

Wrapping Brownian Motion and Heat Kernels II: Symmetric Spaces

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Abstract. In this paper we extend our previous results on wrapping Brownian motion and heat kernels onto compact Lie groups to various symmetric spaces, where a global generalisation of Rouvière’s formula and the e -function are considered. Additionally, we extend some of our results to complex Lie groups, and certain non-compact symmetric spaces.

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1. Introduction

In our previous paper [24], we wrapped Brownian motion and heat kernels from a compact Lie algebra \mathfrak{g} (viewed as a Euclidean vector space) to a compact Lie group G using the wrapping map, Φ , of Dooley and Wildberger [9]. Recall that Φ was defined for a suitable distribution, ν , by

$$\langle \Phi(\nu), f \rangle = \langle \nu, j\tilde{f} \rangle, \quad (1)$$

where $f \in C^\infty(G)$, $\tilde{f} = f \circ \exp$ and j the analytic square root of the determinant of the exponential map. The principal result is the *wrapping formula*, given by

Theorem 1.1. ([9], *Thm. 2*) *Let μ, ν be G -invariant distributions of compact support on \mathfrak{g} or two G -invariant integrable functions, then*

$$\Phi(\mu * \nu) = \Phi(\mu) * \Phi(\nu), \quad (2)$$

where the convolutions are in \mathfrak{g} and G , respectively.

In this paper we consider wrapping Brownian motion and heat kernels in the context of various symmetric spaces, where a global generalisation of the wrapping formula (2) utilising Rouvière’s e -function is considered.

We begin by recalling the theory of e -functions in section 3, firstly examining the wrapping map and the global e -function given by Dooley in [6] and [7], then combine this with the theory of e -functions developed by Rouvière in section 3 in order to state global versions of his formulae concerning differential operators. In sections 4 and 5 we then apply these results to show how to wrap heat kernels onto compact symmetric spaces from their (Euclidean) tangent space, but due to the appearance of the e -function, this requires some different ideas to those in our previous work. Although the complicated nature of the e -function makes explicit calculations difficult, our analysis provides several insights into both compact and non-compact symmetric spaces.

Results concerning Brownian motion and heat kernels on symmetric spaces have been previously given by many authors. Our method differs by using the wrapping map, which can be viewed as a global version of the exponential map. Thus, our results presented in sections 4 and 5 are obtained in the spirit of the tangent space analysis advocated by Helgason ([19], [18]).

In section 4 we consider the case of compact symmetric spaces, where our work explains why the well-known Gaussian approximation - also known as the “sum over classical paths” (see [5], [10]) - does not give exact results for compact symmetric spaces that are not Lie groups. As part of our analysis we also calculate the term $\Omega_* = j^{-1}L_{\mathfrak{p}}j$ used by Helgason in [18] that is not explicitly calculated there.

In section 5 we consider the case of non-compact symmetric spaces, we are able to quickly obtain the heat kernels for complex Lie groups by wrapping. We then discuss extensions of these results to the non-compact symmetric spaces of “split rank” type, in particular the spaces G/K , G complex, where our work implies that the local convolution formula obtained by Torossian ([28]) holds globally.

2. Notation and Formulae for Riemannian Symmetric Spaces

2.1. Introduction.

We follow the notation and conventions of Helgason [17] and Knapp [21], which are essentially standard. Let X be an irreducible, connected Riemannian symmetric space, and let G denote the identity component of the isometry group of X , and K the isotropy subgroup at a chosen origin, o . Note that K is a fixed compact subgroup of G and thus X is diffeomorphic to the quotient G/K . Let \mathfrak{g} be the Lie algebra of G , with exponential map $\exp : \mathfrak{g} \rightarrow G$, and also let \mathfrak{k} be the Lie algebra of K . Let σ denote an involutive automorphism of G with respect to K , such that we can write the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where \mathfrak{k} and \mathfrak{p} are the eigenspaces of σ .

We identify \mathfrak{p} as the tangent space of the Riemannian symmetric space G/K , with exponential map $\text{Exp} : \mathfrak{p} \rightarrow G/K$. Furthermore, if π is the canonical

mapping $\pi : G \rightarrow G/K$, then $\text{Exp} = \pi \circ \exp$. Let \mathfrak{a} denote a maximal abelian subalgebra of \mathfrak{p} , and call the dimension of \mathfrak{a} the *rank* of G/K . Denote Γ as the *integer lattice*, where $\Gamma = \{H \in \mathfrak{a} : \exp(H) = e\}$.

The classification of Riemannian symmetric spaces can be found in [17], Ch. V. We will not give details here, but will briefly outline the structure and relationship of the *compact type*, and the *non-compact type*: Let $\mathfrak{g}_{\mathbb{C}}$ be a complex Lie algebra and \mathfrak{u} a *compact real form*. $\mathfrak{g}_{\mathbb{C}}$ may be regarded as a real Lie algebra $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus i\mathfrak{u}$ with twice the (real) dimension of \mathfrak{u} . We write G for the Lie group corresponding $\mathfrak{g}_{\mathbb{C}}$, and U for the Lie group corresponding to \mathfrak{u} . Thus, if (G, K) is a Riemannian symmetric pair, then (U, K) is called its *compact dual pair*, and U/K is a compact Riemannian symmetric space. For example, let $U = SU(2)$ with $\mathfrak{u} = \mathfrak{su}(2)$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ and $G = SL(2, \mathbb{C})$.

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the set of roots on \mathfrak{g} with respect to \mathfrak{a} , and $W(\mathfrak{g}, \mathfrak{a})$ the corresponding Weyl group. The *restricted roots* of \mathfrak{p} , denoted by Σ_r are the roots of \mathfrak{g} restricted to \mathfrak{a} . Further details of root systems may be found in [21] Ch. VI. We will use Knapp’s definitions and notations for roots, weights, etc. We denote by \mathfrak{a}^+ the positive Weyl chamber with respect to the set of positive restricted roots, and $\bar{\mathfrak{a}}^+$ its closure. We also denote by m_{α} the multiplicity of the root α . A restricted root β is said to be *multipliable* if there exists another restricted root α such that $\beta = k\alpha$ for some integer $k \geq 2$. We denote the set of multipliable roots by Σ_m .

Let $A = \exp(\mathfrak{a})$, and $A^+ = \exp(\bar{\mathfrak{a}}^+)$, and let M denote the centraliser of A in K . Every element in G has a decomposition as k_1ak_2 , with $k_1, k_2 \in K$ and $a \in A$. In this decomposition, a is uniquely determined up to conjugation by a member of the Weyl group (see [21] Thm. 7.39). Additionally, G has a Cartan decomposition KA^+K . We have the following integral formula for symmetric spaces (see [19] Ch. I, Thm. 5.8):

$$\int_{G/K} f(x)dx = c \int_{K/M} \left(\int_{A^+} f(ka \cdot o)\delta(a)da \right) dk_M, \quad f \in C_c(G/K), \quad (3)$$

where c is a suitable constant, dx is the G -invariant measure on G/K , dk_M is the K -invariant measure on K/M , normalised with mass 1, and

$$\delta(\exp H) = \prod_{\alpha \in \Sigma_r^+} (\sinh \alpha(H))^{m_{\alpha}}, \quad H \in \mathfrak{a}^+.$$

We note that $\text{Ad}(k)\mathfrak{p} \subseteq \mathfrak{p}$, and that $\text{Ad} - K$ -invariant functions are determined by their values on \mathfrak{a}^+ (see [19] Ch. I, Thm. 5.17):

$$\int_{\mathfrak{p}} f(X)dX = c \int_{K/M} \left(\int_{\mathfrak{a}^+} f(\text{Ad}(k)H)\delta_0(H)dH \right) dk_M, \quad f \in C_c(\mathfrak{p}),$$

where c is a suitable constant, dk_M is the K -invariant measure on K/M , normalised with measure 1, and

$$\delta_0(H) = \prod_{\alpha \in \Sigma_r^+} \alpha(H)^{m_{\alpha}}, \quad H \in \mathfrak{a}.$$

We denote by J the Jacobian of Exp ,

$$\int_{G/K} f(x)dx = \int_{\mathfrak{p}} f(\text{Exp}X)J(X)dX, \quad f \in C_c(G/K)$$

with J given by $J = \delta/\delta_0$. We also write $J(X) = j^2(X)$, where j is calculated as:

$$j(H) = \prod_{\alpha \in \Sigma_r^+} \left(\frac{\sinh \alpha(H)}{\alpha(H)} \right)^{m_\alpha/2}, \quad H \in \mathfrak{a}.$$

However, we will require that j be smooth and real valued, which is clearly not the case globally for every Riemannian symmetric space. j is smooth and real valued for compact Lie groups and complex Lie groups (since their roots have even multiplicities), but this is not so in general, and some of our results will only be valid within a fundamental domain of the exponential map.

2.2. Differential operators and spherical functions.

We recall from [17] and [19] some notation and basic properties of differential operators on Riemannian symmetric spaces, to which we refer the reader for further details. Let $\mathbf{D}(G/K)$ be the algebra of G -invariant linear differential operators on G/K with complex coefficients. We also let $\mathbf{D}(\mathfrak{p})$ be the algebra of K -invariant constant coefficient linear differential operators on \mathfrak{p} . This is canonically isomorphic to $I(\mathfrak{p})$, the K -invariant subalgebra of the complexification of the symmetric algebra of \mathfrak{p} . Furthermore, $I(\mathfrak{p})$ consists of K -invariant polynomial functions on the dual space \mathfrak{p}^* .

We write $L_{G/K}$ and $L_{\mathfrak{p}}$ for the Laplacians on G/K and \mathfrak{p} , respectively. We will further assume that G/K is *isotropic* (that is, K acts transitively on the unit sphere of \mathfrak{p}), and as a result we have that $\mathbf{D}(G/K)$ and $\mathbf{D}(\mathfrak{p})$ are generated by polynomials of their respective Laplacians. We will also denote the transpose of a differential operator, D , on G/K by D^t . A differential operator D is *symmetric* if it satisfies $D^t = D$, and note that $L_{G/K}$ is a symmetric operator. An important result we will use later is that the Laplacians $L_{G/K}$ and $L_{\mathfrak{p}}$ are related as follows ([19] Ch. II, Prop. 3.15): the image $L_{G/K}^{\text{Exp}^{-1}}$ of $L_{G/K}$ under Exp^{-1} is given by:

$$L_{G/K}^{\text{Exp}^{-1}}f = (j^{-1}L_{\mathfrak{p}} \circ j)f - j^{-1}(L_{\mathfrak{p}}j)f$$

for each K -invariant C^∞ function f on \mathfrak{p} . It is important to note as per our above remarks on the j function, this result holds in general only in a fundamental domain of Exp , but globally in the case of a compact or complex Lie group ([19] Ch. II, §3).

Also recall that a smooth, K -invariant function ϕ on G/K with $\phi(o) = 1$, is said to be a K -spherical function on G/K if it satisfies:

- (i) $\phi(kgK) = \phi(gK) \quad \forall k \in K, g \in G,$
- (ii) $D\phi = \lambda_D\phi$ for each $D \in \mathbf{D}(G/K),$ where $\lambda_D \in \mathbb{C}.$

Property (i) ensures that a spherical function is determined by its values on A^+ . We will also refer to these functions as **bi- K -invariant** functions. We can also view a spherical function ϕ as a K -invariant function on G/K such that $\phi(x) = 1$, where $x = \{K\} \in G/K$.

2.3. Heat kernels and Brownian motion.

Heat equations on Riemannian symmetric spaces have been studied by many authors in a variety of ways. We define the heat equation by:

$$\frac{\partial}{\partial t}u(x, t) = L_{G/K}u(x, t) \tag{4}$$

with initial data $u(x, 0) = f(x)$. The solution on the Cauchy problem is given by

$$u(x, t) = \int_{G/K} h_t(x, y)f(y)dy, \tag{5}$$

where h_t is the heat kernel. We summarise some key properties of h_t :

Theorem 2.1. (c.f [3], [4]) h_t satisfies the following properties: for all $x, y \in G/K$,

- (1) $h_t(x, y) = h_t(y, x) > 0$,
- (2) h_t is the density of a probability measure, with $\lim_{t \rightarrow 0} h_t(x, y) = \delta_x(y)$,
- (3) $(\frac{\partial}{\partial t} - L_{G/K})h_t = 0$,
- (4) $h_{t+s}(x, y) = \int_{G/K} h_t(x, z)h_s(z, y)dz$.

Moreover, if U/K is compact, then h_t can be expressed in terms of the eigenvalues and eigenfunctions of the Laplacian as

$$h_t(x, y) = \sum_{\lambda \in \Lambda^+} e^{-(\|\lambda+\rho\|^2 - \|\rho\|^2)t} \varphi_\lambda(x)\varphi_\lambda(y).$$

These properties of h_t can be shown to hold on more general manifolds - the reader is referred to the listed sources. On G/K , the G -invariance implies that:

Theorem 2.2. (c.f [3], [4]) h_t also satisfies the following properties: for all $x, y \in G/K$,

- (1) $h_t(x, y) = h_t(y^{-1}x)$ is a convolution kernel,
- (2) $x \mapsto h_t(x)$ is K -invariant on G/K , and thus determined by its restriction to the positive Weyl chamber.

Remark 2.3. These formulas can be used to derive the heat kernel on a compact Lie group:

$$H(g, t) = \sum_{\lambda \in \Lambda^+} d_\lambda e^{-(\|\lambda+\rho\|^2 - \|\rho\|^2)t} \chi_\lambda(g), \quad g \in G, t \in \mathbb{R}^+,$$

which follows since the characters are the eigenfunctions of the Laplacian, with eigenvalue $\|\lambda + \rho\|^2 - \|\rho\|^2$, and $\chi_\lambda(e) = d_\lambda$.

We briefly recall the definition of a Brownian motion on a Riemannian symmetric spaces as given in Liao [22], Ch. 2, to which we refer the reader to for further details of diffusion processes on Riemannian manifolds. A diffusion process $(\xi_t)_{t \geq 0}$ in G/K is called a *Brownian motion* if its generator, when restricted to $C_c(G)$, is equal to $\frac{1}{2}L_G$, one-half the Laplacian on G .

Importantly, we note that if $(B_t)_{t \geq 0}$ is a Brownian motion on G/K , then $L_{G/K}$ is the generator of $(B_t)_{t \geq 0}$, and $(B_t)_{t \geq 0}$ satisfies

$$\mathbb{E}(f(B_t)) = \int_{G/K} h_t(x, y) f(y) dy. \quad (6)$$

As we showed in our previous paper [24], wrapping Brownian motion is intimately corrected to wrapping the Laplacian. Hence, in the rest of this paper we shall consider the wrap of the Laplacian for Riemannian symmetric spaces, and focus less on wrapping Brownian motion, as the mechanics of which are essentially identical to that in [24], section 4.2. In the next section we shall consider the Rouvière's e -function, and the global generalisation of Dooley. We combine these in section 3 to show how Rouvière's local formula for the Laplacian holds in a larger setting, that is, how to “wrap” the Laplacian.

3. Rouvière's e -function and the wrapping map for compact Riemannian symmetric spaces

In this section we will consider the so-called e -function introduced by Rouvière in [26] and [27], and the global generalisation given by Dooley in [6] and [7] in terms of the wrapping map. After rephrasing Rouvière's results in the notation of the wrapping map, we will show how his results regarding the local properties of differential operators hold globally (or at least in the larger space of a fundamental domain of the exponential map) by using the results of Dooley. These results will be applied to give our results in section 4.

3.1. The wrapping map.

We now define a version of the wrapping map for compact Riemannian symmetric spaces. Let U/K be a compact Riemannian symmetric space with tangent space \mathfrak{p} . Let ν be a distribution of compact support on \mathfrak{p} , and $f \in C_c^\infty(U)$. We define the *wrapping map*, Φ on U/K by

$$\langle \Phi(\nu), f \rangle_{U/K} = \langle \nu, j \cdot f \circ \text{Exp} \rangle_{\mathfrak{p}}. \quad (7)$$

In the case of a compact symmetric space, the wrapping map is no longer a homomorphism: the convolution on the tangent space becomes “twisted” by a function denoted by e :

Theorem 3.1. ([6]) *Let U/K be a compact symmetric space with tangent space \mathfrak{p} , and μ and ν K -invariant Schwartz functions or distributions of compact support on \mathfrak{p} . There is a function $e : \mathfrak{p} \times \mathfrak{p} \rightarrow \mathbb{C}$ such that*

$$\Phi(\mu) *_{U/K} \Phi(\nu) = \Phi(\mu *_{\mathfrak{p},e} \nu), \tag{8}$$

where

$$(\mu *_{\mathfrak{p},e} \nu)(X) = \int_{\mathfrak{p}} \mu(Y)\nu(X - Y)e(X, Y)dY. \tag{9}$$

We will call the expression $\mu *_{\mathfrak{p},e} \nu$ a *twisted convolution*.

The e -function was introduced by Rouvière in [26], where a local version of Theorem 3.1 was proved for general symmetric spaces (see [26], Prop. 4.1). Efforts to prove a global version of Rouvière’s formula for symmetric spaces have continued in [6] and [7].

The e -function arises from the following: The left hand side of (8) can be written as

$$\begin{aligned} \langle \Phi(\mu) *_{U/K} \Phi(\nu), f \rangle &= \int_{U/K} \int_{U/K} \Phi(\mu)(xK)\Phi(\nu)(yK)f(xyK)dx dy \\ &= \int_{\mathfrak{p}} \int_{\mathfrak{p}} \mu(X)\nu(Y)j(X)j(Y)f(\exp X \exp Y)dX dY. \end{aligned}$$

It is possible to show using the Campbell-Baker-Hausdorff series for Exp (see [6] or [26]), that there exists $h, k \in \mathfrak{h}$ such that

$$\exp X \exp Y = \text{Exp}(h.X + k.Y).$$

We thus make the change of variables $(h.X, k.Y) \mapsto (X, Y)$, and let the Jacobian of this transformation be $\psi(X, Y)$. Our expression then becomes

$$\int_{\mathfrak{p}} \int_{\mathfrak{p}} \mu(h^{-1}X)\nu(k^{-1}Y)j(h^{-1}X)j(k^{-1}Y)f(\text{Exp}(X + Y))\psi(X, Y)dX dY.$$

Since j, μ and ν are all K invariant, this becomes

$$\int_{\mathfrak{p}} \int_{\mathfrak{p}} \mu(X)\nu(Y)\frac{j(X)j(Y)}{j(X + Y)}\psi(X, Y)(j.f \circ \text{Exp})(X + Y)dX dY.$$

Putting

$$e(X, Y) = \frac{j(X)j(Y)}{j(X + Y)}\psi(X, Y)$$

we have

$$\int_{\mathfrak{p}} \int_{\mathfrak{p}} \mu(X)\nu(Y)e(X, Y)(j.f \circ \text{Exp})(X + Y)dX dY$$

which is (9).

We will now briefly show how a global e -function may be constructed for the case of the two-sphere. This construction can be found in [29] and [6] (the latter elucidates as to how this may then be extended to all compact symmetric spaces ([7])).

The global e -function is a ratio g/f that compares the convolution structures of K -orbits of S^2 , to K -orbits of $\mathfrak{p} \cong \mathbb{R}^2$ - the K -orbits in this case being circles centred at the origin. To calculate f , consider two circles centred at the origin of radius r_1 and r_2 on \mathbb{R}^2 . For notational convenience, it is best to consider these circles on the complex plane. We will consider the point r_1 on the first circle (we could take any point, but we could obtain the same result by rotation), and centre the circle of radius r_2 here.

We pick a point on the repositioned circle of radius r_2 . This point can be represented from the above construction as $r_1 + r_2 e^{i\theta}$, or as $r e^{i\psi}$. Therefore,

$$r_1 + r_2 e^{i\theta} = r e^{i\psi}$$

We now vary the point on the circle of radius r_2 (by varying θ), and calculate how r varies. It is not hard to show that

$$\frac{2r}{2r_1 r_2 \sin \theta} \frac{dr}{\pi} = \frac{d\theta}{\pi}$$

The denominator on the left-hand side is the area of the triangle on the complex plane with vertices 0 , r_1 and $r_2 e^{i\theta}$. By Heron's formula, we have:

$$2r_1 r_2 \sin \theta = \left(\prod_{\pm} (r \pm r_1 \pm r_2) \right)^{1/2}$$

where the product is taken over all choices of $+$ and $-$. Thus, the convolution of two circles of radius r_1 and r_2 has density

$$f_{r_1, r_2}(r) = \frac{2r}{\prod_{\pm} (r \pm r_1 \pm r_2)^{1/2}} \chi_{[|r_1 - r_2|, r_1 + r_2]}(r)$$

where χ is the characteristic function for the interval. A similar calculation can be done on the surface of the two-sphere, yielding:

$$g_{r_1, r_2}(r) = \frac{\sin r}{\pi \sin r_1 \sin r_2} \prod_{\pm} 2 \sin \frac{1}{2}(r \pm r_1 \pm r_2)^{1/2} \chi_{[|r_1 - r_2|, r_1 + r_2]}(r)$$

The e -function for the two-sphere is given by

$$(g/f)(r) = \frac{\sin r}{\pi \sin r_1 \sin r_2} \prod_{\pm} \frac{2 \sin \frac{1}{2}(r \pm r_1 \pm r_2)^{1/2}}{(r \pm r_1 \pm r_2)^{1/2}} \chi_{[|r_1 - r_2|, r_1 + r_2]}(r)$$

In [7], it will be shown that a global version of the e -function for all compact symmetric spaces exists. The proof consists of reducing the calculation to the two-dimensional case and using the above ideas. The e -function for the n -dimensional

sphere is:

$$e(X, Y) = \frac{\sin r}{\pi \sin r_1 \sin r_2} \left(\prod_{\pm} \frac{2 \sin \frac{1}{2}(r \pm r_1 \pm r_2)^{1/2}}{(r \pm r_1 \pm r_2)^{1/2}} \chi_{[|r_1-r_2|, r_1+r_2]}(r) \right)^{(n-3)/2}, \quad (10)$$

where $X \in \mathfrak{p}$ is conjugate to r_1H in \mathfrak{a} , $Y \in \mathfrak{p}$ is conjugate to r_2H in \mathfrak{a} , and $X + Y \in \mathfrak{p}$ is conjugate to rH in \mathfrak{a} .

Since the wrapping formula is now a “twisted” homomorphism, it is no longer clear that we may wrap Brownian motion and the heat kernel without some modification. In the next subsection, we will show how the Laplacian and e -function interact.

3.2. Rouvière’s formulae, the wrapping map, and differential operators.

In our previous paper [24], it was shown that the wrap of the Laplacian determines how Brownian motion wraps. However, for general symmetric spaces we do not have such straightforward expressions as we did in [24], due to the twisted convolution involving the e -function (8).

The relationship between the e -function and differential operators is given by Rouvière in [27]. The e -function is written as an infinite series and shown that it converges within a certain neighbourhood of $0 \in \mathfrak{p}$. Therefore, the results in [27] concerning the relationship between the e -function and differential operators only hold within this neighbourhood.

Equipped with the results from [6] and [7] presented in the previous section concerning the global existence of the e -function, we now show that Rouvière’s formulae hold at least within a neighbourhood of $0 \in \mathfrak{p}$ where the j function is smooth and real valued.

We will retain the notations and conventions given in section 2: let G/K be an isotropic Riemannian symmetric space, with tangent space \mathfrak{p} , and $\text{Exp} : \mathfrak{p} \rightarrow G/K$ the exponential map. In the compact case, we will also retain the use of U/K to make this restriction explicit in our formulae, although Rouvière’s results hold for the G/K case, and we will continue to use it in his results.

Remark 3.2. In fact, Rouvière ([26], [27]) considers symmetric spaces which need not be Riemannian, and thus uses the notation $S = G/H$ to denote a symmetric space, and S_0 instead of \mathfrak{p} . We will continue to use G/K and \mathfrak{p} as our analysis will only be considering Riemannian symmetric spaces.

Let \mathfrak{p}' be an K -invariant neighbourhood of $0 \in \mathfrak{p}$ such that Exp is a diffeomorphism. Let Ω_0 be an open neighbourhood of $(0, 0) \in \mathfrak{p}' \times \mathfrak{p}'$ satisfying the condition that if $(X, Y) \in \Omega_0$, then $(k \cdot X, k \cdot Y) \in \Omega_0$ for all $k \in K$, and $(-X, -Y) \in \Omega_0$. Let \mathfrak{p}'' denote the set of $X \in \mathfrak{p}'$ such that $(X, 0) \in \Omega_0$.

We also follow Rouvière's notation for differential operators, writing the elements of $\mathbf{D}(\mathfrak{p}) \cong I(\mathfrak{p})$ of polynomials on \mathfrak{p}^* as $p(\partial_X)$, p , or $p(\xi)$. Recall that the *symbol* of a differential operator on $\mathfrak{p} \cong \mathbb{R}^n$, given by:

$$p(X, \partial_X)f(X) = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} p(X, \xi) \hat{f}(\xi) e^{2\pi i \langle Y, \xi \rangle} dY \right) d\xi$$

We note again that as we assumed G/K is isotropic, $\mathbf{D}(\mathfrak{p})$ and $\mathbf{D}(G/K)$ are polynomial algebras generated, by $L_{\mathfrak{p}}$ and $L_{G/K}$ respectively.

We write ∂_{ξ} for $\partial/\partial\xi$. Let $e_p(X) = (p(\partial_Y)e)(X, 0)$, and $\Delta_Y e(X) = \Delta_Y e(X, 0)$, where Δ_Y is the Laplacian acting on the second variable, evaluated at 0. Let δ_o and δ_0 be the Dirac deltas on G/K and \mathfrak{p} , respectively. The map $p \rightarrow \tilde{p}$ from $\mathbf{D}(\mathfrak{p}) \rightarrow \mathbf{D}(G/K)$ is a linear isomorphism according to

$$(p^t \delta_0)^{\sim} = \tilde{p}^t \delta_o$$

or, more generally,

$$\alpha * (p^t \delta_0)^{\sim} = \tilde{p}^t \alpha$$

for any distribution α on G/K .

Rouvière uses the notation \tilde{f} as the inverse of the map $f \mapsto j(f \circ \exp)$. For the wrapping map (restricted to the fundamental domain) this would read:

$$\langle \Phi(\mu), \tilde{f} \rangle = \langle \mu, f \rangle$$

Our main result provides an analogue of [24] Prop 4.2 for compact Riemannian symmetric spaces:

Proposition 3.3. *Let U/K be a compact Riemannian symmetric space. Suppose μ is a K -invariant Schwartz function on \mathfrak{p} , then on a suitable neighbourhood of $0 \in \mathfrak{p}$,*

$$\Phi((L_{\mathfrak{p}} - \Omega_*)\mu) = L_{G/K}(\Phi(\mu))$$

The quantity Ω_* will be defined below, as well as what constitutes a “suitable neighbourhood”. We will prove Proposition 3.3 by restating a number of results concerning differential operators from [27] in the language of the wrapping map.

Firstly, we recall Theorem 3.1: If μ and ν are K -invariant Schwarz functions or distributions of compact support on \mathfrak{p} , then

$$\Phi(\mu) *_{U/K} \Phi(\nu) = \Phi(\mu *_{\mathfrak{p},e} \nu), \quad (11)$$

where

$$(\mu *_{\mathfrak{p},e} \nu)(X) = \int_{\mathfrak{p}} \mu(Y) \nu(X - Y) e(X, Y) dY. \quad (12)$$

(11) and (12) may be written as follows (see [27] Thm. 2.1, though note we have no restrictions on the supports of μ and ν):

$$\langle \Phi(\mu) *_{U/K} \Phi(\nu), f \rangle = \langle \mu(Y)\nu(X), e(X, Y)j(X + Y)f(\text{Exp}(X + Y)) \rangle. \tag{13}$$

(13) may be applied to give a relationship between wrapping, differential operators, and the e -function.

Rouvière calculates the symbol for general differential operators using the e -function:

Theorem 3.4. ([27] Thm. 3.1) *Let G/K be a symmetric space, and $p \in I(\mathfrak{p})$, and let W be an K -invariant open subset of \mathfrak{p}'' .*

(i) *For any K -invariant distribution u on W , one has*

$$\tilde{p}^t \tilde{u} = ((p^t(X, \partial_X))u)^\sim \quad \text{on Exp } W$$

where $p(X, \xi)$ is the differential operator with analytic coefficients on \mathfrak{p}'' corresponding to the symbol

$$p(X, \partial_X) = e(X, \partial_\xi)p(\xi) = \sum \frac{1}{\alpha!} \partial_Y^\alpha e(X, 0) \cdot \partial_\xi^\alpha p(\xi);$$

here, $X \in \mathfrak{p}''$, $\xi \in \mathfrak{p}^*$ and \sum is a (finite) summation over $\alpha \in \mathbb{N}^n$.

(ii) *If f is any H -invariant C^∞ function on W , one has*

$$\tilde{p} \tilde{f} = ((p(X, \partial_X))f)^\sim \quad \text{on Exp } W$$

We will be specifically considering the Laplacian, where Rouvière proves:

Corollary 3.5. ([27], Cor. 3.6) *With the above notation, we have*

(i) $\tilde{L}_\mathfrak{p} = L_{G/K} + L_\mathfrak{p}j(0),$

(ii) $L_{G/K}\tilde{u} = (L_\mathfrak{p}u - j^{-1}L_\mathfrak{p}j \cdot u)^\sim$ on $\text{Exp } W$.

(iii) $j^{-1}L_\mathfrak{p}j(X) = L_\mathfrak{p}j(0) - L_Y e(X, 0)$ on \mathfrak{p}'' ,

(iv) *If G is complex semisimple, then $L_\mathfrak{p}j(0) = n/12$, where $n = \dim G/K$.*

We now state our version of Theorem 3.4 (with proof almost identical to Rouvière's) using the notation of wrapping:

Theorem 3.6. *For any K -invariant Schwartz functions or distributions of compact support μ on \mathfrak{p} , we have for a suitable neighbourhood of $0 \in \mathfrak{p}$:*

$$\Phi(p)\Phi(\mu) = \Phi((p(X, \partial_X))^t \mu), \tag{14}$$

where $p(X, \partial_X)$ is the differential operator with analytic coefficients on \mathfrak{p} corresponding to the symbol

$$p(X, \xi) = e(X, \partial_\xi)p(\xi) = \sum_{\alpha} \frac{1}{\alpha!} \partial_Y^\alpha e(X, 0) \cdot \partial_\xi^\alpha p(\xi)$$

Moreover, if $f \in C_K^\infty(G/K)$, then

$$\Phi(p)\Phi(f) = \Phi((p(X, \xi))^t f)$$

The next result that we require is:

Proposition 3.7. ([27] Prop. 3.5(i)) *Suppose p is a second order homogeneous operator. Then, with the above notation,*

$$p(X, \partial_X) = p(\partial_X) + e_p(X) = (p(X, \partial_X))^t \quad (15)$$

We now state our version of Corollary 3.5, and define the “suitable neighbourhood” for which it holds:

Corollary 3.8. *Suppose U/K is a compact Riemannian symmetric space with tangent space \mathfrak{p} , and μ is any K -invariant function, or distribution, on \mathfrak{p} . Then for a suitable neighbourhood of $0 \in \mathfrak{p}$ comprising of the domain where j is smooth and real valued, we have:*

$$L_{U/K}\Phi(\mu) = \Phi((L_{\mathfrak{p}} - j^{-1}L_{\mathfrak{p}}j)(\mu))$$

Moreover,

$$j^{-1}L_{\mathfrak{p}}j(X) = L_{\mathfrak{p}}j(0) - L_Y e(X, 0)$$

Proof. Firstly, recall that the Laplacian is a symmetric operator. By Theorem 3.6 and Proposition 3.7 we have

$$\Phi(L_{\mathfrak{p}}\mu + (\Delta_Y e(X, 0) - (L_{\mathfrak{p}}j)(0))\mu) = L_{U/K}\Phi(\mu)$$

where j is smooth and real valued. Taking $\mu = j$, by [8] Prop. 2.4, we have $\Phi(j) = 1$, and so

$$\Phi(L_{\mathfrak{p}}j + (\Delta_Y e(X, 0) - (L_{\mathfrak{p}}j)(0))j) = L_{G/K}1 = 0$$

and thus

$$\Omega_* = j^{-1}L_{\mathfrak{p}}j = (L_{\mathfrak{p}}j)(0) - \Delta_Y e(X, 0)$$

which concludes the proof. ■

Remark 3.9. Corollary 3.8 thus completes the proof of Proposition 3.3, as well as provides Ω_* as promised there.

In the next section, we will compute the term $j^{-1}L_{\mathfrak{p}}j$, which in turn will shed more light on the domain where our results hold.

4. Application to Compact Riemannian Symmetric Spaces

We now apply the above results to analyse compact Riemannian symmetric spaces. However, the situation is more complicated than the case of a compact Lie group since the wrapping map is no longer a homomorphism, and we shall analyze this situation using Rouvière’s e -functions.

We are able to make some general statements about wrapping heat kernels to U/K , but we are unable to explicitly compute the heat kernels due to complicated potential terms arising from the e -function. These complications do enable us to show why the *Gaussian approximation* to the heat kernel does not give exact results for compact symmetric spaces that are not compact Lie groups.

4.1. The Gaussian approximation.

Analysis of the heat kernel K_t on a manifold M of dimension n by considering the heat kernel on its tangent space, combined with information about the exponential map, has been considered by many authors. For small time values, the *Minakshisundaram-Pleijel (M-P) expansion* provides an expansion for the heat kernel on a small neighbourhood of M : if x and y are close, then as $t \rightarrow 0$,

$$K_t(x, y) = (2\pi t)^{-n/2} \exp\left(\frac{-d(x, y)^2}{2t}\right) \times \left(c_0 + tc_1 + \dots t^n c_n + O(t^{n+1})\right), \quad (16)$$

where $d(x, y)$ is the Riemannian distance, and where the c_k ’s are dependent on x and y . Further details of this expansion and related techniques can be found, for example, in [5].

These neighbourhoods are then patched together to approximate the heat kernel on M . This involves using the first order approximation M-P expansion - the term c_0 can be shown to be equal to j^{-1} . The solution which this technique produces is sometimes referred to as the *Gaussian approximation* to the heat kernel on M , which appears to have been first written down in the thesis of Low [23] (see also Camporesi [5], §5). We state the approximation here for the case of a Riemannian symmetric space G/K , using our previous notation:

Definition 4.1. The *Gaussian approximation* to the heat kernel on G/K is given by

$$K_{\text{Gaussian}}(\text{Exp}X, t) = \sum_{\gamma \in \Gamma} \frac{p_t}{j} (X + \gamma), \quad X \in \mathfrak{a}, \quad (17)$$

where p_t is the heat kernel on $\mathfrak{p} \cong \mathbb{R}^n$.

In Dowker [10] (see also Camporesi [5]) the exactness of the Gaussian approximation (17) (except for a so-called “phase factor” of $e^{\|\rho\|^2 t}$) is asserted to hold for compact Lie groups, but not for general compact symmetric spaces.

In the case of compact Lie groups, we assert that the reason why the Gaussian approximation (17) gives an exact expression of the shifted heat kernel, but not in the case of general compact symmetric spaces, is explained by the wrapping

map and the e -function.

Before proceeding further, let us recall the notation for the heat kernels that was used in [24]: Let p_t denote the heat kernel on $\mathfrak{p} \cong \mathbb{R}^n$, let q_t denote the heat kernel on G/K , and q_t^ρ denote the *shifted* heat kernel on G/K , that is, the solution to the heat equation when the Laplacian contains the shift $\|\rho\|^2$.

Firstly, we compute the wrap of a K -invariant Schwartz function as a sum over the integer lattice. The proof is almost identical to the proof of [9], Thm. 1:

Proposition 4.2. *Suppose μ is a K -invariant Schwartz function on \mathfrak{p} , given on \mathfrak{a} . Then,*

$$\Phi(j\mu)(\exp H) = \sum_{\gamma \in \Gamma} \mu(H + \gamma), \quad H \in \mathfrak{a}.$$

Proof. Firstly, we let

$$f^K(g) = \int_K f(k \cdot g) dk$$

Let Ψ be the K -invariant C^∞ function on U/K given on A by $\Psi(\text{Exp}H) = \sum_{\gamma \in \Gamma} \mu(H + \gamma)$, where $H \in \mathfrak{a}$. For $f \in C^\infty(U/K)$,

$$\begin{aligned} (\Phi(j\mu), f) &= (j\mu, j\tilde{f}) = \int_{\mathfrak{p}} j^2(X)\mu(X)\tilde{f}(X)dX \\ &= \int_{\mathfrak{a}^+} \prod_{\alpha \in \Sigma_r^+} \alpha^{m_\alpha}(H)j^2(H)\mu(H) \int_K \tilde{f}(h \cdot H)dh dH \\ &= \int_{\mathfrak{a}^+} \prod_{\alpha \in \Sigma_r^+} |\sin \alpha(\exp H)|^{m_\alpha} \mu(H)(\tilde{f})^K(H)dH \\ &= \frac{1}{|W|} \int_{\mathfrak{a}} \prod_{\alpha \in \Sigma_r^+} |\sin \alpha(\text{Exp}H)|^{m_\alpha} \mu(H)(f^K)^\sim(H)dH. \end{aligned}$$

If $\mathfrak{a}_\Gamma \subseteq \mathfrak{a}$ is a fundamental domain for Γ in \mathfrak{a} , then this becomes

$$\begin{aligned} &\frac{1}{|W|} \int_{\mathfrak{a}_\Gamma} \prod_{\alpha \in \Sigma_r^+} |\sin \alpha(\text{Exp}H)|^{m_\alpha} (f^K)^\sim(H) \sum_{\gamma \in \Gamma} \mu(H + \gamma)dH \\ &= \frac{1}{|W|} \int_A \prod_{\alpha \in \Sigma_r^+} |\sin \alpha(a)|^{m_\alpha} (f^K)(a)da \\ &= \int_{U/K} \Psi(h)f(h)dh \end{aligned}$$

as required. ■

By Proposition 4.2, wrapping the heat kernel p_t on $\mathfrak{p} \cong \mathbb{R}^n$ yields the Gaussian approximation (17). From [24], in the case of a compact Lie group we have

$$\Phi(p_t) = q_t^\rho.$$

In this case, the Gaussian approximation is the shifted heat kernel. Moreover, note that the wrapping formula gives:

$$\Phi(p_{t+s}) = \Phi(p_t * p_s) = \Phi(p_t) * \Phi(p_s) = q_t^\rho * q_s^\rho = q_{t+s}^\rho.$$

However, for compact symmetric spaces that are not Lie groups, we have a non-trivial e -function that is “twisting” the convolution structure on \mathfrak{p} , which “twists” wrapping the heat convolution semigroup:

$$q_{t+s}^\rho = q_t^\rho *_{U/K} q_s^\rho = \Phi(p_t) *_{U/K} \Phi(p_s) = \Phi(p_t *_{\mathfrak{p},e} p_s), \tag{18}$$

but $\Phi(p_t *_{\mathfrak{p},e} p_s)$ is not equal to $\Phi(p_{t+s})$ when e is not identically 1. Thus, the underlying reason that the Gaussian approximation is not exact in this case is that the wrapping map does not preserve convolution here.

Although we have a twisted convolution on the tangent space, the results from section 3 show that the potential term and the e -function are closely related. That is,

$$\Phi((L_{\mathfrak{p}} - \|\rho\|^2 + L_Y e(X, 0))\mu) = L_{U/K}(\Phi(\mu))$$

and

$$-\|\rho\|^2 + L_Y e(X, 0) = (j^{-1}L_{\mathfrak{p}}j)(X)$$

We can now extend the results from [24] to wrapping Brownian motion and the heat kernel onto U/K from \mathfrak{p} . This involves a more complex potential term than the constant $\|\rho\|^2$ term we encountered in our previous results.

A Brownian motion may also be “wrapped” onto U/K using the same mechanics as presented in [24] section 4.2. However, rather than considering a Brownian motion on $\mathfrak{p} \cong \mathbb{R}^n$, it will mean considering a drifted Brownian motion with potential $L_{\mathfrak{p}} + (j^{-1}L_{\mathfrak{p}}j)$.

Therefore, in light of this and Corollary 3.3 we need to consider the heat equation with potential $j^{-1}L_{\mathfrak{p}}j$ given on $\mathfrak{p} \cong \mathbb{R}^n$ by

$$\frac{\partial p_t^e}{\partial t} = (L_{\mathfrak{p}} + (j^{-1}L_{\mathfrak{p}}j))p_t^e, \tag{19}$$

and we shall refer to the fundamental solution of (19), p_t^e , as the *perturbed heat kernel*. Wrapping the perturbed heat kernel from \mathfrak{p} will then yield the heat kernel on U/K since these correspond to Laplacians on \mathfrak{p} and U/K :

Theorem 4.3. *The wrap of the perturbed heat kernel p_t^e on \mathfrak{p} , given by*

$$\Phi(p_t^e)(\text{Exp}H) = \sum_{\gamma \in \Gamma} \frac{p_t^e}{j}(H + \gamma)$$

is the the heat kernel on U/K .

However, the potential term $j^{-1}L_{\mathfrak{p}}j$ is complicated - even for the case of the two-sphere, and finding the perturbed heat kernel appears extremely difficult. As a first step, we now calculate these potentials.

4.2. The quantity $j^{-1}L_{\mathfrak{p}}j$ for the compact case.

In this section we calculate $j^{-1}L_{\mathfrak{p}}j$. In Helgason [18] this is referred to as the quantity Ω_* , but is not explicitly calculated. This calculation is similar in spirit to that of [15] section 2, where the authors calculate $(j \circ \log)L_{G/K}(j \circ \log)^{-1}$ for the non-compact case. However, this quantity in the compact case is not straightforward to handle, as the singular values of j^{-1} would require careful treatment of $(j \circ \log)L_{G/K}(j \circ \log)^{-1}$, and certainly would not lend itself easily to global analysis.

Firstly, note that we are simply taking the derivatives of a function that is invariant under the action of K . Thus, to compute $L_{\mathfrak{p}}j$, we firstly recall the following result from Helgason. Let $\{H_i : i = 1, \dots, l\}$ denote an orthonormal basis of \mathfrak{a} , and D_{H_i} the first-order differential operator in the H_i direction, with $L_{\mathfrak{a}}$ is the Laplacian on \mathfrak{a} .

Proposition 4.4. *([19], Ch. 2, Prop 3.13) For the adjoint action of K on \mathfrak{p} with transversal manifold \mathfrak{a}^+ , the radial part of the Laplacian $L_{\mathfrak{p}}$, denoted by $\Delta(L_{\mathfrak{p}})$, is given by*

$$\begin{aligned} \Delta(L_{\mathfrak{p}}) &= \sum_{i=1}^l \left(D_{H_i}^2 + \sum_{\alpha \in \Sigma_r^+} m_{\alpha} \frac{1}{\alpha(\cdot)} \alpha(H_i) D_{H_i} \right) \\ &= L_{\mathfrak{a}} + \sum_{\alpha \in \Sigma_r^+} m_{\alpha} \frac{1}{\alpha} A_{\alpha} \end{aligned}$$

Here, the vector A_{α} is determined by $\langle A_{\alpha}, H \rangle = \alpha(H)$, $H \in \mathfrak{a}$, and is considered as a differential operator on $A^+ \cdot o$. We also require the following result:

Lemma 4.5. *([25] Thm. 5.1) On the open dense subset of \mathfrak{a} where they are defined,*

(i)
$$\sum_{\alpha \neq k\beta, \alpha, \beta \in \Sigma_r^+} m_{\alpha} m_{\beta} \langle \alpha, \beta \rangle \frac{1}{\alpha\beta} = 0$$

(ii) *In the case of a noncompact symmetric space G/K we have:*

$$\sum_{\alpha \neq k\beta, \alpha, \beta \in \Sigma_r^+} m_{\alpha} m_{\beta} \langle \alpha, \beta \rangle (\coth \alpha \coth \beta - 1) = 0$$

(iii) *In the case of a compact symmetric space U/K we have:*

$$\sum_{\alpha \neq k\beta, \alpha, \beta \in \Sigma_r^+} m_{\alpha} m_{\beta} \langle \alpha, \beta \rangle (\cot \alpha \cot \beta + 1) = 0$$

We now use these expressions to calculate the quantity $(j(H)^{-1}\Delta(L_p j)(H)$, which we will compute for both compact and non-compact symmetric spaces.

Proposition 4.6. *With the above notation,*

$$(j^{-1}L_p j)(H) = -\|\rho\|^2 + F(H)$$

where

$$F(H) = 2 \sum_{\alpha \in \Sigma_m^+} \frac{m_\alpha m_{2\alpha}}{4} \left(\cot \alpha(H) \cot 2\alpha(H) \right) |\alpha|^2 + \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2$$

Proof. Direct computation of the partial derivatives of j yields:

$$D_{H_i} j(H) = j(H) \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\cot \alpha(H) - \frac{1}{\alpha(H)} \right) \alpha(H_i),$$

$$\begin{aligned} \sum_{i=1}^l D_{H_i}^2 j(H) = \\ j(H) \left[\sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) - \frac{1}{\alpha(H)} \right) \left(\cot \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \right. \\ \left. - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right] \end{aligned}$$

and

$$\begin{aligned} \sum_{i=1}^l \sum_{\alpha \in \Sigma_r^+} m_\alpha \frac{1}{\alpha(H)} \alpha(H_i) D_{H_i} j(H) = \\ j(H) \sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{2} \frac{1}{\alpha(H)} \left(\cot \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \end{aligned}$$

Multiplying out and collecting like terms gives

$$\begin{aligned} \Delta(L_p j)(H) = \\ j(H) \left[\sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) + \frac{1}{\alpha(H)} \right) \left(\cot \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \right. \\ \left. - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \right], \end{aligned}$$

that is,

$$\begin{aligned}
 \Omega_*(H) &= (j^{-1}\Delta(L_p)j)(H) \\
 &= \sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) + \frac{1}{\alpha(H)} \right) \left(\cot \beta(H) - \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \\
 &\quad - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \\
 &= \sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) \cot \beta(H) + \cot \beta(H) \frac{1}{\alpha(H)} \right. \\
 &\quad \left. - \cot \alpha(H) \frac{1}{\beta(H)} - \frac{1}{\alpha(H)} \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \\
 &\quad - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2
 \end{aligned}$$

Decomposing the first sum into diagonal and off-diagonal parts we have

$$\begin{aligned}
 \Omega_*(H) &= \sum_{\alpha, \beta \in \Sigma_r^+, \alpha \neq \beta} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) \cot \beta(H) + \cot \beta(H) \frac{1}{\alpha(H)} \right. \\
 &\quad \left. - \cot \alpha(H) \frac{1}{\beta(H)} - \frac{1}{\alpha(H)} \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \\
 &\quad + \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha^2}{4} \left(\operatorname{cosec}^2 \alpha(H) - 1 - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \\
 &\quad - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha}{2} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2. \quad (20)
 \end{aligned}$$

Using Lemma 4.5 (i) on the first sum, and combining the last two, we have

$$\begin{aligned}
 \Omega_*(H) &= \\
 &\sum_{\alpha, \beta \in \Sigma_r^+, \alpha \neq \beta} \frac{m_\alpha m_\beta}{4} \left(\cot \alpha(H) \cot \beta(H) + \cot \beta(H) \frac{1}{\alpha(H)} - \cot \alpha(H) \frac{1}{\beta(H)} \right) \langle \alpha, \beta \rangle \\
 &\quad + \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 - \sum_{\alpha \in \Sigma_r^+} \left(\frac{m_\alpha |\alpha|}{2} \right)^2
 \end{aligned}$$

Since the last two terms of the first sum cancel over the summation for $\alpha \neq 2\beta$, and

$$\|\rho\|^2 = \langle \rho, \rho \rangle = \sum_{\alpha, \beta \in \Sigma_r^+} \frac{m_\alpha m_\beta}{4} \langle \alpha, \beta \rangle = \sum_{\alpha \in \Sigma_r^+} \left(\frac{m_\alpha |\alpha|}{2} \right)^2 + \sum_{\alpha, \beta \in \Sigma_r^+, \alpha \neq \beta} \frac{m_\alpha m_\beta}{4} \langle \alpha, \beta \rangle$$

we have

$$\begin{aligned} \Omega_*(H) = & -\|\rho\|^2 + 2 \sum_{\alpha \in \Sigma_m^+} \frac{m_\alpha m_{2\alpha}}{4} \left(\cot \alpha(H) \cot 2\alpha(H) \right) |\alpha|^2 \\ & + \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 \end{aligned}$$

as claimed. ■

This expression provides us with an alternate proof of the following well-known result:

Corollary 4.7. *Suppose U/K is a compact Lie group. We have*

$$\Omega_*(H) = -\|\rho\|^2$$

Proof. For a compact Lie group, each root (none of which are multipliable) has multiplicity 2. Thus it follows that $F(H) = 0$. ■

For compact symmetric spaces that do not have multipliable roots, we have the following:

Corollary 4.8. *Suppose U/K is a compact symmetric space that does not have multipliable roots, then we have*

$$\lim_{\alpha(H) \rightarrow \mathbf{0}} \Omega_*(H) = \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{12} |\alpha|^2$$

Proof. Consider the individual terms of the sum

$$\sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2$$

Denote by H_0 the elements of \mathfrak{a} where $\alpha(H_0) = 0$. By l'Hôpital's rule,

$$\lim_{H \rightarrow H_0} \left(\operatorname{cosec}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) = \lim_{H \rightarrow H_0} \left(\operatorname{cosec}^2 \alpha(H) - \alpha(H) \cot \alpha(H) \operatorname{cosec}^2 \alpha(H) \right)$$

We re-write the R.H.S. in a neighbourhood of H_0 as

$$\begin{aligned} & \frac{1}{(\alpha(H) - \alpha(H)^3/6 + \mathcal{O}(\alpha(H)^5))^2} \left(1 - \alpha(H) \cdot \frac{1 - \alpha(H)^2/2 + \mathcal{O}(\alpha(H)^4)}{\alpha(H) - \alpha(H)^3/6 + \mathcal{O}(\alpha(H)^4)} \right) \\ &= \frac{1}{\alpha(H)^2 - \alpha(H)^4/3 + \mathcal{O}(\alpha(H)^6)} \left(1 - \frac{1 - \alpha(H)^2/2 + \mathcal{O}(\alpha(H)^4)}{1 - \alpha(H)^2/6 + \mathcal{O}(\alpha(H)^3)} \right) \\ &= \frac{1}{\alpha(H)^2(1 - \alpha(H)^2/3 + \mathcal{O}(\alpha(H)^4))} \left(\frac{\alpha(H)^2/3 + \mathcal{O}(\alpha(H)^4)}{1 - \alpha(H)^2/6 + \mathcal{O}(\alpha(H)^3)} \right) \\ &= \frac{1}{3(1 - \alpha(H)^2/3 + \mathcal{O}(\alpha(H)^4))} \left(\frac{1 + \mathcal{O}(\alpha(H)^2)}{1 - \alpha(H)^2/6 + \mathcal{O}(\alpha(H)^3)} \right) \\ &= \frac{1}{3} \left(\frac{1 + \mathcal{O}(\alpha(H)^2)}{1 - \mathcal{O}(\alpha(H)^2)} \right) \rightarrow \frac{1}{3} \text{ as } H \rightarrow H_0. \end{aligned}$$

The corollary follows. ■

From the classification of symmetric spaces (see [17], Ch X), the only simple, simply connected compact symmetric spaces of rank one are the n -dimensional spheres, S^n . For S^n there is one root α which has multiplicity $n-1$. Normalising $\alpha = 1$, we have the following:

Corollary 4.9. *Suppose $U/K = S^n$. We have*

$$\Omega_*(H) = -\left(\frac{n-1}{2}\right)^2 + \left(\frac{(n-1)(n-3)}{4}\right) \left(\operatorname{cosec}^2 H - \frac{1}{H^2}\right)$$

Furthermore,

$$\lim_{H \rightarrow 0} \Omega_*(H) = \frac{1}{6}n(1-n)$$

However, we have not been able to calculate the heat kernel with these potentials on $\mathfrak{p} \cong \mathbb{R}^n$. Despite this, we have the following theorem guaranteeing the existence of a fundamental solution whose wrap is the heat kernel on U/K :

Theorem 4.10. *There exists a fundamental solution of the semigroup $\exp t(L_{\mathfrak{p}} + \Omega_*)$ in the fundamental domain of U/K whose wrap is the heat kernel on U/K .*

Proof. By Hörmander's theorem,

$$\frac{\partial}{\partial t} - (L_{\mathfrak{p}} + \Omega_*)$$

is hypoelliptic on the set $\{H : |\alpha(H)| < \pi - \epsilon, \forall \alpha \in \Sigma_r^+ \text{ and } \epsilon > 0\}$, that is, the fundamental domain of U/K . Therefore, $\frac{\partial}{\partial t} - (L_{\mathfrak{p}} + \Omega_*)$ has a fundamental solution in the fundamental domain of U/K . This fundamental solution may be wrapped to U/K (since it only takes values in the fundamental domain), which by Theorem 4.3 is the heat kernel on U/K . ■

5. The Non-compact Case

In this section we consider the wrapping map and heat kernels on certain non-compact symmetric spaces, namely complex Lie groups $G_{\mathbb{C}}$, the symmetric spaces $G_{\mathbb{C}}/K$, and the symmetric spaces of split rank.

We extend the wrapping map to complex Lie groups, and then use this to wrap Brownian motion and the heat kernel on these spaces. Our results also hold for the symmetric spaces $G_{\mathbb{C}}/K$, although for this case we have not been able to prove a global wrapping formula. Finally, we conclude by considering the symmetric spaces of split rank, and deduce a similar result to that of above, in that “bending” (in contrast to “wrapping” in the compact case) the heat kernel from the tangent space does not yield the heat kernel on G/K since the convolution

structure is not preserved.

Firstly, we state the quantity $\Omega_* = j^{-1}L_{\mathfrak{p}}j$ for the non-compact case, the calculation for which is similar to that in section 4, and we will not repeat it here. As we also remarked in subsection 4, it is similar as that for the quantity $(j \circ \log)L_{G/K}(j \circ \log)^{-1}$ given in [15] section 2, but performing the calculation on \mathfrak{p} , our results that follow are obtained in the spirit of the tangent space analysis advocated by Helgason ([19], [18]). Furthermore, our results hold globally, rather than only in a neighbourhood of the identity.

Proposition 5.1. *With the above notation,*

$$(j^{-1}L_{\mathfrak{p}}j)(H) = \|\rho\|^2 + F(H)$$

where

$$F(H) = \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2 + 2 \sum_{\alpha \in \Sigma_m^+} \frac{m_\alpha m_{2\alpha}}{4} \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2$$

In the case of a complex Lie group, or a symmetric space $G_{\mathbb{C}}/K$, each root has a multiplicity of two, none of which are multipliable. It is readily seen that $F(H) = 0$, and we consequently have:

Corollary 5.2. *For a complex Lie group, or a symmetric space $G_{\mathbb{C}}/K$, it holds that*

$$(j^{-1}L_{\mathfrak{p}}j)(H) = \|\rho\|^2$$

5.1. Complex Lie Groups.

We briefly consolidate some notations and definitions of complex Lie groups. Let $G_{\mathbb{C}}$ be a complex, connected, semisimple Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}_{\mathbb{C}}$, and K the compact group corresponding to \mathfrak{k} . We denote by $\mathfrak{g}^{\mathbb{R}}$ as $\mathfrak{g}_{\mathbb{C}}$ realised as a real Lie algebra, with the Cartan decomposition given by $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k} + i\mathfrak{k}$, and \mathfrak{k} may be identified as the compact real form, with $\mathfrak{k} \cong \mathfrak{p}$.

We let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra of \mathfrak{p} , with $\mathfrak{a}_{\mathbb{C}}$ its complexification, and \mathfrak{a}^* its dual. The real dimension of each root space is two, ie, $m_\alpha = 2$ for all $\alpha \in \Sigma_r$. Elements of $G_{\mathbb{C}}$ may be written as $k_1 a k_2$, where $k_1, k_2 \in K$ and $a \in \exp(\bar{\mathfrak{a}}^+)$. Since the roots are purely real on \mathfrak{a} , $A^+ = \exp(\bar{\mathfrak{a}}^+)$ is of the form $(\mathbb{R}^+)^l$. Furthermore, from [21] Ch. II, Thm. 2.15, any two Cartan subalgebras are conjugate under the adjoint action of $G_{\mathbb{C}}$.

A real-valued square root of the Jacobian of the exponential map is given by

$$j(H) = \prod_{\alpha \in \Sigma_r^+} \left(\frac{\sinh \alpha(H)/2}{\alpha(H)/2} \right), \quad H \in \mathfrak{a}^+$$

Since each root of a complex Lie group has multiplicity 2 we have (see [18], pp 487) $j^{-1}Lj = \|\rho\|^2 = \frac{\dim G}{12}$. (Compare this to the case of a compact Lie group, where we have $-j^{-1}Lj = \|\rho\|^2 = \frac{\dim G}{24}$.)

We retain the usual notations and definitions of function spaces, as well for spherical functions on G (see also [13]). For example, we let $L^p(K \backslash G / K)$ be the space of bi- K -invariant spherical L^p functions on G , and $\mathcal{S}(K \backslash G / K)$ be the set of bi- K -invariant spherical Schwartz functions on G . For $f \in L^1(K \backslash G / K)$ we define the Fourier transform $\hat{f}(\lambda)$ by

$$\hat{f}(\lambda) = \int_G f(x)\varphi_\lambda(x^{-1})dx, \quad \lambda \in \mathfrak{a}^*$$

where φ_λ is an elementary spherical function corresponding to the weight λ . Harish-Chandra described the following Fourier inversion formula (see [14], Thm. 6.4.2.) for bi- K -invariant Schwartz functions:

$$f(x) = \frac{1}{|W|} \int_\Lambda \hat{f}(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda, \quad \lambda \in \mathfrak{a}^*.$$

The function $c(\lambda)$ on Λ is such that $c(\lambda)^{-1}$ is a tempered distribution, $|c(w\lambda)| = |c(\lambda)|$ for $w \in W$, and we note that $|c(\lambda)|^{-2}d\lambda$ is the Plancherel measure. We also recall that the convolution of two K -invariant distributions, μ and ν , on G is defined by:

$$(\mu * \nu, f) = (\mu(xK) \otimes \nu(yK), f(xyK)), \quad x, y \in G \tag{21}$$

for any test function f on G . The elementary spherical functions on a complex Lie groups are given by (see [19], Ch. IV, Thm. 5.7):

$$\varphi_\lambda(a) = c(\lambda) \frac{\sum_{\omega \in W} \operatorname{sgn} \omega e^{i\omega\lambda(\log a)}}{\sum_{\omega \in W} \operatorname{sgn} \omega e^{\omega\rho(\log a)}}, \quad a \in A$$

where the function c is given by

$$c(\lambda) = \prod_{\alpha \in \Sigma_r^+} \langle \alpha, \rho \rangle / \prod_{\alpha \in \Sigma_r^+} \langle \alpha, \lambda \rangle$$

We also note another formula of Harish-Chandra (see [19], Ch. II, Thm. 5.35): Let du be the normalised Haar measure on G/K , and let π be the product of positive roots. Then, if $H, H' \in \mathfrak{a}_\mathbb{C}$,

$$\pi(H)\pi(H') \int_{G/K} e^{\langle uH, H' \rangle} du = \frac{1}{|W|} \partial(\pi)(\pi) \sum_{\omega \in W} \operatorname{sgn} \omega e^{\langle \omega H, H' \rangle}. \tag{22}$$

Setting $H' = H_\rho$, where H_ρ is the dual element of ρ , we obtain ([19], Ch. II, Cor. 5.36)

$$|W| = \partial(\pi)(\pi)/\pi(H_\rho).$$

The heat kernel of a complex group is given by the following:

Proposition 5.3. ([12]) *Suppose $G_{\mathbb{C}}$ is a complex Lie group. The heat kernel on $G_{\mathbb{C}}$, $q_t(a)$, given on A is*

$$q_t(a) = (2\pi t)^{-n/2} \exp(-t\|\rho\|^2/2) \frac{1}{j(\log a)} \exp(-|\log a|^2/2t), \quad a \in A. \quad (23)$$

We now prove the wrapping theorem for complex groups. The wrapping map Φ is given by

$$\langle \Phi(\nu), f \rangle = \langle \nu, j\tilde{f} \rangle$$

and is well defined if ν is an integrable K -bi-invariant function.

We face a difficulty with the orbital convolution theory with non-compact groups in general, since the co-adjoint orbits can be non-compact, and are not in one-to-one correspondence with the adjoint orbits, as in the compact case. However, the wrapping formula for complex groups does hold for spherical measures, since the Fourier theory is essentially reduced to the abelian case.

Theorem 5.4. *Suppose μ, ν are two K -bi-invariant Schwartz functions on $\mathfrak{g}_{\mathbb{C}}$, then*

$$\Phi(\mu * \nu) = \Phi(\mu) * \Phi(\nu)$$

Proof. The spherical Fourier transform of $\Phi(u)$ at a representation π , indexed by highest weight λ , is given by

$$\langle \Phi(u), \varphi_\lambda \rangle = \langle u, j\tilde{\varphi}_\lambda \rangle = \left\langle u, c(\lambda) \frac{\sum_{\omega \in W} \operatorname{sgn} \omega e^{i\omega\lambda(\cdot)}}{\prod_{\alpha > 0} \alpha(H)} \right\rangle = \frac{\pi(\rho)}{\pi(i\lambda)} \left\langle u, \frac{\sum_{\omega \in W} \operatorname{sgn} \omega e^{i\omega\lambda(\cdot)}}{\prod_{\alpha > 0} \alpha(H)} \right\rangle$$

By Harish-Chandra's formula, this is

$$\begin{aligned} \langle \Phi(u), \varphi_\lambda \rangle &= \left\langle u, \int_K e^{ik\lambda(\cdot)} dk \right\rangle = \langle u, e^{i\lambda(\cdot)} \rangle && \text{(Since } u \text{ is } K\text{-bi-invariant)} \\ &= \hat{u}(\lambda) \end{aligned}$$

Using abelian Fourier analysis, we obtain

$$(\Phi(\mu * \nu))^\wedge = (\mu * \nu)^\wedge = \mu^\wedge \cdot \nu^\wedge = \Phi^\wedge(\mu) \cdot \Phi^\wedge(\nu) = (\Phi(\mu) * \Phi(\nu))^\wedge. \quad \blacksquare$$

Remark 5.5. For Lie groups, Rouvière's conjecture that $e = 1$ for Lie groups is equivalent to the Kashiwara-Vergne conjecture ([20]). This was recently proven in [1]. For the symmetric spaces $G_{\mathbb{C}}/K$, $e = 1$ was proven in a neighbourhood of $0 \in \mathfrak{p}$ by Torossian in [28].

5.2. The wrap of Brownian motion and heat kernels on complex Lie groups.

We prove the following general result for computing the wrap of a K -bi-invariant Schwartz function:

Proposition 5.6. *Let G/K be a semisimple Riemannian symmetric space of the non-compact type, with tangent space \mathfrak{p} . Let $\varphi \in \mathcal{S}(\mathfrak{p})$ be K -bi-invariant. Then $\Phi(j\varphi)$ is a C^∞ K -bi-invariant function, given on A , by*

$$\Phi(j\varphi)(\exp H) = \varphi(H), \quad H \in \mathfrak{a}.$$

Proof. Firstly, we let

$$f^K(g) = \int \int_{K \times K} f(k_1 g k_2) dk_1 dk_2$$

Since $\text{Exp} : \mathfrak{p} \rightarrow G/K$ is a global diffeomorphism, it follows that for any $f \in C^\infty(G/K)$ that

$$\langle \Phi(j\varphi), f \rangle = \langle j\varphi, j\tilde{f} \rangle = \int_{\mathfrak{p}} j^2(X) \varphi(X) \tilde{f}(X) dX \tag{24}$$

$$= \frac{1}{|W|} \int_{\mathfrak{a}} \prod_{\alpha \in \Sigma_r^+} (\sinh \alpha(H)/2) \varphi(H) \tilde{f}^K(H) dH \tag{25}$$

$$= \frac{1}{|W|} \int_A \prod_{\alpha \in \Sigma_r^+} (\sinh \alpha(\log a)/2) \varphi(\log a) \tilde{f}^K(a) da \tag{26}$$

$$= \int_{G/K} \varphi(\log g) f(g) dg \tag{27}$$

The result follows from (3). ■

Using Proposition 5.6 for complex Lie groups, we immediately recover the results of Proposition 5.3:

Corollary 5.7. *Let $p_t(H) = (2\pi t)^{-n/2} e^{-|H|^2/2t}$ be the heat kernel on \mathfrak{g} (given on \mathfrak{a}). Then*

$$\Phi(p_t)(\exp H) = (2\pi t)^{-n/2} \frac{1}{j(H)} \exp(-|H|^2/2t), \quad H \in \mathfrak{a}$$

which is the shifted heat kernel, $q_t^\rho(g)$.

Remark 5.8. At this point we identify two critical steps in the proof of Proposition 5.6, namely (25) to (26), and (26) to (27). In [9], the step (25) to (26) is achieved for a compact Lie group through a function Ψ , which is a sum over the integer lattice of the function φ . Thus (25) may be given on a fundamental domain, and hence (26) obtained. The step in going from (26) to (27) relies on the fact that all Cartan subalgebras in \mathfrak{g} are conjugate to each other under the adjoint group.

We now consider the wrap of the Laplacian. It follows from Corollary 5.2:

Proposition 5.9. *Let G be a complex connected Lie group with Lie algebra \mathfrak{g} . Then for any $u \in S(\mathfrak{g})$*

$$\Phi(L_{\mathfrak{g}}(u)) = (L_G + \|\rho\|^2)(\Phi u)$$

where Φ is the wrapping map, $L_{\mathfrak{g}}$ is the Laplacian on \mathfrak{g} (regarded as a Euclidean vector space), and ρ the half sum of positive roots.

We can now extend the results from [24] to wrapping Brownian motion and the heat kernel onto G from \mathfrak{g} . The potential term in this case is just the constant $\|\rho\|^2$, and thus a Brownian motion may be “wrapped” onto G using the same mechanics as presented in [24] section 4.2. We will not reproduce the method here again, but following [24], we apply the Feynman-Kač formula to obtain:

Proposition 5.10. *The density of $(\xi_t)_{t \geq 0}$ under $\tilde{\mathbb{P}}$, denoted by $\tilde{\mathbb{E}}$, is*

$$\tilde{\mathbb{E}}(\xi_t) = (2\pi t)^{-n/2} e^{-\|\rho\|^2 t/2} \frac{1}{j(H)} \exp(-|H|^2/2t), \quad t \in \mathbb{R}^+, H \in \mathfrak{a}.$$

which is the standard heat kernel on G .

5.3. Wrapping for certain non-compact symmetric spaces.

There are many examples of non-compact symmetric spaces. In this section we conjecture that our results on wrapping Brownian motion and heat kernel may be extended to other symmetric spaces - more specifically, our results on wrapping Brownian motion and heat kernels should hold for every symmetric space for which the wrapping theorem holds. The wrapping theorem is conjectured to hold for all Lie groups in [9].

Firstly, we will consider the symmetric spaces G/K , G complex. The wrapping theorem was proved to hold locally for these spaces in [28]. We have not been able to prove a global wrapping theorem, but analogous results of section 3 hold.

Recall from Corollary 5.2 that $(j^{-1}L_{\mathfrak{p}}j) = \|\rho\|^2$. We apply Proposition 5.6 to find the shifted heat kernel, and wrap Brownian motion and the heat kernel as we did in the complex case, yielding the following analogue of Corollary 5.7:

Proposition 5.11. *The density of $(\xi_t)_{t \geq 0}$ under $\tilde{\mathbb{P}}$, denoted by $\tilde{\mathbb{E}}$, is*

$$\tilde{\mathbb{E}}(\xi_t) = (2\pi t)^{-n/2} e^{-\|\rho\|^2 t/2} \frac{1}{j(H)} \exp(-|H|^2/2t), \quad t \in \mathbb{R}^+, H \in \mathfrak{a}.$$

which is the standard heat kernel on G .

Remark 5.12. Proposition 5.11 holds with $\|\rho\|^2 = \frac{n}{12}$, which agrees with the expression for the heat kernel on the symmetric spaces G/K , G complex, given by Arede in [2], §2.2.

We now consider the symmetric spaces of *split-rank* type:

Definition 5.13. A symmetric space G/K is said to be of *split-rank* if $\text{rank } G = \text{rank } G/K + \text{rank } K$.

Proposition 5.14. ([17], Ch. IX, Thm. 6.1) *The following are equivalent:*

- (i) G/K has split rank,
- (ii) Each restricted root has even multiplicity,
- (iii) All Cartan subalgebras of \mathfrak{g} are conjugate under the adjoint action.

It follows from (ii) and (iii) that the roots of G/K with respect to $\mathfrak{a}_{\mathbb{C}}$ are real on \mathfrak{a} . Thus, the maximal abelian subgroup corresponding to the Cartan subalgebra is of the form $(e^{\alpha_1(H)}, \dots, e^{\alpha_i(H)})$, where the α_i 's are all real, so the subgroup is of the form $(\mathbb{R}^+)^n$.

By [17], Ch. IX, §4, it is known that the non-compact, simple, simply connected symmetric spaces of split-rank consist of the following:

- (i) $\mathbb{R}^{2n} \times \mathbb{R}^+ \cong SO_0(2n+1, 1)/SO(2n+1)$, the odd dimensional hyperbolic spaces,
- (ii) $G_{\mathbb{C}}/K$, where K is a maximal compact subgroup of $G_{\mathbb{C}}$,
- (iii) $SU^*(2n)/Sp(n)$,
- (iv) $E_{6(-26)}/F_4$.

We apply proposition 5.6 for these spaces, and the heat kernel p_t may be wrapped from \mathfrak{p} to give

$$\Phi(p_t)(\text{Exp}H) = (2\pi t)^{-n/2} e^{-\|H\|^2 t/2} \frac{1}{j(H)} \exp(-|H|^2/2t), \quad t \in \mathbb{R}^+, H \in \mathfrak{a}. \quad (28)$$

However, the e -function is not identically 1 for all the spaces of split-rank type. We now make a similar observation to that for compact symmetric spaces: The Gaussian approximation to the heat kernel, (28), is not the heat kernel for the spaces of split-rank type when the e -function is not identically 1.

For non-compact symmetric spaces that do not have multipliable roots, we have the following:

Proposition 5.15. *Suppose G/K is a non-compact symmetric space that does not have multipliable roots, then Ω_* is a bounded C^∞ function on \mathfrak{p} . Furthermore,*

$$\lim_{\alpha(H) \rightarrow \mathbf{0}} \Omega_*(H) = - \sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{12} |\alpha|^2$$

Proof. It is clear that Ω_* is a bounded C^∞ function on \mathfrak{p} , except on the hyperplane in \mathfrak{p} where $\alpha(H) = 0$, which we will denote H_0 . To show that Ω_* is bounded on H_0 , we consider the individual terms of the sum

$$\sum_{\alpha \in \Sigma_r^+} \frac{m_\alpha(m_\alpha - 2)}{4} \left(\operatorname{csch}^2 \alpha(H) - \frac{1}{\alpha(H)^2} \right) |\alpha|^2$$

The proof is analogous to Corollary 4.8, replacing cosec by csch, and noting the change in sign. ■

From the classification of symmetric spaces (see [17], Ch X), the only simple, simply connected non-compact symmetric spaces of rank one with no multipliable roots are the n -dimensional hyperbolic spaces, $H^n \cong SO_0(n, 1)/SO(n)$, $n \geq 2$. Since H^n , is a symmetric space of rank 1, $\dim \mathfrak{a} = 1$ and so $\mathfrak{a} \cong \mathbb{R}$. Therefore there is only one root $\alpha \in \mathfrak{a}^*$ which has multiplicity $n - 1$. We normalise α such that $\alpha = 1$, thus $\rho = (n - 1)/2$, giving us the non-compact analogue of Proposition 4.9:

Corollary 5.16. *Suppose $G/K = H^n$. We have*

$$\Omega_*(H) = \left(\frac{n - 1}{2} \right)^2 + \left(\frac{(n - 1)(n - 3)}{4} \right) \left(\operatorname{csch}^2 H - \frac{1}{H^2} \right)$$

Furthermore,

$$\lim_{H \rightarrow 0} \Omega_*(H) = \frac{1}{3}(2n^2 + 5n + 3)$$

The e -function for $H^n \cong SO_0(n, 1)/SO(n)$, $n \geq 2$, has been calculated by M. Flensted-Jensen ([26], pp. 258) to be

$$e(X, Y) = \left(4 \frac{u}{\operatorname{sh} u} \frac{v}{\operatorname{sh} v} \frac{w}{\operatorname{sh} w} \frac{\operatorname{ch} w - \operatorname{ch}(u - v)}{w^2 - (u - v)^2} \frac{\operatorname{ch} w - \operatorname{ch}(u + v)}{w^2 - (u + v)^2} \right)^{(n-3)/2} \tag{29}$$

where u, v and w are the norms of X, Y and $X + Y$, respectively. Note that this expression is symmetric in X and Y , and is only equal to 1 when $n = 3$. (29) is also the non-compact analogue for the expression for the (globally defined) e -function given for n -dimensional sphere in [6] and [7]. We have not been able to find a proof of Flensted-Jensen’s calculation, though it should be analogous to that in [7] by replacing \sin with \sinh .

In light of our results in for compact symmetric spaces, we conjecture that wrapping the solution to the heat equation with this potential yields the heat kernel on the hyperbolic spaces. However, we have not been able to calculate the heat kernel with these potentials on $\mathfrak{p} \cong \mathbb{R}^n$. Despite this, we are guaranteed the existence of a fundamental solution whose wrap is the heat kernel on X by Hörmander’s theorem, since

$$\frac{\partial}{\partial t} - (L_{\mathfrak{p}} + \Omega_*)$$

is hypoelliptic on \mathfrak{p} . Therefore, $\frac{\partial}{\partial t} - (L_{\mathfrak{p}} + \Omega_*)$ has a fundamental solution \mathfrak{p} . This fundamental solution may be wrapped to G/K , which by Theorem 4.3 is the heat kernel on G/K . \square

5.4. Other non-compact semisimple Lie groups.

Extending the wrapping theorem and our results on wrapping Brownian motion and heat kernels to other spaces may prove quite difficult. Here, we cite the example of $SL(2, \mathbb{R})$. The (co)adjoint orbits for $SL(2, \mathbb{R})$ consist of one- and two-sheeted hyperboloids - the reader is referred to [16] for a detailed survey of $SL(2, \mathbb{R})$ and the orbit method. Convolving these orbits as was shown for the case of a 2-sphere in subsection 3 is therefore more difficult to describe.

For the moment, let us consider how one would wrap a function or distribution from \mathfrak{g} to G . The Iwasawa decomposition for $SL(2, \mathbb{R})$ is given by $G = KAN$ where

$$K = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

(see, for example, [16], or [21] Ch. VI, §6). All the elements may be conjugated into the two distinct abelian subgroups K and A , isomorphic to the circle \mathbb{T} , and \mathbb{R}^+ , respectively. This creates a problem when one asks how to “wrap” a function or distribution.

In [9], the wrap of j times an Ad-invariant Schwartz function to a compact Lie group was calculated by summing over an integer lattice:

$$\Phi(j\mu)(\exp H) = \sum_{\gamma \in \Gamma} \mu(H + \gamma), \quad H \in \mathfrak{a}. \quad (30)$$

We showed in section 5 that the wrap of j times an Ad-invariant Schwartz function to a complex Lie group may be calculated by “bending” it:

$$\Phi(j\mu)(\exp H) = \mu(H), \quad H \in \mathfrak{a}^+. \quad (31)$$

These follow from the fact that in the case of a compact Lie group, $A \cong \mathbb{T}^l$, and for a complex Lie group, $A^+ \cong (\mathbb{R}^+)^l$. Since all the elements of $SL(2, \mathbb{R})$ may be conjugated into the two distinct abelian subgroups, isomorphic to \mathbb{T} and \mathbb{R}^+ , do we sum over a lattice, or do we “bend” to compute the wrap? Arguably, we would need to do some combination of both.

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