \pmod{4}$. We consider the basis element $e_1^{\otimes r-1} \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. In (4), they appear in $\Lambda_{[r-a,1]}$ for $a = 1$ with the coefficient -1 . Then the sum of coefficients of them in (4) is equal to $2r > 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(iv) Suppose both k and r are even such that $k - r \equiv 2 \pmod{4}$. Then $r \geq 2$. We consider the basis element $e_1^{\otimes r} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. In $4\Lambda_{[r,k-r]}$, they only appear as $e_1^{\otimes r} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ with the coefficient $-4 \neq 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(v) Suppose k is odd and r is even such that $k - r \equiv 1 \pmod{4}$. Note that $k - r \geq 3$. Similarly as (iii), we consider the basis element $e_1^{\otimes r-1} \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. They only appear in $\Lambda_{[r-a,1]}$ for $a = 1$ with the coefficient -1 . Then the sum of coefficients of them in (5) is equal to $2r > 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(vi) If k is odd and r is even such that $k - r \equiv 3 \pmod{4}$, then $c_k(v_\lambda \theta \zeta_{k+2}) = 0$.

(vii) Suppose k is even and r is odd such that $k - r \equiv 3 \pmod{4}$. If $r = 1$, $c_k(v_\lambda \theta \zeta_{k+2})$ is presented by

$$\pi_k \left(-2(\Lambda_{[0,k-2]} - \Lambda_{[0,1]}) - 2 \sum_{j=2}^{\frac{k}{2}-1} (-1)^{j-1} C_{j-1} (\Lambda_{[0,2j-2]} - \Lambda_{[0,2j-1]}) \right)$$

by (6). But

$$\pi_k(\Lambda_{[0,k-2]}) = \pi_k(\Lambda_{[0,1]}) = \pi_k(\Lambda_{[0,2j-2]}) = \pi_k(\Lambda_{[0,2j-1]}) = \pi_k(e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1})).$$

Then $c_k(v_\lambda \theta \zeta_{k+2}) = 0$.

If $r \geq 3$, first, we consider the basis element $e_1^{\otimes r} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. In (6), they only appears as $e_1^{\otimes r-1} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r} \otimes e_1$ in $\Lambda_{[r-1,1]}$ with the coefficient $(-1)^{k-r-1} = 1$, in $\Lambda_{[r-a,1]}$ for $a = 1$ with coefficients $(-1)^{k-r-1} = 1$ and $\Lambda_{[r-1,2j-2]}$ for $j = 2$ with the coefficient 1. Then the sum of coefficients of them in (6) is equal to $2 + (-2)(-1)^{\frac{k-r-1}{2}} + 2(-1)r = k - 3r + 1$. Thus if $k \neq 3r - 1$, $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

Suppose $k = 3r - 1$. If $(r, k) = (3, 8)$, (6) is equal to $4\pi_k(\Lambda_{[2,1]} - \Lambda_{[2,2]} + \Lambda_{[2,3]} - \Lambda_{[2,4]})$. This becomes 0 by direct computations.

If $(r, k) \neq (3, 8)$, then $r \geq 5$ and $k - r = 2r - 1 \geq 9$. We consider the basis element

$$e_1^{\otimes r-1} \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_6 \otimes \cdots \otimes e_{k-r-1} \otimes e_1 \otimes e_{k-r}$$

and its cyclic permutations. In (6), they appear as

$$e_1^{\otimes r-1} \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_6 \otimes \cdots \otimes e_{k-r-1} \otimes e_1 \otimes e_{k-r}$$

in $\Lambda_{[r-1,k-r-1]}$ with the coefficient $(-1)^4 = 1$, as $e_1 \otimes e_{k-r} \otimes e_1^{\otimes r-1} \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_6 \otimes \cdots \otimes e_{k-r-1}$ in $\Lambda_{[r-a,1]}$ for $a = r - 1$ with the coefficient $(-1)^{k-r-1}(-1)^4 = 1$ and as $e_1^{\otimes r-1} \otimes e_2 \otimes e_3 \otimes e_4 \otimes e_5 \otimes e_1 \otimes e_6 \otimes \cdots \otimes e_{k-r-1} \otimes e_1 \otimes e_{k-r}$ in $\Lambda_{[r-1,2j-2]}$ for $j = 3$ with the coefficient $(-1)^{k-r-2} = -1$. Thus the sum of

coefficients of them in (6) is equal to $(-2) + 2(-1)^{r-1}r + (-2)(-1)^{\frac{k-r-1}{2}}C_2(-1) = -2 + 2r + (r-1)(r-2) = r(r-1) > 0$. Therefore if $k = 3r - 1$ and $k \geq 5$, then $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$.

(viii) Suppose k is even and r is odd such that $k - r \equiv 3 \pmod{4}$. If $r = 1$, similarly as above, $c_k(v_\lambda \theta \zeta_{k+2})$ is $(4g - 4)\pi_k(e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1}))$. We consider the basis element $e_1 \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_{k-1}$ and its cyclic permutations. They appear in $e_1 \otimes (e_1 \wedge \cdots \wedge e_{k-1})$ as $e_1 \otimes e_1 \otimes e_2 \otimes \cdots \otimes e_{k-1}$ with coefficient 1 and as $e_1 \otimes e_2 \otimes \cdots \otimes e_{k-1} \otimes e_1$ with the coefficient $(-1)^{k-2} = 1$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$. If $r \geq 3$, we consider the basis element $e_1^{\otimes r} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ and its cyclic permutations. In (7), they only appear as $e_1^{\otimes r} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r}$ in $\Lambda_{[r, k-r]}$ with the coefficient -1 and as $e_1^{\otimes r-1} \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{k-r} \otimes e_1$ in $\Lambda_{[r-1, 1]}$ with coefficients $(-1)^{k-r-1} = 1$ and in $\Lambda_{[r-1, 2j-2]}$ for $j = 2$ with the coefficient 1. Then the sum of coefficients of them in (7) is equal to $(4g - 4)(-1) + 2 \cdot 1 + (-2)(-1)^{\frac{k-r-1}{2}} = 5 + k - r - 4g \leq 5 + k - r - 4(k + 2) = -3k - r + 1 = -2k + (-k - r + 1) < 0$. Thus $c_k(v_\lambda \theta \zeta_{k+2}) \neq 0$. ■

Let us recall our Theorem 1.1.

Theorem 3.9. *Suppose $\lambda = (r + 1, 1^{k-r-1})$ for $r \geq 1$ and $k - r - 1 \geq 2$. Then the multiplicity of the Sp -irreducible modules $[\lambda]$ in the k -th Johnson cokernels are larger than or equal to 1 if none of the following conditions are satisfied: (i) k is odd and r is even such that $k - r \equiv 3 \pmod{4}$, (ii) $r = 1$ and k is odd, (iii) $r = 1$ and $k - r \equiv 1 \pmod{4}$.*

Proof. We obtain the existence of $[\lambda]$ except for the cases (i),(ii),(iii) and $(r, k) = (3, 8)$ by Proposition 3.8.

When $(r, k) = (3, 8)$, by direct computations, we can obtain that $c_k(v_\lambda s_6 \theta \zeta_{k+2}) = \pi_k(v_{3,8})$, where

$$v_{3,8} = 24 \left\{ \begin{array}{l} [5,7,8] + [5,6,7] + [4,6,8] + [3,6,7] + [3,5,6] + [2,6,8] + \\ [2,5,7] + [2,3,7] - [4,5,7] - [3,7,8] - [3,5,8] - [3,4,7] - [3,4,5] - [2,4,8] - \\ [2,4,6] - [2,3,5] + 2[3,4,8] + 2[2,4,5] - 2[5,6,8] - 2[2,6,7] \end{array} \right\} \in H^{\otimes 8}.$$

Here the elements $[h, i, j]$ ($1 < h < i < j \leq 8$) are

$$e_1^{\otimes 3} \otimes (e_1 \wedge \cdots \wedge e_5) \cdot (s_3 s_4 \cdots s_{j-1} \cdot s_2 s_3 \cdots s_{i-1} \cdot s_1 s_2 \cdots s_{h-1}).$$

Since the sum of coefficients of the element $e_1 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_1 \otimes e_3 \otimes e_4 \otimes e_5$ and its cyclic permutations in $v_{3,8}$ is equal to $24 \times (-8)$, then we have $c_k(v_\lambda s_6 \theta \zeta_{k+2}) \neq 0$. Hence we obtain the existence of $[4, 1^4]$. The proof is completed. ■

A. Tables of multiplicities

In [ES1] and [ES2], to compute multiplicities of $[k]$ and $[1^k]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$, we used a major index for a standard tableau of shape λ . But by using Zhuravlev’s results in [Zh], we compute multiplicities of $[\lambda]$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ more systematically. In this appendix, we explain this method.

A.1. Representation theory for $\mathrm{GL}(2g, \mathbb{Q})$. Let us briefly recall some results for the representation theory of $\mathrm{GL}(2g, \mathbb{Q})$.

An irreducible polynomial representation of $\mathrm{GL}(2g, \mathbb{Q})$ is uniquely determined by its highest weight. Then the set of isomorphism classes of irreducible polynomial representations is parametrized by

$$P_{\mathrm{GL}}^+ := \{\lambda_1 \geq \dots \geq \lambda_{2g} \geq 0 \mid \lambda_i \in \mathbb{Z}, 1 \leq i \leq 2g\}.$$

This is the set of all partitions such that $\ell(\lambda) \leq 2g$. We denote by (λ) the corresponding irreducible polynomial representation of $\mathrm{GL}(2g, \mathbb{Q})$. Note that (1) is a natural representation of $\mathrm{GL}(2g, \mathbb{Q})$ on H .

For two partitions λ and μ satisfying $\lambda \supset \mu$, the skew shape $\lambda \setminus \mu$ is a vertical strip if there is at most one box in each row.

Proposition A.1 (Pieri's formula). *Let μ be a partition such that $\ell(\mu) \leq n$. Then*

$$(1^k) \otimes (\mu) \cong \bigoplus_{\lambda} (\lambda),$$

where λ runs over the set of partitions obtained by adding a vertical k -strip to μ such that $\ell(\lambda) \leq n$.

We regard $\mathrm{Sp}(2g, \mathbb{Q})$ as a subgroup of $\mathrm{GL}(2g, \mathbb{Q})$. We consider the restriction of an irreducible polynomial representation (λ) to $\mathrm{Sp}(2g, \mathbb{Q})$. We can give its irreducible decomposition using the Littlewood-Richardson coefficients $\mathrm{LR}_{\lambda\mu}^{\nu}$ as follows.

Proposition A.2 (Branching rules for GL to Sp [FH, 25.39],[KT, Proposition 2.5.1]). *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_g \geq 0)$ be a partition such that $\ell(\lambda) \leq g$. Then we have*

$$\mathrm{Res}_{\mathrm{Sp}(2g, \mathbb{Q})}^{\mathrm{GL}(2g, \mathbb{Q})}(\lambda) \cong \bigoplus_{\mu} N_{\lambda\mu}[\mu]$$

where μ runs over all partitions such that $\ell(\mu) \leq g$. Here

$$N_{\lambda\mu} = \sum_{\eta} \mathrm{LR}_{\eta\mu}^{\lambda}$$

where η runs over all partitions $\eta = (\eta_1 = \eta_2 \geq \eta_3 = \eta_4 \geq \dots)$ with each part occurring an even number of times, namely η' even. Here η' is the conjugate partition of η .

Remark A.3. We have a combinatorial description of the Littlewood-Richardson coefficients. The Littlewood-Richardson coefficients $\mathrm{LR}_{\mu\nu}^{\lambda}$ is the number of semi-standard tableaux T on $\lambda \setminus \mu$ with weight ν such that $w(T)$ is a lattice permutation. Here, for a semi-standard tableau T on $\lambda \setminus \mu$, we define a sequence $w(T)$ of integers by reading the numbers inserted in $\lambda \setminus \mu$ from right to left in successive rows, starting with top row. (For more details, see, e.g. [FH], [Mac].)

A.2. Zhuravlev's formula. We denote by $\mathrm{Möb}$ the Möbius function. For a

composition (m_1, m_2, \dots, m_r) of m , we define

$$l_{m_1, m_2, \dots, m_r} = \frac{1}{m} \sum_{d|m_1, m_2, \dots, m_r} \text{Möb}(d) \frac{(m/d)!}{(m_1/d)!(m_2/d)! \cdots (m_r/d)!}.$$

In [Zh], Zhuravlev gives the following multiplicity formula of irreducible representation $(\lambda)_{\text{GL}}$ of $\text{GL}(2g, \mathbf{Q})$ corresponding to a partition λ in the degree m part of the free Lie algebra $\mathcal{L}_{2g}(m)$.

Theorem A.4 ([Zh, (21)]). *Let $\lambda = (\lambda_1, \dots, \lambda_g)$ be a partition of m . Then, the multiplicity $\sharp_{(\lambda)}$ of irreducible representation (λ) of GL_{2g} in $\mathcal{L}_{2g}(m)$ is given by*

$$\sum_{\sigma \in \mathfrak{S}_g} \text{sgn}(\sigma) l_{\lambda_1 - 1 + \sigma(1), \lambda_2 - 2 + \sigma(2), \dots, \lambda_g - g + \sigma(g)},$$

where σ runs over the set of all $\sigma \in \mathfrak{S}_g$ such that $\lambda_i - i + \sigma(i) \geq 0$ for any $1 \leq i \leq g$.

We obtain the following three corollaries.

Corollary A.5 ([Zh, (22)]). *If $\lambda = (n - r + 1, 1^{r-1})$, then*

$$\sharp_{(\lambda)} = \frac{1}{n} \sum_{1 \leq i \leq r-1, d|n, i} \text{Möb}(d) (-1)^{r-1-i+(d-1)i/d} \frac{(n/d)!}{((n-i)/d)!(i/d)!}.$$

Corollary A.6 ([Zh, (23)]). *If $\lambda = (\lambda_1, \lambda_2)$ satisfies $\lambda_1 + \lambda_2 = n, \lambda_2 \geq 1$, then*

$$\sharp_{(\lambda)} = l_{\lambda_1, \lambda_2} - l_{\lambda_1+1, \lambda_2-1}.$$

Corollary A.7 ([Zh, Proposition 4.1]). *For a partition λ , we denote by λ' its conjugate partition. Let χ^λ be the irreducible character of \mathfrak{S}_g corresponding to λ . We denote by $\chi^\lambda(\sigma)$ its value at the element $\sigma \in \mathfrak{S}_g$. Then we have*

$$\sharp_{(\lambda')} = \sharp_{(\lambda)} + \delta_{n \equiv 2 \pmod{4}} \frac{2}{n} \sum_{d|(n/2)} \text{Möb}(d) \chi^\lambda((12 \dots n)^{n/2d}).$$

A.3. Nakayama-Murnaghan’s formula. We denote by χ_ν^λ the character value of irreducible character χ^λ at the conjugacy class corresponding to a partition ν . We have the following inductive formula for irreducible character values.

Theorem A.8 (Nakayama-Murnaghan’s formula). *Let λ, ν be partitions. We assume $|\lambda| = |\nu| + r$ for an integer $r \geq 1$. We have*

$$\chi_{\nu \cup (r)}^\lambda = \sum_{\mu} (-1)^{\text{ht}(\lambda \setminus \mu)} \chi_\nu^\mu,$$

where μ runs over the set of partitions satisfying that $\lambda \setminus \mu$ is a border strip of λ of size r . Here, $\text{ht}(\lambda \setminus \mu)$ is (depth of the border strip $\lambda \setminus \mu$) $- 1$.

A.4. Tables of multiplicities in the free Lie algebra $\mathcal{L}_{2g}^{\mathbb{Q}}(m)$. In this subsection, we suppose $g \geq m$. We have the following table of multiplicities $\sharp_{(\lambda)}$ in $\mathcal{L}_{2g}^{\mathbb{Q}}(m)$ of GL-irreducible representations (λ) for a partition $\lambda \vdash m$.

(λ)	$\sharp_{(\lambda)}$
$(m-1, 1)$	1
$(m-2, 1^2)$	$\frac{m-3}{2} + \frac{1}{2}\delta_{m:\text{even}}$
$(m-3, 1^3)$	$\frac{m^2-6m+11}{6} - \frac{1}{2}\delta_{m \equiv 2,4(\text{mod } 6)} - \frac{1}{3}\delta_{m \equiv 3(\text{mod } 6)} - \frac{5}{6}\delta_{m \equiv 0(\text{mod } 6)}$
$(m-4, 1^4)$	$\frac{m^3-10m^2+35m-50}{24} - \frac{m-6}{8}\delta_{m \equiv 2,4(\text{mod } 6)} + \frac{1}{3}\delta_{m \equiv 3(\text{mod } 6)} - \frac{3m-26}{24}\delta_{m \equiv 0(\text{mod } 6)}$
$(m-2, 2)$	$\frac{m-3}{2} - \frac{1}{2}\delta_{m:\text{even}}$
$(m-3, 3)$	$\frac{(m-1)(m-5)}{6} - \frac{1}{3}\delta_{m \equiv 0(\text{mod } 3)} + \frac{1}{2}\delta_{m:\text{even}}$
$(m-4, 4)$	$\frac{(m-1)(m-2)(m-7)}{24} + \frac{1}{3}\delta_{m \equiv 0(\text{mod } 3)} - \frac{m-2}{8}\delta_{m:\text{even}}$
$(m-3, 2, 1)$	$\frac{m^2-6m+8}{3} + \frac{1}{3}\delta_{m \equiv 0(\text{mod } 3)}$
$(m-4, 2^2)$	$\frac{m^3-10m^2+26m-8}{4} + \frac{m-4}{4}\delta_{m \equiv 1,5(\text{mod } 6)} - \frac{1}{3}\delta_{m \equiv 0(\text{mod } 6)} + \frac{3m-16}{12}\delta_{m \equiv 3(\text{mod } 6)}$
$(m-4, 2, 1^2)$	$\frac{m^3-10m^2+31m-30}{8} + \frac{m-2}{8}\delta_{m:\text{even}}$
$(m-4, 3, 1)$	$\frac{m^3-10m^2+27m-18}{8} + \frac{m-6}{8}\delta_{m:\text{even}}$
$(m-5, 1^5)$	$\frac{m^4-15m^3+85m^2-225m+274}{120} + \frac{m-6}{8}\delta_{m:\text{even}} - \frac{1}{3}\delta_{m \equiv 0(\text{mod } 3)} - \frac{1}{5}\delta_{m \equiv 0(\text{mod } 5)}$
$(m-5, 2, 1^3)$	$\frac{m^4-15m^3+80m^2-180m+144}{30} + \frac{1}{5}\delta_{m \equiv 0(\text{mod } 5)}$
$(m-5, 2^2, 1)$	$\frac{m^4-15m^3+77m^2-153m+90}{24} + \frac{m-6}{8}\delta_{m:\text{even}}$
$(m-5, 3, 2)$	$\frac{m^4-15m^3+73m^2-129m+70}{24} + \frac{m-2}{8}\delta_{m:\text{even}} + \frac{1}{3}\delta_{m \equiv 0(\text{mod } 3)}$

In the table above, many of the $m_{(\lambda)}$'s can be computed by Corollary A.5 and Corollary A.6. If not, we use Theorem A.4. For example, if $\lambda = (m, 2, 1)$, the elements $\sigma \in \mathfrak{S}_3$ which appear in the righthand side in Theorem A.4 satisfy $\sigma(3) = 2, 3$. Then we have $\sharp_{(m,2,1)} = \ell_{m,2,1} - \ell_{m+1,1,1} - \ell_{m,3,0} + \ell_{m+2,1,0}$. If $\lambda = (m, 2, 2)$, all elements in \mathfrak{S}_3 appear in the righthand side. Then we have $\sharp_{(m-4,2,2)} = \ell_{m-4,2,2} - \ell_{m-4,3,1} - \ell_{m-3,1,2} + \ell_{m-3,3,0} + \ell_{m-2,1,1} - \ell_{m-2,2,0}$. If the length of λ is large, we use Corollary A.7 and Nakayama-Murnaghan's formula. We have the following table.

(λ)	$\sharp_{(\lambda)}$
(1^m)	0 for $m \geq 3$
$(2, 1^{m-2})$	1 for $m \geq 3$
$(2^2, 1^{m-4})$	$\sharp_{(n-2,2)} + \delta_{m \equiv 2(\text{mod } 4)}$ for $m \geq 4$
$(3, 1^{m-3})$	$\sharp_{(m-2,1^2)} - \delta_{m \equiv 2(\text{mod } 4)}$
$(2^3, 1^{m-6})$	$\sharp_{(m-3,3)} - \delta_{m \equiv 2(\text{mod } 4)}$
$(3, 2, 1^{m-5})$	$\sharp_{(m-3,2,1)}$
$(2^4, 1^{m-8})$	$\sharp_{(m-4,4)} + \frac{m-2}{4}\delta_{m \equiv 2(\text{mod } 4)}$
$(3^2, 1^{m-6})$	$\sharp_{(m-4,2^2)} + \frac{m-4}{2}\delta_{m \equiv 2(\text{mod } 4)}$
$(4, 2, 1^{m-6})$	$\sharp_{(m-4,2,1^2)} - \frac{m-2}{4}\delta_{m \equiv 2(\text{mod } 4)}$
$(3, 2^2, 1^{m-7})$	$\sharp_{(m-4,3,1)} - \frac{m-6}{4}\delta_{m \equiv 2(\text{mod } 4)}$

By Corollary A.7, to prove these formulas, we only consider the case $m \equiv 2 \pmod{4}$. The element $(12 \cdots m)^{m/2d}$ in \mathfrak{S}_m belongs to the conjugate class corresponding to the partition $((2d)^{m/2d})$ for each odd integer d .

For example, if $\lambda = (2, 1^{m-3})$, then $\lambda' = (m-2, 1^2)$. By Theorem A.8, we have $\chi_{(2)^{m/2}}^{(m-2,1^2)} = \chi_{(2^{m/2-1})}^{(m-4,1^2)} - \chi_{2^{m/2-1}}^{(m-2)} = \chi_{(2^{m/2-1})}^{(m-4,1^2)} - 1 = \cdots \chi_{(2^2)}^{(2,1^2)} - \frac{m-4}{2} = -\chi_{(2)}^{(2)} - \frac{m-4}{2} =$

$\frac{-m+2}{2}$ for $d = 1$, and $\chi_{((2d)^{n/2d})}^{(n-2,1^2)} = \chi_{((2d)^{(n/2d)-1})}^{(n-2-2d,1^2)} = \dots = \chi_{(d)}^{(d-2,1^2)} = 1$ for $d \geq 3$. Thus the second term in Corollary A.7 becomes $\frac{2}{m} \left(\frac{-m+2}{2} + \sum_{3 \leq d|(m/2)} \text{Möb}(d) \right) = -1$. Here, we use $\sum_{d|n} \text{Möb}(d) = \delta_{n=1}$. We show two more examples. If $\lambda = (3, 2, 1^{m-6})$, then $\lambda' = (m-3, 2, 1)$. Similarly as above, we obtain $\chi_{(2d)^{m/2d}}^{(m-3,2,1)} = \delta_{d=1} \chi_{(2^3)}^{(3,2,1)} + \delta_{d \geq 3} 3 \chi_{(2d)}^{(2d-3,2,1)} = 0$. If $\lambda = (2^3, 1^{m-6})$, then $\lambda' = (m-3, 3)$. Similarly as above, we obtain $\chi_{(2d)^{m/2d}}^{(m-3,3)} = \delta_{d=1} (\chi_{(2^{m/2-1})}^{(m-5,3)} + \chi_{(2^{m/2})}^{(m-3,1)}) + \delta_{d \geq 3} 3 \chi_{(2d)}^{(2d-3,3)} = \delta_{d=1} (\chi_{(2^{m/2-1})}^{(m-5,3)} - 1) = \delta_{d=1} (\chi_{(2^3)}^{(3^2)} - \frac{m-6}{2}) = \delta_{d=1} \frac{m}{2}$. Here, we use $\chi^{(3^2),(2^3)} = \chi_{(2^2)}^{(3,1)} - \chi_{(2^2)}^{(2^2)} = \chi_{(2)}^{(1^2)} - 1 = -2$.

A.5. Table of multiplicities in the Lie algebra $\mathfrak{h}_{g,1}(k)$. In this subsection, we suppose $g \geq k + 2$. We have the following table of multiplicities $b_{[\lambda]}$ in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$ of Sp-irreducible representations $[\lambda]$ for a partition λ .

$[\lambda]$	$b_{[\lambda]}$
$[k]$	$\delta_{k:\text{odd}}$
$[1^k]$	$\delta_{k \equiv 1,2 \pmod{4}}$
$[k-1, 1]$	$\frac{k-1}{2} + \frac{1}{2} \delta_{k:\text{even}}$
$[2, 1^{k-2}]$	$b_{[k-1,1]} - \delta_{k \equiv 2 \pmod{4}}$
$[k-2, 1^2]$	$\frac{k^2-2k}{4} - \frac{2k-3}{4} \delta_{k:\text{odd}}$
$[3, 1^{k-3}]$	$b_{[k-2,1^2]} - \frac{k-2}{2} \delta_{k \equiv 2 \pmod{4}} - \frac{k+1}{2} \delta_{k \equiv 3 \pmod{4}}$
$[k-2]$	$\frac{k^2+2k}{8} - \frac{6k-3}{8} \delta_{k:\text{odd}}$
$[1^{k-2}]$	$b_{[k-2]} - \frac{3k}{4} \delta_{k \equiv 0 \pmod{4}} + \frac{3k-3}{4} \delta_{k \equiv 1 \pmod{4}}$
$[k-3, 1]$	$\frac{3k^3-12k^2+k+24}{24} - \frac{3k}{8} \delta_{k:\text{even}} + \frac{1}{3} \delta_{k \equiv 1,4 \pmod{6}} - \frac{1}{3} \delta_{k \equiv 2,5 \pmod{6}}$

For example, if $\lambda = [1^{k-2}]$, first we have $b_{[1^{k-2}]} = b_{(3^2, 1^{k-4})} + b_{(3, 2, 1^{k-3})} + b_{(2^4, 1^{k-6})} + b_{(2^3, 1^{k-4})} + 2b_{(2^2, 1^{k-2})} + b_{(2, 1^k)} + b_{(1^{k+2})}$ by the branching rules for GL to Sp (Proposition A.2). Here we denote by $b_{(\lambda)}$ the multiplicity of irreducible GL-module (λ) in $\mathfrak{h}_{g,1}^{\mathbb{Q}}(k)$. By the Pieri rule (Proposition A.1), the righthand side becomes $-\{\#_{(3^2, 1^{k-4})} + \#_{(3, 2, 1^{k-3})} + \#_{(2^4, 1^{k-6})} + \#_{(2^3, 1^{k-4})} + 2\#_{(2^2, 1^{k-2})} + \#_{(2, 1^k)} + \#_{(1^{k+2})}\} + \{\#_{(3^2, 1^{k-5})} + 2\#_{(3, 2, 1^{k-4})} + \#_{(3, 1^{k-2})} + \#_{(2^4, 1^{k-7})} + 2\#_{(2^3, 1^{k-5})} + 4\#_{(2^2, 1^{k-3})} + 3\#_{(2, 1^{k-1})} + 2\#_{(1^{k+1})}\}$. We can compute these terms by using the tables in subsection A.

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