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The Derived Algebra of a Stabilizer, Families of Coadjoint Orbits, and Sheets

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Abstract. Let $\mathfrak{g}$ be a finite-dimensional real or complex Lie algebra, and let $\mu \in \mathfrak{g}^*$. In the first part of the paper, we discuss the relation between the derived Lie algebra of the stabilizer of μ and the set of coadjoint orbits which have the same dimension as the orbit of μ. In the second part, we consider semisimple Lie algebras and discuss the relation between the derived algebra of a centralizer and sheets.
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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional real or complex Lie algebra. The adjoint group G of $\mathfrak{g}$ acts on the dual space $\mathfrak{g}^*$ via the coadjoint action, and $\mathfrak{g}^*$ is foliated into the orbits of this action. Consider the union of all orbits which have the same codimension k. Denote this union by $\mathfrak{g}^*_k$. Each of the sets $\mathfrak{g}^*_k$ is a quasi-affine algebraic variety, and $\mathfrak{g}^*$ is a disjoint union of all $\mathfrak{g}^*_k$. The study of the varieties $\mathfrak{g}^*_k$ was initiated by A. Kirillov in connection with the orbit method [6], which relates the unitary dual of G to the set of coadjoint orbits $\mathfrak{g}^*/G$. The sets $\mathfrak{g}^*_k/G$ appear in this picture as natural strata of $\mathfrak{g}^*/G$, therefore it is important to understand the geometry of $\mathfrak{g}^*_k$ for each k.

Let $\mathfrak{g}_\mu = \{x \in \mathfrak{g} \mid \mathrm{ad}^* x(\mu) = 0\}$ be the stabilizer of an element $\mu \in \mathfrak{g}^*$ with respect to the coadjoint representation of $\mathfrak{g}$. In the present note, the following simple geometric fact is proved: any element $\xi \in \mathfrak{g}^*$ which is tangent to the variety $\mathfrak{g}^*_k$ at a point $\mu \in \mathfrak{g}^*$ vanishes on the derived algebra of $\mathfrak{g}_\mu$. As a corollary, the

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codimension of the set $\{\mu \in \mathfrak{g}^* \mid \dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \geq k\}$ is at least k, which can be viewed as a generalization of the well-known fact that the stabilizer of a generic element $\mu \in \mathfrak{g}^*$ is Abelian.

In the second part of the note, we assume that $\mathfrak{g}$ is semisimple. In this case, the set $\mathfrak{g}^*_k$ can be identified with the variety of adjoint orbits of codimension k. The irreducible components of this latter variety are called sheets. Let $a \in \mathfrak{g}$, and let $\mathfrak{g}^a = \{x \in \mathfrak{g} \mid [x, a] = 0\}$ be the centralizer of a in $\mathfrak{g}$. Then the above-formulated statement becomes the following: the derived algebra of the centralizer of a is orthogonal to any sheet passing through a. It was conjectured by the author that if $\mathfrak{g}$ is a classical simple Lie algebra, and if there is a unique sheet S passing through $a \in \mathfrak{g}$, then $[\mathfrak{g}^a, \mathfrak{g}^a]$ is exactly the orthogonal complement to S. As it has been recently shown by A. Premet and L. Topley [12], this conjecture is true for any algebraically closed ground field of characteristic zero. In the present paper, we formulate a more general conjecture that if $S_1, \dots, S_k$ are sheets passing through a, then $[\mathfrak{g}^a, \mathfrak{g}^a]$ is the orthogonal complement to $\sum \mathrm{T}_a S_i$.

2. The derived algebra of a stabilizer and families of coadjoint orbits

In this section, $\mathfrak{g}$ is, as in the introduction, a finite-dimensional real or complex Lie algebra.

For any $\mu \in \mathfrak{g}^*$, denote $\mathfrak{g}^*(\mu) = \mathfrak{g}^*_k$ where $k = \dim \mathfrak{g}_\mu$. In other words, $\mathfrak{g}^*(\mu)$ is $\mathfrak{g}^*_k$ passing through μ.

Lemma 2.1. *Let $\mu \in \mathfrak{g}^*$, and let γ be a smooth curve in $\mathfrak{g}^*$ such that $\gamma(0) = \mu$ and $\gamma(t) \in \mathfrak{g}^*(\mu)$ for all t. Then the tangent vector $\dot\gamma(0)$ vanishes on the derived algebra of $\mathfrak{g}_\mu$:*

$$\langle \dot\gamma(0), [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \rangle = 0.$$

Proof. Since $\dim \mathfrak{g}_{\gamma(t)} = \dim \mathfrak{g}_\mu$ for all t, it is possible to choose a basis $e_1(t), \dots, e_k(t)$ in $\mathfrak{g}_{\gamma(t)}$ such that $e_i(t)$ depends smoothly on t. Since $e_i(t) \in \mathfrak{g}_{\gamma(t)}$, the following equality holds:

$$\langle \gamma(t), [e_i(t), e_j(t)] \rangle = 0.$$

Differentiating with respect to t at $t = 0$, we obtain

$$\langle \dot\gamma(0), [e_i(0), e_j(0)] \rangle + \langle \mu, [\dot e_i(0), e_j(0)] \rangle + \langle \mu, [e_i(0), \dot e_j(0)] \rangle = 0.$$

Since $e_i(t)$ are elements of the stabilizer, the last two terms vanish, and

$$\langle \dot\gamma(0), [e_i(0), e_j(0)] \rangle = 0,$$

which implies that $\dot\gamma(0)$ vanishes on the derived algebra of $\mathfrak{g}_\mu$. ∎

Remark 2.2. The lemma remains true if $\gamma(t)$ is only defined for $t \geq 0$ and the right derivative $\dot{\gamma}_+(0)$ exists. This may happen if $\mathfrak{g}^*(\mu)$ has a singularity at μ.

The main result of the paper is the following.

Theorem 2.3. *Let $\mu \in \mathfrak{g}^*$, and assume that $\mathfrak{g}^*(\mu)$ is smooth at the point μ. Then*

(i) Each element of the tangent space $\mathrm{T}_\mu \mathfrak{g}^*(\mu)$ vanishes on the derived algebra of $\mathfrak{g}_\mu$:

$$\langle \mathrm{T}_\mu \mathfrak{g}^*(\mu), [\mathfrak{g}_\mu, \mathfrak{g}_\mu] \rangle = 0.$$

(ii) The following inequality is satisfied:

$$\dim\,[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \leq \operatorname{codim}_\mu \mathfrak{g}^*(\mu). \tag{1}$$

(iii) The equality

$$\dim\,[\mathfrak{g}_\mu, \mathfrak{g}_\mu] = \operatorname{codim}_\mu \mathfrak{g}^*(\mu)$$

is satisfied if and only if $\mathrm{T}_\mu \mathfrak{g}^*(\mu)$ is exactly the annihilator of $[\mathfrak{g}_\mu, \mathfrak{g}_\mu]$.

Proof. Use Lemma 2.1. ■

Remark 2.4. The notation codim_μ stands for the codimension at the point μ. Note that different irreducible components of $\mathfrak{g}^*(\mu)$ may have different dimensions, so $\operatorname{codim}_\mu \mathfrak{g}^*(\mu)$ may depend on μ.

Remark 2.5. Inequality (1) shows that the derived algebra of a stabilizer cannot be too big. It resembles the following inequality for the index of a stabilizer: $\operatorname{ind} \mathfrak{g}_\mu \geq \operatorname{ind} \mathfrak{g}$ (Vinberg, see [10]). Recall that the index of a Lie algebra $\mathfrak{g}$ is the dimension of the stabilizer of a regular element $\mu \in \mathfrak{g}^*$.

Corollary 2.6. *The codimension of the set of elements $\mu \in \mathfrak{g}^*$ such that*

$$\dim\,[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \geq k$$

is at least k.

Example 2.7. Since the set of regular elements is dense in $\mathfrak{g}^*$, Corollary 2.6 implies a well-known fact: for regular $\mu \in \mathfrak{g}^*$, its stabilizer $\mathfrak{g}_\mu$ is Abelian [4]. Corollary 2.6 can be viewed as a natural generalization of this fact. It says that for a "not too singular" μ, its stabilizer is almost Abelian.

Example 2.8. Suppose that the set of singular elements in $\mathfrak{g}^*$ is a hypersurface. Then the stabilizer $\mathfrak{g}_\mu$ of a generic singular element $\mu \in \mathfrak{g}^*$ has a one-dimensional derived algebra. Therefore, $\mathfrak{g}_\mu$ is isomorphic to one of the following algebras:

1. Abelian;
2. $\mathfrak{aff}(1) \oplus \mathrm{Abelian}$, where $\mathfrak{aff}(1)$ is the Lie algebra of affine transformations of the line;
3. $\mathfrak{h}_{2n+1} \oplus \mathrm{Abelian}$, where $\mathfrak{h}_{2n+1}$ is the $2n+1$-dimensional Heisenberg algebra.

The class of Lie algebras for which the set of singular elements in $\mathfrak{g}^*$ is a hypersurface is particularly important for integrable systems [1, 2].

Remark 2.9. Note that (1) can be rewritten as

$$\dim_\mu \mathfrak{g}^*(\mu) - \dim O(\mu) \leq \dim \mathfrak{g}_\mu - \dim\,[\mathfrak{g}_\mu, \mathfrak{g}_\mu] \tag{2}$$

where $O(\mu)$ is the coadjoint orbit of μ.

Example 2.10. Let $\mathfrak{g} = \mathfrak{gl}(n)$,

$$\mu = \mathrm{diag}(\underbrace{\lambda_1, \dots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2}, \dots, \underbrace{\lambda_s, \dots \lambda_s}_{k_s}).$$

Then $\mathfrak{g}_\mu \simeq \mathfrak{gl}(k_1) \oplus \dots \oplus \mathfrak{gl}(k_s)$, so $\dim \mathfrak{g}_\mu - \dim\,[\mathfrak{g}_\mu, \mathfrak{g}_\mu] = s$. On the other hand, the intersection of $\mathfrak{g}^*(\mu)$ with a sufficiently small neighborhood of $O(\mu)$ is parameterized by the eigenvalues $\lambda_1, \dots, \lambda_s$, so $\dim_\mu \mathfrak{g}^*(\mu) - \dim O(\mu)$ is also equal to s, and inequalities (1), (2) turn into equalities.

This example is a particular case of a more general situation when the transverse Poisson structure at the point μ is linearizable. Recall the definition of a transverse Poisson structure. Let M be a Poisson manifold, and let $\mu \in M$. By Weinstein's splitting theorem [14], M can be locally decomposed into the direct product of a symplectic manifold and a manifold with a Poisson structure vanishing at μ. This latter Poisson structure is unique up to a diffeomorphism and is called the transverse Poisson structure at the point μ. More details can be found in [3].

In the case when M is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$, the linear part of the transverse Poisson structure at a point μ is the Lie-Poisson structure of the stabilizer $\mathfrak{g}_\mu$. Consequently, if the transverse Poisson structure at a point μ is linearizable, then the Lie-Poisson structure on $\mathfrak{g}^*$ can be locally decomposed into the direct product of a symplectic structure and the Lie-Poisson structure of the stabilizer $\mathfrak{g}_\mu$, which allows to prove the following.

Proposition 2.11. *Assume that the transverse Poisson structure at a point* μ *is linearizable. Then* $\mathfrak{g}^*(\mu)$ *is smooth at the point* μ*, and*

$$\dim_\mu \mathfrak{g}^*(\mu) - \dim O(\mu) = \dim \mathfrak{g}_\mu - \dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu].$$

Proof. Since the transverse Poisson structure at μ is linearizable, the Poisson manifold $\mathfrak{g}^*$ can be locally decomposed into a direct product of $\mathfrak{g}^*_\mu$ and a symplectic manifold W. Consequently, each element ν sufficiently close to μ may be written as a pair (ξ, ψ), where $\xi \in \mathfrak{g}^*_\mu$ and $\psi \in W$.

Since W is symplectic, the dimension of $\mathfrak{g}_\nu$ is equal to the dimension of the stabilizer of ξ in $\mathfrak{g}_\mu$. Consequently, $\mathfrak{g}^*(\mu)$ consists of pairs (ξ, ψ) such that the stabilizer of ξ in $\mathfrak{g}_\mu$ coincides with $\mathfrak{g}_\mu$, which means that ξ vanishes on the derived algebra of $\mathfrak{g}_\mu$. Therefore,

$$\dim \mathfrak{g}^*(\mu) = \dim W + \dim \mathfrak{g}_\mu - \dim[\mathfrak{g}_\mu, \mathfrak{g}_\mu].$$

Since $\dim W = \dim O(\mu)$, this proves the proposition. ■

Example 2.12 (M. Duflo, see [3, 13])**.** Let $\mathfrak{g}$ be a Lie algebra given by the following linear Poisson structure

$$\frac{\partial}{\partial x_1} \wedge (x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + 2x_4 \frac{\partial}{\partial x_4}) + x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}.$$

Consider $\mu \in \mathfrak{g}^*$ with $x_4 = 0$ and $x_2^2 + x_3^2 > 0$. Then the stabilizer of μ is Abelian: $\dim [\mathfrak{g}_\mu, \mathfrak{g}_\mu] = 0$. On the other hand, $\operatorname{codim} \mathfrak{g}^*(\mu) = 1$. Consequently, the transverse Poisson structure at μ is not linearizable.

3. Semisimple case: the derived algebra of a centralizer and sheets

In the semisimple case, the coadjoint and the adjoint actions can be identified by the means of the Killing form. This identification maps the variety $\mathfrak{g}^*_k$ to the variety

$$\mathfrak{g}^{(k)} = \{a \in \mathfrak{g} \mid \dim \mathfrak{g}^a = k\}$$

where $\mathfrak{g}^a = \{x \in \mathfrak{g} \mid [a, x] = 0\}$ is the centralizer of a in $\mathfrak{g}$. The irreducible components of the varieties $\mathfrak{g}^{(k)}$ are called the sheets of $\mathfrak{g}$. Recall some facts about the topology of sheets.

1. Sheets are not necessarily smooth. However, if $\mathfrak{g}$ is a *classical* simple Lie algebra, then the sheets are smooth (Im Hof [5]).
2. Sheets are not necessarily disjoint even in the classical case. However, they are for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ (Kraft and Luna [7], Peterson [11]).

We study the relation between sheets and the derived algebra of a centralizer.

Using Theorem 2.3, we obtain the following.

Proposition 3.1. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. Suppose that $a \in \mathfrak{g}$ belongs to a sheet S, and that S is smooth at the point a. Then the derived algebra of the centralizer of a is orthogonal to S at the point a:*

$$\langle\, [\mathfrak{g}^a, \mathfrak{g}^a], \mathrm{T}_a S \,\rangle = 0.$$

Corollary 3.2. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra. Suppose that $a \in \mathfrak{g}$ belongs to a sheet S, and that S is smooth at the point a. Then the following three statements are equivalent.*

(i) The derived algebra of the centralizer of a is exactly the orthogonal complement to S at the point a:

$$[\mathfrak{g}^a, \mathfrak{g}^a] = (\mathrm{T}_a S)^{\perp}. \tag{3}$$

(ii) The dimension of the derived algebra of the centralizer of μ is equal to the codimension of S:

$$\dim\, [\mathfrak{g}^a, \mathfrak{g}^a] = \operatorname{codim} S.$$

(iii) The codimension of $[\mathfrak{g}^a, \mathfrak{g}^a]$ in $\mathfrak{g}^a$ is equal to the codimension of $O(a)$ in S:

$$\dim\, \mathfrak{g}^a - \dim\, [\mathfrak{g}^a, \mathfrak{g}^a] = \dim\, S - \dim O(a).$$

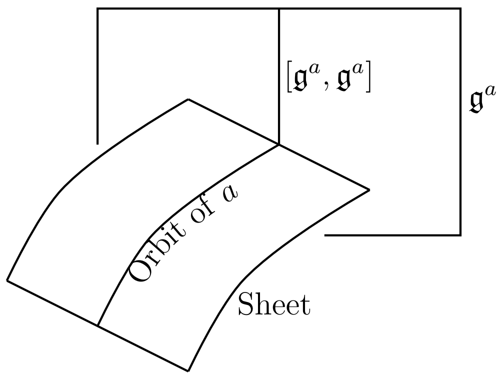

Figure 1

If one of these three conditions is satisfied, the picture is the following: the centralizer is the orthogonal complement to the orbit while its derived algebra is the orthogonal complement to the sheet (Figure 1).

Proposition 3.3. *If $\mathfrak{g}$ is a real or complex semisimple Lie algebra, and $a \in \mathfrak{g}$ is semisimple, then there is only one sheet S passing through a, S is smooth at a, and equality (3) holds.*

Proof. Since the transverse Poisson structure at a semisimple point is linearizable [8], the proof follows from Proposition 2.11. ■

Corollary 3.4. *Let $\mathfrak{g}$ be a compact real Lie algebra. Then all sheets of $\mathfrak{g}$ are smooth, disjoint, and the equality (3) holds for every $a \in \mathfrak{g}$.*

Proof. The proof follows from the fact that all elements of a compact algebra are semisimple. ■

Now assume that $\mathfrak{g} \simeq \mathfrak{sl}(n, \mathbb{C})$.

Proposition 3.5. *The equality (3) holds for every $a \in \mathfrak{g}$.*

Proof. We can assume that a is block-diagonal, with Jordan blocks of size $\lambda_1, \dots$, and we denote by $h(\lambda)$ the size of the largest Jordan block of a with the eigenvalue λ. Prove that

$$\dim \mathfrak{g}^a - \dim [\mathfrak{g}^a, \mathfrak{g}^a] = \dim S - \dim O(a) = \sum h(\lambda) - 1.$$

1. $\dim \mathfrak{g}^a - \dim [\mathfrak{g}^a, \mathfrak{g}^a] = \sum h(\lambda) - 1$.

   It suffices to prove this equality for the case when a is nilpotent. This can be easily done by studying the commutation relations for $\mathfrak{g}^a$ found by O. Yakimova [15].

2. $\dim S - \dim O(a) = \sum h(\lambda) - 1$.

   This fact is known in the case when a is nilpotent (A. Moreau [9]). The idea of the proof for an arbitrary element is as follows. For each eigenvalue λ, take a sequence of complex numbers $\varepsilon_1(\lambda), \dots, \varepsilon_{h(\lambda)}(\lambda)$. To each Jordan block of a with the eigenvalue λ, add a diagonal matrix $\mathrm{diag}(\varepsilon_1(\lambda), \dots, \varepsilon_k(\lambda))$ where k is the size of the block. We assume that the numbers $\varepsilon_1(\lambda), \dots, \varepsilon_{h(\lambda)}(\lambda)$ verify one additional condition which guarantees that the perturbed a has trace zero. This gives a family $a_\varepsilon \in \mathfrak{sl}(n, \mathbb{C})$ of dimension $\sum h(\lambda) - 1$. It is easy to check that the dimension of the centralizer of each $x \in a_\varepsilon$ is equal to the dimension of $\mathfrak{g}^a$, so $a_\varepsilon \subset S$ where S is the sheet passing through a. At the same time, the family a_ε is transversal to the orbit $O(a)$, so

$$\dim S \geq \dim O(a) + \sum h(\lambda) - 1.$$

On the other hand, by Proposition 3.1,

$$\dim S - \dim O(a) \leq \dim \mathfrak{g}^a - \dim [\mathfrak{g}^a, \mathfrak{g}^a] = \sum h(\lambda) - 1,$$

so

$$\dim S - \dim O(a) = \sum h(\lambda) - 1. \qquad \blacksquare$$

It was conjectured by the author that if $\mathfrak{g}$ is a complex classical simple Lie algebra, and if there is only one sheet S passing through $a \in \mathfrak{g}$, then equality (3) holds. Recently, A. Premet and L. Topley [12] have proved that this is indeed so for any algebraically closed ground field of characteristic 0. Moreover, they showed that equality (3) holds *if and only if* there is a unique sheet passing through a, and gave a combinatorial description of elements $a \in \mathfrak{g}$ for which this is so. To summarize:

Theorem 3.6 (A. Premet and L. Topley [12]). *Let $\mathfrak{g}$ be a complex classical simple Lie algebra over and algebraically closed field of characteristic 0, and let $a \in \mathfrak{g}$. Then there is a unique sheet S passing through a if and only if equality (3) holds, i.e. $[\mathfrak{g}^a, \mathfrak{g}^a] = (\mathrm{T}_a S)^{\perp}$.*

Note that this theorem was previously proved by O. Yakimova for the case of rigid nilpotent elements [15].

Conjecture 3.7. Let $\mathfrak{g}$ be a complex classical simple Lie algebra. If $a \in \mathfrak{g}$ belongs to several sheets $S_1, \dots S_k$, then

$$[\mathfrak{g}^a, \mathfrak{g}^a] = \left(\sum_{i=1}^{k} \mathrm{T}_a S_i \right)^{\perp}.$$

Note that this conjecture is false for the exceptional Lie algebra G_2, as it follows from Remark 3 of [15].

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