On the Compact Space of Closed Subgroups
of Locally Compact Groups

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Abstract.   Let G be a locally compact topological group. We denote by $\mathscr{C}(G)$ the hyperspace of all closed subgroups of G equipped with the Chabauty topology; this is a compact space. The main result of this paper is to prove that the assignment $\mathscr{C} : G \longmapsto \mathscr{C}(G)$ is functorial from the category of locally compact groups and proper morphisms to the category of compact spaces and preserves projective limits of projective systems of topological groups in which all bonding maps are surjective and proper.

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1.   Introduction

Let G be a locally compact group and $\mathscr{C}(G)$ the hyperspace of all closed subgroups of G. The Chabauty topology on $\mathscr{C}(G)$ has the sets

$$\mathcal{O}_1(K) = \{H \in \mathscr{C}(G): \ H \cap K = \emptyset\},$$
$$\mathcal{O}_2(V) = \{H \in \mathscr{C}(G): \ H \cap V \neq \emptyset\},$$

as an open subbase, where V and K run respectively, over all open and compact subsets of G.

The hyperspace $\mathscr{C}(G)$ endowed with the Chabauty topology is called the Chabauty space of the group G. The following well-known result can be found in [3, Théorème 1, page 181] (see also [1, Lemma E.1.1]).

Proposition 1.1.   *The Chabauty space of a locally compact group is compact.*

Let **CompGr** be the category of compact topological groups and continuous morphisms and **CompTop** the category of compact spaces and continuous maps. Then,

$$\mathscr{C} : \textbf{CompGr} \longrightarrow \textbf{CompTop}$$

defined by

$$\mathscr{C}\left(G_1 \xrightarrow{\phi} G_2\right) \ : \ \mathscr{C}(G_1) \xrightarrow{\mathscr{C}(\phi)} \mathscr{C}(G_2),$$

where $\mathscr{C}(\phi)(H) = \phi(H)$, is a covariant functor ([6, Theorem 2]). Our first result in this paper is to show that in the case of locally compact groups it remains functorial on the category **LCGr** of locally compact topological groups and proper morphisms (i.e., closed morphisms ϕ such that inverse images $\phi^{-1}(K)$ of compact subsets of the range are compact).

     A projective system of topological groups is called a strong projective system and its limit is called a strong projective limit if the bonding maps are surjective and proper. We shall prove the following main result.

Main Theorem. *The functor $\mathscr{C} : G \longmapsto \mathscr{C}(G)$ from the category of locally compact topological groups and proper maps to the category of compact spaces preserves strong projective limits.*

2. Main results

We start with the following proposition.

Proposition 2.1. *If $\phi : G_1 \longrightarrow G_2$ is a proper morphism between locally compact topological groups, then the following mapping*

$$\mathscr{C}(\phi) : \mathscr{C}(G_1) \longrightarrow \mathscr{C}(G_2)$$
$$H \longmapsto \phi(H)$$

is continuous. Moreover, if ϕ is surjective, then $\mathscr{C}(\phi)$ is surjective.

Proof. It is clear that for every compact subset K and any open subset U of G_2, we have

$$\mathscr{C}(\phi)^{-1}\big(\mathcal{O}_1(K)\big) = \mathcal{O}_1\big(\phi^{-1}(K)\big),$$

and

$$\mathscr{C}(\phi)^{-1}\big(\mathcal{O}_2(U)\big) = \mathcal{O}_2\big(\phi^{-1}(U)\big).$$

As $\{\mathcal{O}_1(K)|\ K$ is a compact subset of $G_2\} \cup \{\mathcal{O}_2(U)|\ U$ is an open subset of $G_2\}$ is a prebase of $\mathscr{C}(G_2)$, then $\mathscr{C}(\phi)$ is continuous. The second statement is obvious. ∎

     In the language of category, we have

Proposition 2.2. *Let **LCGr** be the category of locally compact topological groups and proper morphisms and let **CompTop** be the category of compact spaces and continuous maps. Then, the assignment*

$$\mathscr{C} : \mathbf{LCGr} \longrightarrow \mathbf{CompTop}$$

defined by

$$\mathscr{C}\left(G_1 \xrightarrow{\phi} G_2\right) \ : \ \mathscr{C}(G_1) \xrightarrow{\mathscr{C}(\phi)} \mathscr{C}(G_2)$$

is a covariant functor.

Proof.   It is obvious that $\mathscr{C}$ preserves composition and identity morphisms; that is, $\mathscr{C}(\phi_1 \circ \phi_2) = \mathscr{C}(\phi_1) \circ \mathscr{C}(\phi_2)$ whenever $\phi_1 \circ \phi_2$ is defined, and $\mathscr{C}(\mathrm{id}_G) = \mathrm{id}_{\mathscr{C}(G)}$, for each **LCGr**-object G.   ■

Remark 2.3.   For later use, observe that we have proved that, if $\phi : G_1 \longrightarrow G_2$ is an isomorphism between locally compact topological groups, then $\mathscr{C}(\phi)$ is a homeomorphism and $\mathscr{C}(\phi)^{-1} = \mathscr{C}(\phi^{-1})$.

Let K be a normal compact subgroup of a locally compact group G. Then the natural map $p : G \longrightarrow G/K$ is a proper map ([14, Lemma 32.8]) and therefore we obtain the following corollary.

Corollary 2.4.   *Let G be a locally compact topological group and K a normal compact subgroup of G. Then the following mapping*

$$\mathscr{C}(p) : \mathscr{C}(G) \longrightarrow \mathscr{C}(G/K)$$
$$H \longmapsto p(H)$$

is continuous, where $p : G \longrightarrow G/K$ is the canonical projection.

Proposition 2.5.   *Let G_1 and G_2 be two locally compact topological groups. Let $\phi : G_1 \longrightarrow G_2$ be a continuous open homomorphism. Then the following mapping*

$$\mathscr{C}^*(\phi) : \mathscr{C}(G_2) \longrightarrow \mathscr{C}(G_1)$$
$$K \longmapsto \phi^{-1}(K)$$

is continuous.

Proof.   See [7, Proposition 1.2].   ■

Remark 2.6.   The assignment

$$\mathscr{C}^*\left(G_1 \xrightarrow{\phi} G_2\right) \ : \ \mathscr{C}(G_2) \xrightarrow{\mathscr{C}^*(\phi)} \mathscr{C}(G_1)$$

is a contravariant functor from the category of locally compact topological groups and continuous and open maps into the category of compact spaces.

Corollary 2.7.   *Let G be a locally compact topological group and K a compact normal subgroup. Then the following mapping*

$$\mathscr{C}^*(p) : \mathscr{C}(G/K) \longrightarrow \mathscr{C}(G)$$
$$H \longmapsto p^{-1}(H)$$

is continuous, where $p : G \longmapsto G/K$ is the canonical projection.

Proposition 2.8.   *If $\phi : G_1 \longrightarrow G_2$ is a proper quotient morphism between locally compact topological groups, then $\mathscr{C}(G_2)$ is a retract of $\mathscr{C}(G_1)$; that is, the following diagram*

$$
\begin{array}{ccc}
 & \mathscr{C}(G_2) & \\
\mathscr{C}^*(\phi) \swarrow & & \searrow \mathrm{id}_{\mathscr{C}(G_2)} \\
\mathscr{C}(G_1) & \xrightarrow{\ \mathscr{C}(\phi)\ } & \mathscr{C}(G_2)
\end{array}
$$

is commutative.

The following definition is introduced in ([9, Definition 1.24]).

Definition 2.9 (Strong projective limit)**.**   A projective system of topological groups in which all bonding maps are surjective and proper is called a strong projective system and its limit is called a strong projective limit.

In [9], the following result is established in Corollary 1.25, page 86.

Proposition 2.10.   *The limit maps of a strong projective system of locally compact topological groups are proper and surjective.*

The main results of this paper is the following.

Main Theorem.   *Let $\{\phi_{\alpha\beta} : G_\beta \longrightarrow G_\alpha \mid (\alpha,\beta) \in I \times I, \alpha \leq \beta\}$ be a strong projective system of locally compact topological groups with a strong projective limit $G = \varprojlim_{\alpha \in I} G_\alpha$. Then $\{\mathscr{C}(\phi_{\alpha\beta}) : \mathscr{C}(G_\beta) \longrightarrow \mathscr{C}(G_\alpha) \mid (\alpha,\beta) \in I \times I, \alpha \leq \beta\}$ is a projective system of compact spaces and continuous maps with surjective bonding maps $\mathscr{C}(\phi_{\alpha\beta})$, and*

$$\mathscr{C}(G) = \varprojlim_{\alpha \in I} \mathscr{C}(G_\alpha).$$

Proof.   The first statement follows from Corollary 2.2. Next, We observe that in view of Proposition 2.10, the limit morphisms

$$\phi_\alpha : G \longrightarrow G_\alpha$$

are proper. The family $\{\mathscr{C}(\phi_\alpha) : \mathscr{C}(G) \longrightarrow \mathscr{C}(G_\alpha) \mid \alpha \in I\}$ is compatible with the projective system $\{\mathscr{C}(\phi_{\alpha\beta}) : \mathscr{C}(G_\beta) \longrightarrow \mathscr{C}(G_\alpha) \mid (\alpha,\beta) \in I \times I, \alpha \leq \beta\}$; that is, for every $(\alpha,\beta) \in I \times I$ such that $\alpha \leq \beta$ the following diagram

$$
\begin{array}{ccc}
 & \mathscr{C}(G) & \\
\mathscr{C}(\phi_\beta) \swarrow & & \searrow \mathscr{C}(\phi_\alpha) \\
\mathscr{C}(G_\beta) & \xrightarrow{\ \mathscr{C}(\phi_{\alpha\beta})\ } & \mathscr{C}(G_\alpha)
\end{array}
$$

is commutative. Then by the universal property of the projective limit there exists a unique morphism

$$\lambda : \mathscr{C}(G) \longrightarrow \varprojlim_{\alpha \in I} \mathscr{C}(G_\alpha),$$

such that
$$\forall \alpha \in I, \ \mathscr{C}(\phi_\alpha) = \lambda_\alpha \circ \lambda,$$
where $\lambda_\alpha : \varprojlim_{\beta \in I} \mathscr{C}(G_\beta) \longrightarrow \mathscr{C}(G_\alpha)$ are the limit morphisms. We now show that λ is an isomorphism. Since bijective continuous functions between compact spaces are homeomorphisms, the bijectivity of the function λ suffices for a proof of its being a homeomorphism. The surjectivity of λ follows from the surjectivity of each $\mathscr{C}(\phi_\alpha)$ which, in turn, is a consequence of the fact that each limit map $\phi_\alpha : G \longrightarrow G_\alpha$ is surjective (see Proposition 2.10 and Proposition 2.1). Next, we prove that λ is injective. Let H, K be two closed subgroups of G such that $\lambda(H) = \lambda(K)$. Then, for every $\alpha \in I$ we have
$$H \ker(\phi_\alpha) = K \ker(\phi_\alpha)$$
and therefore
$$\bigcap_{\alpha \in I} H \ker(\phi_\alpha) = \bigcap_{\alpha \in I} K \ker(\phi_\alpha).$$
Since the set $\{\ker(\phi_\alpha)| \ \alpha \in I\}$ is a filter basis of closed normal subgroups converging to the identity element e of G ([8, Theorem 2.1] or [14, Lemma 18.2]) then $H = K$ and the theorem is proved. ∎

Remark 2.11. In the special case where G is compact, this previous theorem was proved by S. Fisher and P. Gartside in [6, Proposition 13].

3. Spaces of closed normal subgroups of strong projective limits

We need the following in the proof of the main result of this section.

Proposition 3.1. *If a net $(H_\alpha)_{\alpha \in J}$ of closed subgroups of a locally compact group G converges to a closed subgroup H, then the following assertions are equivalent.*

(1) *The points $x_1, x_2, \ldots, x_n$ belong to H.*

(2) *There exists a subnet $(H_{\alpha_\beta})_{\beta \in I}$ of the net $(H_\alpha)_{\alpha \in J}$ and there exist nets $\{x_1^\beta\}_{\beta \in I}, \ldots, \{x_n^\beta\}_{\beta \in I}$ of G converging, respectively, to $x_1, \ldots, x_n$ in the topology of G, with $x_1^\beta, \ldots, x_n^\beta \in H_{\alpha_\beta}$ for all $\beta \in I$.*

Proof. This follows from Remark 1.1, page 628 and Lemma 1.2, page 629 of [11]. ∎

Now, let us introduce the following definition.

Definition 3.2 (Closed property). Let G be a locally compact group. Let $\mathcal{P}$ be a property of subgroups of a group. We say that $\mathcal{P}$ is closed if the set
$$\{H \in \mathscr{C}(G) : H \text{ satisfies } \mathcal{P}\}$$
is closed in $\mathscr{C}(G)$.

Let G be topological group. Let $\mathrm{End}(G)$ denotes the space of endomorphisms of G. For $\Delta \subset \mathrm{End}(G)$ we put

$$I(\Delta) = \{H \in \mathscr{C}(G) : (\forall\ f \in \Delta)\ f(H) \le H\} = \bigcap_{f \in \Delta}\{H \in \mathscr{C}(G) : f(H) \le H\}.$$

Proposition 3.3.   *The space $I(\Delta)$ is closed in $\mathscr{C}(G)$.*

Proof.   For every $f \in \Delta$ we set

$$I_f = \{H \in \mathscr{C}(G) : f(H) \le H\}.$$

Let $(H_i)_{i \in I}$ be a net of elements of I_f converging to a closed subgroup H. In view of Proposition 3.1, for every $x \in H$ there exists a subnet $(H_{i_j})_{j \in J}$ and elements $x_j \in H_{i_j}$ for each $j \in J$ such that $x = \lim_{j \in J} x_j$. Since f is continuous, $f(x) = \lim_{j \in J}(f(x_j))$, and since $x_j \in H_{i_j} \in I_f$ we have $f(x_j) \in H_{i_j}$ for all $j \in J$. Accordingly, by Proposition 3.1 again, $f(x) = \lim_{j \in J}(f(x_j)) \in \lim_{j \in J} H_{i_j} = H$. Therefore $f(H) \le H$. Consequently, I_f is closed. As

$$I(\Delta) = \bigcap_{f \in \Delta} I_f,$$

then $I(\Delta)$ is closed in $\mathscr{C}(G)$.   ∎

As an immediate consequence of Proposition 3.3, we have the following result.

Corollary 3.4.   *The property of being a normal (resp. a characteristic) subgroup is closed.*   ∎

For a topological group G, let $\mathscr{C}_{\lhd}(G)$ be the set of all closed normal subgroups of the group G. As an immediate consequence we get the following proposition.

Proposition 3.5.   *Let $G = \varprojlim_{\alpha \in I} G_\alpha$ be a strong limit of a strong projective system of topological groups. Then*

$$\mathscr{C}_{\lhd}(G) = \varprojlim_{\alpha \in I} \mathscr{C}_{\lhd}(G_\alpha).$$

Proof.   Let $\phi_\alpha : G \longrightarrow G_\alpha$ be the limit morphisms. By Corollary 3.4, $\mathscr{C}_{\lhd}(G)$ is a closed subset of $\mathscr{C}(G)$. Then in view of [12, Corollary 1.1.8] we obtain

$$\mathscr{C}_{\lhd}(G_\alpha) = \mathscr{C}(\phi_\alpha)\,(\mathscr{C}_{\lhd}(G))\,.$$

Then

$$\mathscr{C}_{\lhd}(G) = \varprojlim_{\alpha \in I} \mathscr{C}_{\lhd}(G_\alpha).$$ ∎

4. An application to the connectedness of hyperspaces of pro-Lie groups

In this section we are interested in the following question:
Under which conditions is the hyperspace $\mathscr{C}(G)$ connected?

Before proceeding, let us recall the following well-known results.

Theorem 4.1. *For every locally compact abelian group G, the hyperspace $\mathscr{C}(\mathbb{R} \times G)$ is connected.*

Proof. See [10, Theorem 1] or [5, Theorem 1.3]. ∎

In this context it may be instructive to the reader to consider the following

Exercise. The function $\alpha\colon [0,1] \to \mathscr{C}(\mathbb{R})$,

$$\alpha(r) = \begin{cases} \frac{1}{r}\cdot\mathbb{Z} & \text{if } 0 < r \le 1, \\ \{0\} & \text{for r=0,} \end{cases}$$

is an arc connecting $\{0\}$ and $\mathbb{Z}$ in $\mathscr{C}(\mathbb{R})$. ∎

Theorem 4.2. *Let $\mathbb{H}_3$ be the 3-dimensional Heisenberg group. Then the hyperspace $\mathscr{C}(\mathbb{H}_3)$ is arcwise connected.*

Proof. See [4, Theorem 1.3]. ∎

The following definitions are introduced in [9, page 148-149].

Definition 4.3 (Co-Lie group). A subgroup N of a topological group G is called a co-Lie subgroup if it is normal and G/N is a Lie group.

For a topological group G, let $\mathcal{N}(G)$ be the set of all co-Lie subgroups.

Definition 4.4. We say that a topological group G has arbitrarily small co-Lie subgroups if every identity neighborhood of G contains a member of $\mathcal{N}(G)$.

Definition 4.5 (Pro-Lie group). A topological group G is called a pro-Lie group if it is complete and has arbitrarily small co-Lie subgroups.

A topological group G is said to be almost connected if the factor group G/G_0 of G modulo the identity component G_0 is compact. The following theorem is proved in [15] (see also [8, page 647]).

Theorem 4.6 (Yamabe's Theorem). *Every almost connected locally compact topological group is a pro-Lie group.*

Every locally compact pro-Lie group G gives rise to a strong projective system

$$\{p_{NM} : G/M \longrightarrow G/N \mid (M, N) \in \mathcal{N}_c(G) \times \mathcal{N}_c(G),\ N \le M\}$$

whose projective limit is isomorphic to G, where

$$\mathcal{N}_c(G) = \{N \in \mathcal{N}(G) \mid N \text{ is a compact subgroup of } G\}.$$

The main theorem reduces the study of connectedness of the hyperspace $\mathscr{C}(G)$ of arbitrary pro-Lie group G to that of Lie groups. More precisely, we have the following result:

Theorem 4.7. *Let G be a locally compact pro-Lie group. Then the following statements are equivalent.*

(1) *The space $\mathscr{C}(G)$ is connected;*

(2) *for every $N \in \mathcal{N}(G)$, the space $\mathscr{C}(G/N)$ is connected.*

Proof. This follows from the fact that $G = \varprojlim\limits_{N \in \mathcal{N}_c(G)} G/N$, from our Main Theorem, and from Theorem 6.1.20 of [13]. ∎

Knowing that for a compact Lie group G, the trivial subgroup $\{\{e\}\}$ is clopen in $\mathscr{C}(G)$ (See the proof of Lemma 4 in [10]), we immediately derive the interesting well-known theorem (see [10, Lemma 4]).

Theorem 4.8. *If G is a nontrivial compact group, then $\mathscr{C}(G)$ is disconnected.* ∎

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