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Quadratic Leibniz Algebras

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Abstract. Left (or right) Leibniz algebras endowed with symmetric non-degenerate and associative bilinear forms (called quadratic Leibniz algebras) are investigated. In particular, we prove that left (resp. right) Leibniz algebras that carry this structure are also right (resp. left) Leibniz algebras. Moreover, we construct several examples of this type of algebras. Next, we prove that any solvable quadratic Leibniz algebra is a T^*-extension (see M. Bordemann, Nondegenerate associative bilinear forms on nonassociative algebras, Acta Math. Univ. Com. LXIV 2 (1997), 151–201) of a solvable Lie algebra in the category of Leibniz algebras. In addition, we reduce the study of quadratic Leibniz algebras to that of quadratic Lie algebras by introducing some extensions of Leibniz algebras. Finally, we give an inductive description of quadratic Leibniz algebras by using T^*-extensions and double extensions (central extension followed by generalized semi-direct product).
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Introduction

In 1993, J.L. Loday introduced Leibniz algebras which are a generalization of Lie algebras ([14], [15]). More precisely, let $\mathfrak{L}$ be a vector space endowed with a product $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \longmapsto \mathfrak{L}$ such that for all $x, y, z \in \mathfrak{L}$, $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$ (resp. $[x,[y,z]] = [[x,y],z] - [[x,z],y]$). Then, $(\mathfrak{L}, [\ ,\ ])$ is called a left Leibniz algebra (resp. a right Leibniz algebra). If $\mathfrak{L}$ is, at the same time, a left and a right Leibniz algebra, then we say that $\mathfrak{L}$ is a symmetric Leibniz algebra [11]. In the last years, the theory of Leibniz algebras has been extensively studied. Many results on the theory of Lie algebras have been generalized to the case of Leibniz algebras ([6], [7], [8], [9], [10], [11], [12], [13], [17]).

The goal of this paper is the study of the structure of finite dimensional (Left or right) Leibniz algebra $(\mathfrak{L}, [\ ,\ ])$ over a commutative field $\mathbb{K}$ of characteristic zero endowed with a symmetric non-degenerate bilinear form B such that B is associative (ie. $B([x,y],z) = B(x,[y,z]), \forall x, y, z \in \mathfrak{L}$). We call $(\mathcal{L}, B)$ a quadratic Leibniz algebra.

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It is well known that the existence of symmetric non-degenerate and invariant bilinear forms on a non-associative algebra $(A, .)$ is a very important tool to study the structure of A (for examples: Killing form on semi-simple Lie algebras, Albert form on semi-simple Jordan algebras, invariant (resp. associative) scalar products on Lie (resp. Jordan) algebras[16], [1], [4]). Hence it seems to be interesting to investigate the structure of quadratic Leibniz algebras. The main tools used for our purpose are T^*-extension and double extension.

Quadratic Lie algebras were described by A. Medina and Ph. Revoy in [16] using the concept of double extension of quadratic Lie algebras. The notion of T^*-extension was introduced by M. Bordemann in [4] in order to study structures of non-associative algebras that carry symmetric non-degenerate associative bilinear forms. He proved that a non-associative algebra A with a symmetric non-degenerate and associative bilinear form is a T^*-extension (or a non-degenerate ideal of codimension 1 of a T^*-extension) of a non-associative algebra H if and only if A contains a totally isotropic ideal of dimension $[\frac{Dim(A)}{2}]$, where $[\frac{Dim(A)}{2}]$ is the integer part of $\frac{Dim(A)}{2}$. He showed also two interesting results, more precisely:

1. All nilpotent associative algebras with symmetric non-degenerate associative bilinear forms admits a totally isotropic ideal of dimension $[\frac{Dim(A)}{2}]$.

2. All solvable Lie algebras over a field of caracteristic zero endowed with symmetric non-degenerate associative bilinear forms admits a totally isotropic ideal of dimension $[\frac{Dim(A)}{2}]$.

A natural question arise: What about the situation in the case of quadratic solvable (left or right) Leibniz algebras over a commutative field of characteristic zero?

In the second section, we prove that every quadratic Leibniz algebras $\mathfrak{L}$ is symmetric. Let us note, in particular, that symmetric Leibniz algebras appear in the study of some bi-invariant connections on Lie groups [3]. Next, we construct several interesting examples of quadratic Leibniz algebras. In the third section, we study quadratic Leibniz algebras via the notion of T^*-extension and we give an answer of the above question. More precisely, we prove that every quadratic solvable Leibniz algebra $\mathfrak{L}$ over a field of characteristic zero admits a totally isotropic ideal of dimension $[\frac{Dim(\mathfrak{L})}{2}]$ which contain the Leibniz kernel of $\mathfrak{L}$. Consequently, all quadratic solvable Leibniz algebras over a field of characteristic zero are T^*-extension (or a non-degenerate ideal of codimension 1 of a T^*-extension) of a Lie algebra in the category of symmetric Leibniz algebras.

In order to have more information on the structure of quadratic (non necessarily solvable) Leibniz algebras we generalize, in the fourth and the fifth sections, the concept of double extension to quadratic Leibniz algebras. More precisely, we introduce a double extension of quadratic Leibniz algebra by one dimensional Lie algebra. Next, we give some interesting results on the structure of this type of algebras. What allowed us to reduce the study of quadratic Leibniz algebras to that of quadratic Lie algebras. Finally, by using our results and those of A. Medina and Ph. Revoy [16], we give an inductive description of quadratic Leibniz algebras.

1. Definitions and preliminary results

Definition 1.1. Let $(\mathfrak{L}, .)$ be a non-associative algebra. Then, for all $x \in \mathfrak{L}$ we define the endomorphisms L_x, R_x of $\mathfrak{L}$ by $L_x(y) := x.y$, $R_x(y) := y.x$, $\forall y \in \mathfrak{L}$. The map L_x (resp . R_x) is called the left multiplication by x (resp. right multiplication by x).

Definition 1.2. Let $\mathfrak{L}$ be a vector space and let $[\ ,\ ] : \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ be a bilinear map on $\mathfrak{L}$.

1. If L_x is a derivation of $\mathfrak{L}$, $\forall x \in \mathfrak{L}$, ie. $[x, [y, z]] = [[x, y], z]+[y, [x, z]], \forall x, y, z \in \mathfrak{L}$, then $\mathfrak{L}$ is called a left Leibniz algebra.
2. If R_x is a derivation of $\mathfrak{L}$, $\forall x \in \mathfrak{L}$, ie. $[x, [y, z]] = [[x, y], z]-[[x, z], y], \forall x, y, z \in \mathfrak{L}$, then $\mathfrak{L}$ is called a right Leibniz algebra.

Remark 1.3. If $(\mathfrak{L}, [\ ,\ ])$ is a right (resp. left) Leibniz algebra , then $\mathfrak{L}$ endowed with the product $\{\ ,\ \} : (x, y) \longmapsto \{x, y\} = [y, x]$ is a left (resp. right)Leibniz algebra.

Proposition 1.4. *Let $(\mathfrak{L}, [\ ,\ ])$ be a left Leibniz algebra. Then, $(\mathfrak{L}, [\ ,\ ])$ is a right Leibniz algebra if and only if $[x, [y, z]] = -[[y, z], x], \forall x, y, z \in \mathfrak{L}$.*

Proof. Since $\mathfrak{L}$ is a left Leibniz algebra, then $[[y, z], x] - [[y, x], z] - [y, [z, x]] = [[y, z], x] - [[y, x], z] - [[y, z], x] - [z, [y, x]] = -\Big([[y, x], z] + [z, [y, x]]\Big), \forall x, y, z \in \mathfrak{L}$. Which proves the Proposition. ∎

Definition 1.5. Let $(\mathfrak{L}, [\ ,\ ])$ be a (left or right) Leibniz algebra .

1. We denote by $[U, V]$ the vector space spanned by the set $\{[u, v], u \in U, v \in V\}$, where U and V are two sub-spaces of $\mathfrak{L}$.
2. If U is a sub-space of $\mathfrak{L}$, then U is said to be an ideal of $\mathfrak{L}$ if $[U, \mathfrak{L}] \subseteq \mathfrak{L}$ and $[\mathfrak{L}, U] \subseteq \mathfrak{L}$.

Corollary 1.6. *If $(\mathfrak{L}, [\ ,\ ])$ is a left and a right Leibniz algebra, then the vector space $\mathfrak{L}^2 = [\mathfrak{L}, \mathfrak{L}] := span\{[x, y] : x, y \in \mathfrak{L}\}$ endowed with $[\ ,\ ]_{|[\mathfrak{L},\mathfrak{L}]\times[\mathfrak{L},\mathfrak{L}]}$ is a Lie algebra.*

Definition 1.7. Let $(\mathfrak{L}, [\ ,\ ])$ be a (left or right) Leibniz algebra. The vector space $\mathcal{I}_{\mathfrak{L}}$ spanned by the set $\{[x, x], x \in \mathfrak{L}\}$ is called the Leibniz kernel of $\mathfrak{L}$.

Proposition 1.8. *[8] Let $(\mathfrak{L}, [\ ,\ ])$ be a (left or right) Leibniz algebra. Then, $\mathcal{I}_{\mathfrak{L}}$ is an ideal of $\mathcal{L}$. Moreover, if $(\mathfrak{L}, [\ ,\ ])$ is a left (resp. right) Leibniz algebra, then $[\mathcal{I}_{\mathfrak{L}}, \mathfrak{L}] = \{0\}$ (resp. $[\mathfrak{L}, \mathcal{I}_{\mathfrak{L}}] = \{0\}$).*

Remark 1.9. Let $(\mathfrak{L}, [\ ,\ ])$ be a (left or right) Leibniz algebra. It is clear that $\mathfrak{L}$ is a Lie algebra if and only if $\mathcal{I}_{\mathfrak{L}} = \{0\}$. Therefore, the quotient algebra $\mathfrak{L}/\mathcal{I}_{\mathfrak{L}}$ is a Lie algebra.

In the following, we recall some results about representations of left and right Leibniz algebras.

Definition 1.10. Let $\mathfrak{L}$ be a non-associative algebra, V be a vector space and $r, l : \mathfrak{L} \to End(V)$ be two linear maps.

(i) If $\mathfrak{L}$ is a left Leibniz algebra, then we say that (r, l) is a left representation of $\mathfrak{L}$ in V if for all $x, y \in \mathfrak{L}$:

$$l([x,y]) = [l(x), l(y)];\ r([x,y]) = r(y)r(x) + l(x)r(y);\ r([x,y]) = [l(x), r(y)].$$

(ii) If $\mathfrak{L}$ is a right Leibniz algebra, then we say that (r, l) is a right representation of $\mathfrak{L}$ in V if for all $x, y \in \mathfrak{L}$:

$$l[x,y]) = [r(y), l(x)];\ l([x,y]) = l(x)l(y) + r(y)l(x);\ r([x,y]) = [r(y), r(x)].$$

Example 1.11. Let $\mathfrak{L}$ be a left (resp. right) Leibniz algebra. Then, we consider the maps $L : \mathfrak{L} \to End(\mathfrak{L})$ and $R : \mathfrak{L} \to End(\mathfrak{L})$ defined by: $L(x) := L_x, R(x) := R_x, \forall x \in \mathfrak{L}$. Therefore, (R, L) is a left representation (resp. a right representation) of $\mathfrak{L}$ in $\mathfrak{L}$ called the left (resp. right) adjoint representation of $\mathfrak{L}$.

Proposition 1.12. *Let $\mathfrak{L}$ be a left (resp. right) Leibniz algebra and $r, l : \mathfrak{L} \to End(V)$ be two linear maps. Then, the vector space $\mathfrak{L}_1 = \mathfrak{L} \oplus V$ endowed with the product defined by:*

$$[x + u, y + v] = [x, y] + l(x)v + r(y)u, \forall x, y \in \mathfrak{L}, u, v \in V$$

is a left (resp. right) Leibniz algebra if and only if (r, l) is a left (resp. right) representation of $\mathfrak{L}$ in V.

Remark 1.13. Let $\mathfrak{L}$ be a left (resp. right) Leibniz algebra, V be a vector space and (r, l) a left representation (resp. right representation) of $\mathfrak{L}$ in V. Let us consider two linear maps $l^*, r^* : \mathfrak{L} \to End(V^*)$ defined by

$$l^*(x)(f) = f \circ r(x), \quad r^*(x)(f) = f \circ l(x),\ \forall x \in \mathfrak{L}, f \in V^*.$$

In general, (r^*, l^*) is not a left nor a right representation of $\mathfrak{L}$ in V^*.

Proposition 1.14. *Let $\mathfrak{L}$ be a left (resp. right) Leibniz algebra and (R, L) ft (resp. right) be the adjoint representation of $\mathfrak{L}$. Let us consider the linear maps $L^*, R^* : \mathfrak{L} \to End(\mathfrak{L}^*)$ defined by:*

$$L^*(x)(f) = f \circ R(x), \quad R^*(x)(f) = f \circ L(x),\ \forall x \in \mathfrak{L}, f \in \mathfrak{L}^*.$$

(R^*, L^*) is a left (resp. right) representation of $\mathfrak{L}$ in $\mathfrak{L}^*$ if and only if $\mathfrak{L}$ is a right (resp. left) Leibniz algebra.

Proof. Suppose that $(\mathfrak{L}, [\,,\,])$ is a left Leibniz algebra. A simple calculation proves that (L^*, R^*) is a left representation of $\mathfrak{L}$ in $\mathfrak{L}^*$ if and only if

$[x, [y, z]] = -[[y, z], x]$ and $[y, [x, z]] = -[y, [z, x]] \forall x, y, z \in \mathfrak{L}$. Proposition 1.4 gives the result. ■

The previous Proposition leads us to study the notion of symmetric Leibniz algebras introduced in [11].

Definition 1.15. [11]
If $\mathfrak{L}$ is a left and a right Leibniz algebra, then $\mathfrak{L}$ is called a symmetric Leibniz algebra.

Examples 1.16. [8] If $(\mathfrak{L} := \langle x, y\rangle, [\ ,\ ])$ is a two dimensional symmetric Leibniz algebra, then $\mathfrak{L}$ is isomorphic to one of the following Leibniz algebras:

1. The two dimensional abelian Lie algebra.
2. The two dimensional non abelian Lie algebra , endowed with the product defined by:
   $[x, y] = y$.
3. The symmetric Leibniz algebra $\mathfrak{L}_1$, endowed with the product defined by:
   $[x, x] = y, [x, y] = [y, x] = [y, y] = 0$.

In the following, we introduce the representation of symmetric Leibniz algebras and give some related results.

Definition 1.17. Let $\mathfrak{L}$ be a symmetric Leibniz algebra, V be a vector space and $r, l : \mathfrak{L} \to End(V)$ be two linear maps. Then, we say that (r, l) is a representation of $\mathfrak{L}$ in V if (r, l) is a left and a right representation of $\mathfrak{L}$ in V. We denote by $Rep(\mathfrak{L}, V)$ the set of all representations of $\mathfrak{L}$ in V.

Example 1.18. (R, L) (see. Example 1.11) is a representation of $\mathfrak{L}$ in $\mathfrak{L}$ called the adjoint representation of $\mathfrak{L}$.

Remark 1.19. Let $\mathfrak{L}$ be a symmetric Leibniz algebra, V be a vector space and $r, l : \mathfrak{L} \to End(V)$ be two linear maps. Then, The vector space $\overline{\mathfrak{L}} = \mathfrak{L} \oplus V$ endowed with the product defined by: $[x+u, y+v] = [x, y]+l(x)v+r(y)u, \forall x, y \in \mathfrak{L}, u, v \in V$, is a symmetric Leibniz algebra if and only if $(r, l) \in Rep(\mathfrak{L}, V)$.

Proposition 1.20. *Let $\mathfrak{L}$ be a symmetric Leibniz algebra, V be a vector space and $r, l : \mathfrak{L} \to End(V)$ be two bilinear maps. Then, (r, l) is a representation of $\mathfrak{L}$ in V if and only if for all $x, y \in \mathfrak{L}$:*

$$l([x, y]) = [l(x), l(y)],\ r([x, y]) = [l(x), r(y)],\ r([x, y]) = r(y)r(x) + l(x)r(y)$$
$$l(x)l(y) = -r(x)l(y), l(x)r(y) = -r(x)r(y), r([y, x]) = -l([y, x]).$$

In this case, we say that V is an $\mathfrak{L}$-module.

Proof. By the previous Remark $(r, l) \in Rep(\mathfrak{L}, V)$ if and only if the vector space $\overline{\mathfrak{L}} = \mathfrak{L} \oplus V$ endowed with the product defined by: $[x + u, y + v] = [x, y] + l(x)v + r(y)u, \forall x, y \in \mathfrak{L}, u, v \in V,$ is a symmetric Leibniz algebra. That is, $\overline{\mathfrak{L}}$ is a left Leibniz algebra satisfying the condition

$$[x+u,[y+v,z+w]] = -[[y+v,z+w],x+u], \forall x,y,z \in \mathfrak{L}, u,v,w \in V$$

(see. Proposition 1.4). So, $\overline{\mathfrak{L}}$ is a left Leibniz algebra if and only if (r,l) is left representation of $\mathfrak{L}$ in V. Moreover, a simple computation proves that $[x+u,[y+v,z+w]] = -[[y+v,z+w],x+u], \forall x,y,z \in \mathfrak{L}, u,v,w \in V$ if and only if $l(x)l(y) = -r(x)l(y)$; $l(x)r(y) = -r(x)r(y)$; $r([y,x]) = -l([y,x]), \forall x,y \in \mathfrak{L}$. ∎

Corollary 1.21. *Let $(\mathfrak{L},[\ ,\ ])$ be a symmetric Leibniz algebra. Then, for all $x,y,z \in \mathfrak{L}$*

$$L([x,y]) = [L(x),L(y)];\ R([x,y]) = [R(y),R(x)];\ L([x,y]) = R([y,x])$$
$$L(x)L(y) = -R(x)L(y);\ R(x)R(y) = -L(x)R(y);\ L(x)L(y) = R(x)R(y).$$

Proof. Since $(R,L) \in Rep(\mathfrak{L},\mathfrak{L})$, then by the previous Proposition we get: $L([x,y]) = [L(x),L(y)]; R([x,y]) = [L(x),R(y)]; R([x,y]) = R(y)R(x) + L(x)R(y)$ $L(x)L(y) = -R(x)L(y); L(x)R(y) = -R(x)R(y); R([y,x]) = -L([y,x])$. Therefore, $R([x,y]) = R(y)R(x) + L(x)R(y) = R(y)R(x) - R(x)R(y) = [R(y),R(x)]$. Moreover, since $[\mathfrak{L},\mathcal{I}_{\mathfrak{L}}] = \{0\}$, then $L(x)L(y) = -L(x)R(y)$. So, $L(x)L(y) = R(x)R(y)$. Consequently, $L([x,y]) = [L(x),L(y)] = [R(x),R(y)] = R([y,x])$. ∎

Example 1.22. (Co-adjoint representation of symmetric Leibniz algebras)

If $\mathfrak{L}$ is a symmetric Leibniz algebra, then by Proposition 1.14, (R^*,L^*) is a representation of $\mathfrak{L}$ in $\mathfrak{L}^*$ called the co-adjoint representation of $\mathfrak{L}$, ,where $L^*,R^* : \mathfrak{L} \to End(\mathfrak{L}^*)$ are the linear maps defined by:

$$L^*(x)(f) = f \circ R(x), \quad R^*(x)(f) = f \circ L(x),\ \forall x \in \mathfrak{L}, f \in \mathfrak{L}^*.$$

2. Quadratic Leibniz algebras

This section is devoted to study the structures of Leibniz algebras endowed with an associative scalar product. Moreover, we construct many interesting examples of quadratic Leibniz algebras.

Definition 2.1. Let $(\mathfrak{L},[\ ,\ ])$ be a left (or a right) Leibniz algebra and $B\colon \mathfrak{L}\times\mathfrak{L} \to \mathbb{K}$ be a bilinear form on $\mathfrak{L}$. If $B([x,y],z) = B(x,[y,z]), \forall x,y,z \in \mathfrak{L}$, then we say that B is associative (or invariant) on $\mathfrak{L}$. If moreover B is symmetric and non-degenerate, then B is called an associative scalar product on $\mathfrak{L}$. In this case, $(\mathfrak{L},B)$ is called a quadratic Leibniz algebra.

Example 2.2. Let $\mathfrak{L}_1 := \langle x,y\rangle$, $[x,x] = y$ be the two dimensional non Lie Leibniz algebra given in Examples 1.16. Then, $\mathfrak{L}_1$ endowed with the symmetric bilinear form B defined by $B(x,y) = 1$, $B(x,x) = B(y,y) = 0$ is a quadratic Leibniz algebra.

Remark 2.3. There exists non-quadratic symmetric Leibniz algebras. For example, the vector space $\mathfrak{L} := \langle x,y,z\rangle$ endowed with the product $[\ ,\ ] : \mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}$ defined by $[x,x] = y$, $[x,z] = -[z,x] = y$ is a non Lie symmetric Leibniz algebra. Moreover, $\mathfrak{L}$ is not a quadratic Leibniz algebra.

Theorem 2.4. *Let $(\mathfrak{L}, [\ ,\ ])$ be a left (resp. right) Leibniz algebra. If $\mathfrak{L}$ is quadratic, then $\mathfrak{L}$ is symmetric.*

Proof. Suppose that $(\mathfrak{L}, [\ ,\ ])$ is a left Leibniz algebra. Then, $[\mathcal{I}_{\mathfrak{L}}, \mathfrak{L}] = \{0\}$. So, $0 = B\big([[x,y]+[y,x],z],u\big) = B\big(x,[y,[z,u]]+[[z,u],y]\big), \forall x,y,z,u \in \mathfrak{L}$. The fact that B is non-degenerate implies that $[y,[z,u]]+[[z,u],y] = 0, \forall y,z,u \in \mathfrak{L}$. Consequently, $\mathfrak{L}$ is a right Leibniz algebra (Proposition 1.4). By the same way, we prove the result for right Leibniz algebra. ■

Proposition 2.5. *Let $\mathfrak{L}$ be a symmetric Leibniz algebra. Then, $\mathfrak{L}$ is quadratic if and only if the adjoint and the co-adjoint representations of $\mathfrak{L}$ are equivalent.*

Proof. Suppose that $\mathfrak{L}$ is quadratic. Let B be an associative scalar product on $\mathfrak{L}$. Then, we consider the map $\phi : \mathfrak{L} \to \mathfrak{L}^*; \phi(x)(y) = B(x,y), \forall x,y \in \mathfrak{L}$. Since B is non-degenerate, then ϕ is an isomorphism of vector spaces. Moreover, the fact that B is associative implies that $\Big(\phi \circ L_x(y)\Big)(z) = B\Big([x,y],z\Big) = B\Big(y,[z,x]\Big) = \phi(y) \circ R_x(z) = \Big(L_x^* \circ \phi(y)\Big)(z), \forall x,y,z \in \mathfrak{L}$. Similarly, we prove that $\Big(\phi \circ R_x(y)\Big)(z) = \Big(R_x^* \circ \phi(y)\Big)(z)$.
Conversely , if there exists an isomorphism of vector spaces $\phi : \mathfrak{L} \to \mathfrak{L}^*$ such that

$$\Big(\phi \circ L_x(y)\Big)(z) = \Big(L_x^* \circ \phi(y)\Big)(z) \text{ and } \Big(\phi \circ R_x(y)\Big)(z) = \Big(R_x^* \circ \phi(y)\Big)(z).$$

Then, we define the bilinear form $T : \mathfrak{L} \times \mathfrak{L} \to \mathbb{K}$ by $T(x,y) = \phi(x)(y), \forall x,y \in \mathfrak{L}$. Since ϕ est invertible, then T is non-degenerate. Moreover,

$$T\Big([x,y],z\Big) = \phi \circ R_y(x)(z) = R_y^* \circ \phi(x)(z) = \phi(x) \circ L_y(z) = T(x,[y,z]), \forall x,y,z \in \mathfrak{L}.$$

So, the bilinear form T is associative but T is not necessarily symmetric. Let us consider the symmetric (resp. anti-symmetric) part T_s (resp. T_a) of T defined by: $T_s(x,y) = \frac{1}{2}(T(x,y)+T(y,x))$, (resp. $T_a(x,y) = \frac{1}{2}(T(x,y)-T(y,x))$, $\forall x,y \in \mathfrak{L}$. Since $\phi \circ L_x(y)(z) = L_x^* \circ \phi(y)(z) = \phi(y) \circ R_x(z), \forall x,y,z \in \mathfrak{L}$, then $T([x,y],z) = T(y,[z,x]), \forall x,y,z \in \mathfrak{L}$. Therefore, T is associative if and only if T_s and T_a are associative. Let $\mathfrak{L}_s = \{x \in \mathfrak{L}; T_s(x,y) = 0, \forall y \in \mathfrak{L}\}$ and $\mathfrak{L}_a = \{x \in \mathfrak{L}; T_a(x,y) = 0, \forall y \in \mathfrak{L}\}$. Since T is non-degenerate, then $\mathfrak{L}_s \cap \mathfrak{L}_a = \{0\}$. For all $x,y,z \in \mathfrak{L}$, we get $T_a([x,y],z) = T_a(x,[y,z]) = -T_a(y,[z,x]) = T_a(z,[x,y]) = -T_a([x,y],z)$. Consequently, $T_a([x,y],z) = 0, \forall x,y,z \in \mathfrak{L}$. So, $[\mathfrak{L},\mathfrak{L}] \subseteq \mathfrak{L}_a$. Since $\mathfrak{L}_s$ is an ideal of $\mathcal{L}$, then $[\mathfrak{L}_s,\mathfrak{L}_s] \subseteq \mathfrak{L}_s \cap \mathfrak{L}_a = \{0\}$.

It is clear that $\mathfrak{L} = \mathcal{U} \oplus \mathfrak{L}_s$, where $\mathcal{U}$ is a sub-vector space of $\mathfrak{L}$ such that $\mathfrak{L}_a \subseteq \mathcal{U}$. Since $\mathcal{U}$ and $\mathfrak{L}_s$ are ideals of $\mathfrak{L}$, then $[\mathcal{U},\mathfrak{L}_s] = [\mathfrak{L}_s,\mathcal{U}] = \{0\}$. Let F be a symmetric non-degenerate bilinear form on $\mathcal{L}_s$. Then, the bilinear form F is associative because $[\mathfrak{L}_s,\mathfrak{L}_s] = \{0\}$. Consequently, the bilinear form $B : \mathfrak{L} \times \mathfrak{L} \to \mathbb{K}$ defined by:
$B_{|\mathcal{U}\times\mathcal{U}} = T_{s_{|\mathcal{U}\times\mathcal{U}}}$, $B_{|\mathfrak{L}_s\times\mathfrak{L}_s} = F$, $B(\mathcal{U},\mathfrak{L}_s) = B(\mathfrak{L}_s,\mathcal{U}) = \{0\}$ is symmetric, associative and non-degenerate on $\mathfrak{L}$. ■

Now, we give some Properties of quadratic Leibniz algebras proved in the general case of non-associative algebras ([4]) as well as other new results.

Definition 2.6. Let $(\mathfrak{L}, [\ ,\ ], B)$ be a quadratic Leibniz algebra and $\mathcal{U}$ an ideal of $\mathfrak{L}$. Then, we say that $\mathcal{U}$ is non-degenerate if $B_{|\mathcal{U}\times\mathcal{U}}$ is non-degenerate.

Proposition 2.7. *[4] Let $(\mathfrak{L}, [\ ,\ ], B)$ be a quadratic Leibniz algebra and $\mathcal{U}$ an ideal of $\mathfrak{L}$. Then,*

(a) $\mathcal{U}^\perp = \{x \in \mathfrak{L},\ B(x,u) = 0,\ \forall u \in \mathcal{U}\}$ *is an ideal of* $\mathfrak{L}$.

(b) *If $\mathcal{U}$ is a non-degenerate ideal of $\mathcal{L}$, then $\mathfrak{L} = \mathcal{U} \oplus \mathcal{U}^\perp$.*

Definition 2.8. Let $(\mathfrak{L}, [\ ,\ ])$ be a symmetric Leibniz algebra.

1. The vector space $Z(\mathfrak{L}) = \{x \in \mathfrak{L}, [x,y] = [y,x] = 0, \forall y \in \mathfrak{L}\}$ is called the center of $\mathfrak{L}$.

2. The vector space $\mathcal{R} = \{x \in \mathfrak{L}, [x,y] + [y,x] = 0, \forall y \in \mathfrak{L}\}$ is called the Leibniz radical of $\mathfrak{L}$.

Proposition 2.9. *Let $(\mathfrak{L}, [\ ,\ ], B)$ be a quadratic Leibniz algebra. Then,*

1. $\mathcal{I}_\mathfrak{L} \subseteq [\mathfrak{L}, \mathfrak{L}] \subseteq \mathcal{I}_\mathfrak{L}^\perp$,

2. $\mathcal{I}_\mathfrak{L} \subseteq Z(\mathfrak{L})$,

3. $Z(\mathfrak{L}) = [\mathfrak{L}, \mathfrak{L}]^\perp([4])$,

4. $\mathcal{I}_\mathfrak{L} \subseteq \mathcal{R}$ *and* $\mathcal{I}_\mathfrak{L} \subseteq \mathcal{R}^\perp \subseteq Z(\mathfrak{L})$.

Proof. 1. It is clear that $\mathcal{I}_\mathfrak{L} \subseteq [\mathfrak{L}, \mathfrak{L}]$. Moreover, since $[\mathfrak{L}, \mathcal{I}_\mathfrak{L}] = \{0\}$, then $B([x,y],i) = B(x,[y,i]) = 0, \forall x,y \in \mathfrak{L}, i \in \mathcal{I}_\mathfrak{L}$. Consequently, $[\mathfrak{L}, \mathfrak{L}] \subseteq \mathcal{I}_\mathfrak{L}^\perp$.
2. The assertion follows from Proposition 1.8 and Theorem 2.4.
3. It is easy to see that $Z(\mathfrak{L}) \subseteq \mathcal{R}$. So, $\mathcal{I}_\mathfrak{L} \subseteq \mathcal{R}$ and $\mathcal{R}^\perp \subseteq Z(\mathfrak{L})$. Moreover, if $x, y \in \mathfrak{L}$ and $z \in \mathcal{R}$, then $B([x,y] + [y,x], z) = B(x, [y,z] + [z,y]) = 0$. Therefore, $\mathcal{I}_\mathfrak{L} \subseteq \mathcal{R}^\perp$. ■

Remark 2.10. Let $(\mathfrak{L}, [\ ,\ ])$ be a quadratic Leibniz algebra. Then, $\mathcal{W} = \mathcal{I}_\mathfrak{L}^\perp/\mathcal{I}_\mathfrak{L}$ endowed with the bilinear form $B_\mathcal{W} : \mathcal{W} \times \mathcal{W} \to \mathbb{K}$ defined by: $B_\mathcal{W}(\overline{x}, \overline{y}) = B(x,y), \forall x, y \in \mathfrak{L}$ is a quadratic Lie algebra.

During this section, we are going to construct many examples of quadratic Leibniz algebras.

Definition 2.11. [18] Let $(\mathfrak{L}, [\ ,\ ])$ be a non associative algebra. For $n \geq 1$, we define the following sequences of subspaces $\mathfrak{L}$:

$$\mathfrak{L}^0 = \mathfrak{L}, \qquad \mathfrak{L}^n = [\mathfrak{L}^{n-1}, \mathfrak{L}^{n-1}],$$

$$\mathfrak{L}^{(0)} = \mathfrak{L}, \qquad \mathfrak{L}^{(n)} = [\mathfrak{L}, \mathfrak{L}^{(n-1)}],$$

$$\mathfrak{L}^{\langle 0\rangle} = \mathfrak{L}, \qquad \mathfrak{L}^{\langle n\rangle} = [\mathfrak{L}^{\langle n-1\rangle}, \mathfrak{L}].$$

(i) If there exists $n \in \mathbb{N}$ such that $\mathfrak{L}^n = \{0\}$, then we say that $\mathfrak{L}$ is solvable.

(ii) If there exists $n \in \mathbb{N}$ such that $\mathfrak{L}^{(n)} = \{0\}$, then we say that $\mathfrak{L}$ is left nilpotent.

(iii) If there exists $n \in \mathbb{N}$ such that $\mathfrak{L}^{<n\rangle} = \{0\}$, then we say that $\mathfrak{L}$ is right nilpotent.

(iv) $\mathfrak{L}$ is said to be nilpotent, if there exists $p \in \mathbb{N}$ such that every product $[\ldots[x_1, x_2]\ldots x_p]$ of p elements of $\mathfrak{L}$, no matter how associated, is zero. We call the index of nilpotency of $\mathfrak{L}$, the smallest integer p satisfying this Property.

Remark 2.12. It is well known that $\mathfrak{L}$ is right (resp. left) nilpotent if and only if R_x (resp. L_x) is nilpotent for all $x \in \mathfrak{L}$.

Proposition 2.13. *Let $(\mathfrak{L}, [\ ,\ ])$ be a symmetric Leibniz algebra. Then, the following assertions are equivalent:*

1. *$\mathfrak{L}$ is nilpotent*
2. *$\mathfrak{L}$ is left nilpotent*
3. *$\mathfrak{L}$ is right nilpotent.*

Proof. Since $L_xL_y = -R_xL_y, R_xR_y = -L_xR_y$, and $L_xL_y = R_xR_y, \forall x, y \in \mathfrak{L}$ (Corollary 1.21), then $\forall k \in \mathbb{N}^*, \forall\{x_1, \ldots, x_k\} \in \mathfrak{L}, \forall S_{x_i} \in \{L_{x_i}, 1 \leq i \leq k\} \cup \{R_{x_i}, 1 \leq i \leq k\}$ we have $S_{x_1} \circ \cdots \circ S_{x_k} = (-1)^j L_{x_1} \circ \cdots \circ L_{x_k} = (-1)^t R_{x_1} \circ \cdots \circ R_{x_k}$, where j and $t \in \mathbb{N}$. Which proves that $1, 2$ and 3 are equivalent. ■

Example 2.14. In [6], the authors classified solvable right Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra. We use the same notations as in [6]. Let $\mathfrak{L}$ be a right solvable Leibniz algebra.

(i) If the nilpotent radical of $\mathfrak{L}$ is a non Lie graded filiform Leibniz algebra. Then, $\mathfrak{L}$ is not a symmetric Leibniz algebra.

(ii) If the nilpotent radical is a graded filiform Lie algebra, then
$\mathfrak{L}$ is symmetric and $Dim(\mathfrak{L}) = n + 1$ if and only if $\mathfrak{L}$ is isomorphic to the nilpotent algebra, $R_{n+1,1}(0, 0, 1)$;
$\mathfrak{L}$ is symmetric and $Dim(\mathfrak{L}) = 2n + 1$ if and only if $\mathfrak{L}$ is isomorphic to the nilpotent algebra $R_{2n+1,1}(0, 0, 1)$.

Now, for every fixed $p \in \mathbb{N}$, we are going to construct non Lie nilpotent symmetric Leibniz algebras with index of nilpotency p.

Definition 2.15. Let $(A, .)$ be an algebra. Then,

(a) If $(a.b).c = (b.a).c$ and $c.(b.a) = c.(a.b), \forall a, b \in A$, then we say that A is semi-commutative.

(b) If $a.(b.c) = b.(a.c)$ and $(a.b).c = (a.c).b, \forall a, b, c \in A$, then we say that A is an LR-algebra (see. [5]).

Remark 2.16. If $(A, .)$ is an associative algebra, then A is semi-commutative if and only if A is an LR-algebra.

The set of associative commutative algebras is strictly contained in the set of associative LR-algebras. Indeed, Let $A = \langle a, b, c, d \rangle$ be a four dimensional vector space. Then, A endowed with the product defined by $a.b = c.a = d$, is an associative non commutative LR-algebra.

Proposition 2.17. *Let $(\mathfrak{L}, [\ ,\ ]_{\mathfrak{L}})$ be a symmetric Leibniz algebra and $(A, .)$ be an associative LR- algebra. Then, The vector space $\mathfrak{L} \otimes A$ endowed with the product defined by*

$$[x \otimes a, y \otimes b] = [x, y]_{\mathfrak{L}} \otimes a.b, \forall x, y \in \mathfrak{L}, a, b \in A$$

is a symmetric Leibniz algebra. Moreover, if $\mathfrak{L}$ is a Lie algebra then $(\mathfrak{L} \otimes A, [\ ,\ ])$ is a Lie algebra if and only if $(A, .)$ is a commutative algebra.

Proof. Since, $(a.b).c = (b.a).c, \forall a, b \in A$, then $[x \oplus a, [y \oplus b, z \oplus c]] = [[x, y], z] \oplus a.(b.c) + [y, [x, z]] \oplus a.(b.c) = [[x, y], z] \oplus (a.b).c + [y, [x, z]] \oplus b.(a.c) = [[x \oplus a, y \oplus b], z \oplus c] + [y \oplus b, [x \oplus a, z \oplus c]]$. Therefore, $\mathfrak{L} \otimes A$ is a left Leibniz algebra. Moreover, the fact that $c.(b.a) = c.(a.b), \forall a, b \in A$ implies that $a.(b.c) = (a.b).c = (b.a).c = b.(a.c) = b.(c.a) = (b.c).a$. Consequently, $[x \oplus a, [y \oplus b, z \oplus c]] + [[y \oplus b, z \oplus c], x \oplus a] = ([x, [y, z]] + [[y, z], x]) \oplus a.(b.c) = 0$. So, $\mathfrak{L} \otimes A$ is a symmetric Leibniz algebra (Proposition 1.4).

Now, suppose that $\mathfrak{L}$ is a Lie algebra. If $(A, .)$ is commutative, then $[x \oplus a, y \oplus b] + [y \oplus b, x \oplus a] = ([x, y] + [y, x]) \oplus a.b = 0$. So, $\mathfrak{L} \otimes A$ is a Lie algebra. Conversely, if $\mathfrak{L} \otimes A$ is a Lie algebra then $0 = [x \oplus a, y \oplus b] + [y \oplus b, x \oplus a] = [x, y] \oplus (a.b - b.a)$. So, A is commutative. ■

Remark 2.18. It is clear that if A is nilpotent of index of nilpotency $p \in \mathbb{N}$ and $\mathfrak{L}$ is not nilpotent, then $\mathfrak{L} \otimes A$ is a nilpotent symmetric Leibniz algebra of index of nilpotency p.

Proposition 2.19. *Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a Lie algebra, $(A, .)$ be an associative non-commutative LR-algebra and $\mathfrak{g} \otimes A$ be the symmetric Leibniz algebra constructed in Proposition 2.17. Then, the vector space*

$$L(\mathfrak{g}, A) = \Big(\mathfrak{g} \otimes A\Big) \oplus \Big(\mathfrak{g}^* \otimes A^*\Big)$$

endowed with the product defined for all $x, y \in \mathfrak{g}, a, b \in A, f, g \in \mathfrak{g}^*, f', g' \in A^*$ by:

$$[x \otimes a + f \otimes f', y \otimes b + g \otimes g'] = [x, y]_{\mathfrak{g}} \otimes a.b + (f \circ L_y)(f' \circ L_b) + (g \circ R_x)(g' \circ R_a)$$

is a non Lie symmetric Leibniz algebra. Moreover, the bilinear form

$$\begin{aligned} B_{L(\mathfrak{g},A)} : L(\mathfrak{g}, A) \times L(\mathfrak{g}, A) &\to \mathbb{K} \\ (x \otimes a + f \otimes f', y \otimes b + g \otimes g') &\longmapsto g(x)g'(a) + f(y)f'(b) \end{aligned}$$

is an associative scalar product on $L(\mathfrak{g}, A)$.

The preceding Lemma shows that we can construct non Lie quadratic Leibniz algebras from Lie algebras and associatives non-commutatives LR-algebras. So, it is important to construct associative non-commutative LR-algebras.

Proposition 2.20. *If* $(A, .)$ *is an associative commutative algebra,* $\mathbb{K}e$ *is a one dimensional vector space and* $w : A \times A \to \mathbb{K}$ *is a bilinear form. Then, the vector space* $\overline{A} = A \oplus \mathbb{K}e$ *endowed with the following product:*

$$(a + \lambda e) \circ (b + \lambda' e) = a.b + w(a, b)e, \forall a, b \in A, \lambda, \lambda' \in \mathbb{K}$$

is a semi-commutative algebra. Moreover,

(i) $(\overline{A}, \circ)$ *is associative if and only if* $w(a.b, c) = w(a, b.c), \forall a, b, c \in A$.

(ii) $(\overline{A}, \circ)$ *is commutative if and only if* $w(a, b) = w(b, a), \forall a, b \in A$.

Proof. Only computaion. ∎

Now, let $\mathbb{K}^n$ be the vector space of dimension $n \in \mathbb{N}^*$ and $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{K}^n$.
We define $e_i.e_j := \begin{cases} e_{i+j+1}, & \text{si} \quad i + j + 1 \le n \\ 0, & \text{otherwise.} \end{cases}$
The algebra $(\mathbb{K}^n, .)$ will be noted, $A_n = (\mathbb{K}^n, .)$.

Proposition 2.21. *The algebra* A_n *is an associative commutative algebra. In addition, if* $n = 2p$ $(resp.\, 2p + 1)$, *then* A_n *is nilpotent with index of nilpotency* $p + 1\, (resp.\, p + 2)$.

Proof. It is easy to see that A_n is associative commutative. Moreover, for all $k \ge 2$, ${A_n}^{(k)}$ is spanned by the set $\{e_{2k-1}, \ldots, e_n\}$. Therefore, if $n = 2p$ then, ${A_n}^{(p)}$ is spanned by the set $\{e_{n-1}, \ldots, e_n\}$. Consequently ${A_n}^{(p+1)} = \{0\}$. If $n = 2p + 1$ then, ${A_n}^{(p+1)}$ is spanned by the set $\{e_n\}$. So, $A^{(p+2)} = \{0\}$. ∎

Proposition 2.22. *Let* $n \ge 3, n \in \mathbb{N}$. *Consider the algebra* A_n *of dimension* n. *Let* $\mathbb{K}e$ *be a one-dimensional vector space and define the bilinear map* $w\colon A_n \times A_n \to \mathbb{K}$ *by* $w(e_i, e_j) = 0, \forall (i, j) \in \{1, \ldots, n\}^2 \backslash \{(1, 2), (2, 2)\}$ *and* $w(e_1, e_2) = \alpha, w(e_2, e_2) = \beta, \alpha, \beta \in \mathbb{K}^*$ *such that* $\alpha \ne \beta$. *The vector space* $\overline{A}_n = A_n \oplus \mathbb{K}e$ *endowd with the product defined by*

$$(a + \lambda e) \circ (b + \lambda' e) = a.b + w(a, b)e, \forall a, b \in A_n, \lambda, \lambda' \in \mathbb{K}$$

is a nilpotent, non-commutative, associative LR-algebra. In addition, $\overline{A}_n$ *and* A_n *have the same index of nilpotency.*

Proof. Since ${A_n}^2$ is spanned by the set $\{e_3, \ldots, e_n\}$, then $w({A_n}^2, A_n) = w(A_n, {A_n}^2) = \{0\}$. Therefore, $w(a.b, c) = w(a, b.c), \forall a, b, c \in A_n$. Moreover, $w(e_1, e_1) = \alpha \ne 0$. So, w is not symmetric. The Proposition 2.20 implies that $(\overline{A}, \circ)$ is an associative non-commutative LR -algebra. It is clear that $\overline{A}^2 \subseteq {A_n}^2 + w(A_n, A_n)$. Since $w({A_n}^2, A_n) = w(A_n, {A_n}^2) = \{0\}$, then $\overline{A}^3 \subseteq {A_n}^3$. Consequently, for all $k \geq 3$ $\overline{A}^k \subseteq {A_n}^k$. But $n \geq 3$. So, by Proposition 2.21, A_n is nilpotent with index of nilpotency grater or equal to 3. We conclude that $\overline{A}$ is nilpotent and the index of nilpotency of $\overline{A}$ is equal to the index of nilpotency of A_n. ■

Now, we are in position to construct non-Lie, nilpotent, quadratic Leibniz algebras. Let $p \geq 3$. Consider the algebra A_n of dimension $n \in \{2(p-1), 2p-3\}$. Let $(\mathfrak{g}, [\ ,\ ])$ be a perfect Lie algebra (i.e $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$). Then, the symmetric Leibniz algebra $\mathfrak{g} \otimes \overline{A_n}$ constructed in Proposition 2.17 is a non Lie, nilpotent, symmetric Leibniz with index of nilpotency p. Next, applying the Proposition 2.19, we get a nilpotent quadratic Leibniz algebra $L(\mathfrak{g}, \overline{A_n})$ with index of nilpotency p.

Remark 2.23. The examples that we constructed so far are nilpotent. But there are also non nilpotent symmetric Leibniz algebras (see. Examples 3.6).

It is clear that a non Lie symmetric Leibniz algebra is not simple in the classical sens, because the Leibniz Kernel is a non trivial ideal. In [13], the authors define the notions of simple and semi-simple Leibniz algebras as follow:

Definition 2.24. Let $(\mathfrak{L}, [\ ,\ ])$ be a right Leibniz algebra. Then ,

1. We say that $\mathfrak{L}$ is simple if $[\mathfrak{L}, \mathfrak{L}] \neq \mathcal{I}_{\mathfrak{L}}$ and $\mathcal{L}$ has no ideals but $\{0\}$, $\mathcal{L}$ and $\mathcal{I}_{\mathfrak{L}}$.
2. We say that $\mathfrak{L}$ is semi simple if the maximal solvable ideal of $\mathfrak{L}$ is $\mathcal{I}_{\mathfrak{L}}$.

In [9], it is shown that if a left Leibniz algebra $\mathfrak{L}$ is semi simple, then it's Killing form $\kappa : \mathfrak{L} \times \mathfrak{L} \to \mathbb{K}; (x, y) := Trace(L_x L_y)$ is non-degenerate. That is $Rad(\kappa) = \mathcal{I}_{\mathfrak{L}}$. Unlike the case of Lie algebras, the converse is not true.
In the following Proposition, we prove that all quadratic simple (resp. semi-simple) Leibniz algebras are simple (resp. semi-simple) Lie algebras.

Proposition 2.25. *Let $(\mathfrak{L}, B)$ be a quadratic Leibniz algebra. Then,*

(i) *$\mathfrak{L}$ is simple if and only if $\mathfrak{L}$ is a Lie algebra.*

(ii) *$\mathfrak{L}$ is semi simple if and only if $\mathfrak{L}$ is a semi simple Lie algebra.*

Proof. (i) If $(\mathfrak{L}, B)$ is quadratic, then $\mathcal{I}_{\mathfrak{L}}^{\perp}$ is an ideal $\mathfrak{L}$. Consequently, there are only three cases:

1. If $\mathcal{I}_{\mathfrak{L}}^{\perp} = \{0\}$, then $\mathcal{I}_{\mathfrak{L}} = \{0\}$. Therefore, $\mathfrak{L}$ is a simple Lie algebra.
2. If $\mathcal{I}_{\mathfrak{L}}^{\perp} = \mathfrak{L}$, then $\mathcal{I}_{\mathfrak{L}} = \{0\}$. Consequently, $\mathfrak{L}$ is a simple Lie algebra.
3. If $\mathcal{I}_{\mathfrak{L}}^{\perp} = \mathcal{I}_{\mathfrak{L}}$, then $\mathcal{I}_{\mathfrak{L}} = [\mathfrak{L}, \mathfrak{L}]$ because $\mathcal{I}_{\mathfrak{L}} \subseteq [\mathfrak{L}, \mathfrak{L}] \subseteq \mathcal{I}_{\mathfrak{L}}^{\perp}$. Which contradicts the fact that $[\mathfrak{L}, \mathfrak{L}] \neq \mathcal{I}_{\mathfrak{L}}$.

(ii) By [2], we get $\mathfrak{L} = \mathcal{S} \oplus \mathcal{I}_{\mathfrak{L}}$, where $\mathcal{S}$ is a sub-algebra of $\mathfrak{L}$. More precisely $\mathcal{S}$ is a semi simple Lie algebra. So, $[\mathcal{S},\mathcal{S}] = \mathcal{S} \subseteq [\mathfrak{L},\mathfrak{L}] \subseteq \mathcal{I}_{\mathfrak{L}}^{\perp}$. Since $\mathcal{I}_{\mathfrak{L}} \subseteq \mathcal{I}_{\mathfrak{L}}^{\perp}$, then $B(\mathfrak{L},\mathcal{I}_{\mathfrak{L}}) = \{0\}$. Consequentely, $\mathcal{I}_{\mathfrak{L}} = \{0\}$. ■

3. Quadratic solvable Leibniz algebras and T^*-extension

In this section, we study solvable quadratic Leibniz algebras via the concept of T^*-extension introduced by M. Bordmann for non-associative algebras [4]. Let us first recall this concept.

Definition 3.1. [4] Let $(A, .)$ be a non-associative algebra and $B : A \times A \to \mathbb{K}$ be a bilinear form. We say that B is invariant if $B(a.b, c) = B(a, b.c)$, $\forall a, b, c \in A$. If moreover B is symmetric and non-degenerate, then we say that B is an invariant scalar product on A. In this case, (A, B) is called a metrised algebra.

Remark 3.2. The terminology metrised is used by M. Bordemann ([4]) for non-associative algebras. In this paper, we use the terminology quadratic instead of metrised in the case of Leibniz algebra.

Theorem 3.3. *[4] Let $(A, .)$ a non-associative algebra, A^* be the dual space of A and $w : A \times A \to A^*$ be a bilinear map. We define the following multiplication on the vector space $A \oplus A^*$:*

$$(a + f) \bullet (b + g) = (a.b) + w(a, b) + g \circ R_a + f \circ L_b,\ \forall a, b \in A, f, g \in A^*.$$

The bilinear form $Q\colon A \times A \to \mathbb{K}$ defined by

$$Q(a + f, b + g) = f(b) + g(a), a, b \in A, f, g \in A^*,$$

is invariant if and only if $w(a, b)(c) = w(c, a)(b) = w(b, c)(a), \forall a, b, c \in A$. In this case, the metrised algebra $(A \oplus A^*, Q)$ is called the T^*-extension of A by means of w. We denote, $(A \oplus A^*, Q) = T^*_w(A)$.

Definition 3.4. Let $\mathfrak{L}$ be a left (resp. right) Leibniz algebra, V be a vector space and (r, l) be a left (resp. right) representation of $\mathfrak{L}$ in V. Let $w : \mathfrak{L}\times\mathfrak{L} \to V$ be a bilinear map, then we say that w is a left (resp. right) 2-cocycle of $\mathfrak{L}$ if $\forall x, y, z \in \mathfrak{L}$:

$w([x, y], z)(+)w(y, [x, z])-w(x, [y, z])-l(x)(w(y, z))+l(y)(w(x, z))+r(z)(w(x, y)) = 0$,

(respectively,

$w([x, y], z)-w([x, z], y)-w(x, [y, z])-l(x)(w(y, z)) - r(y)\Big(w(x, z)\Big)$
$+ r(z)\Big(w(x, y)\Big) = 0$). In particular, if $\mathfrak{L}$ is a symmetric Leibniz algebra, (r, l) is a representation of $\mathfrak{L}$ in V and $w : \mathfrak{L} \times \mathfrak{L} \to V$ is a left and a right 2-cocycle of $\mathfrak{L}$ then we say that w is a bi-2-cocycle of $\mathfrak{L}$.

Proposition 3.5. *Let $\mathfrak{L}$ be a symmetric Leibniz algebra and $w : \mathfrak{L}\times\mathfrak{L} \to \mathfrak{L}^*$ be a bilinear map. Then, the vector space $T^*_w(\mathfrak{L}) = \mathfrak{L} \oplus \mathfrak{L}^*$ endowd with the product defined by :*

$$[x + f, y + g] = [x, y] + w(x, y) + g \circ R_x + f \circ L_y, \forall x, y \in \mathfrak{L}, f, g \in \mathfrak{L}^*,$$

is a symmetric Leibniz algebra if and only if w is a bi-2-cocycle of $\mathfrak{L}$.
Moreover, the bilinear form $T^*_w(\mathfrak{L}) \times T^*_w(\mathfrak{L}) \to \mathbb{K}; (x+f, y+g) \longmapsto f(y)+g(x)$ is symmetric and non-degenerate on $T^*_w(\mathfrak{L})$. The bilinear form B is associative on $T^*_w(\mathfrak{L})$ if and only if

$$w(x,y)(z) = w(y,z)(x), \forall x,y,z \in \mathfrak{L}.$$

In this case, the quadratic Leibniz algebra $(T^*_w(\mathfrak{L}), B)$ is called the T^*-extension of $\mathfrak{L}$ by means of w.

The T^*-extension given in the previous Proposition allows us to construct several examples of quadratic Leibniz algebras.

Examples 3.6. 1. Let $(\mathfrak{g}, [\,,\,])$ be an abelian Lie algebra and $w : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}^*$ be a bilinear map. Then, we can consider the quadratic Leibniz algebra ${T^*}_w(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$, T^*-extension of $\mathfrak{g}$ by means of w. It is clear that the product on ${T^*}_w(\mathfrak{g})$ is defined for all $x, y \in \mathfrak{g}, f, h \in \mathfrak{g}^*$ by: $[x+f, y+g] = w(x,y)$. Therefore, ${T^*}_w(\mathfrak{g})$ is nilpotent with index of nilpotency 2. Moreover, if w is not anti-symmetric, then ${T^*}_w(\mathfrak{g})$ is not a Lie algebra.

2. Let $p \geq 3$. Let $\mathfrak{g}$ be a perfect Lie algebra (i.e. $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$) and A_n be the algebra of dimension $n \in \{2(p-1), 2(p-3)\}$. Then, the quadratic Leibniz algebra $L(\mathfrak{g}, \overline{A_n})$ constructed in Proposit2.19 is nilpotent with index of nilpotency p. This algebra is the T^*-extension of the symmetric Leibniz algebra $\mathfrak{g} \otimes \overline{A}_n$ constructed in Proposition 2.17 by means of $w = 0$.

3. Let $\mathfrak{h} = \langle a,b,c,d\rangle$ be a vector space of dimension 4. We define on $\mathfrak{h}$ the following product: $[a,a] = c\,[a,b] = -[b,a] = b, \quad [d,a] = -[a,d] = d, \quad [b,d] = -[d,b] = c$. It is clear that $\mathfrak{h}$ is a non Lie, solvable and non nilpotent symmetric Leibniz algebra. Moreover, the bilinear form
$$B : \mathfrak{h}\times\mathfrak{h} \to \mathbb{K} \text{ defined by } B(a,c) = B(b,d) = 1$$
is an associative scalar product on $\mathfrak{h}$. Consequentely, $(\mathfrak{h}, B)$ is a solvable, non nilpotent quadratic Leibniz algebra. The algebra $\mathfrak{h}$ is the T^*-extension of the non abelian two dimensional Lie algebra $\mathfrak{g} = \langle a,b\rangle$ by means of the cocycle w defined by $w(a,a) = c$.

In the following theorem, M. Bordemann gave a necessarily and sufficient condition to a metrised non-associative algebra to be a T^*-extension or an ideal of codimension one in a T^*-extension.

Theorem 3.7. *[4] Let (A,Q) be a metrised algebra of dimension n over a field $\mathbb{K}$ of characteristic not equal to two. Then, A is isometric to a T^*-extension $(T^*_w(H), Q_H)$ if and only if A contains an isotropic ideal I of dimension $\frac{n}{2}$. In this case, $H \cong A/I$. Note that every isotropic vector space I of A of dimension $\frac{n}{2}$ is an ideal of A if and only if it is abelian .*

The previous theorem shows that if the metrised non-associative algebra A admits an isotropic ideal of dimension $[\frac{Dim(A)}{2}]$, then A is constructed by a T^*-

extension. Until now, this situation is true for metrised, nilpotent non-associative algebras and quadratic, solvable Lie algebras (see [4]).

A natural question: What about the situation in the case of solvable quadratic Leibniz algebras? In the following theorem, we give an answer to this question.

Theorem 3.8. *Let $(\mathfrak{L}, B)$ be a solvable quadratic Leibniz algebra of dimension n. Then, $\mathfrak{L}$ has a maximal isotropic ideal $\mathfrak{H}$ of dimension $[\frac{n}{2}]$ containing $\mathcal{I}_{\mathfrak{L}}$. If n is even , then $\mathfrak{L}$ is isomorphic to a T^*-extension of the Lie algebra $\mathfrak{L}/\mathfrak{H}$. If n is odd, then $\mathfrak{L}$ is isomorphic to a non-degenerate ideal of codimension one of a T^*-extension of the Lie algebra $\mathfrak{L}/\mathfrak{H}$.*

Proof. It is well known that $(\mathcal{W} = \mathcal{I}_{\mathfrak{L}}^{\perp}/\mathcal{I}_{\mathfrak{L}}, [\ ,\ ]_{\mathcal{W}})$ and $(\mathfrak{g} = \mathfrak{L}/\mathcal{I}_{\mathfrak{L}}, [\ ,\ ]_{\mathfrak{g}})$ are Lie algebras. The product $[\ ,\ ]_{\mathcal{W}}$ (resp. $[\ ,\ ]_{\mathfrak{g}}$) is defined by $[P_{\mathcal{W}}(i_1), P_{\mathcal{W}}(i_2)]_{\mathcal{W}} = P_{\mathcal{W}}\Big([i_1, i_2]\Big), \forall i_1, i_2 \in \mathcal{I}_{\mathfrak{L}}^{\perp}$ (resp. $[P_{\mathfrak{g}}(x), P_{\mathfrak{g}}(y)]_{\mathfrak{g}} = P_{\mathfrak{g}}\Big([x, y]\Big), \forall x, y \in \mathfrak{L}$), where $P_{\mathcal{W}} : \mathcal{I}_{\mathfrak{L}}^{\perp} \to \mathcal{W}$ and $P_{\mathfrak{g}} : \mathfrak{L} \to \mathfrak{g}$ are the canonical surjections. It is clear that $(\mathcal{W}, [\ ,\ ]_{\mathcal{W}})$ and $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ are two solvable Lie algebras. Moreover the bilinear form $B_{\mathcal{W}} := (P_{\mathcal{W}}(i), P_{\mathcal{W}'}(i')) \longmapsto B(i, i')$ is an associative scalar product on $\mathcal{W}$. Therefore, $(\mathcal{W}, [\ ,\ ]_{\mathcal{W}}, B_{\mathcal{W}})$ is a quadratic solvable Lie algebra. Now, we define the map

$$\begin{aligned} \Pi : \mathfrak{g} &\to gl(\mathcal{W}) \\ P_{\mathfrak{g}}(x) &\longmapsto \Pi(P_{\mathfrak{g}}(x))\Big(P_{\mathcal{W}}(i)\Big) = P_{\mathcal{W}}\Big([x, i]\Big). \end{aligned}$$

The map Π is well defined because $\mathcal{I}_{\mathfrak{L}}$ et $\mathcal{I}_{\mathfrak{L}}^{\perp}$ are two ideals of $\mathfrak{L}$. Moreover, for all $x, y \in \mathfrak{L}$

$\Big[\Pi(P_{\mathfrak{g}}(x)), \Pi(P_{\mathfrak{g}}(y))\Big](P_{\mathcal{W}}(i)) = P_{\mathcal{W}}\Big([x, [y, i]] - [y, [x, i]]\Big) = P_{\mathcal{W}}\Big([[x, y], i]\Big) = \Pi\Big(P_{\mathfrak{g}}([x, y])\Big)\Big(P_{\mathcal{W}}(i)\Big)$. Therefore, the map Π is a morphism of Lie algebras. Consequently, the Lie algebra $L := \Pi(\mathfrak{g})$ is isomorphic to $\mathfrak{g}/Ker(\Pi)$. This implies that L is a solvable sub-Lie algebra of $gl(\mathcal{W})$. Since $B\Big([x, i], i'\Big) = B\Big(i, [i', x]\Big), \forall x \in \mathfrak{L}, i, i' \in \mathcal{I}_{\mathfrak{L}}^{\perp}$, then

$$B_{\mathcal{W}}\Big(\Pi(P_{\mathfrak{g}}(x))(P_{\mathcal{W}}(i)), P_{\mathcal{W}}(i')\Big) = B_{\mathcal{W}}\Big(P_{\mathcal{W}}(i), P_{\mathcal{W}}([i', x])\Big).$$

Let $x \in \mathfrak{L}, i \in \mathcal{I}_{\mathfrak{L}}^{\perp}$. Since $[x, i] + [i, x] \in \mathcal{I}_{\mathfrak{L}}$, then $P_{\mathcal{W}}([x, i]) = -P_{\mathcal{W}}([i, x])$. Therefore,

$$\begin{aligned} B_{\mathcal{W}}\Big(\Pi(P_{\mathfrak{g}}(x))(P_{\mathcal{W}}(i)), P_{\mathcal{W}}(i')\Big) &= -B_{\mathcal{W}}\Big(P_{\mathcal{W}}(i), P_{\mathcal{W}}([x, i'])\Big) \\ &= -B_{\mathcal{W}}\Big(P_{\mathcal{W}}(i), \Pi(P_{\mathfrak{g}}(x))(P_{\mathcal{W}}(i'))\Big). \end{aligned}$$

So, applying the Lemma 3.2 [4] to the quadratic Lie algebra $(\mathcal{W}, B_{\mathcal{W}})$ and the solvable sub-Lie algebra L of $gl(\mathcal{W})$, we conclude that $\mathcal{W}$ has a maximal isotropic L-stable sub-vector space $\mathcal{J}$ of dimension $[\frac{Dim(\mathcal{W})}{2}]$.

Let $\mathfrak{H} = P_{\mathcal{W}}^{-1}(\mathcal{J})$. Then, $\mathcal{I}_{\mathfrak{L}} \subseteq \mathfrak{H}$. Since $\mathcal{J}$ is an isotropic sub-vector space $\mathcal{W}$, then $\mathfrak{H}$ is an isotropic sub-vector space $\mathfrak{L}$. In fact,

$B(x,y) = B_{\mathcal{W}}\Big(P_{\mathcal{W}}(x), P_{\mathcal{W}}(y)\Big) = 0, \forall x, y \in \mathfrak{H}$.
Moreover, the fact that $\mathcal{J}$ is L-stable implies that for all $x \in \mathfrak{H}, z \in \mathfrak{L}$,
$P_{\mathcal{W}}([x,z]) = -P_{\mathcal{W}}([z,x]) = \Pi_{\mathcal{W}}(P_{\mathfrak{g}}(z))(P_{\mathcal{W}}(x)) \in \mathcal{J}$. So, $\mathfrak{H}$ is an ideal of $\mathfrak{L}$.
In addition, $Dim(\mathfrak{H}) = Dim(\mathcal{J}) + Dim(\mathcal{I}_{\mathfrak{L}}) = [\frac{Dim(\mathcal{W})}{2}] + Dim(\mathcal{I}_{\mathfrak{L}}) = [\frac{Dim(\mathfrak{L})}{2}]$.
We conclude that $\mathfrak{H}$ is a maximal isotropic ideal of $\mathfrak{L}$ of dimension $[\frac{Dim \mathfrak{L}}{2}]$ containing $\mathcal{I}_{\mathfrak{L}}$.

Now, if n is even, then the Theorem 3.2 [4] imply that $\mathfrak{L}$ is isomorphic to a T^*-extension $(T^*_w(H), Q)$ of the algebra $H \cong \mathfrak{L}/\mathfrak{H}$. Since $\mathcal{I}_{\mathfrak{L}} \subseteq \mathfrak{H}$, then H is a Lie algebra. If n is even, then by Lemma 3.2 [4], $\mathfrak{L}$ is isomorphic to a non-degenerate ideal of codimension one in a T^*-extension of the Lie algebra $\mathfrak{L}/\mathfrak{H}$. ■

At present, we can not have information about non-solvable quadratic Leibniz algebras using the T^*-extension. In order to have more information about quadratic Leibniz algebras, we introduce other types of extension of Leibniz algebras. Let us indicate that this new extensions are also going to give us additional results on quadratic solvable Leibniz algebras.

4. Extensions of Leibniz algebras

Let $\mathfrak{L}$ be a symmetric Leibniz algebra, V be a vector space and $w : \mathfrak{L} \times \mathfrak{L} \to V^*$ be a bilinear map. Then, the vector space $\mathfrak{L}_1 = \mathfrak{L} \oplus V^*$ endowed with the product defined by:

$$[x+f, y+g] = [x,y] + w(x,y), \forall x, y \in \mathfrak{L}, f, g \in v^*$$

is a symmetric Leibniz algebra if and only if w is a bi-2-cocycle of $\mathfrak{L}$. The symmetric Leibniz algebra $\mathfrak{L}_1$ is called the central extension of $\mathfrak{L}$ by V by means of w.

Proposition 4.1. *Let $\mathfrak{L}$ be a symmetric Leibniz algebra, V be a vector space and (r,l) be a representation of $\mathfrak{L}$ in V. Then, $w : \mathfrak{L} \times \mathfrak{L} \to V$ is a bi-2-cocycle of $\mathfrak{L}$ if and only if:*

$$w([x,y],z) + w(y,[x,z]) - w(x,[y,z]) - l(x)\Big(w(y,z)\Big) + l(y)\Big(w(x,z)\Big) + r(z)\Big(w(x,y)\Big) = 0,$$

$$w(x,[y,z]) + w([y,z],x) + r(x)\Big(w(y,z)\Big) + l(x)\Big(w(y,z)\Big) = 0, \forall x, y, z \in \mathfrak{L}.$$

Proof. The bilinear map w is a bi-2-cocycle of $\mathfrak{L}$, if and only if the vector space $\mathfrak{L}_1 = \mathfrak{L} \oplus V^*$ central extension of $\mathfrak{L}$ by V by means of w is a symmetric Leibniz algebra. In particular, the fact that $\mathfrak{L}_1$ is a left Leibniz algebra is equivalent to the fact that w is a left 2-cocycle of $\mathfrak{L}$. Moreover, $[[a,b],c] + [c,[a,b]] = 0, \forall a, b, c \in \mathfrak{L}_1$ if and only if $w(x,[y,z]) + w([y,z],x) + r(x)\Big(w(y,z)\Big) + l(x)\Big(w(y,z)\Big) = 0, \forall x, y, z \in \mathfrak{L}$. ■

In the next Proposition, we are going to introduce certain operators allowing to construct central extensions by a one dimensional Leibniz algebra. This construction plays a role in the double extension which will be defined in the following.

Definition 4.2. Let $(\mathfrak{L}, B)$ be a quadratic Leibniz algebra f be an endomorphism of $\mathfrak{L}$. Then, the endomorphism f^* de $\mathfrak{L}$ defined by $B(f(x), y) = B(x, f^*(y)), \forall x, y \in \mathfrak{L}$ is called the adjoint of f with respect to B.

Proposition 4.3. *Let $(\mathfrak{L}, B)$ is a quadratic Leibniz algebra and $w : \mathfrak{L} \times \mathfrak{L} \to \mathbb{K}$ be a bilinear map. Then, there exists an endomorphism δ of $\mathfrak{L}$ such that $w(x, y) = B(\delta(x), y), \forall x, y \in \mathfrak{L}$. The map w is a bi-2-cocycle of $\mathfrak{L}$ if and only if*

$$(\delta + \delta^*)(\mathfrak{L}) \subseteq Z(\mathfrak{L}) \text{ and } \quad \delta([x, y]) = [\delta(x), y] - [\delta(y), x], \forall x, y \in \mathfrak{L}.$$

Moreover, δ is a derivation of $\mathfrak{L}$ if and only if $[\delta(x), y] = -[y, \delta(x)], \forall x, y \in \mathfrak{L}$. (i. e. $\delta(\mathfrak{L}) \subseteq \mathcal{R}$).

Proof. (i) $w([x, y], z) + w(y, [x, z]) - w(x, [y, x]) = B(z, \delta([x, y]) + [\delta(y), x] - [\delta(x), y])$,

(ii) $w(x, [y, z]) + w([y, z], x) = B(y, [z, (\delta + \delta^*)(x)]), \forall x, y, z \in \mathfrak{L}$. ■

Lemma 4.4. *Let $(\mathfrak{L}, B)$ be a quadratic Leibniz algebra and f be an endomorphism of $\mathfrak{L}$. Then,*

1. *$(f + f^*)([\mathfrak{L}, \mathfrak{L}]) = \{0\}$ if and only if $(f + f^*)(\mathfrak{L}) \subseteq Z(\mathfrak{L})$.*
2. *$f^*(\mathcal{I}_\mathfrak{L}) = f(\mathcal{I}_\mathfrak{L}) = \{0\}$ if and only if $f(\mathfrak{L}) \subseteq \mathcal{R}$ and $f^*(\mathfrak{L}) \subseteq \mathcal{R}$.*

Proof. Let $x, y, z \in \mathfrak{L}$. Then

1. $B\Big((f+f^*)([x, y]), z\Big) = B\Big([x, y], (f+f^*)(z)\Big) = B\Big(y, [(f+f^*)(z), x]\Big)$. Since B is non-degenerate, then $(f+f^*)([x, y]) = 0$ if and only if $[y, (f+f^*)(z)] = [(f + f^*)(z), x] = 0$.
2. $B\Big(z, f([x, y]+[y, x])\Big) = B\Big(x, [y, f^*(z)]+[f^*(z), y]\Big)$. Therefore, $f(\mathcal{I}_\mathfrak{L}) = \{0\}$ if and only if $f^*(\mathfrak{L}) \subseteq \mathcal{R}$. Similarely, we prove that $f^*(\mathcal{I}_\mathfrak{L}) = \{0\}$ if and only if $f(\mathfrak{L}) \subseteq \mathcal{R}$. ■

Now, let $\mathfrak{L}$ be a symmetric Leibniz algebra and $V = \mathbb{K}d$ be the one dimensional Lie algebra. Let δ_1, δ_2 be two derivations of $\mathfrak{L}$ and $a_0 \in Z(\mathfrak{L})$.
Consider the vector space $\widetilde{\mathfrak{L}} = \mathfrak{L} \oplus V$ on which we define the following product :

$$[x + \alpha d, y + \beta d] = [x, y] + \alpha\delta_1(y) + \beta\delta_2(x) + \alpha\beta a_0, \forall x, y \in \mathfrak{L}, \alpha, \beta \in \mathbb{K} \qquad (1)$$

Proposition 4.5. *The vector space $\widetilde{\mathfrak{L}}$ endowed with the product (1) is a symmetric Leibniz algebra if and only if*

$$(\delta_1, \delta_2) \in Rep(V, \mathfrak{L}); \;\; \delta_1(\mathfrak{L}) \subseteq \mathcal{R}; \;\; \delta_2(\mathfrak{L}) \subseteq \mathcal{R};$$
$$(\delta_1 + \delta_2)(\mathfrak{L}) \subseteq Z(\mathfrak{L}); \;\; \delta_1(a_0) = \delta_2(a_0) = 0.$$

In this case, $(\delta_1, \delta_2, a_0)$ is called "an admissible triple" and the symmetric Leibniz algebra $\widetilde{\mathfrak{L}}$ is called the generalized semi-direct product of $\mathfrak{L}$ by V by means of $(\delta_1, \delta_2, a_0)$.

Proof. $(\widetilde{\mathfrak{L}}, [\ ,\ ])$ is a symmetric Leibniz algebra if and only if

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] \text{ and } [a,[b,c]] = -[[b,c],a], \forall a,b,c \in \widetilde{\mathfrak{L}}.$$

These conditions are satisfied if and only if
$\delta_1 \circ \delta_2 = \delta_2 \circ \delta_1; {\delta_1}^2 + \delta_2 \circ \delta_1 = {\delta_2}^2 + \delta_1 \circ \delta_2 = 0; \delta_1(a_0) = \delta_2(a_0) = 0;$
$[\delta_1(x), y] = -[y, \delta_1(x)]; [\delta_2(x), y] = -[y, \delta_2(x)]; [\delta_1(x), y] = -[\delta_2(x), y];$
$[x, \delta_1(y)] = -[x, \delta_2(y)].$ ■

Now, we are in position to introduce the double extension of quadratic Leibniz algebras.
Let $(\mathfrak{L}, [\ ,\ ]_{\mathfrak{L}}, B)$ be a quadratic Leibniz algebra, $V = \mathbb{K}d$ be the one dimensional Lie algebra, (δ, δ^*, a_0) is an admissible triple and $B(a_0, a_0) = 0$. The Proposition 4.3 implies that the map $\phi : \mathfrak{L} \times \mathfrak{L} \to V^*; (x,y) \longmapsto \phi(x,y) = B(\delta(x), y)d^*$, is a bi-2-cocycle of $\mathfrak{L}$. Then, $\mathfrak{L}_1 = \mathfrak{L} \oplus V^*$ endowed with the product defined for all by:

$$[x+f, y+g] = [x,y] + B(\delta(x), y)d^*, \forall x,y \in \mathfrak{L}, f,g \in V^*,$$

is a symmetric Leibniz algebra.
Now, we define two endomorphisms δ_1 and δ_2 of $\mathfrak{L}_1$ as follow:

$$\delta_1(x+\lambda d^*) = \delta(x) + B(x, a_0)d^*, \qquad \delta_2(x+\lambda d^*) = \delta^*(x) + B(x,a_0)d^*,\ \forall x \in \mathfrak{L}, \lambda \in \mathbb{K}.$$

Since $(\delta, \delta^*) \in Rep(V, \mathfrak{L})$ and $\delta(a_0) = \delta^*(a_0) = 0$, then $(\delta_1, \delta_2) \in Rep(V, \mathfrak{L}_1)$. Moreover, $\delta_1 \in Der(\mathfrak{L}_1)$. In fact, for all $x, y \in \mathfrak{L}, f, g \in V^*$
$\delta_1\Big([x+f, y+g]\Big) - \Big[\delta_1(x+f), y+g\Big] - \Big[x+f, \delta_1(y+g)\Big] = B\Big((\delta^2 + \delta^* \circ \delta)(x), y\Big)d^* = 0.$
Similarly, we prove that δ_2 is a derivation of $\mathfrak{L}_1$. Since $\Big(\delta + \delta^*\Big)\Big([\mathfrak{L}, \mathfrak{L}]\Big) = \{0\}$ and $a_0 \in Z(\mathfrak{L})$, then $\Big(\delta_1 + \delta_2\Big)\Big([x+f, y+g]\Big) = \Big(\delta + \delta^*\Big)([x,y]) + 2B\Big([x,y], a_0\Big)d^* = 0.$
Therefore, $(\delta_1 + \delta_2)\Big([\mathfrak{L}_1, \mathfrak{L}_1]\Big) = \{0\}$. Moreover, since $\delta(\mathfrak{L}) \subseteq \mathcal{R}$ then

$$[\delta_1(x+\lambda d^*), y + \lambda' d^*] + [y + \lambda' d^*, \delta_1(x + \lambda d^*)] = B\Big((\delta^2 + \delta^* \circ \delta)(x), y\Big) = 0$$

Consequently, $\delta_1(\mathfrak{L}_1) \subseteq \mathcal{R}_1$, where $\mathcal{R}_1 = \{x_1 \in \mathfrak{L}_1, [x_1, y_1] + [y_1, x_1] = 0, \forall y_1 \in \mathfrak{L}_1\}$. Similarly, we show that $\delta_2(\mathfrak{L}_1) \subseteq \mathcal{R}_1$. Let us consider the element $a_1 = a_0 + \alpha d^* \in \mathfrak{L}_1$ where α is a fixed scalar in $\mathbb{K}$. So , $a_1 \in Z(\mathfrak{L}_1)$. Since $B(a_0, a_0) = 0$, then $\delta_1(a_1) = {\delta^*}_1(a_1) = 0$.
We conclude that the triple $(\delta_1, \delta_2, a_1)$ is an admissible triple of $\mathfrak{L}_1$. Consequently, we can consider the symmetric Leibniz algebra $\overline{\mathfrak{L}} = \mathfrak{L}_1 \oplus V = V^* \oplus \mathfrak{L} \oplus V$, generalized semi-direct product of $\mathfrak{L}_1$ by V. In addition, it is easy to show that the bilinear form $\overline{B} : \overline{\mathfrak{L}} \times \overline{\mathfrak{L}} \to \mathbb{K}$ defined by

$$\overline{\mathfrak{B}}_{|\mathfrak{L}\times\mathfrak{L}} = B; \quad B(d, d^*) = 1$$

is an associative scalar product on $\overline{\mathfrak{L}}$. Thus, we have proved the following theorem:

Theorem 4.6. *Let $(\mathfrak{L}, [\ ,\ ]_{\mathfrak{L}}, B)$ be a quadratic Leibniz algebra, $V = \mathbb{K}d$ be the one dimensional Lie algebra and $(\delta, a_0) \in Der(\mathfrak{L}) \times Z(\mathfrak{L})$ such that (δ, δ^*, a_0) is* *an " admissible triple " and* $B(a_0,a_0)=0$. *Then, the vector space* $\overline{\mathfrak{L}}=V^*\oplus\mathfrak{L}\oplus V$ *endowed with the following product:*

$$[d,d]=\alpha d^*+a_0;\qquad [x,y]=[x,y]_{\mathfrak{L}}+B(\delta(x),y)d^*;$$
$$[d,x]=B(x,a_0)d^*+\delta(x);\qquad [x,d]=B(x,a_0)d^*+\delta^*(x),$$

is a symmetric Leibniz algebra. Moreover, the bilinear form $\overline{B}:\overline{\mathfrak{L}}\times\overline{\mathfrak{L}}\to\mathbb{K}$ *defined by*

$$\overline{\mathfrak{B}}_{|\mathfrak{L}\times\mathfrak{L}}=B;\quad B(d,d^*)=1$$

is an associative scalar product on $\overline{\mathfrak{L}}$.

Definition 4.7. The quadratic Leibniz algebra $(\overline{L},\overline{B})$ constructed in the previous theorem is called the double extension of $\mathfrak{L}$ by V by means of (δ,δ^*,a_0).

Example 4.8. Let $\mathfrak{g}=\langle x,y\rangle$ be the abelian Lie algebra of dimension 2 over $\mathbb{C}$. Then, the bilinear form $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ defined by $B(x,x)=B(y,y)=1$ is an associative scalar product on $\mathfrak{g}$. Let $a_0=x+iy$, where $i^2=-1$. It is clear that $a_0\in Z(\mathfrak{g})$ and that $B(a_0,a_0)=0$.
Let us consider the endomorphism δ of $\mathfrak{g}$ such that $\delta(x)=x+iy$ and $\delta(y)=ix-y$.

It is clear that δ is a symmetric derivation of $\mathfrak{g}$. Besides, $\delta(a_0)=0$.
Now, let us consider $\alpha\in\mathbb{C}$. Let $V=\mathbb{C}d$ be a vector space of dimension 1. Since $\delta^2=0$, then (δ,δ) is a representation of V in $\mathfrak{g}$. To conclude, (δ,δ,a_0) is an admissible triple of $\mathfrak{g}$. Consequently, we can consider the symmetric Leibniz algebra $\mathfrak{L}=V^*\oplus\mathfrak{g}\oplus V$ double extension of $\mathfrak{g}$ by V by means of (δ,δ,a_0). The product on $\mathfrak{L}$ is given by:

$$[d,d]=\alpha d^*+x+iy\,;\ [x,x]=-[y,y]=d^*\,;\ [x,y]=[y,x]=id^*$$
$$[x,d]=[d,x]=d^*+x+iy\,;\ [y,d]=[d,y]=id^*+ix-y.$$

The bilinear form $B_{\mathfrak{L}}$ defined on $\mathfrak{L}$ by $B_{\mathfrak{L}}(x,x)=B_{\mathfrak{L}}(y,y)=B_{\mathfrak{L}}(d,d^*)=1$, is an associatif scalar product on $\mathfrak{L}$. Then, the resulting quadratic Leibniz algebra is commutative 3-nilpotente of dimension 4 over $\mathbb{C}$.

Since the double extension depends on the choice of an admissible triple, then we may ask the following question: Under which conditions two admissible triples give equivalent double extensions?
First, we investigate the case of central extension. Let $(\mathfrak{L},B)$ be a quadratic Leibniz algebra and $\mathbb{K}d$ be a one dimensional Lie algebra. let us consider two derivations δ and δ' of $\mathfrak{L}$ such that the maps $w,w':\mathfrak{L}\times\mathfrak{L}\to\mathbb{K}$, defined by $w(x,y)=B(\delta(x),y)$; $w'(x,y)=B(\delta'(x),y),\forall x,y\in\mathfrak{L}$,are bi-2-cocycles of $\mathfrak{L}$.

Definition 4.9. If $\mathfrak{L}_c$ (resp. $\mathfrak{L}'_c$) is the symmetric Leibniz algebras central extension of $\mathfrak{L}$ by $\mathbb{K}d$ by means of w (resp. w') , then $\mathfrak{L}_c$ and $\mathfrak{L}'_c$ are said to be equivalent if there exists an isomorphism of Leibniz algebras $\phi:\mathfrak{L}_c\to\mathfrak{L}'_c$ satisfying:

$$\phi(d^*)=d^*,\phi(x)=x+\lambda(x)d^*,\forall x\in\mathfrak{L}\text{ where }\lambda\in\mathfrak{L}^*.$$

Let $u_o \in \mathfrak{L}$ such that $\lambda(x) = B(u_0, x), \forall x \in \mathfrak{L}$. The fact that ϕ is an isomorphism of Leibniz algebras implies that $w'(x,y) - w(x,y) = \lambda([x,y]), \forall x, y \in \mathfrak{L}$. That is, $\delta' - \delta = L_{u_0}$. Consequently, we have proved the following Proposition:

Proposition 4.10. *Let $\mathfrak{L}$ is a quadratic Leibniz algebra, $\mathbb{K}d$ be a one dimensional Lie algebra and $w, w' : \mathfrak{L} \times \mathfrak{L} \to \mathbb{K}$ be two bi-2-cocycles of $\mathfrak{L}$. Then, the central extensions $\mathfrak{L}_c$ and $\mathfrak{L}'_c$ of $\mathfrak{L}$ by $\mathbb{K}d$ by means of w and w' respectively are equivalent if and only if there exists a linear form $\lambda \in \mathfrak{L}^*$ satisfying: $w'(x,y) - w(x,y) = \lambda([x,y]), \forall x, y \in \mathfrak{L}$. (ie. $w' - w = -\xi\lambda$ where ξ is the Leibniz-coboundary defined in [10].)*

Now, let $(\mathfrak{L}, B)$ be a quadratic Leibniz algebra, $\mathbb{K}d$ be a one dimensional Lie algebra and $(\delta, \delta^*, a_0), (\delta', \delta'^*, a'_0)$ be two admissible triples of $\mathfrak{L}$. Let us fix $\alpha, \alpha' \in \mathbb{K}$.

Definition 4.11. If $\mathfrak{L}_\alpha$ (resp. $\mathfrak{L}_{\alpha'}$) is the double extension of $\mathfrak{L}$ by $\mathbb{K}d$ by means of (δ, δ^*, a_0) (resp. $(\delta', \delta'^*, a'_0)$). Then, $\mathfrak{L}_\alpha$ and $\mathfrak{L}_{\alpha'}$ are called equivalent if there exists an isomorphism of Leibniz algebras $\psi : \mathfrak{L}_\alpha \to \mathfrak{L}_{\alpha'}$ verifying:

$$\psi(x) = x + \lambda(x)d^*, \psi(d) = d + \alpha_0 + l_0 d^*, \psi(d^*) = d^*, \text{ where } \lambda \in \mathfrak{L}^*, \alpha_0 \in \mathfrak{L}, l_0 \in \mathbb{K}.$$

Proposition 4.12. *The doubles extensions $\mathfrak{L}_\alpha$ and $\mathfrak{L}_{\alpha'}$ are equivalent if and only if there exists $\alpha_0, u_0 \in \mathfrak{L}$ such that*

$$\begin{aligned}
&\delta' - \delta = L_{u_0} = -L_{\alpha_0} \\
&a_0 - a'_0 = (\delta + \delta^*)(\alpha_0) - [\alpha_0, \alpha_0] \\
&\delta(\alpha_0) = -\delta(u_0),\ \delta^*(\alpha_0) = -\delta^*(u_0) \\
&\alpha - \alpha' = 2B(\alpha_0, a'_0) + B(\delta'(\alpha_0), \alpha_0) - B(a_0, u_0).
\end{aligned}$$

Proof. Suppose that there exists an isomorphism of Leibniz algebras $\psi : \mathfrak{L}_\alpha \to \mathfrak{L}_{\alpha'}$ verifying:

$$\psi(x) = x + \lambda(x)d^*, \psi(d) = d + \alpha_0 + l_0 d^*, \psi(d^*) = d^*, \text{ where } \lambda \in \mathfrak{L}^*, \alpha_0 \in \mathfrak{L}, l_0 \in \mathbb{K}.$$

Let $u_0 \in \mathfrak{L}$ such that $\lambda(x) = B(u_0, x), \forall x \in \mathfrak{L}$. The fact that $\psi([x,y]) = [\psi(x), \psi(y)]$ implies that $\delta' - \delta = L_{u_0}$. Since $\psi([d,x]) = [\psi(d), \psi(x)]$, then $\delta - \delta' = L_{\alpha_0}$ and $a_0 - a'_0 = \delta'(\alpha_0) - \delta^*(u_0)$. Moreover, $\psi([d,d]) = [\psi(d), \psi(d)]$. Therefore, $a_0 - a'_0 = \delta'(\alpha_0) + \delta'^*(\alpha_0) + [\alpha_0, \alpha_0]$ and $\alpha - \alpha' = 2B(\alpha_0, a'_0) + B(\delta'(\alpha_0), \alpha_0) - B(a_0, u_0)$. In addition, $\delta^* - \delta'^* = R_{\alpha_0}$ and $a_0 - a'_0 = \delta'^*(\alpha_0) - \delta(u_0)$ because $\psi([x,d]) = [\psi(x), \psi(d)]$. Consequently,

$$\begin{aligned}
a_0 - a'_0 &= \delta'(\alpha_0) + \delta'^*(\alpha_0) + [\alpha_0, \alpha_0] \\
&= \delta(\alpha_0) - [\alpha_0, \alpha_0] + \delta^*(\alpha_0) - [\alpha_0, \alpha_0] + [\alpha_0, \alpha_0] \\
&= (\delta + \delta^*)(\alpha_0) - [\alpha_0, \alpha_0].
\end{aligned}$$

Or $a_0 - a'_0 = \delta'^*(\alpha_0) - \delta(u_0) = \delta'(\alpha_0) - \delta^*(u_0)$, then $\delta(\alpha_0) = -\delta(u_0), \delta^*(\alpha_0) = -\delta^*(u_0)$.

Conversely, consider the map $\psi : \mathfrak{L}_\alpha \to \mathfrak{L}_{\alpha'}$ defined by: $\psi(x) = x + B(u-0,x)d^*$, $\psi(d) = d + \alpha_0 + l_0 d^*, \psi(d^*) = d^*,$ where $\lambda \in \mathfrak{L}^*, \alpha_0 \in \mathfrak{L}, l_0 \in \mathbb{K}$. Then, by direct computations we prove that ψ is an isomorphism of vector spaces. ■

Using extensions developed in the previous section, we are going to give an inductive description of quadratic Leibniz algebras.

Theorem 4.13. *Let $(\mathfrak{L}, B)$ be a non Lie quadratic Leibniz algebra. Then, $\mathfrak{L}$ is isomorphic to a double extension of a symmetric Leibniz algebra by the one dimensional Lie algebra.*

Proof. Since $\mathfrak{L}$ is not a Lie algebra, then $\mathcal{I}_\mathfrak{L} \neq \{0\}$. Let $e \in \mathcal{I}_\mathfrak{L} \backslash \{0\}$ and $\mathcal{J} = \mathbb{K}e$. Then, $\mathcal{J} \subseteq \mathcal{J}^\perp$. Since B is non-degenerate, then there exists $d \in \mathfrak{L}\backslash\{0\}$ such that $B(e,d) = 1$ and $B(d,d) = 0$. Let $\mathcal{V} = \mathbb{K}d$ and $\mathfrak{H} = (\mathcal{J} \oplus \mathcal{V})^\perp$, then

$$\mathfrak{L} = \mathcal{J} \oplus \mathfrak{H} \oplus \mathcal{V} \qquad \text{and} \qquad \mathcal{J}^\perp = \mathcal{J} \oplus \mathfrak{H} \quad \text{ is an ideal of } \mathfrak{L}.$$

Let $x, y \in \mathfrak{H}$, then $[x,y] = \phi(x,y)e + [x,y]_\mathfrak{H}$ where $\phi : \mathfrak{H} \times \mathfrak{H} \to \mathbb{K}$ is a bilinear form and $[\ ,\ ]_\mathfrak{H} : \mathfrak{H} \to \mathfrak{H} \to \mathfrak{H}$ is a bilinear map. It is easy to show that $(\mathfrak{H}, [\ ,\ ]_\mathfrak{H})$ is a symmetric Leibniz algebra and that the bilinear form $B_\mathfrak{H} = B_{|_{\mathfrak{H}\times\mathfrak{H}}}$ is an associative scalar product on $(\mathfrak{H}, [\ ,\ ]_\mathfrak{H})$. Therefore, the bilinear map ϕ is a bi-2-cocycle on $(\mathfrak{H}, [\ ,\ ]_\mathfrak{H})$. The fact that $\mathcal{J}^\perp = \mathcal{J} \oplus \mathfrak{H}$ is an ideal of $\mathfrak{L}$ implies that

$$\begin{aligned}
&[d,d] = \alpha e + a_0 + \lambda d, \text{ where } \alpha, \lambda \in \mathbb{K}, a_0 \in \mathfrak{H},\\
&[x,d] = \varphi(x)e + D(x), \text{where } D \in End(\mathfrak{H}), \varphi \in \mathfrak{H}^*,\\
&[d,x] = \psi(x)e + \delta(x), \text{ where } \delta \in End(\mathfrak{H}), \psi \in \mathfrak{H}^*.
\end{aligned}$$

Since B is symmetric and associative, then $\phi(x,y) = B([x,y],d) = B(y,[d,x]) = B(x,[y,d]), \forall x,y \in \mathfrak{H}$. It follows that $B(D(y),x) = B(y,\delta(x))$. So, $D = \delta^*$. Moreover, $\varphi(x) = B([x,d],d) = B(d,[d,x]) = \psi(x) = B(x,a_0)$
and $\lambda = B([d,d],e) = B(d,[d,e]) = 0$. Then, $[d,d] = \alpha e + a_0$. Consequently, $a_0 \in \mathcal{I}_\mathfrak{L} \subseteq Z(\mathfrak{L})$. So, $B(a_0,a_0) = 0$ and $\delta(a_0) = D(a_0) = 0$. Therefore, $a_0 \in Z(H)$. In fact, for all $x \in \mathfrak{H}$ we have:
$[a_0,x]_\mathfrak{H} = [a_0,x] - B(\delta(a_0),x)e = 0,\quad [x,a_0]_\mathfrak{H} = [x,a_0] - B(x,\delta^*(a_0)) = 0$. Since $\mathfrak{L}$ is a symmetric Leibniz algebra, then by Proposition 1.4 $\mathfrak{L}$ is a left Leibniz algebra and the product on $\mathfrak{L}$ satisfies: $[a,[b,c]] + [[b,c],a] = 0, \forall a,b,c \in \mathfrak{L}$. Consequently, an easy calculation proves that:

$$\begin{aligned}
&\delta \in Der(\mathfrak{H}),\ \delta^2 + D \circ \delta = 0,\ \delta \circ D = D \circ \delta, (\delta + D)([\mathfrak{H},\mathfrak{H}]_\mathfrak{H}) = \{0\},\\
&\delta(\mathfrak{H}) \subseteq \mathcal{R}_\mathfrak{H},\ \delta^*(\mathfrak{H}) \subseteq \mathcal{R}_\mathfrak{H}, \quad \text{where } \mathcal{R}_\mathfrak{H} = \{y \in \mathfrak{H}, [x,y]_\mathfrak{H} + [y,x]_\mathfrak{H} = 0, \forall x \in \mathfrak{H}\}.
\end{aligned}$$

Therefore, we can consider the quadratic Leibniz algebra $\overline{\mathfrak{L}} = \mathcal{V}^* \oplus \mathfrak{H} \oplus \mathcal{V}$ double extension of $(\mathfrak{H}, B_\mathfrak{H})$ by V by means of the admissible triple (δ, δ^*, a_0).
It is clear that the map $F : \mathfrak{L} \to \overline{\mathfrak{L}}; \lambda e + x + \lambda' d \longmapsto \lambda d^* + x + \lambda' d$ is an isomorphism of Leibniz algebra. Then, $\mathfrak{L}$ is isomorphic to the double extension of $\mathfrak{H}$ by $\mathcal{V}$. ■

The following corollary reduces the description of quadratic Leibniz algebras to that of quadratic Lie algebras.

Corollary 4.14. *Let $(\mathfrak{L}, B)$ be a non Lie quadratic Leibniz algebra. Then, $\mathfrak{L}$ is obtained from a quadratic Lie algebra by a finite number of "Leibniz" double extensions by the one dimensional Lie algebra.*

Proof. We proceed by induction on the dimension $n \in \mathbb{N}$ of $\mathfrak{L}$. If $n = 2$, then $\mathfrak{L} = \mathfrak{L}_1$ (Examples 1.16), which is the "Leibniz" double extension of $\{0\}$ by the one dimensional Lie algebra.
Suppose that the result is true for every $k < n$. Then, by the preceding theorem, $\mathfrak{L}$ is a double extension of a symmetric Leibniz algebra $\mathfrak{H}$ by the one dimensional Lie algebra. Since $dim(\mathfrak{H}) = n - 2 < n$, then $\mathfrak{H}$ is obtained from a quadratic Lie algebra by a finite number of "Leibniz " doubles extensions by the one dimensional Lie algebra. ∎

This corollary and the description of quadratic Leibniz algebras (see [16]) give the following inductive description of quadratic Lie algebras.
Let Σ be the set constituted by $\{0\}$, the one dimensional Lie algebra and all simple Lie algebras.

Corollary 4.15. *Let $(\mathfrak{L}, B)$ be a quadratic Leibniz algebra. If $\mathfrak{L} \notin \Sigma$, then $\mathfrak{L}$ is obtained from a finite number of elements of Σ by a finite number of orthogonal direct sums of quadratic Lie algebras and/or "Lie" double extensions by the one dimensional Lie algebra and/or "Leibniz" double extensions by the one dimensional Lie algebra and/or "Lie" double extensions by a simple Lie algebra.*

Proof. The corollary 4.14 shows that $\mathfrak{L}$ is obtained from a quadratic Lie algebra $\mathfrak{g}$ by "Leibniz" doubles extensions by the one dimensional Lie algebra. Theorem 1 [16] proves that $\mathfrak{g}$ is obtained by a finite number of orthogonal direct sums and/or "Lie" doubles extensions by a simple Lie algebra and/or "Lie" doubles extensions ≪de Lie≫by the one dimensional Lie algebra. ∎

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