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A Geometric Mean for Symmetric Spaces of Noncompact Type

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Abstract. The concept of the t-geometric mean of two positive definite matrices is extended to symmetric spaces of noncompact type. The t-geometric mean of two points in such a symmetric space yields the unique geodesic joining the points and the geometric mean is the midpoint. A parametrization of the geodesic in terms of the two points is given. Inequalities about geometric mean and geodesic triangle are given in terms of Kostant's pre-order on semisimple Lie groups as well as on their Lie algebras.
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1. Introduction

Let $\mathbb{P}_n$ be the set of $n\times n$ positive definite matrices over $\mathbb{C}$. If $A\in\mathbb{P}_n$, then there exists a unique Hermitian matrix H such that $A=e^H$, so $A^t:=e^{tH}$ for $t\in\mathbb{R}$. For $t\in[0,1]$, the t*-geometric mean* of $A,B\in\mathbb{P}_n$ is

$$A\#_tB=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^tA^{1/2}. \tag{1.1}$$

When $t=1/2$, $A\#_{1/2}B$ is called the geometric mean of A and B, and it was first introduced in [21] and is often denoted by $A\#B$ in the literature. Recently the t-geometric mean has been gaining intensive interest (e.g., see [1, 3, 5, 6, 8, 9, 16, 18, 20, 23] and the references therein), partially because of its connection with Riemannian geometry. More precisely, $\mathbb{P}_n$ may be equipped with a suitable Riemannian metric (see (2.4)) so that the curve $\gamma(t)=A\#_tB$, $0\leqslant t\leqslant 1$, is the unique geodesic joining A and B in $\mathbb{P}_n$ ([5, p.205]).

Let $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$ be in $\mathbb{R}^n$. Let $x^\downarrow=(x_{[1]},x_{[2]},\ldots,x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$. We say that x is *majorized* by y, denoted by $x\prec y$, if

$$\sum_{i=1}^k x_{[i]}\leqslant\sum_{i=1}^k y_{[i]},\quad k=1,2,\ldots,n-1,\quad\text{and}\quad \sum_{i=1}^n x_{[i]}=\sum_{i=1}^n y_{[i]}.$$

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An equivalent condition for $x \prec y$ is

$$\operatorname{conv} S_n \cdot x \subset \operatorname{conv} S_n \cdot y,$$

where $\operatorname{conv} S_n \cdot x$ denotes the convex hull of the orbit of x under the action of the symmetric group S_n ([11, 19]). If $x > 0$ (i.e., $x_i > 0$ for $i = 1, \dots, n$) and $y > 0$, we say that x is *log majorized* by y, denoted by $x \prec_{\log} y$, if

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \dots, n-1, \quad \text{and} \quad \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

In other words, $x \prec_{\log} y$ if and only if $\log x \prec \log y$.

For an $n \times n$ matrix H with positive eigenvalues, let

$$\lambda(H) = (\lambda_1(H), \dots, \lambda_n(H))$$

denote the vector of eigenvalues of H such that $\lambda_1(H) \geqslant \dots \geqslant \lambda_n(H)$. The following interesting results appeared in a very recent paper of Bhatia and Grover [7, p.730]. The first inequality is a result of Ando and Hiai [2, Corollary 2.3] as the complementary counterpart of the famous Golden-Thompson inequality: $\operatorname{tr} e^{A+B} \leqslant \operatorname{tr} e^A e^B$ for Hermitian matrices A and B. The second inequality follows from a result of Araki [4].

Theorem 1.1. Let $A, B \in \mathbb{P}_n$. For any $t \in [0, 1]$ and $s > 0$,

$$\begin{aligned} \lambda(A \#_t B) &\prec_{\log} \lambda\left(e^{(1-t)\log A + t \log B}\right) \\ &\prec_{\log} \lambda\left(B^{ts/2} A^{(1-t)s} B^{ts/2}\right)^{1/s} \\ &= \lambda\left(A^{(1-t)s} B^{ts}\right)^{1/s}. \end{aligned} \tag{1.2}$$

It is known that $\mathbb{P}_n$ is a Riemannian manifold of nonpositive curvature (i.e., all sectional curvatures are $\leqslant 0$). In general, the distance on a Riemannian manifold of nonpositive curvature has some nice convexity property ([15, IX.Theorem 4.3]): For any two geodesics $\alpha(t)$ and $\beta(t)$ with $t \in \mathbb{R}$, the Riemannian distance $d(\alpha(t), \beta(t))$ between $\alpha(t)$ and $\beta(t)$ is a convex function of t. In particular, for two geodesics $\alpha(t)$ and $\beta(t)$ with $\alpha(0) = A$, $\alpha(1) = B$, $\beta(0) = C$ and $\beta(1) = D$, we have

$$d(\alpha(t), \beta(t)) \leqslant (1-t)d(A, C) + td(B, D), \quad 0 \leqslant t \leqslant 1. \tag{1.3}$$

When $C = D$, for any geodesic $\alpha(t)$ joining A to B and not containing C, we have

$$d(\alpha(t), C) \leqslant (1-t)d(A, C) + td(B, C). \tag{1.4}$$

In particular, we have for A and B in $\mathbb{P}_n$

$$d(A \#_t B, C) \leqslant (1-t)d(A, C) + td(B, C), \tag{1.5}$$

where $d(A, B) = \|\lambda(\log BA^{-1})\|$ (see [5, p.205-206]) and $\|\cdot\|$ is the Euclidean norm (the distance d is denoted as δ_2 in [5]).

Kostant [14] introduced a pre-order relation on semisimple Lie groups. This relation can be carried over to symmetric spaces of noncompact type. Such spaces also have nonpositive curvature. Besides extending some majorization results for matrices, such as the Golden-Thompson inequality, to symmetric spaces, Kostant also obtained a nice geometric property (see Theorem 3.9) of geodesic triangle. Although $\mathbb{P}_n$ is not a symmetric space of noncompact type, the space $\mathbb{P}_n^1$ of matrices in $\mathbb{P}_n$ of determinant 1 is. The order relation $\lambda(A) \prec_{\log} \lambda(B)$ on $\mathbb{P}_n^1$ is precisely Kostant's pre-order. More details will be given later.

In this paper, we will extend, namely in Theorem 3.5, the inequalities on $\mathbb{P}_n$ given in Theorem 1.1 to Kostant's pre-order on symmetric spaces of noncompact type. We will note, in Remark 3.12, that Theorem 3.5 is stronger than the distance convexity (1.5). Some preliminaries of symmetric spaces will be given next, and the main results will be established in Section 3. We note that an abstract extension of symmetric spaces of noncompact type which preserve the distance convexity, called lineated symmetric spaces, has been studied in [17].

2. Preliminaries and Notations

2.1. Symmetric spaces. The reader is referred to [10] for the standard notation and facts on symmetric spaces to be reviewed here.

Let G be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and identity element e, let $\Theta\colon\ G \to G$ be a nontrivial Lie group automorphism with $\Theta^2 = \mathrm{id}_G$ (the identity map on G), and let K be a compact subgroup of G such that K is the fixed point set of Θ. Then the homogeneous space G/K is a symmetric space.

The differential map $d\Theta\colon\ \mathfrak{g} \to \mathfrak{g}$ of Θ has eigenvalues ± 1. The eigenspace of 1 is the Lie algebra $\mathfrak{k}$ of K, and the eigenspace of -1 is an $\mathrm{Ad}\, K$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ complementary to $\mathfrak{k}$. Let

$$\pi : G \to G/K, \quad g \mapsto gK,$$

be the natural projection. Then $\mathfrak{p}$ may be identified with the tangent space $T_o(G/K)$ of G/K at the origin $o = eK$ via $d\pi$. Thus any $\mathrm{Ad}\, K$-invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ induces a unique G-invariant Riemannian metric on G/K, i.e., a Riemannian metric invariant under the natural action of G on G/K given by $(g, xK) \mapsto gxK$.

Since G is semisimple, the Killing form B on $\mathfrak{g}$ is nondegenerate. If B is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, then the symmetric space G/K is said to be of noncompact type, and the automorphism Θ is called a Cartan involution.

¿From now on, we will assume that G/K is a symmetric space of noncompact type, unless explicitly stated otherwise. Then the map $\mathfrak{p} \times K \to G$, $(X, k) \mapsto g = e^X k$, is a diffeomorphism, where $e^X := \exp X$ is the Lie group exponential map ([10, VI.Theorem 1.1]).

Let $P := \{e^X : X \in \mathfrak{p}\}$. Then every $g \in G$ can be uniquely written as $g = pk$ with $p \in P$ and $k \in K$. The decomposition $G = PK$ is called a Cartan decomposition of G.

Let $* : G \to G$ be the diffeomorphism defined by $*(g) = g^* = \Theta(g^{-1})$. Then $k^* = k^{-1}$ for $k \in K$ and $p^* = p$ for $p \in P$. The map $G \to P$, $g \mapsto gg^*$, is onto. Because for any $g \in G$, it maps gK to a single point gg^*, it follows that the map

$$\psi : \ G/K \ \to P, \quad gK \mapsto gg^*, \tag{2.1}$$

is a bijection. It is in fact a diffeomorphism by the Cartan decomposition $G = PK$. Via ψ, P may be identified with G/K, and so may be regarded as a symmetric space of noncompact type. Note that for $p \in P$, $\psi^{-1}(p) = p^{1/2}K$, and G acts on P by $(g,p) \mapsto gpg^*$.

2.2. Complete multiplicative Jordan decomposition. An element $X \in \mathfrak{g}$ is called real semisimple (resp., nilpotent) if $\operatorname{ad} X$ is diagonalizable over $\mathbb{R}$ (resp., $\operatorname{ad} X$ is nilpotent). An element $g \in G$ is called *hyperbolic* (resp., *unipotent*) if $g = \exp X$ for some real semisimple (resp., nilpotent) $X \in \mathfrak{g}$; in either case X is unique and we write $X = \log g$. An element $g \in G$ is called *elliptic* if $\operatorname{Ad} g$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The following important result, due to Kostant [14], is called the *complete multiplicative Jordan decomposition*, abbreviated as CMJD.

Theorem 2.1. ([14, Proposition 2.1]) *Each $g \in G$ can be uniquely written as $g = ehu$, where e is elliptic, h is hyperbolic, u is unipotent, and the three elements e, h and u commute.*

2.3. Kostant's pre-order. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let A be the analytic subgroup generated by $\mathfrak{a}$. We have $\mathfrak{p} = \operatorname{Ad}(K)\mathfrak{a}$ ([12, p.378]). The Weyl group W of $(\mathfrak{g}, \mathfrak{a})$ acts simply transitively on $\mathfrak{a}$, and also on A through the exponential map $\exp : \mathfrak{a} \to A$.

Let L and $\mathfrak{l}$ denote the set of hyperbolic elements in G and the set of real semisimple elements in $\mathfrak{g}$, respectively. It is known that ([14, Proposition 6.2])

$$L = P^2 := \{pq : p, q \in P\}.$$

According to [14, Proposition 2.4], $h \in L$ if and only if h is conjugate to an element in A, and $X \in \mathfrak{l}$ if and only if X is conjugate to an element in $\mathfrak{a}$ (i.e., $\operatorname{Ad} g(X) \in \mathfrak{a}$ for some $g \in G$).

For any $X \in \mathfrak{l}$, let $W(X)$ denote the set of elements in $\mathfrak{a}$ that are conjugate to X:

$$W(X) = \operatorname{Ad} G(X) \cap \mathfrak{a}.$$

It is known that ([14, Proposition 2.4]) $W(X)$ is a single W-orbit in $\mathfrak{a}$. Let $\operatorname{conv} W(X)$ be the convex hull in $\mathfrak{a}$ generated by $W(X)$. For any $g \in G$, define

$$A(g) := \exp \operatorname{conv} W(\log h(g)),$$

where $h(g)$ is the hyperbolic component of g in its CMJD.

Kostant's pre-order $\prec$ on G ([14, p.426]) is defined by setting $f \prec g$ if

$$A(f) \subset A(g).$$

This pre-order induces a partial order on the conjugacy classes of G. Though Kostant's pre-order appears to depend on the choice of $\mathfrak{a}$, it actually does not, due to the following fact.

Theorem 2.2. ([14, Theorem 3.1]) *Let $f, g \in G$. Then $f \prec g$ if and only if $|\pi(f)| \leqslant |\pi(g)|$ for any finite dimensional representation π of G, where $|\pi(g)|$ is the spectral radius of $\pi(g)$.*

Kostant's pre-order $\prec$ on G can be defined on $\mathfrak{g}$ as well: for $X, Y \in \mathfrak{g}$, define $X \prec Y$ if $\exp X \prec \exp Y$. This pre-order for $X, Y \in \mathfrak{l}$ takes the form

$$X \prec Y \quad \Longleftrightarrow \quad \operatorname{conv} W(X) \subset \operatorname{conv} W(Y). \tag{2.2}$$

2.4. Geodesics. Any geodesic in G/K emitting from the origin $K = eK \in G/K$ has the form $e^{tX}K$ for $X \in \mathfrak{p}$ ([10, IV.Theorem 3.3]). By the G-invariance of the Riemannian metric, any geodesic from gK has the form $ge^{tX}K$. By the Cartan decomposition $G = PK$, any two points in G/K are joined by a unique geodesic of this form.

Proposition 2.3. *Let $p, q \in P$. The unique geodesic joining p and q in P has the following parametrization*

$$\gamma(t) = p\#_t q = p^{1/2}\left(p^{-1/2} q p^{-1/2}\right)^t p^{1/2}, \quad 0 \leqslant t \leqslant 1. \tag{2.3}$$

Proof. Using the identification of P and G/K via the map ψ defined in (2.1), the unique geodesic in P from p (at $t = 0$) to q (at $t = 1$) is given by $\gamma(t) = p^{1/2}e^{tY}p^{1/2}$ for some $Y \in \mathfrak{p}$. Because $q = \gamma(1) = p^{1/2}e^{Y}p^{1/2}$, $\gamma(t) = p^{1/2}(p^{-1/2}qp^{-1/2})^t p^{1/2}$. ∎

The parametrization (2.3) has the same form as the t-geometric mean (1.1) on $\mathbb{P}_n$. Many properties of the t-geometric mean on $\mathbb{P}_n$ can be extended. For examples, $p\#_t q = q\#_{1-t}p$ and $(p\#_t q)^{-1} = p^{-1}\#_t q^{-1}$ for all $0 \leqslant t \leqslant 1$.

2.5. Examples. Let G be the general linear group $\mathrm{GL}_n(\mathbb{C})$ of $n \times n$ nonsingular complex matrices, and $K = \mathrm{U}(n)$ be the unitary group. Then G/K is a symmetric space with the Cartan involution $A \mapsto (A^*)^{-1}$, where $*$ denotes conjugate transpose, but it agrees with the $*$-map on a Lie group defined earlier. However, G/K is not of noncompact type because G is not semisimple. Let $P = \mathbb{P}_n$, the space of positive definite matrices. The map $\psi\colon G/K \to P$ is defined as before. The preceding discussion of G-invariant Riemannian structures is still valid as it does not rely on G being semisimple. For $X \in \mathfrak{g}$ and $g, h \in G$, the matrix product gXh represents naturally the tangent vector on G at gh obtained from X by a left g-translation and a right h-translation. Then the tangent space T_pP of P at p can be expressed as $T_pP = \{p^{1/2}Xp^{1/2} : X \in \mathfrak{p}\}$. Now take an $\operatorname{Ad} K$-invariant inner product on $\mathfrak{p}$ given by $\langle X, Y\rangle = \operatorname{tr}(XY)$ for $X, Y \in \mathfrak{p}$. Because the G-action on P is given by $(g, p) \mapsto gpg^*$, the induced Riemannian inner product on T_pP is

$$\langle X, Y\rangle_p = \langle p^{-1/2}Xp^{-1/2}, p^{-1/2}Yp^{-1/2}\rangle = \operatorname{tr}(p^{-1}Xp^{-1}Y) \tag{2.4}$$

for $X, Y \in T_pP$. This agrees with the Riemannian structure on $\mathbb{P}_n$ defined in [5].

The discussion of geodesics on G/K and P is also valid without the semisimple assumption. Indeed, by [13, X.Theorem 3.3 (2)], $t \mapsto e^{tX}K$, $X \in \mathfrak{p}$, is a geodesic in G/K if for $X, Y \in \mathfrak{p}$, $[X, Y]$ has zero $\mathfrak{p}$-component in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which holds for $G = \mathrm{GL}_n(\mathbb{C})$. Therefore, by (2.3), the geodesic on $\mathbb{P}_n$ is the t-geometric mean (1.1).

Note that the special linear group $\mathrm{SL}_n(\mathbb{C})$ is semisimple, and the corresponding P is the space $\mathbb{P}_n^1$ of positive definite matrices of determinant 1, which is a symmetric space of noncompact type.

3. Main Results

3.1. Inequalities on geometric means.. The symmetric space $P \subset G$ inherits the pre-order $\prec$ on G. We are going to extend Theorem 1.1 and, to do so, we need the following lemmas.

Lemma 3.1. (Golden-Thompson-Kostant [14, Theorem 6.3])

$$e^{X+Y} \prec e^X e^Y, \qquad X, Y \in \mathfrak{p}.$$

For $X, Y \in \mathfrak{p}$, let $e^X e^Y = e^Z k$ be the Cartan decomposition with $Z \in \mathfrak{p}$ and $k \in K$. By [14, Lemma 4.3], $e^Z k \prec e^Z$, and then by Lemma 3.1, we obtain $e^{X+Y} \prec e^Z$. This is recorded below and may be regarded as a Lie algebra version of the Golden-Thompson-Kostant inequality.

Corollary 3.2. *For $X, Y \in \mathfrak{p}$, let $e^X e^Y = e^Z k$ for $Z \in \mathfrak{p}$ and $k \in K$. Then $X + Y \prec Z$.*

Lemma 3.3. ([22, Theorem 4.3]) *If $X, Y \in \mathfrak{p}$, then*

$$e^{rX/2} e^{rY} e^{rX/2} \prec (e^{X/2} e^Y e^{X/2})^r, \quad 0 \leqslant r \leqslant 1, \tag{3.1}$$

$$(e^{X/2} e^Y e^{X/2})^r \prec e^{rX/2} e^{rY} e^{rX/2}, \qquad r \geqslant 1. \tag{3.2}$$

For $x, y \in G$, we will write $x \overset{c}{=} y$ if x and y are conjugate, i.e., if $y = gxg^{-1}$ for some $g \in G$. Note that if $x \overset{c}{=} x'$ and if $x \prec y$ (resp., $y \prec x$), then $x' \prec y$ (resp., $y \prec x'$).

Lemma 3.4. *If $p, q \in P$, then*

$$p^r q^r \prec (pq)^r, \quad 0 \leqslant r \leqslant 1, \quad \text{and} \quad (pq)^r \prec p^r q^r, \qquad r \geqslant 1.$$

Proof. Let $p = e^X$ and $q = e^Y$ for some $X, Y \in \mathfrak{p}$. For $0 \leqslant r \leqslant 1$,

$$\begin{aligned} p^r q^r &= e^{rX} e^{rY} \overset{c}{=} e^{-rX/2} \left(e^{rX} e^{rY}\right) e^{rX/2} = e^{rX/2} e^{rY} e^{rX/2} \\ &\prec (e^{X/2} e^Y e^{X/2})^r \quad \text{by (3.1)} \\ &\overset{c}{=} (e^X e^Y)^r = (pq)^r. \end{aligned}$$

That is, $p^r q^r \prec (pq)^r$ for $0 \leqslant r \leqslant 1$. The other inequality can be established similarly. ■

Now we extend Theorem 1.1 in terms of Kostant's pre-order.

Theorem 3.5. *Let $p, q \in P$ and $t \in [0,1]$. If $0 < r \leqslant 1$, we have*

$$\begin{aligned} p\#_t q &\prec e^{(1-t)\log p + t\log q} \\ &\prec \left(p^{(1-t)r} q^{tr}\right)^{1/r} \stackrel{c}{=} (q^{tr/2} p^{(1-t)r} q^{tr/2})^{1/r} \stackrel{c}{=} (p^{(1-t)r/2} q^{tr} p^{(1-t)r/2})^{1/r} \\ &\prec p^{1-t} q^t. \end{aligned} \tag{3.3}$$

If $r > 1$, we have

$$\begin{aligned} p\#_t q &\prec e^{(1-t)\log p + t\log q} \\ &\prec p^{1-t} q^t \\ &\prec \left(p^{(1-t)r} q^{tr}\right)^{1/r} \stackrel{c}{=} (q^{tr/2} p^{(1-t)r} q^{tr/2})^{1/r} \stackrel{c}{=} (p^{(1-t)r/2} q^{tr} p^{(1-t)r/2})^{1/r}. \end{aligned}$$

Proof. Let $\pi : G \to \mathrm{GL}(V)$ be any finite dimensional representation of G. There exists an inner product on V such that $\pi(z)$ is positive definite for all $z \in P$ ([14, p.435]). Then we have

$$\begin{aligned} |\pi(p\#_t q)| &= |\pi(p^{1/2}(p^{-1/2} q p^{-1/2})^t p^{1/2})| \\ &= |\pi(p)^{1/2}(\pi(p)^{-1/2}\pi(q)\pi(p)^{-1/2})^t \pi(p)^{1/2}| \\ &= |\pi(p)\#_t \pi(q)| \\ &\leqslant |e^{(1-t)\log \pi(p) + t\log \pi(q)}| \qquad \text{by Theorem 1.1} \\ &= |e^{(1-t)d\pi(\log p) + t d\pi(\log q)}| \qquad \text{by } \log\circ\pi = d\pi\circ\log \\ &= |e^{d\pi((1-t)\log p + t\log q)}| \\ &= |\pi(e^{(1-t)\log p + t\log q})| \qquad \text{by } \exp\circ d\pi = \pi\circ\exp, \end{aligned}$$

where $d\pi$ denotes the differential of π. So the relation (3.3) is valid by Theorem 2.2.

Similarly, $e^{(1-t)\log p + t\log q} \prec (q^{tr/2} p^{(1-t)r} q^{tr/2})^{1/r}$ for all $r > 0$ is established by applying the corresponding majorization relation in Theorem 1.1. Other inequalities follow similarly from Lemma 3.1 and Lemma 3.4. ■

Many inequalities for $\mathbb{P}_n$ are also true for the symmetric space P and can be proved by the technique in the proof of Theorem 3.5. For examples, Theorem 3.6 and Theorem 3.7 below are extensions of [2, Theorem 2.1] and [2, Theorem 3.1], respectively.

Theorem 3.6. *Let $p, q \in P$ and $0 \leqslant t \leqslant 1$. Then the following statements are true:*

$$p^r \#_t q^r \prec (p\#_t q)^r, \quad r \geqslant 1,$$
$$(p\#_t q)^r \prec p^r \#_t q^r, \quad 0 < r \leqslant 1,$$
$$(p^r \#_t q^r)^{1/r} \prec (p^s \#_t q^s)^{1/s}, \quad 0 < s \leqslant r.$$

Theorem 3.7. *For any* $p, q \in P$,

$$\{p^{r-s}\#_t(p^{(1-t)s/2t}q^r p^{(1-t)s/2t})\}^{1/r} \prec (p^{(1-t)/2}q^t p^{(1-t)/2}$$

whenever $0 < t \leqslant 1$, $r \geqslant 0$, *and* $s \leqslant \min\{t, tr\}$.

The following result extends [9, Lemma 3.3], whose proof works in our context as well.

Theorem 3.8. *For any* $0 \leqslant t \leqslant 1$ *and* $H, K \in \mathfrak{p}$,

$$\exp\{(1-t)H + tK\} = \lim_{r\to 0}\{\exp(rH)\#_t \exp(rK)\}^{1/r}.$$

3.2. Geodesic triangle. We will consider a geodesic triangle formed by three arbitrary points $o, r, s \in G/K$. Because the geometry on G/K is invariant under the G-action, we may assume $o = K$ is one point, and let r and s be the other two points.

For $r = gK$, let $K_r = gKg^{-1}$ be the subgroup of G that fixes r, and let $\mathfrak{p}_r = \mathrm{Ad}\,(g)\mathfrak{p} \subset \mathfrak{l}$. As in [14, p.449-450], let $x(r, s)$ be the unique vector in $\mathfrak{p}_r$ such that $s = e^{x(r,s)}r$. Note that $x(s, r) = -x(r, s)$ and by the G-invariance,

$$\mathrm{Ad}\,(h)x(r, s) = x(hr, hs), \quad \text{for all } h \in G. \tag{3.4}$$

Now assume that the inner product on $\mathfrak{p}$ is the Killing form B restricted to $\mathfrak{p}$. For $X \in \mathfrak{l}$ with $X = \mathrm{Ad}\,(g)Y$, $g \in G$ and $Y \in \mathfrak{p}$, define $\|X\| = \|Y\|$. This is well defined because B is $\mathrm{Ad}\,G$-invariant. Because the geodesics from $o = K$ take the form $e^{tY}o$ for $Y \in \mathfrak{p}$, the geodesics from $r = gK$ take the form $e^{tX}r = ge^{tY}o$ for $X \in \mathfrak{p}_r$, where $Y = \mathrm{Ad}\,(g^{-1})X \in \mathfrak{p}$. This implies $\|X\| = d(r, e^X r)$ (the Riemannian distance between r and $e^X r$), and hence,

$$\|x(r, s)\| = d(r, s). \tag{3.5}$$

As a general property of nonpositive curvature ([15, IX.Corollary 3.10]),

$$\|x(o, s) - x(o, r)\| \leqslant \|x(r, s)\|, \tag{3.6}$$

in contrast to $x(o, s) - x(o, r) = x(r, s)$ on a Euclidean space. However, the geodesic arc $t \mapsto e^{tx(r,s)}r$ with $t \in [0, 1]$ contains more information than merely its length $\|x(r, s)\|$. In fact, Kostant obtained the following interesting result via the Golden-Thompson-Kostant inequality (Lemma 3.1). We will give a simpler proof based on Corollary 3.2.

Theorem 3.9. ([14, Theorem 7.2]) *Let* $o, r, s \in G/K$. *Then* $x(o, s) - x(o, r) \prec x(r, s)$.

Proof. By the definition of $x(r, s)$,

$$e^{x(o,s)}o = s = e^{x(r,s)}r = e^{x(r,s)}e^{x(o,r)}o.$$

Let $X := x(o,r)$, $Y := x(o,s)$ and $Z := x(r,s)$. Then $e^Y = e^Z e^X k$ for some $k \in K$. This is $e^Y = e^X \exp[\mathrm{Ad}\,(e^{-X})Z]k$, and hence $e^{-X}e^Y = \exp[\mathrm{Ad}\,(e^{-X})Z]k$. Because $\mathrm{Ad}\,(e^{-X})Z \in \mathfrak{p}$, by Corollary 3.2, we have $-X+Y \prec \mathrm{Ad}\,(e^{-X})Z$, which implies $-X+Y \prec Z$. ∎

By the identification of G/K and P via map ψ, the points o, r and s in G/K correspond to e, p and q in P with $r = p^{1/2}K$ and $s = q^{1/2}K$. By (7.2.6) in [14],

$$e^{2x(r,s)} = qp^{-1}. \tag{3.7}$$

Recall that $p\#_t q$, $0 \leqslant t \leqslant 1$, is the geodesic in P joining p to q by Proposition 2.3. Regarded as a geodesic in G/K, this may also be denoted as $r\#_t s$.

Theorem 3.10. *Let $o, r, s \in G/K$. Then $x(o\#_t r, o\#_t s) \prec t\,x(r,s)$ for $0 \leqslant t \leqslant 1$.*

Proof. Because of (3.4), we may assume that $o = K$. Write $r = p^{1/2}K$ and $s = q^{1/2}K$ with $p, q \in P$. By (2.3), $o\#_t r = p^{t/2}K$ and $o\#_t s = q^{t/2}K$. So by (3.7), we have

$$\exp 2x(o\#_t r, o\#_t s) = q^t p^{-t}, \quad \exp 2tx(r,s) = (qp^{-1})^t.$$

Since $q^t p^{-t} \prec (qp^{-1})^t$ by Lemma 3.4, we have $x(o\#_t r, o\#_t s) \prec t\,x(r,s)$. ∎

Taking norm in Theorem 3.10, we obtain

$$d(o\#_t r, o\#_t s) \leqslant td(r,s). \tag{3.8}$$

This is precisely the distance convexity (1.3) when $A = C = o$, $B = r$ and $D = s$. In fact, Theorem 3.10 may be regarded as a stronger form of the distance convexity (1.3) on G/K because the latter can be derived from (3.8) as follows. Also see [5, Theorem 6.1.12].

Corollary 3.11. *Let $r, s, r', s' \in G/K$. For each $t \in [0,1]$, we have*

$$d(r'\#_t r, s'\#_t s) \leqslant (1-t)d(r',s') + td(r,s).$$

Proof. Consider the geodesic triangle with vertices r', r, s. By (3.8), we have

$$d(r'\#_t r, r'\#_t s) \leqslant td(r,s).$$

Similarly, by considering the geodesic triangle with vertices s, r', s', we have

$$d(r'\#_t s, s'\#_t s) = d(s\#_{1-t} r', s\#_{1-t} s') \leqslant (1-t)d(r',s').$$

By the triangular inequality, we get

$$d(r'\#_t r, s'\#_t s) \leqslant d(r'\#_t r, r'\#_t s) + d(r'\#_t s, s'\#_t s) \leqslant td(r,s) + (1-t)d(r',s'). \ ∎$$

Remark 3.12. Both $p\#_t q$ and $e^{(1-t)\log p + t\log q}$ are curves in P joining p and q. The former is the unique geodesic and the latter is the exponential of the line segment $(1-t)\log p + t\log q$ in the Euclidean space $\mathfrak{p}$. Because of (3.7) and (3.5),

$d(p,q) = \|\frac{1}{2}\log qp^{-1}\|$ is the Riemannian distance between $p, q \in P$. The relation $p\#_t q \prec \exp[(1-t)\log p + t\log q]$ of (3.3) is equivalent to

$$\log(p\#_t q) \prec (1-t)\log p + t\log q,$$

which implies $d(p\#_t q, e) \leqslant (1-t)d(p,e) + td(q,e)$ after taking norm. Because of (3.4), we have for all $p, q, r \in P$

$$d(p\#_t q, r) \leqslant (1-t)d(p,r) + td(q,r). \tag{3.9}$$

This is precisely (1.5), so (3.3) may be regarded as a stronger form of this special case of distance convexity. Needless to say, (3.3) is related to the pre-order of G and thus does not exist in general for Riemannian manifolds of nonpositive curvature. We finally note that (3.3) is equivalent to

$$x(o, r\#_t s) \prec (1-t)x(o,r) + tx(o,s), \quad \text{for all } o, s, r \in G/K. \tag{3.10}$$

This is because, under the assumption of $o = K$, by definition $2x(o,r) = \log p$ for any $r = p^{1/2}K$ and thus, on the group level, (3.10) means

$$\exp x(o, r\#_r s) \prec \exp[(1-t)x(o,r) + tx(o,s)].$$

One may compare (3.10) with [5, Exercise 6.1.13].

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