Globalizations of Infinitesimal Actions on Supermanifolds

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Abstract. Let $\mathcal{G}$ be a Lie supergroup with Lie superalgebra $\mathfrak{g}$, $\mathcal{M}$ a supermanifold and $\mathrm{Vec}(\mathcal{M})$ the set of vector fields on $\mathcal{M}$. Let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be an infinitesimal action, i.e. a homomorphism of Lie superalgebras. We show the existence of a local $\mathcal{G}$-action on $\mathcal{M}$ inducing the infinitesimal action λ and find necessary and sufficient conditions for the existence of a globalization in the sense of Palais.
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1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra, $\mathcal{G}$ any Lie supergroup with $\mathfrak{g}$ as its Lie algebra of right-invariant vector fields, $\mathcal{M}$ a supermanifold and denote the set of vector fields on $\mathcal{M}$ by $\mathrm{Vec}(\mathcal{M})$. Any action $\varphi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, or local action, of the Lie supergroup $\mathcal{G}$ on $\mathcal{M}$ induces an infinitesimal action on $\mathcal{M}$, i.e. a homomorphism of Lie superalgebra $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$, by setting

$$\lambda(X) = (X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*.$$

The vector field $(X \otimes \mathrm{id}_{\mathcal{M}}^*)$ denotes the extension of the right-invariant vector field X on $\mathcal{G}$ to a vector field on $\mathcal{G} \times \mathcal{M}$, and $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*)$ is its evaluation in the neutral element e of $\mathcal{G}$. Starting with an infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ of $\mathcal{G}$ on $\mathcal{M}$, it is a natural question to ask in which cases this infinitesimal action is induced, in the just describes way, by a local or global action of $\mathcal{G}$ on $\mathcal{M}$, or some larger supermanifold $\mathcal{M}^*$ containing $\mathcal{M}$ as an open subsupermanifold.

In the case of a smooth manifold M and a Lie group G, Palais studied these questions in detail ([Pal57]). Concerning the existence of local actions, he showed that every infinitesimal action λ is induced by a local action of G on M. This generalizes the fact that the flow of any vector field on M defines a local $\mathbb{R}$-action on M.

In the case of one (not necessarily homogeneous) vector field X on a supermanifold, Monterde and Sánchez-Valenzuela, and Garnier and Wurzbacher proved that the flow $\varphi : \mathcal{W} \subseteq \mathbb{R}^{1|1} \times \mathcal{M} \to \mathcal{M}$ of X defines a local $\mathbb{R}^{1|1}$-action on $\mathcal{M}$ if and only if X is contained in a $1|1$-dimensional Lie subsuperalgebra $\mathfrak{g}$ of $\mathrm{Vec}(\mathcal{M})$ (see [MSV93] and [GW13]). In [GW13] the same is also shown for a holomorphic vector field on a complex supermanifold.

In [Pal57], Palais also found necessary and sufficient conditions for the existence of a globalization, i.e. a (possibly non-Hausdorff) manifold M^*, containing on M as an open submanifold, with an G-action on M^* that induces λ and satisfies $G \cdot M = M^*$.

In this paper, we extend these results to the case of (real or complex) supermanifolds and (real or complex) Lie supergroups. The existence of a local actions with a given infinitesimal actions and conditions for the existence of globalizations are proven. A key point in the proof is, as in the classical case in [Pal57], the study of the distribution $\mathcal{D} = \mathcal{D}_\lambda$ on the product $\mathcal{G} \times \mathcal{M}$ spanned by vector fields of the form

$$X + \lambda(X) \text{ for } X \in \mathfrak{g},$$

considering X and $\lambda(X)$ as vector fields on the product $\mathcal{G} \times \mathcal{M}$. Also, the fact that the flow of one even vector field on a supermanifold defines a local $\mathbb{R}$-action, or $\mathbb{C}$-action in the complex case, is used (see [MSV93] and [GW13]).

The outline of this paper is the following:
First, some notations are introduced and a few basic definitions are given. In Section §3, facts about distributions on supermanifolds are collected. Then, the relation between infinitesimal and local actions on supermanifolds is studied in §4. The main result there is the equivalence of infinitesimal and local actions up to shrinking:

Theorem 1.1.   *Let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be an infinitesimal action. Then there exists a local $\mathcal{G}$-action $\varphi : \mathcal{W} \subseteq \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ with induced infinitesimal action λ.*

Moreover, any local action $\varphi : \mathcal{W} \to \mathcal{M}$ is uniquely determined by its induced infinitesimal action and domain of definition.

In Section §5, conditions for the existence of globalizations are studied. To generalize the classical result of Palais, the notion of univalence of an infinitesimal action λ is extended to the case of supermanifolds. In the case of an infinitesimal action λ whose underlying action is globalizable, the obstruction for λ to be globalizable is a holonomy phenomenon. The main result for the conditions of globalizability is the following:

Theorem 1.2.   *The infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ is globalizable if and only if one of the following equivalent conditions is satisfied:*

(i) *The restricted infinitesimal action $\lambda|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \mathrm{Vec}(\mathcal{M})$ is globalizable to an action of G, where $\mathfrak{g}_0$ denotes the even part of $\mathfrak{g}$.*

(ii) *The infinitesimal action λ is univalent.*

(iii) The underlying infinitesimal action is globalizable, and all leaves $\Sigma \subset G \times M$ of the distribution $\mathcal{D}_\lambda$ are "holonomy free".

Condition *(iii)* of this theorem together with the appropriate classical results of Palais yields the following in Section §6:

Corollary 1.3. *Let $\mathcal{G}$ be a simply-connected Lie supergroup and $\lambda : \mathfrak{g} \rightarrow \mathrm{Vec}(\mathcal{M})$ an infinitesimal action whose support is relatively compact in M. Then there exists a global $\mathcal{G}$-action on $\mathcal{M}$ which induces λ.*

In particular, there is a one-to-one correspondence between infinitesimal and global actions of a simply-connected Lie supergroup on supermanifold with compact underlying manifold.

Corollary 1.4. *Let $\lambda : \mathfrak{g} \rightarrow \mathrm{Vec}(\mathcal{M})$ be an infinitesimal action of a simply-connected Lie supergroup $\mathcal{G}$ and let $\{X_i\}_{i \in I}$, $X_i \in \mathfrak{g}_0$ be a set of generators of $\mathfrak{g}_0$ such that underlying vector fields of $\lambda(X_i)$, $i \in I$, on M have global flows. Then there exists a global $\mathcal{G}$-action on $\mathcal{M}$ which induces λ.*

A special version of Corollary 1.4 in the context of DeWitt supermanifolds was published by Tuynman in [Tuy13].

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2. Notation

Throughout, we work with the "Berezin-Leites-Kostant"-approach to supermanifolds (see, e.g. [Ber87], [Leĭ80], or [Kos77]). A supermanifold is denoted by a calligraphic letter, e.g. $\mathcal{M}$, and the underlying manifold by the corresponding standard uppercase letters, e.g. M. The structure sheaf of a supermanifold $\mathcal{M}$ shall be denoted by $\mathcal{O}_\mathcal{M}$. If not otherwise mentioned, the considered supermanifolds are assumed to be real supermanifolds, i.e. the structure sheaf is locally isomorphic to $\mathcal{C}_U^\infty \otimes \bigwedge \mathbb{R}^n$ for appropriate open subsets $U \subset M$, where $\mathcal{C}_U^\infty$ denotes the sheaf of smooth functions on U. By a complex supermanifold a supermanifold whose structure sheaf is locally isomorphic to $\mathcal{O}_U \otimes \bigwedge \mathbb{C}^n$ is meant, where $\mathcal{O}_U$ is the sheaf of holomorphic functions on U.

If $\varphi : \mathcal{M} \rightarrow \mathcal{N}$ is a morphism between supermanifolds $\mathcal{M}$ and $\mathcal{N}$, let $\tilde{\varphi} : M \rightarrow N$ be the underlying map and $\varphi^* : \mathcal{O}_\mathcal{N} \rightarrow \tilde{\varphi}_* \mathcal{O}_\mathcal{M}$ its pullback, i.e. $\varphi = (\tilde{\varphi}, \varphi^*)$.

A (smooth/holomorphic) vector field or derivation X on a (real/complex) supermanifold $\mathcal{M}$ is a graded (real, respectively, complex-linear) derivation of sheaves $X : \mathcal{O}_\mathcal{M} \rightarrow \mathcal{O}_\mathcal{M}$. The notation for the sheaf of derivations or tangent sheaf will be $\mathrm{Der}\,\mathcal{O}_\mathcal{M}$ or $\mathcal{T}_\mathcal{M}$, and $\mathrm{Vec}(\mathcal{M}) = \mathrm{Der}\,\mathcal{O}_\mathcal{M}(M)$. $\mathrm{Vec}(\mathcal{M})$ is a Lie superalgebra, possibly of infinite dimension.

A (real/complex) Lie supergroup can be defined to be a group object in the category of (real/complex) supermanifolds, i.e. a (real/complex) supermanifold $\mathcal{G} = (G, \mathcal{O}_\mathcal{G})$ together with morphisms for multiplication, inversion and the neutral

element such that the usual group axioms are satisfied. The underlying manifold G is a classical Lie group. In the following, we always assume that G is connected. A vector field X on a Lie supergroup $\mathcal{G}$ is called right-invariant if

$$\mu^* \circ X = (X \otimes \mathrm{id}_{\mathcal{G}}^*) \circ \mu^{*(1)},$$

where $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denotes the multiplication on $\mathcal{G}$. We define the Lie superalgebra $\mathfrak{g}$ of $\mathcal{G}$ to be the set of right-invariant vector fields on $\mathcal{G}$. Its even part $\mathfrak{g}_0$ can be identified with the Lie algebra (of right-invariant vector fields) of G.

Definition 2.1. A local action of a Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ on a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ is a morphism $\varphi : \mathcal{W} \to \mathcal{M}$, $\mathcal{W} = (W, \mathcal{O}_{\mathcal{G} \times \mathcal{M}}|_W)$, where W is an open neighbourhood of $\{e\} \times M$ in $G \times M$ such that $W_p = \{g \in G \,|\, (g, p) \in W\}$ is connected for each $p \in M$, satisfying the action properties:

(i) If $\iota_e : \mathcal{M} \to \{e\} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{M}$ is the canonical inclusion, then $\varphi \circ \iota_e = \mathrm{id}_{\mathcal{M}}$.

(ii) If $\mu : \mathcal{G} \times \mathcal{G} \to G$ is the multiplication on $\mathcal{G}$, then the equation

$$\varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}}) = \varphi \circ (\mathrm{id}_{\mathcal{G}} \times \varphi)$$

holds on the open subsupermanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$ where both sides are defined.

3. Distributions on supermanifolds

Commuting vector fields Let $\mu : \mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \to \mathbb{R}^{m|n}$ denote the addition on $\mathbb{R}^{m|n}$ which is given by $((s, \sigma), (t, \tau)) \mapsto (s + t, \sigma + \tau)$ in coordinates. The goal of this section is the proof of the following result:

Theorem 3.1. Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold, $X_1, \ldots, X_m$ be even and $Y_1, \ldots, Y_n$ odd vector fields on $\mathcal{M}$ which all commute. Then, for any $p \in M$ there exists an open connected neighbourhood U of 0 in $\mathbb{R}^m$, an open neighbourhood V of p in M and a morphism $\varphi = (\tilde{\varphi}, \varphi^*) : \mathcal{U} \times \mathcal{V} \to \mathcal{M}$, where $\mathcal{U} = (U, \mathcal{O}_{\mathbb{R}^{m|n}}|_U)$ and $\mathcal{V} = (V, \mathcal{O}_{\mathcal{M}}|_V)$, such that:

(i) The map φ has the action property, i.e. we have $\varphi \circ \iota_0 = \mathrm{id}_{\mathcal{M}}$ if $\iota_0 : \mathcal{M} \hookrightarrow \{0\} \times \mathcal{M} \subset \mathbb{R}^{m|n} \times \mathcal{M}$ denotes the canonical inclusion (in particular the evaluation in $(t, \tau) = (0, 0)$ is the pullback ι_0^*), and the equality

$$\varphi \circ (\mathrm{id}_{\mathbb{R}^{m|n}} \times \varphi) = \varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}})$$

holds on the open subsupermanifold of $\mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \times \mathcal{M}$ where both sides are defined.

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[1] The vector field $(X \otimes \mathrm{id}_{\mathcal{G}}^*)$ is the extension of the vector field X on $\mathcal{G}$ to a vector field on the product $\mathcal{G} \times \mathcal{G}$ such that $X = (X \otimes \mathrm{id}_{\mathcal{G}}^*) \circ \pi_1^*$ and $0 = (X \otimes \mathrm{id}_{\mathcal{G}}^*) \circ \pi_2^*$ if $\pi_i : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $i = 1, 2$, is the projection onto the i-th component.

(ii) *If (t, τ) are coordinates on $\mathbb{R}^{m|n}$, then for all $i = 1, \ldots, m$, $j = 1 \ldots, n$ we have*

$$\frac{\partial}{\partial t_i} \circ \varphi^* = \varphi^* \circ X_i \quad and \quad \frac{\partial}{\partial \tau_j} \circ \varphi^* = \varphi^* \circ Y_j.^{(2)}$$

By replacing $\mathbb{R}^{m|n}$ by $\mathbb{C}^{m|n}$ an analogous result also holds true for a complex supermanifold $\mathcal{M}$ and holomorphic vector fields $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$.

Remark 3.2. Note that φ locally defines a local action of $\mathbb{R}^{m|n}$ on $\mathcal{M}$ in the sense that φ would be a local action if the assumption that $\{0\} \times \mathcal{M}$ is contained in the domain of definition was dropped.

The proof makes use of the flows of vector fields on supermanifolds.

Definition 3.3. Let X be an even vector field on a supermanifold $\mathcal{M}$. A flow of X (with respect to the initial condition $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ and $t_0 = 0 \in \mathbb{R}$) is a morphism $\varphi = \varphi^X : \mathcal{W} \to \mathcal{M}$, where $\mathcal{W} = (W, \mathcal{O}_{\mathbb{R} \times \mathcal{M}}|_W)$ and $W \subseteq \mathbb{R} \times M$ is an open neighbourhood of $\{0\} \times M$ such that $W_p = \{t \in \mathbb{R} | (t, p) \in W\} \subseteq \mathbb{R}$ is connected for each $p \in M$, with

(i) $\varphi \circ \iota_0 = \mathrm{id}_{\mathcal{M}}$, and

(ii) $\frac{\partial}{\partial t} \circ \varphi^* = \varphi^* \circ X$.

The flow $\varphi : \mathcal{W} \to \mathcal{M}$ is called maximal if for any flow $\varphi' : \mathcal{W}' = (W', \mathcal{O}_{\mathbb{R} \times \mathcal{M}}|_{W'}) \to \mathcal{M}$ of X we have $W' \subseteq W$ and $\varphi' = \varphi|_{W'}$.

Remark 3.4. Any vector field X on a supermanifold $\mathcal{M}$ induces a vector field $\tilde{X}$ on the underlying manifold M. The reduced vector field $\tilde{X}$ can be defined by $\tilde{X}(f) = \mathrm{ev}(X(F))$, if F is a function on $\mathcal{M}$ with $\mathrm{ev}(F) = \tilde{F} = f$, where $\mathrm{ev} : \mathcal{O}_{\mathcal{M}} \to \mathcal{C}_M^{\infty}$ denotes the evaluation map.

As in the classical case, there exists a unique maximal flow of an even vector field on a supermanifold.

Theorem 3.5 (see [MSV93], Theorem 3.5/3.6, or [GW13], Theorem 2.3 and Theorem 3.4)**.** *Let X be an even vector field on a supermanifold $\mathcal{M}$. Then there exists a unique maximal flow φ of X. The reduced map $\tilde{\varphi} : W \to M$ is then the unique maximal flow of the reduced vector field $\tilde{X}$ on M. Moreover, the flow φ defines a local $\mathbb{R}$-action on $\mathcal{M}$.*

Remark 3.6. For any even holomorphic vector field X on a complex supermanifold $\mathcal{M}$ there also exists a holomorphic flow $\varphi : \mathcal{W} \subseteq \mathbb{C} \times \mathcal{M} \to \mathcal{M}$ (see [GW13], Theorem 5.4). But there are differences to the real case in terms of the possible domains of definitions of the flow; see [GW13] for details and examples.

As in the classical case, a flow $\varphi : \mathcal{W} \to \mathcal{M}$ of an even holomorphic vector field may not satisfy the equation $\varphi \circ (\mathrm{id}_{\mathbb{C}} \times \varphi) = \varphi \circ (\mu_{\mathbb{C}} \times \mathrm{id}_{\mathcal{M}})$ on all of the open

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$^{(2)}$Considering $\frac{\partial}{\partial t_i}$ and $\frac{\partial}{\partial \tau_j}$ as vector fields on the product $\mathbb{R}^{m|n} \times \mathcal{M}$.

subsupermanifold of $\mathbb{C} \times \mathbb{C} \times \mathcal{M}$ on which both sides of the equation are defined, denoting the multiplication on $\mathbb{C}$ by $\mu_{\mathbb{C}}$, and thus not define a local $\mathbb{C}$-action on its domain of definition. One obtains a local $\mathbb{C}$-action only after shrinking its domain of definition in a suitable way; for example after shrinking W such that each $W_p = \{z \in \mathbb{C} | (z,p) \in W\} \subseteq \mathbb{C}$ is convex.

Lemma 3.7. *Let X and Y be vector fields on $\mathcal{M}$ and let Y be even with flow φ^Y. If $\iota_t : \mathcal{M} \hookrightarrow \{t\} \times \mathcal{M} \subset \mathbb{R} \times \mathcal{M}$ is the canonical inclusion, denote by φ_t^Y the composition $\varphi^Y \circ \iota_t$. Then*

$$[X,Y] = \left. \frac{\partial}{\partial t} \right|_0 (\varphi_t^Y)_* X$$

for $(\varphi_t^Y)_* X = (\varphi_{-t}^Y)^* \circ X \circ (\varphi_t^Y)^*$, $\left. \frac{\partial}{\partial t} \right|_s = \iota_s^* \circ \frac{\partial}{\partial t}$ for $s \in \mathbb{R}$.

Proof. The lemma can be proven in a very similar way as the analogous classical result (see e.g. [KN63], Proposition 1.9). A key point is the Taylor expansion $(\varphi_t^Y)^*(f) = f + t\hat{g}_t$ with $\hat{g}_0 = Y(f)$, where $\hat{g} \in \mathcal{O}_{\mathbb{R} \times \mathcal{M}}(I \times V)$, $I \times V \subseteq \mathbb{R} \times M$ open, $\hat{g}_t(x) = \hat{g}(t,x)$. ∎

As a corollary we get, as in the classical case (cf. [KN63], Corollary 1.10 and 1.11):

Corollary 3.8. *Under the assumptions of the above lemma, we have:*

(i) $\left. \frac{\partial}{\partial t} \right|_s (\varphi_t^Y)_* X = (\varphi_s^Y)_* [X,Y]$

(ii) If $[X,Y] = 0$, then $(\varphi_t^Y)_* X = X$, i.e. $(\varphi_t^Y)^* \circ X = X \circ (\varphi_t^Y)^*$, for all t.

(iii) If $[X,Y] = 0$ and X is also even, then the flows of X and Y commute, i.e. $\varphi_t^X \circ \varphi_s^Y = \varphi_s^Y \circ \varphi_t^X$ for all s, t.

Proof of Theorem 3.1. Let $X_1, \ldots, X_m$ be even and $Y_1, \ldots, Y_n$ odd vector fields on $\mathcal{M}$ which all commute. Furthermore, let $\varphi^{X_i} : \mathcal{W}_i \to \mathcal{M}$ denote the flow of X_i for any i. By Corollary 3.8 *(iii)* these flows all commute. Given $p \in M$, there exist an open neighbourhood $V \subseteq M$ of p and an open connected neighbourhood $U \subseteq \mathbb{R}^m$ of 0 such that

$$\beta : U \times \mathcal{V} \to \mathcal{M}, \ \beta_t = \varphi_{t_1}^{X_1} \circ \ldots \circ \varphi_{t_m}^{X_m},$$

is defined, where $\mathcal{V} = (V, \mathcal{O}_\mathcal{M}|_V)$. Since the flows φ^{X_i} commute and each flow defines a local $\mathbb{R}$-action on $\mathcal{M}$, the map β has the action property, i.e. $\beta_0^* = \iota_0^* \circ \beta^* = \mathrm{id}_\mathcal{M}^*$ and $\beta_s \circ \beta_s = \beta_{s+t}$ for all s,t such that both sides of the equation are defined.

Let $\tau_1, \ldots, \tau_n$ be coordinates on $\mathbb{R}^{0|n}$ and define

$$\alpha : \mathbb{R}^{0|n} \times \mathcal{M} \to \mathcal{M} \text{ by } \alpha^*(f) = \exp\left(\sum_{j=1}^n \tau_j Y_j \right)(f).^{(3)}$$

The underlying map is $\tilde{\alpha} = \mathrm{id}_M$. The sum

$$\exp(\sum_{j=1}^{n} \tau_j Y_j)(f) = \sum_{k=0}^{\infty} \frac{1}{k!}(\sum_{j=1}^{n} \tau_j Y_j)^k(f)$$

is finite because $(\sum_{j=1}^{n} \tau_j Y_j)^{n+1} = 0$. Since the odd vector fields Y_j all commute, i.e. $[Y_i, Y_j] = Y_i Y_j + Y_j Y_i = 0$, we get $(\sigma_i Y_i)(\tau_j Y_j) = (\tau_j Y_j)(\sigma_i Y_i)$, and thus $\exp(\sum_i \sigma_i Y_i)\exp(\sum_j \tau_j Y_j) = \exp(\sum_k (\sigma_k + \tau_k)Y_k)$ if $(\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n)$ are coordinates on $\mathbb{R}^{0|n} \times \mathbb{R}^{0|n}$. Consequently, the map α defines an $\mathbb{R}^{0|n}$-action on $\mathcal{M}$.

Now, let $\mathcal{W} = (U \times V, \mathcal{O}_{\mathbb{R}^{m|n} \times \mathcal{M}}|_{U \times V})$ and define

$$\varphi : \mathcal{W} \to \mathcal{M}, \quad \varphi = \beta \circ (\mathrm{id}_{\mathbb{R}^m} \times \alpha) \text{ on } \mathcal{W}.$$

Then $\varphi^*(f) = \exp(\sum_j \tau_j Y_j)(\varphi_{t_1}^{X_1})^* \circ \ldots \circ (\varphi_{t_m}^{X_m})^*(f)$. The map φ satisfies $\varphi \circ \iota_0 = \mathrm{id}_{\mathcal{M}}$ and $\frac{\partial}{\partial t_i} \circ \varphi^* = X_i \circ \varphi^*, \frac{\partial}{\partial \tau_j} \circ \varphi^* = Y_j \circ \varphi^*$ for all i, j, making use of the commutativity of the vector fields and Corollary 3.8. A calculation using the action properties of α and β and again Corollary 3.8 shows that $\varphi \circ (\mathrm{id}_{\mathbb{R}^{m|n}} \times \varphi) = \varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}})$ on the open subsupermanifold of $\mathbb{R}^{m|n} \times \mathbb{R}^{m|n} \times \mathcal{M}$ on which both $\varphi \circ (\mathrm{id}_{\mathbb{R}^{m|n}} \times \varphi)$ and $\varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}})$ are defined. $\blacksquare$

Remark 3.9. The complex version of the result can be proven along the lines of the real case using holomorphic flows and an analogue of Corollary 3.8 for holomorphic vector fields.

Distributions and Frobenius' theorem

Definition 3.10. A (smooth or holomorphic) distribution $\mathcal{D}$ on a (real or complex) supermanifold $\mathcal{M}$ is a graded $\mathcal{O}_{\mathcal{M}}$-subsheaf of the tangent sheaf $\mathcal{T}_{\mathcal{M}} = \mathrm{Der}\,\mathcal{O}_{\mathcal{M}}$ of $\mathcal{M}$ which is locally a direct factor, i.e. for each point $p \in M$ there exists an open neighbourhood U of p in M and a subsheaf $\mathcal{E}$ of $\mathrm{Der}\,\mathcal{O}_{\mathcal{M}}|_U$ on U such that $\mathcal{D}|_U \oplus \mathcal{E} = \mathrm{Der}\,\mathcal{O}_{\mathcal{M}}|_U$.

Remark 3.11 (cf. [Var04], Section 4.7). Locally there exist independent vector fields $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ spanning a distribution $\mathcal{D}$, where the X_i are even and the Y_j are odd. Moreover, $(r|s) = \dim \mathcal{D}(p)$ for any $p \in M$ (if M is connected) and $(r|s)$ is called the rank of the distribution $\mathcal{D}$.

If $\psi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism and $\mathcal{D}$ is a distribution on $\mathcal{M}$, then there is a distribution $\psi_*(\mathcal{D})$ on $\mathcal{N}$ which is spanned by vector fields of the form $\psi_*(X) = (\psi^{-1})^* \circ X \circ \psi^*$ for vector fields X on $\mathcal{M}$ belonging to $\mathcal{D}$.

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[3]The sum $\exp(\sum_{j=1}^{n} \tau_j Y_j)(f) = \sum_{k=0}^{\infty} \frac{1}{k!}(\sum_{j=1}^{n} \tau_j Y_j)^k(f)$ is to be understood in the following way: The vector fields Y_j, which are a priori vector fields on $\mathcal{M}$, are considered as vector fields on the product $\mathbb{R}^{0|n} \times \mathcal{M}$ and similarly f is considered as a function on this product. Hence, $\tau_j Y_j(f)$ here in fact means $\tau_j \cdot (\mathrm{id}_{\mathbb{R}^{0|n}}^* \otimes Y_j)(\pi_{\mathcal{M}}^*(f))$, where τ_j is now considered as a coordinate on $\mathbb{R}^{0|n} \times \mathcal{M}$, $\pi_{\mathcal{M}} : \mathbb{R}^{0|n} \times \mathcal{M} \to \mathcal{M}$ is the canonical projection onto $\mathcal{M}$ and $\mathrm{id}_{\mathbb{R}^{0|n}}^* \otimes Y_j$ is the extension of Y_j to a vector field on $\mathbb{R}^{0|n} \times \mathcal{M}$.

Remark 3.12. A distribution $\mathcal{D}$ of rank $(r|s)$ on a supermanifold $\mathcal{M}$ induces a distribution $\widetilde{\mathcal{D}}$ of rank r on the underlying classical manifold M by defining $\widetilde{\mathcal{D}}(p) = \{\tilde{X}(p)| X \in \mathcal{D}\} \subseteq T_pM$ for $p \in M$. A vector field on M belongs to the reduced distribution $\widetilde{\mathcal{D}}$ if and only if it is the reduced vector field $\tilde{X}$ of some vector field X on $\mathcal{M}$ belonging to $\mathcal{D}$.

As in the classical case, one can define the notion of an involutive distribution.

Definition 3.13. A distribution $\mathcal{D}$ is called involutive if $\mathcal{D}_p$ is a Lie subsuperalgebra of $(\mathrm{Der}\,\mathcal{O}_\mathcal{M})_p$ for each $p \in M$, i.e. if for any two vector fields X and Y on $\mathcal{M}$ belonging to $\mathcal{D}$ their commutator $[X, Y]$ also belongs to $\mathcal{D}$.

The local structure of an involutive distribution on a supermanifold is described by the following version of Frobenius' theorem.

Theorem 3.14 (Local Frobenius' Theorem, cf. [Var04] Theorem 4.7.1). *A distribution $\mathcal{D}$ on a real (resp. complex) supermanifold $\mathcal{M}$ is involutive if and only if there are local coordinates $(x, \theta) = (x_1, \ldots, x_m, \theta_1, \ldots, \theta_n)$ around each point $p \in M$ such that $\mathcal{D}$ is locally spanned by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_s}$.*

If $\mathcal{D}$ is locally spanned by $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r}, \frac{\partial}{\partial \theta_1}, \ldots, \frac{\partial}{\partial \theta_s}$, the involutiveness of $\mathcal{D}$ follows directly. A key point in the proof of the local Frobenius' theorem is the existence of homogeneous commuting vector fields which locally span the distribution. Their existence can be proven as in the classical case (as for example in [Lee03], Theorem 19.10). Then the desired coordinates can be found by applying Theorem 3.1.

Definition 3.15. Let $\mathcal{D}$ be a distribution of rank $(r|s)$ on a supermanifold $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$. An $(r|s)$-dimensional subsupermanifold $j : \mathcal{N} \to \mathcal{M}$, $\mathcal{N} = (N, \mathcal{O}_\mathcal{N})$, of $\mathcal{M}$ is called an integral manifold of $\mathcal{D}$ through $p \in M$ if

(i) the point p is contained in $\tilde{\jmath}(N)$, and

(ii) the distribution $\mathcal{D}$ is tangent to $\mathcal{N}$, i.e. for any vector field X on $\mathcal{M}$ belonging to $\mathcal{D}$ there exists a vector field $\bar{X}$ on $\mathcal{N}$ such that $\bar{X} \circ j^* = j^* \circ X$, or equivalently $X(\ker j^*) \subseteq \ker j^*$, and all vector fields on $\mathcal{N}$ arise in this way.

Remark 3.16. In the case of supermanifolds, integral manifolds of a distribution do not provide as much information about the distribution as in the classical case.

Contrary to the classical case, there is no global version of a Frobenius' theorem for the above defined notion of an integral manifold; the existence of integral manifolds through every point is not equivalent to the involutiveness of a distribution. Nevertheless, the local Frobenius' theorem still guarantees the existence of integral manifolds through every point for an involutive distribution.

Example 3.17 (A non-involutive distribution with integral manifolds). Let $\mathcal{M} = \mathbb{R}^{0|2}$, with coordinates θ_1 and θ_2, and let $\mathcal{D}$ be the distribution on $\mathcal{M}$ spanned by the odd vector field

$$X = \frac{\partial}{\partial \theta_1} + \theta_1 \theta_2 \frac{\partial}{\partial \theta_2}.$$

The distribution $\mathcal{D}$ is not involutive since $[X, X] = 2XX = 2\theta_2 \frac{\partial}{\partial \theta_2} \notin \mathcal{D}$, but $\mathcal{N} = \mathbb{R}^{0|1} \times \{0\} \subset \mathcal{M}$ is an integral manifold of $\mathcal{D}$, and thus there exist integral manifolds through each point (which is only 0). Moreover, remark that the involutive distribution spanned by $\frac{\partial}{\partial \theta_1}$ has the same integral manifolds as $\mathcal{D}$.

4. Infinitesimal and local group actions

In the classical case, the flow $\varphi^X : \Omega \subseteq \mathbb{R} \times M \to M$ of a vector field X on a manifold M defines a local $\mathbb{R}$-action on M. More generally, as proven in [Pal57], Chapter II, if we have a Lie algebra homomorphism $\lambda : \mathfrak{g} \to \mathrm{Vec}(M)$ of a finite dimensional Lie algebra $\mathfrak{g}$ into the Lie algebra of vector fields on M, there is a local action $\varphi : \Omega \subseteq G \times M \to M$ of a Lie group G, with Lie algebra of right-invariant vector fields $\mathfrak{g}$, such that its induced infinitesimal action

$$\mathfrak{g} \to \mathrm{Vec}(M),\ X \mapsto \left.\frac{\partial}{\partial t}\right|_0 \varphi(\exp(tX), -) = (X(e) \otimes \mathrm{id}_M^*) \circ \varphi^*$$

coincides with λ. A typical example is the case where $X_1, \ldots, X_n$ are vector fields on M whose $\mathbb{R}$-span $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{X_1, \ldots, X_n\}$ is a Lie subalgebra of $\mathrm{Vec}(M)$ and λ is the inclusion $\mathfrak{g} \hookrightarrow \mathrm{Vec}(M)$.

The goal in this section is the proof of an analogous theorem on supermanifolds. For that purpose a suitable distribution on $\mathcal{G} \times \mathcal{M}$ is introduced, as in the classical case in [Pal57].

This also generalizes the result in [MSV93] and [GW13] that the flow of one vector field X on a supermanifold $\mathcal{M}$ is a local $\mathbb{R}^{1|1}$-action if and only if X is contained in a $1|1$-dimensional Lie subsuperalgebra $\mathfrak{g} \subset \mathrm{Vec}(\mathcal{M})$.

The results in this section are formulated for the real case, but are equally true in the complex case, i.e. for complex supermanifolds $\mathcal{M}$, complex Lie supergroups $\mathcal{G}$ and morphisms $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ of complex Lie superalgebras.

Proposition 4.1. *Let φ be a local action of a Lie supergroup $\mathcal{G}$, with Lie superalgebra $\mathfrak{g}$, on a supermanifold $\mathcal{M}$. Then $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*$ [4] is a vector field on $\mathcal{M}$ for any $X \in \mathfrak{g}$.*

The map

$$\lambda_\varphi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M}) = \mathrm{Der}\,\mathcal{O}_\mathcal{M}(M),\ X \mapsto (X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*$$

is a homomorphism of Lie superalgebras.

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[4] Here, and in the following, $(X \otimes \mathrm{id}_{\mathcal{M}}^*)$ denotes again the canonical extension of the vector field X on $\mathcal{G}$ to a vector field on $\mathcal{G} \times \mathcal{M}$, and $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*)$ is its evaluation in e, i.e. $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) = \iota_e^* \circ (X \otimes \mathrm{id}_{\mathcal{M}}^*)$.

Proof.   Let $X, Y \in \mathfrak{g}$ be homogeneous. The vector field $X \otimes \mathrm{id}_{\mathcal{M}}^*$ has the parity $|X|$ of X. Let $f, g \in \mathcal{O}_{\mathcal{M}}(M)$ and let f be homogeneous with parity $|f|$. Then, using $\varphi \circ \iota_e = \mathrm{id}_{\mathcal{M}}$, we get

$$((X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*)(fg) = \iota_e^*(X \otimes \mathrm{id}_{\mathcal{M}}^*)(\varphi^*(f)\varphi^*(g))$$
$$= \Big(((X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*)(f)\Big)g + (-1)^{|X||f|}f\Big(((X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*)(g)\Big).$$

Since φ is a local action, we have $\varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}}) = \varphi \circ (\mathrm{id}_{\mathcal{G}} \times \varphi)$. A calculation using this identity then gives $\lambda_\varphi(X)\lambda_\varphi(Y) = (XY(e) \otimes \mathbb{1}) \circ \varphi^*$. Thus

$$[\lambda_\varphi(X), \lambda_\varphi(Y)] = \Big(\Big(XY(e) - (-1)^{|X||Y|}YX(e)\Big) \otimes \mathbb{1}\Big) \circ \varphi^* = \lambda_\varphi([X,Y]). \quad \blacksquare$$

Definition 4.2.   Let $\mathcal{M}$ be a supermanifold and $\mathcal{G}$ a Lie supergroup with Lie superalgebra $\mathfrak{g}$. An infinitesimal action of $\mathcal{G}$ on $\mathcal{M}$ is a homomorphism $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ of Lie superalgebras.

The induced infinitesimal action of a local $\mathcal{G}$-action φ on $\mathcal{M}$ is the homomorphism

$$\lambda_\varphi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M}), \; \lambda_\varphi(X) = (X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \varphi^*.$$

As in the classical case, there is an equivalence of infinitesimal actions and local actions up to shrinking. This is is the content of the following theorem, whose proof is carried out in the remainder of this section.

Theorem 4.3.   *Let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be an infinitesimal action. Then there exists a local $\mathcal{G}$-action $\varphi : \mathcal{W} \subseteq \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$ such that its induced infinitesimal action λ_φ equals λ.*

Moreover, any local action $\varphi : \mathcal{W} \to \mathcal{M}$ is uniquely determined by its induced infinitesimal action and domain of definition.

Distributions associated to infinitesimal actions   Given an infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$, define the distribution $\mathcal{D} = \mathcal{D}_\lambda$ on $\mathcal{G} \times \mathcal{M}$ as the distribution spanned by vector fields of the form $X + \lambda(X)^{(5)}$ for $X \in \mathfrak{g}$ (cf. [Pal57], Chapter II, Definition 7). The rank of the distribution $\mathcal{D}$ equals the dimension of the Lie superalgebra $\mathfrak{g}$.

If the homomorphism $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ is given, we can take $\mathcal{G}$ to be any Lie supergroup with Lie superalgebra $\mathfrak{g}$. One choice is for example the unique Lie supergroup with simply-connected underlying Lie group and Lie superalgebra $\mathfrak{g}$ (for the existence of such $\mathcal{G}$ see, e.g., [Kos83], and in the complex case [Vis11]).

In the following, properties of the distribution associated to an infinitesimal action are studied.

Lemma 4.4.   *The distribution $\mathcal{D} = \mathcal{D}_\lambda$ associated to an infinitesimal action λ is involutive.*

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[5] The vector field $X + \lambda(X)$ is here considered as a vector field on $\mathcal{G} \times \mathcal{M}$, so more formally one should write $X \otimes \mathrm{id}_{\mathcal{M}}^* + \mathrm{id}_{\mathcal{G}}^* \otimes \lambda(X)$ for $X + \lambda(X)$.

Proof. Since $\mathcal{D}$ is spanned by vector fields of the form $X + \lambda(X)$ for $X \in \mathfrak{g}$, it is enough to check that the bracket of two such vector fields belongs again to $\mathcal{D}$. Using that λ is a homomorphism of Lie superalgebras we get

$$[X + \lambda(X), Y + \lambda(Y)] = [X, Y] + [\lambda(X), \lambda(Y)] = [X, Y] + \lambda([X, Y])$$

for any $X, Y \in \mathfrak{g}$ and thus $[X + \lambda(X), Y + \lambda(Y)] = [X, Y] + \lambda([X, Y])$ also belongs to $\mathcal{D}$. ∎

The local Frobenius' theorem now yields that the distribution $\mathcal{D}$ associated to an infinitesimal action locally looks like a standard distribution on a product. In the following, local charts for the distribution $\mathcal{D}$ which locally transform $\mathcal{D}$ to a standard distribution and satisfy a few more properties with respect to the product structure of $\mathcal{G} \times \mathcal{M}$ are of special interest.

Definition 4.5 (flat chart). Let $\mathcal{G}$ be a Lie supergroup, $\mathcal{M}$ a supermanifold and $\mathcal{D}$ the distribution on $\mathcal{G} \times \mathcal{M}$ associated to an infinitesimal action on $\mathcal{M}$. Let $U \subseteq G$ be an open connected neighbourhood of a point $g \in G$, $\mathcal{U} = (U, \mathcal{O}_{\mathcal{G}}|_U)$, and denote by $\iota_g : \mathcal{M} \hookrightarrow \{g\} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{M}$ the canonical inclusion. Moreover, let $V \subseteq M$ be open, $\mathcal{V} = (V, \mathcal{O}_{\mathcal{M}}|_V)$, and let $\rho : \mathcal{V} \to \mathcal{M}$ be a diffeomorphism onto its image. Denote by $\mathcal{D}_{\mathcal{G}}$ the standard distribution on $\mathcal{G} \times \mathcal{M}$ in $\mathcal{G}$-direction, which is spanned by vector fields $X \otimes \mathrm{id}_{\mathcal{M}}^*$ on $\mathcal{G} \times \mathcal{M}$, where X is an arbitrary vector field on $\mathcal{G}$.

A diffeomorphism onto its image $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ is called a flat chart with respect to $(\mathcal{D}, U, V, g, \rho)$, or simply a flat chart (in g), if the following conditions are satisfied:

(i) $\psi_*(\mathcal{D}_{\mathcal{G}}|_W) = \mathcal{D}|_{\tilde{\psi}(W)}$ for each open subset $W \subseteq U \times V$,

(ii) $\pi_{\mathcal{G}}|_{U \times V} = \pi_{\mathcal{G}} \circ \psi$ for the projection $\pi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{G}$,

(iii) $\psi \circ \iota_g|_V = \iota_g|_{\tilde{\rho}(V)} \circ \rho$.

Remark 4.6. If $\psi : \mathcal{U} \times \mathcal{V}_1 \to \mathcal{G} \times \mathcal{M}$ is a flat chart with respect to $(\mathcal{D}, U, V_1, g, \rho_1)$ and $\rho_2 : \mathcal{V}_2 \to \mathcal{M}$ is diffeomorphism onto its image with $\tilde{\rho}_2(V_2) \subseteq V_1$, then the map $\psi' = \psi \circ (\mathrm{id}_{\mathcal{U}} \times \rho_2)$ is a flat chart with respect to
$$(\mathcal{D}, U, V_2, g, \rho_1 \circ \rho_2).$$

Lemma 4.7. *Let $\mathcal{D}$ be the distribution on $\mathcal{G} \times \mathcal{M}$ associated to the infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$, $\lambda_0 = \lambda|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \mathrm{Vec}(\mathcal{M})$ the restriction of λ to the even part $\mathfrak{g}_0$ of $\mathfrak{g}$, and $\mathcal{D}_0$ the distribution on $G \times \mathcal{M}$ associated to λ_0. Let $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ be a flat chart with respect to $(\mathcal{D}, U, V, g, \rho)$ and define $\psi_0 : U \times \mathcal{V} \to G \times \mathcal{M}$ by $\psi_0 = (\mathrm{id}_G \times \varphi_0) \circ (\mathrm{diag} \times \mathrm{id}_{\mathcal{V}})$, where $\mathrm{diag} : U \to U \times U$ denotes the diagonal and φ_0 is the composition of $\pi_{\mathcal{M}} \circ \psi$ and the canonical inclusion $U \times \mathcal{V} \hookrightarrow \mathcal{U} \times \mathcal{V}$. Then ψ_0 is a flat chart with respect to $(\mathcal{D}_0, U, V, g, \rho)$.*

Proof. It can be checked by direct calculations that ψ_0 is a flat chart, using that the even right-invariant vector fields on $\mathcal{G}$ can be identified with the right-invariant vector fields on G, i.e. $\mathrm{Lie}(G) \cong \mathfrak{g}_0$ if $\mathfrak{g}_0$ denotes the even part of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ of $\mathcal{G}$. ∎

Proposition 4.8 (Local existence of flat charts). *Let $\mathcal{D}$ be the distribution associated to the infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ on the supermanifold $\mathcal{M}$ and let $\mathcal{G}$ be a Lie supergroup with $\mathfrak{g}$ as its Lie superalgebra of right-invariant vector fields. For any point $(g, p) \in G \times M$ there are an open connected neighbourhood U of g in G and an open neighbourhood V of p in M such that there exists a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, g, \rho = \mathrm{id}_{\mathcal{V}})$.*

By the above remark this implies moreover the existence of U and V and a flat chart with respect to $(\mathcal{D}, U, V, g, \rho)$ for arbitrary ρ.

Proof. Let $X_1, \ldots, X_k, Y_1, \ldots, Y_l$ be a basis of $\mathfrak{g}$ such that $X_1, \ldots, X_k$ are even and $Y_1, \ldots, Y_l$ odd vector fields. Then the tangent vectors $X_1(g'), \ldots, X_k(g')$, $Y_1(g'), \ldots, Y_l(g') \in T_{g'}\mathcal{G} = \{X(g') \,|\, X \in \mathfrak{g}\}$ are linearly independent for all $g' \in G$. Since the distribution $\mathcal{D}$ is spanned by vector fields of the form $X + \lambda(X)$ for $X \in \mathfrak{g}$, there exist local coordinates (t, τ) for $\mathcal{G}$ on an open connected neighbourhood $U \subseteq G$ of g and local coordinates (x, θ) for $\mathcal{M}$ on an open neighbourhood $V \subseteq M$ of p so that $\mathcal{D}$ is locally spanned by the commuting vector fields

$$A_i = \frac{\partial}{\partial t_i} + \sum_{u=1}^{m} a_{iu} \frac{\partial}{\partial x_u} + \sum_{v=1}^{n} b_{iv} \frac{\partial}{\partial \theta_v} \quad \text{and} \quad B_j = \frac{\partial}{\partial \tau_j} + \sum_{u=1}^{m} c_{ju} \frac{\partial}{\partial x_u} + \sum_{v=1}^{n} d_{jv} \frac{\partial}{\partial \theta_v}$$

for $i = 1 \ldots, k$ and $j = 1, \ldots, l$, where $a_{iu}, b_{iv}, c_{ju}, d_{jv} \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}(U \times V)$.

After shrinking U and V, $\mathcal{U} = (U, \mathcal{O}_{\mathcal{G}}|_U)$ can be assumed to be an open subsupermanifold of $\mathbb{R}^{k|l}$ with $g = 0$ and there is a morphism $\varphi : \mathcal{U} \times (\mathcal{U} \times \mathcal{V}) \to \mathcal{G} \times \mathcal{M}$ associated to the above defined commuting vector fields, satisfying the properties $\varphi \circ \iota_0 = \mathrm{id}$ and $\varphi \circ (\mathrm{id}_{\mathbb{R}^{k|l}} \times \varphi) = \varphi \circ (\mu_{\mathbb{R}^{k|l}} \times \mathrm{id}_{\mathcal{M}})$ (cf. Theorem 3.1). Since

$$\frac{\partial}{\partial t_i} \circ \varphi^* = \varphi^* \circ A_i \quad \text{and} \quad \frac{\partial}{\partial \tau_j} \circ \varphi^* = \varphi^* \circ B_j,$$

the subsupermanifold $\mathcal{V} \cong \{0\} \times \mathcal{V} \subset \mathcal{U} \times \mathcal{V}$ is transversal to $\varphi(\mathcal{U} \times \{(0, p)\})$ in $\tilde{\varphi}(0, 0, p) = (0, p)$. The map $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$, $\psi = \varphi|_{\mathcal{U} \times \{0\} \times \mathcal{V}}$, identifies the standard distribution $\mathcal{D}_{\mathcal{U}}$ on $\mathcal{U} \times \mathcal{V}$ with $\mathcal{D}$, i.e. $\psi_*(\mathcal{D}_{\mathcal{G}}|_W) = \psi_*(\mathcal{D}_{\mathcal{U}}|_W) = \mathcal{D}|_{\tilde{\psi}(W)}$ for all open subsets $W \subseteq U \times V$. The action property of the map φ moreover implies that

$$\psi \circ \iota_g|_V = \psi \circ \iota_0|_V = \varphi|_{\mathcal{U} \times \{0\} \times \mathcal{V}} \circ \iota_0|_V = \mathrm{id}_{\mathcal{U} \times \mathcal{V}}|_{\{0\} \times V} = \iota_0|_V = \iota_g|_V.$$

Hence, it only remains to show that $\pi_{\mathcal{G}}|_{U \times V} = \pi_{\mathcal{G}} \circ \psi$ in order to prove that ψ is a flat chart. This is equivalent to showing $\psi^*(t_i) = t_i$ and $\psi^*(\tau_j) = \tau_j$ for all i, j, where t_i and τ_j are now considered as local coordinate functions on $\mathcal{G} \times \mathcal{M}$. Since $\frac{\partial}{\partial t_r}(\psi^*(t_i)) = \psi^*(\psi_*(\frac{\partial}{\partial t_r})(t_i)) = \psi^*(0) = 0$ for all $r \neq i$, $\frac{\partial}{\partial t_i}(\psi^*(t_i)) = 1$ and $\frac{\partial}{\partial \tau_s}(\psi^*(t_i)) = \psi^*(\psi_*(\frac{\partial}{\partial \tau_s})(t_i)) = \psi^*(0) = 0$ for all s, the function $\psi^*(t_i) \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}(U \times V)$ is of the form

$$\psi^*(t_i) = (t_i + c_i(x)) + \sum_{\nu \neq 0} c_\nu(x) \theta^\nu$$

for some smooth functions $c_i, c_\nu (\nu \neq 0)$ on V. The property $\psi \circ \iota_0 = \iota_0$ now implies $0 = \iota_0^*(t_i) = \iota_0^* \circ \psi^*(t_i) = (0 + c_i(x)) + \sum_{\nu \neq 0} c_\nu(x) \theta^\nu$ and therefore $c_i = c_\nu = 0$. Hence $\psi^*(t_i) = t_i$ as required. A similar argument yields $\psi^*(\tau_j) = \tau_j$. ∎

Lemma 4.9.   *Let $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ be a diffeomorphism onto its image such that*

(i) *$\psi_*(\mathcal{D}_\mathcal{G}|_W) = \mathcal{D}|_{\tilde{\psi}(W)}$ for a distribution $\mathcal{D}$ associated to an infinitesimal action and any open subset $W \subseteq U \times V$, and*

(ii) *$\pi_\mathcal{G}|_{U \times V} = \pi_\mathcal{G} \circ \psi$.*

Then given any element $g \in U$ there exists a diffeomorphism onto its image $\rho : \mathcal{V} \to \mathcal{M}$ such that $\psi \circ \iota_g|_V = \iota_g|_{\tilde{\rho}(V)} \circ \rho$. Hence ψ is a flat chart with respect to $(\mathcal{D}, U, V, g, \rho)$.

Proof.   The property $\pi_\mathcal{G} \circ \psi = \pi_\mathcal{G}|_{U \times V}$ implies $(\tilde{\psi} \circ \tilde{\iota}_g)(V) \subset \{g\} \times M$. To show that $\psi \circ \iota_g|_V : \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ induces a map $\rho : \mathcal{V} \to \mathcal{M} \cong \{g\} \times \mathcal{M}$, it is enough to check that $(\psi \circ \iota_g|_V)^* \circ \pi_\mathcal{G}^* = \mathrm{ev}_g$, where $\mathrm{ev}_g : \mathcal{O}_\mathcal{G}(G) \to \mathbb{R}$ denotes the evaluation in g.

The fact $\pi_\mathcal{G} \circ \psi = \pi_\mathcal{G}|_{U \times V}$ implies $(\psi \circ \iota_g|_V)^* \circ \pi_\mathcal{G}^* = (\pi_\mathcal{G} \circ \psi \circ \iota_g|_V)^* = (\pi_\mathcal{G}|_{U \times V} \circ \iota_g|_V)^* = \mathrm{ev}_g$ since $(\pi_\mathcal{G} \circ \iota_g)$ is the unique map $\mathcal{M} \to \{g\} \subset \mathcal{G}$.

The map ρ satisfies $\psi \circ \iota_g|_V = \iota_g|_{\tilde{\rho}(V)} \circ \rho$ by definition and is a diffeomorphism onto its image since ψ is a diffeomorphism onto its image and $\mathcal{V}$ and $\mathcal{M}$ have the same dimension.   ∎

Proposition 4.10 (Uniqueness of flat charts).   *If $\mathcal{D}$ is a distribution on $\mathcal{G} \times \mathcal{M}$ associated to an infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$, then a flat chart*
$$\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M} \quad \text{with respect to } (\mathcal{D}, U, V, g, \rho)$$
is unique.

The proof of this proposition is carried out in two steps:

(i) First, it is shown that two flat charts ψ_1 and ψ_2 with respect to $(\mathcal{D}, U, V, g, \rho)$, coincide on an open neighbourhood of $\{g\} \times V$ in $U \times V$.

(ii) Second, the local statement and the connectedness of U are used to globally get $\psi_1 = \psi_2$.

Proof.   (i) Consider first the case where $\rho = \mathrm{id}$ and $\mathcal{D} = \mathcal{D}_\mathcal{G}$, i.e. the infinitesimal action λ is the zero map. Now, let ψ be a flat chart with respect to $(\mathcal{D}_\mathcal{G}, U, V, g, \mathrm{id})$. Since the identity $\mathrm{id}_{\mathcal{G} \times \mathcal{M}}|_{U \times V}$ is also a flat chart with respect to $(\mathcal{D}_\mathcal{G}, U, V, g, \mathrm{id})$, the equality of ψ and the identity in a neighbourhood of any point (g, p) for $p \in V$ needs to be shown.

Let (t, τ) be local coordinates on a connected neighbourhood U' of $g \in G$ such that $g = 0$ in the coordinates and let (x, θ) be local coordinates on a neighbourhood V' of $p \in V$. Then (t, τ, x, θ) are local coordinates for $\mathcal{G} \times \mathcal{M}$ in a neighbourhood of $(0, p) = \tilde{\psi}(0, p)$ and therefore on a neighbourhood of $\tilde{\psi}(U'' \times V'') \subseteq U' \times V'$ for appropriate subsets $U'' \subseteq U'$ and $V'' \subseteq V'$. Since $\pi_\mathcal{G}|_{U \times V} = \pi_\mathcal{G} \circ \psi$, we have

$$\psi^*(t, \tau) = \psi^*(\pi_\mathcal{G}^*(t, \tau)) = \pi_\mathcal{G}^*(t, \tau) = (t, \tau).$$

Moreover, $\psi_*(\mathcal{D}_\mathcal{G}) = \mathcal{D}_\mathcal{G}$ implies

$$\psi_* \left(\frac{\partial}{\partial t_r} \right), \psi_* \left(\frac{\partial}{\partial \tau_s} \right) \in \mathrm{span} \left\{ \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_k}, \frac{\partial}{\partial \tau_1}, \ldots, \frac{\partial}{\partial \tau_l} \right\}$$

for all r, s. Hence

$$\psi_* \left(\frac{\partial}{\partial t_r} \right)(x, \theta) = \psi_* \left(\frac{\partial}{\partial \tau_s} \right)(x, \theta) = (0, 0),$$

and then $\frac{\partial}{\partial t_r}(\psi^*(x, \theta)) = \psi^*(\psi_*(\frac{\partial}{\partial t_r})(x, \theta)) = \psi^*(0, 0) = (0, 0)$ and similarly $\frac{\partial}{\partial \tau_s}(\psi^*(x, \theta)) = (0, 0)$. Consequently, we have

$$\psi^*(x, \theta) = \iota_0^* \circ \psi^*(x, \theta) = \iota_0^*(x, \theta) = (x, \theta)$$

and thus $\psi = \mathrm{id}$ on $U'' \times V''$.

Let ψ_1 and ψ_2 now be flat charts with respect to $(\mathcal{D}, U, V, g, \rho)$ for a distribution $\mathcal{D}$ associated to an arbitrary infinitesimal action and arbitrary ρ. For any $p \in V$, we have

$$\tilde{\psi}_1(g, p) = \tilde{\psi}_1 \circ \tilde{\iota}_g(p) = \tilde{\iota}_g(\tilde{\rho}(p)) = (g, \tilde{\rho}(p)) = \tilde{\psi}_2 \circ \tilde{\iota}_g(p) = \tilde{\psi}_2(g, p).$$

Now let U' and U'' be open connected neighbourhoods of g and V' and V'' open neighbourhoods of p with $U'' \subseteq U'$ and $V'' \subseteq V'$ such that the associated open subsupermanifolds $\mathcal{U}'$ of $\mathcal{G}$ and $\mathcal{V}'$ of $\mathcal{M}$ are isomorphic to superdomains and $\tilde{\psi}_1(U'' \times V'')$ is contained in $\tilde{\psi}_2(U' \times V')$. The composition $\psi_2^{-1} \circ \psi_1$ is defined on $U'' \times V''$ and a flat chart with respect to $(\mathcal{D}_\mathcal{G}, U'', V'', g, \mathrm{id})$. By the above argumentation, $\psi_2^{-1} \circ \psi_1 = \mathrm{id}$ and thus $\psi_1 = \psi_2$ on $U'' \times V''$.

(ii) Let ψ_1 and ψ_2 be two flat charts with respect to $(\mathcal{D}, U, V, g, \rho)$. For each $p \in V$ define the subset $W_p \subseteq U$ containing the points $t \in U$ such that $\psi_1 = \psi_2$ on an open neighbourhood of (t, p) in $U \times V$. The sets W_p are open by definition and contain g as a consequence of (i). To prove $\psi_1 = \psi_2$, i.e. $W_p = U$ for each p, it is therefore enough to show that each W_p is also closed in U due to the connectedness of U.

If W_p is not closed, then there is a point $t_0 \in U \setminus W_p$ such that $W_p \cap \Omega \neq \emptyset$ for every open neighbourhood Ω of t_0. The continuity of the underlying maps implies $\tilde{\psi}_1(t_0, p) = \tilde{\psi}_2(t_0, p)$. Let U' be an open neighbourhood of t_0 in U and V' and open neighbourhood of p in V such that the associated open subsupermanifolds $\mathcal{U}'$ and $\mathcal{V}'$ are isomorphic to superdomains. Now let $U'' \subseteq U'$ be an open connected neighbourhood of t_0 and $V'' \subseteq V'$ an open neighbourhood of p such that $\tilde{\psi}_1(U'' \times V'') \subseteq \tilde{\psi}_2(U' \times V')$. Moreover, let s_0 be an element of $U'' \cap W_p$, which exists by assumption on the choice of t_0. After shrinking V' and V'', the maps ψ_1 and ψ_2 coincide on an open neighbourhood of $\{s_0\} \times V'$. By Lemma 4.9 there exists a diffeomorphism onto its image $\rho_0 : \mathcal{V}' \to \mathcal{M}$ such that the restrictions of ψ_1 and ψ_2 to $U' \times V'$ are flat charts with respect to $(\mathcal{D}, U', V', s_0, \rho_0)$. The same argument as given in (i) then shows that ψ_1 and ψ_2 coincide on $U'' \times V''$ which is a contradiction to the assumption $t_0 \notin W_p$. ∎

Remark 4.11. Let ψ and ψ' be flat charts and denote by ψ_0 and ψ'_0 the associated flat charts for the even part (cf. Lemma 5.43). Then $\psi = \psi'$ if and only if $\psi_0 = \psi'_0$.

Moreover, if $\psi_0 : U \times V \to G \times M$ is a flat chart with respect to $(\mathcal{D}_0, U, V, g, \rho)$ then the uniqueness and local existence of flat charts imply that there is a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, g, \rho)$.

Uniqueness of local actions

Lemma 4.12. *Let* $\varphi : \mathcal{W} \subseteq \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ *be the local action of the Lie supergroup* $\mathcal{G}$ *on a supermanifold* $\mathcal{M}$ *with induced infinitesimal action* λ_φ. *Let* $\mathcal{D}$ *denote the distribution associated to* λ_φ. *Define*

$$\psi = (\mathrm{id}_\mathcal{G} \times \varphi) \circ (\mathrm{diag} \times \mathrm{id}_\mathcal{M}) : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$$

where $\mathrm{diag} : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ *denotes the diagonal.*

Let $p \in M$ *and* $U \subset W_p = \{g \in G | (g, p) \in W\}$ *be a relatively compact connected open neighbourhood of* $e \in W_p \subset G$. *Then there exists an open neighbourhood* V *of* $p \in M$ *with* $U \times V \subset W$. *The restriction* $\psi|_{U \times V}$ *is a flat chart with respect to* $(\mathcal{D}, U, V, e, \mathrm{id})$.

Proof. Since $\overline{U} \times \{p\} \subset W$ is compact, we can find an open neighbourhood V of p with $U \times V \subseteq W$. By definition of ψ, we have $\pi_\mathcal{G} \circ \psi = \pi_\mathcal{G}$. Moreover, $\psi \circ \iota_e = \iota_e$ since φ is a local action.

Let $X \in \mathfrak{g}$, then $X \circ \mathrm{diag}^* = \mathrm{diag}^* \circ (X \otimes \mathrm{id}_\mathcal{G}^* + \mathrm{id}_\mathcal{G}^* \otimes X)$ since X is a derivation. If $\iota_e^\mathcal{G} : \mathcal{G} \hookrightarrow \{e\} \times \mathcal{G} \subset \mathcal{G} \times \mathcal{G}$ is the inclusion, then $\mu \circ \iota_e^\mathcal{G} = \mathrm{id}_\mathcal{G}$ for the multiplication μ on $\mathcal{G}$. Since X is right-invariant, we have $X = (X(e) \otimes \mathrm{id}_\mathcal{G}^*) \circ \mu^*$. By a calculation, using these facts and that φ is a local action, we obtain

$$\psi_*(X \otimes \mathbb{1}) = X \otimes \mathbb{1} + \mathbb{1} \otimes \lambda_\varphi(X),$$

which yields $\psi_*(\mathcal{D}_\mathcal{G}) = \mathcal{D}$. ∎

The lemma implies that every local action of a Lie supergroup is uniquely determined by its domain of definition and its induced infinitesimal action:

Corollary 4.13. *The domain of definition and the induced infinitesimal action uniquely determine a local action, i.e. if* $\varphi_1 : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$ *and* $\varphi_2 : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$ *are two local actions of the Lie supergroup* $\mathcal{G}$ *on the supermanifold* $\mathcal{M}$ *with the same induced infinitesimal action* $\lambda = \lambda_{\varphi_1} = \lambda_{\varphi_2}$, *then* $\varphi_1 = \varphi_2$.

Proof. By the preceding lemma and the uniqueness of flat charts we have $\psi_1 = (\mathrm{id}_\mathcal{G} \times \varphi_1) \circ (\mathrm{diag} \times \mathrm{id}_\mathcal{M})$ and $\psi_2 = (\mathrm{id}_\mathcal{G} \times \varphi_2) \circ (\mathrm{diag} \times \mathrm{id}_\mathcal{M})$ and hence $\varphi_1 = \pi_\mathcal{M} \circ \psi_1 = \pi_\mathcal{M} \circ \psi_2 = \varphi_2$. ∎

Construction of a local action In the following, let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be a fixed infinitesimal action and $\mathcal{G}$ a Lie supergroup with multiplication $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and Lie superalgebra of right-invariant vector fields $\mathfrak{g}$.

The goal is now to find a local $\mathcal{G}$-action on $\mathcal{M}$ with induced infinitesimal action λ. Such a local action of $\mathcal{G}$ on $\mathcal{M}$ is constructed using flat charts for the distribution $\mathcal{D}$ associated to λ. The domain of definition of the constructed action depends, in general, on some choices. After restricting two local actions with a the same infinitesimal action a neighbourhood of $\{e\} \times M$ in $G \times M$ the local actions coincide as proven in the previous paragraph. Nevertheless, in general there is, as in the classical case (cf. [Pal57] Chapter III.4), no unique maximal domain of definition on which the local action can be defined.

Definition of a local action:
Choose a neighbourhood basis $\{U_\alpha\}_{\alpha \in A}$ of the identity $e \in G$ such that (cf. [Pal57], Chapter II, §7):

(i) Each U_α is connected and $U_\alpha = (U_\alpha)^{-1} = \{g \in G \mid g^{-1} \in U_\alpha\}$.

(ii) For $\alpha, \beta \in A$ either $U_\alpha \subseteq U_\beta$ or $U_\beta \subseteq U_\alpha$ holds.

Note that the two conditions guarantee the connectedness of $U_\alpha \cap U_\beta$ for arbitrary $\alpha, \beta \in A$.

For each $p \in M$ choose $\alpha(p) \in A$ and a neighbourhood $V_p \subseteq M$ of p such that there is a flat chart

$$\psi_p : \mathcal{U}^2_{\alpha(p)} \times \mathcal{V}_p \to \mathcal{G} \times \mathcal{M}$$

with respect to $(\mathcal{D}, U^2_{\alpha(p)}, V_p, e, \mathrm{id})$, where $\mathcal{U}^2_{\alpha(a)}$ and $\mathcal{V}_p$ denote again the open subsupermanifolds of $\mathcal{G}$ and $\mathcal{M}$ with underlying sets $U^2_{\alpha(p)} = \{gh \mid g, h \in U_\alpha\}$ and V_p. For two elements $p, q \in M$, we may assume $U_{\alpha(p)} \subseteq U_{\alpha(q)}$. Therefore, if the intersection

$$(U^2_{\alpha(p)} \times V_p) \cap (U^2_{\alpha(q)} \times V_q) = U^2_{\alpha(p)} \times (V_p \cap V_q)$$

is non-empty, then the restrictions of ψ_p and ψ_q to their common domain of definition are both flat charts with respect to $(\mathcal{D}, U^2_{\alpha(p)}, (V_p \cap V_q), e, \mathrm{id})$ and hence coincide. Let

$$W = \bigcup_{p \in M} (U_{\alpha(p)} \times V_p) \subseteq G \times M.$$

The set W is open by definition and contains $\{e\} \times M$. Furthermore, for each $p \in M$ the subset $W_p = \{g \in G \mid (g, p) \in W\} \subseteq G$ is connected since all $U_{\alpha(q)}$ are connected. Let $\mathcal{W} = (W, \mathcal{O}_{G \times \mathcal{M}}|_W)$ and define a morphism $\psi : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$ by $\psi|_{U_{\alpha(p)} \times V_p} = \psi_p$ for each $p \in M$. Let

$$\varphi : \mathcal{W} \to \mathcal{M}, \; \varphi = \pi_{\mathcal{M}} \circ \psi.$$

We now show that φ defines a local group action with induced infinitesimal action λ.

Proposition 4.14.   *The map φ defines a local action of $\mathcal{G}$ on the supermanifold $\mathcal{M}$.*

Proof. Since the map ψ defining $\varphi = \pi_{\mathcal{M}} \circ \psi$ is locally given by flat charts with respect to $(\mathcal{D}, U_{\alpha(p)}, V_p, e, \mathrm{id})$, we have $\psi \circ \iota_e = \iota_e$ and therefore $\varphi \circ \iota_e = \pi_{\mathcal{M}} \circ \psi \circ \iota_e = \pi_{\mathcal{M}} \circ \iota_e = \mathrm{id}_{\mathcal{M}}$. Thus it remains to show that

$$\varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}}) = \varphi \circ (\mathrm{id}_{\mathcal{G}} \times \varphi) \tag{$\star$}$$

on the open subsupermanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$ where both sides are defined. To prove $(\star)$ the special form of the distribution associated to an infinitesimal action with respect to the group structure of $\mathcal{G}$ is used. The commutativity of the following diagram will be shown:

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{G} \times \mathcal{M} & \xrightarrow{\ \chi \times \mathrm{id}_{\mathcal{M}}\ } & \mathcal{G} \times \mathcal{G} \times \mathcal{M} \\
{\scriptstyle \mathrm{id}_{\mathcal{G}} \times \psi} \downarrow & & \downarrow {\scriptstyle (\chi^{-1} \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi)} \\
\mathcal{G} \times \mathcal{G} \times \mathcal{M} & \xrightarrow[(\tau \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi) \circ (\tau \times \mathrm{id}_{\mathcal{M}})]{} & \mathcal{G} \times \mathcal{G} \times \mathcal{M}
\end{array}
$$

In the above diagram all maps are only defined on appropriate open subsupermanifolds of $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$. The map $\tau : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ denotes the map which interchanges the two components. Moreover, the map $\chi : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ is defined by

$$\chi = (\mathrm{id}_{\mathcal{G}} \times \mu) \circ (\mathrm{diag} \times \mathrm{id}_{\mathcal{G}}),$$

such that $\mu = \pi_2 \circ \chi$ and $\pi_1 = \pi_1 \circ \chi$, where π_i, $i = 1, 2$, is the projection onto the i-th factor. The underlying map is given by $\tilde{\chi}(g, h) = (g, gh)$.

Note that if $\mathcal{G} = \mathcal{M}$ and the infinitesimal action λ is the canonical inclusion $\mathfrak{g} \hookrightarrow \mathrm{Vec}(\mathcal{G})$ of the right-invariant vector fields, then χ is a flat chart for the distribution $\mathcal{D}$ with respect to $(\mathcal{D}, G, M, e, \mathrm{id})$, χ and ψ coincide (on their common domain of definition) and $(\star)$ is equivalent to the associativity of the multiplication μ.

Let

$$\Psi_1 = (\tau \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi) \circ (\tau \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi)$$

and

$$\Psi_2 = (\chi^{-1} \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi) \circ (\chi \times \mathrm{id}_{\mathcal{M}}).$$

The underlying maps are $\tilde{\Psi}_1(g, h, p) = (g, h, \tilde{\varphi}(g, \tilde{\varphi}(h, p)))$ and $\tilde{\Psi}_2(g, h, p) = (g, h, \tilde{\varphi}(gh, p))$. The open subsupermanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$ on which both morphisms Ψ_1 and Ψ_2 are defined is exactly the open subsupermanifold which is the common domain of definition of $\varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}})$ and $\varphi \circ (\mathrm{id}_{\mathcal{G}} \times \varphi)$. A calculation shows $\pi_{\mathcal{M}} \circ \Psi_1 = \varphi \circ (\mathrm{id}_{\mathcal{G}} \times \varphi)$ and $\pi_{\mathcal{M}} \circ \Psi_2 = \varphi \circ (\mu \times \mathrm{id}_{\mathcal{M}})$ and the commutativity of the above diagram directly implies $(\star)$.

To show the equality of Ψ_1 and Ψ_2, we consider the distribution $\mathcal{D}_{\mathbb{1} \otimes \lambda}$ on $\mathcal{G} \times (\mathcal{G} \times \mathcal{M})$ associated to the infinitesimal action $\mathbb{1} \otimes \lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{G} \times \mathcal{M})$, $X \mapsto (\mathrm{id}_{\mathcal{G}}^* \otimes \lambda(X))$, of $\mathcal{G}$ on $\mathcal{G} \times \mathcal{M}$ and show that Ψ_1 and Ψ_2 are locally both flat charts for this distribution.

Let $\mathcal{D}_{\mathcal{G}}$ be the distribution on $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$ which is spanned by vector fields of the form $X \otimes \mathrm{id}_{\mathcal{G}}^* \otimes \mathrm{id}_{\mathcal{M}}^*$ for $X \in \mathfrak{g}$. The distribution $\mathcal{D}_{\mathbb{1} \otimes \lambda}$ is spanned by vector

fields of the form $X \otimes \mathrm{id}_{\mathcal{G}}^* \otimes \mathrm{id}_{\mathcal{M}}^* + \mathrm{id}_{\mathcal{G}}^* \otimes \mathrm{id}_{\mathcal{G}}^* \otimes \lambda(X)$. For $X \in \mathfrak{g}$ and writing $\mathbb{1}$ for $\mathrm{id}_{\mathcal{G}}^*$ or $\mathrm{id}_{\mathcal{M}}^*$, we have

$$
\begin{aligned}
(\Psi_1)_*(X \otimes \mathbb{1} \otimes \mathbb{1}) &= (\tau \times \mathrm{id}_{\mathcal{M}})_*(\mathrm{id}_{\mathcal{G}} \times \psi)_*(\tau \times \mathrm{id}_{\mathcal{M}})_*(\mathrm{id}_{\mathcal{G}} \times \psi)_*(X \otimes \mathbb{1} \otimes \mathbb{1}) \\
&= (\tau \times \mathrm{id}_{\mathcal{M}})_*(\mathrm{id}_{\mathcal{G}} \times \psi)_*(\mathbb{1} \otimes X \otimes \mathbb{1}) = (\tau \times \mathrm{id}_{\mathcal{M}})_*(\mathbb{1} \otimes X \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda(X)) \\
&= X \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda(X),
\end{aligned}
$$

where the fact that ψ is a flat chart for the distribution $\mathcal{D}$ associated λ is used. Similarly, using the fact that $\chi_*(X \otimes \mathbb{1}) = (X \otimes \mathbb{1} + \mathbb{1} \otimes X)$, we get $(\Psi_2)_*(X \otimes \mathbb{1} \otimes \mathbb{1}) = X \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \lambda(X)$. Hence, Ψ_1 and Ψ_2 both transform $\mathcal{D}_{\mathcal{G}}$ into $\mathcal{D}_{\mathbb{1} \otimes \lambda}$, i.e. $(\Psi_i)_*(\mathcal{D}_{\mathcal{G}}) = \mathcal{D}_{\mathbb{1} \otimes \lambda}$.

Remark that $\pi_{\mathcal{G}} \circ \Psi_1 = \pi_{\mathcal{G}} = \pi_{\mathcal{G}} \circ \Psi_2$ if $\pi_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \times \mathcal{M} \to \mathcal{G}$ denotes the projection onto the first component. Now, let $\iota_e^{\mathcal{G} \times \mathcal{M}} : \mathcal{G} \times \mathcal{M} \hookrightarrow \{e\} \times \mathcal{G} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{G} \times \mathcal{M}$ and $\iota_e^{\mathcal{M}} : \mathcal{M} \hookrightarrow \{e\} \times \mathcal{M} \subset \mathcal{G} \times \mathcal{M}$ denote the inclusions. Then a calculation shows

$$
\Psi_1 \circ \iota_e^{\mathcal{G} \times \mathcal{M}} = \iota_e^{\mathcal{G} \times \mathcal{M}} \circ \psi \quad \text{and} \quad \Psi_2 \circ \iota_e^{\mathcal{G} \times \mathcal{M}} = \iota_e^{\mathcal{G} \times \mathcal{M}} \circ \psi
$$

since $\psi \circ \iota_e^{\mathcal{M}} = \iota_e^{\mathcal{M}}$ and $(\chi \times \mathrm{id}_{\mathcal{M}}) \circ \iota_e^{\mathcal{G} \times \mathcal{M}} = \iota_e^{\mathcal{G} \times \mathcal{M}}$ using that e is the identity element of $\mathcal{G}$ and the definition of χ. This shows that Ψ_1 and Ψ_2 are both locally flat charts in e.

In order to check that Ψ_1 and Ψ_2 coincide everywhere, the special form of the sets in $\{U_\alpha\}_{\alpha \in A}$ is important. Let $(g, g', p) \in G \times G \times M$ such that Ψ_1 and Ψ_2 are defined on a neighbourhood of (g, g', p), i.e. such that (g', p), (gg', p), $(g, \tilde{\varphi}(g, p)) \in W$. Then by definition of ψ, there exists $U_\alpha \in \{U_\gamma\}_{\gamma \in A}$ containing g, a neighbourhood V_α of $q = \tilde{\varphi}(g', p)$ in M and a flat chart $\psi_\alpha : \mathcal{U}_\alpha^2 \times \mathcal{V}_\alpha \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_\alpha^2, V_\alpha, e, \mathrm{id})$ and with $\psi|_{U_\alpha \times V_\alpha} = \psi_\alpha|_{U_\alpha \times V_\alpha}$. Furthermore, choose $U_\beta \in \{U_\gamma\}_{\gamma \in A}$ containing g' and gg' and a neighbourhood V_β of p in M such that there exists a flat chart $\psi_\beta : \mathcal{U}_\beta^2 \times \mathcal{V}_\beta \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_\beta^2, V_\beta, e, \mathrm{id})$ and with $\psi|_{U_\beta \times V_\beta} = \psi_\beta|_{U_\beta \times V_\beta}$.

Shrink V_β and choose a neighbourhood U of g' in G with $U \subseteq U_\beta$ such that $\tilde{\varphi}(U \times V_\beta) \subseteq V_\alpha$. By the special choice of the neighbourhood basis $\{U_\gamma\}_{\gamma \in A}$ of e in G either $U_\alpha \subseteq U_\beta$ or $U_\beta \subseteq U_\alpha$ is true.

First, let $U_\alpha \subseteq U_\beta$. The map Ψ_1 is defined on $U_\alpha \times U \times V_\beta$ and the restriction of Ψ_1 to $U_\alpha \times U \times V_\beta$ is a flat chart with respect to $(\mathcal{D}_{\mathbb{1} \otimes \lambda}, U_\alpha, U \times V_\beta, e, \psi)$. Moreover, we have $\tilde{\mu}(U_\alpha \times U) = U_\alpha \cdot U \subseteq U_\beta^2$ and thus $\Psi_{2,\beta} = (\chi^{-1} \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi_\beta) \circ (\chi \times \mathrm{id}_{\mathcal{M}})$ is also defined on $U_\alpha \times U \times V_\beta$ and a flat chart with respect to $(\mathcal{D}_{\mathbb{1} \otimes \lambda}, U_\alpha, U \times V_\beta, e, \psi)$. By the uniqueness of flat charts, Ψ_1 and $\Psi_{2,\beta}$ coincide on $U_\alpha \times U \times V_\beta$ so that $\Psi_1 = \Psi_2$ near $(g, g', p) \in U_\alpha \times U \times V_\beta$.

Consider now the case $U_\beta \subseteq U_\alpha$. The map
$$
\Psi_{1,\alpha} = (\tau \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi_\alpha) \circ (\tau \times \mathrm{id}_{\mathcal{M}}) \circ (\mathrm{id}_{\mathcal{G}} \times \psi)
$$
is defined on $U_\alpha^2 \times U \times V_\beta$ and is a flat chart with respect to $(\mathcal{D}_{\mathbb{1} \otimes \lambda}, U_\alpha^2, U \times V_\beta, e, \psi)$. The set $U_\beta(g')^{-1}$ contains e and g. Since $U_\beta(g')^{-1} \subset U_\beta \cdot U_\beta^{-1} = U_\beta^2 \subseteq U_\alpha^2$, the map $\Psi_{1,\alpha}$ may be restricted to $U_\beta(g')^{-1} \times U \times V_\beta$ and is then a flat chart with respect to $(\mathcal{D}_{\mathbb{1} \otimes \lambda}, U_\beta(g')^{-1}, U \times V_\beta, e, \psi)$. After possibly shrinking U, we may assume $\tilde{\mu}(U_\beta(g')^{-1} \times U) = U_\beta(g')^{-1} \cdot U \subseteq U_\beta^2$. Then Ψ_2 is defined on $U_\beta(g')^{-1} \times U \times V_\beta$ and is a flat chart with respect to $(\mathcal{D}_{\mathbb{1} \otimes \lambda}, U_\beta(g')^{-1}, U \times V_\beta, e, \psi)$. Again, the uniqueness of flat charts implies $\Psi_1 = \Psi_2$ near $(g, g', p) \in U_\beta(g')^{-1} \times U \times V_\beta$. ∎

Proposition 4.15. *The infinitesimal action*

$$\lambda_\varphi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M}), \; X \mapsto (X(e) \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \varphi^*$$

induced by the local $\mathcal{G}$-action φ is λ.

Proof. Since the map ψ is locally a flat chart, ψ is a local diffeormorphism, $\pi_{\mathcal{G}} \circ \psi = \pi_{\mathcal{G}}$ and $\psi \circ \iota_e = \iota_e$. Moreover, we have locally $\psi_*(\mathcal{D}_{\mathcal{G}}) = \mathcal{D}$. Therefore the vector field $\psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}})$ on $\mathcal{G} \times \mathcal{M}$ belongs to the distribution $\mathcal{D}$ for each vector field $X \in \mathfrak{g}$. We have

$$\psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \pi^*_{\mathcal{G}} = (\psi^{-1})^* \circ (X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \psi^* \circ \pi^*_{\mathcal{G}} = (\psi^{-1})^* \circ (X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \pi^*_{\mathcal{G}}$$
$$= (\psi^{-1})^* \circ \pi^*_{\mathcal{G}} \circ X = \pi^*_{\mathcal{G}} \circ X = (X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \pi^*_{\mathcal{G}}.$$

Let $X_1, \ldots, X_{k+l}$ be a basis of $\mathfrak{g}$ and $a_1, \ldots, a_{k+l}$ local functions on $\mathcal{G} \times \mathcal{M}$ such that

$$\psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}}) = \sum_{i=1}^{k+l} a_i(X_i \otimes \mathrm{id}^*_{\mathcal{M}} + \mathrm{id}^*_{\mathcal{G}} \otimes \lambda(X_i)),$$

which is possible since $\psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}})$ belongs to $\mathcal{D}$. Then, combining the above, we have

$$(X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \pi^*_{\mathcal{G}} = \left(\sum_{i=1}^{k+l} a_i(X_i \otimes \mathrm{id}^*_{\mathcal{M}} + \mathrm{id}^*_{\mathcal{G}} \otimes \lambda(X_i)) \right) \circ \pi^*_{\mathcal{G}}$$
$$= \left(\left(\sum_{i=1}^{k+l} a_i X_i \right) \otimes \mathrm{id}^*_{\mathcal{M}} \right) \circ \pi^*_{\mathcal{G}},$$

which implies $X = \sum_{i=1}^{k+l} a_i X_i$. Since $X \in \mathfrak{g}$ and $X_1, \ldots, X_{k+l}$ is a basis of $\mathfrak{g}$, the a_i's are all constants and hence

$$\psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}}) = X \otimes \mathrm{id}^*_{\mathcal{M}} + \mathrm{id}^*_{\mathcal{G}} \otimes \lambda(X).$$

Therefore,

$$\lambda_\varphi(X) = (X(e) \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \varphi^* = \iota^*_e \circ (X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \psi^* \circ \pi^*_{\mathcal{M}}$$
$$= \iota^*_e \circ \psi^* \circ \psi_*(X \otimes \mathrm{id}^*_{\mathcal{M}}) \circ \pi^*_{\mathcal{M}} = (\psi \circ \iota_e)^* \circ (X \otimes \mathrm{id}^*_{\mathcal{M}} + \mathrm{id}^*_{\mathcal{G}} \otimes \lambda(X)) \circ \pi^*_{\mathcal{M}}$$
$$= \iota^*_e \circ (0 + \pi^*_{\mathcal{M}} \circ \lambda(X)) = \lambda(X). \qquad \blacksquare$$

5. Globalizations of infinitesimal actions

After we proved the existence of a local action of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$ with a given induced infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$, it is natural to ask in which cases this extends a global $\mathcal{G}$-action on $\mathcal{M}$.

A simple way to obtain examples of a local action which is not global is to start with an action on a supermanifold $\mathcal{M}'$. This action then induces a local action, which is not global, on every non-invariant open subsupermanifold $\mathcal{M} \subset \mathcal{M}'$

The aim of this section is to characterize all infinitesimal action which "arise" in the just described way from a global action. These infinitesimal actions are called globalizable.

In the classical case (see [Pal57], Chapter III), Palais found necessary and sufficient conditions for an infinitesimal action to be globalizable, allowing the larger manifold M', a globalization of the infinitesimal action, to be a possibly non-Hausdorff manifold.

In this section, similar conditions for the existence of globalizations of infinitesimal actions of Lie supergroups on supermanifolds are proven, and differences to the classical case are pointed out. It is also shown by an example that an infinitesimal action on a supermanifold may not be globalizable even if its underlying infinitesimal action is.

In analogy to the classical case (cf. [Pal57], Chapter III, Definition II), we define the notion of a globalization of an infinitesimal action.

Definition 5.1. A globalization of an infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$ is a pair $(\mathcal{M}', \varphi')$ with the following properties:

(i) $\mathcal{M}'$ is a supermanifold, whose underlying manifold M' is allowed to be a non-Hausdorff manifold, and $\mathcal{M}$ is an open subsupermanifold of $\mathcal{M}'$, and

(ii) $\varphi' : \mathcal{G} \times \mathcal{M}' \to \mathcal{M}'$ is an action of the Lie supergroup $\mathcal{G}$ on $\mathcal{M}'$ such that its infinitesimal action restricted to $\mathcal{M}$ coincides with λ, and

(iii) $\tilde{\varphi}'(G \times M) = M'$.

If there is no chance of confusion the supermanifold $\mathcal{M}'$ is also called a globalization. The infinitesimal action λ is called globalizable if there exists a globalization $(\mathcal{M}', \varphi')$ of λ.

Definition 5.2. The domain of definition $\mathcal{W} = (W, \mathcal{O}_{\mathcal{W}})$ of a local $\mathcal{G}$-action $\varphi : \mathcal{W} \to \mathcal{M}$ is called maximally balanced if $W_p = W_{\tilde{\varphi}(h,p)} h = \{gh \mid g \in W_{\tilde{\varphi}(h,p)}\}$ for all $(h,p) \in W$, where again $W_q = \{g \in G \mid (g,q) \in W\}$ for any $q \in M$.

Remark 5.3. In [Pal57] a maximally balanced domain of definition of local action is called maximum (see Definition VII in Chapter III, [Pal57]). Here, the term maximally balanced is used instead in order to avoid any confusion with maximal domains of definition.

In the classical case, the following theorem states necessary and sufficient conditions for the existence of a globalization of an infinitesimal action.

Theorem 5.4 (see [Pal57], Chapter III, Theorem X). *Let* $\lambda : \mathfrak{g} \to \mathrm{Vec}(M)$ *be an infinitesimal action of a Lie group* G *on a manifold* M. *Then the following statements are equivalent:*

(i) *The infinitesimal action* λ *is globalizable.*

(ii) *There exists a local action $\varphi : W \to M$ of G on M with infinitesimal action λ whose domain of definition W is maximally balanced; this local action with a maximally balanced domain of definition is then unique and any other local action with the same infinitesimal action λ is a restriction of φ.*

(iii) *Let $\mathcal{D}$ be the distribution on $G \times M$ associated to the infinitesimal action λ and $\Sigma \subset G \times M$ any leaf of $\mathcal{D}$, i.e. a maximal connected integral manifold. Then the map $\pi_G|_\Sigma : \Sigma \to M$ is injective, where $\pi_G : G \times M \to G$ denotes the projection onto G.*

Remark 5.5. If $\varphi' : \mathcal{G} \times \mathcal{M}' \to \mathcal{M}'$ is a globalization of the infinitesimal action λ, then the underlying action $\tilde{\varphi}' : G \times M' \to M'$ is a globalization of the reduced infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$. Therefore, a necessary condition for an infinitesimal action to be globalizable is the existence of a globalization M' of the reduced infinitesimal action.

Univalent leaves and holonomy Throughout the rest of this section, let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be a fixed infinitesimal action of a Lie supergroup $\mathcal{G}$ and let $\mathcal{D}$ be the associated distribution on $\mathcal{G} \times \mathcal{M}$. In the following, conditions for the existence of a globalization of λ are studied.

Trying to generalize the classical result (Theorem 5.4) to supermanifolds, the question of an appropriate formulation of condition (*iii*) arises since integral manifolds do not uniquely determine a distribution on a supermanifold (cf. Example 3.17). For that purpose the notion of a univalent leaf $\Sigma \subset G \times M$, extending the classical notion, is introduced.

Then occurring holonomy phenomena of the distribution $\mathcal{D}$ are studied and a connection between absence of such phenomena and the notion of univalent leaves is established.

Remark 5.6. The infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ induces an infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ of the classical Lie group G on the manifold M, where the Lie algebra of G is identified with the even part $\mathfrak{g}_0$ of $\mathfrak{g}$. For any vector field $X \in \mathfrak{g}_0 (\subseteq \mathfrak{g})$ we define $\tilde{\lambda}(X)$ to be the reduced vector field $\tilde{Y}$ of $Y = \lambda(X)$.

Lemma 5.7. *Let $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ denote the induced infinitesimal action of G on M and $\mathcal{D}_{\tilde{\lambda}}$ the associated distribution on $G \times M$. Then we have $\widetilde{\mathcal{D}} = \mathcal{D}_{\tilde{\lambda}}$, where $\widetilde{\mathcal{D}}$ denotes the distribution on $G \times M$ induced by $\mathcal{D}$ (cf. Remark 3.12).*

Proof. For any odd vector field Y on a supermanifold we have $\tilde{Y} = 0$. Since $X + \lambda(X)$ has the same parity as X if X is homogeneous, the reduced distribution $\widetilde{\mathcal{D}}$ is spanned by vector fields of the form $\tilde{X} + \tilde{\lambda}(X)$ for $X \in \mathfrak{g}_0$. These vector fields also generate $\mathcal{D}_{\tilde{\lambda}}$ and thus $\widetilde{\mathcal{D}} = \mathcal{D}_{\tilde{\lambda}}$. ∎

As a corollary of the identity $\widetilde{\mathcal{D}} = \mathcal{D}_{\tilde{\lambda}}$, we get the following relation between integral manifolds of the involutive distributions $\mathcal{D}$ on $\mathcal{G} \times \mathcal{M}$ and $\mathcal{D}_{\tilde{\lambda}}$ on $G \times M$.

Corollary 5.8. *Every integral manifold $N \subset G \times M$ of the distribution $\mathcal{D}_{\tilde{\lambda}}$ is*

the underlying manifold of some integral manifold $\mathcal{N} \subset \mathcal{G} \times \mathcal{M}$ of the distribution $\mathcal{D}$ and conversely the underlying manifold of every integral manifold of $\mathcal{D}$ is an integral manifold of $\mathcal{D}_{\tilde{\lambda}}$.

In the following, by a leaf $\Sigma \subset G \times M$ a leaf, i.e. a maximal connected integral manifold, of the distribution $\mathcal{D}_{\tilde{\lambda}} = \widetilde{\mathcal{D}}$ is meant. By the preceding corollary every leaf Σ is as well the underlying manifold of an integral manifold of the distribution $\mathcal{D}$.

The involutiveness of the distribution $\mathcal{D}_{\tilde{\lambda}}$ guarantees the existence of a leaf through each point $(g, p) \in G \times M$. This leaf is denoted by $\Sigma_{(g,p)}$.

Definition 5.9.   A leaf $\Sigma \subset G \times M$ is called univalent (with respect to $\mathcal{D}$) if for every path $\gamma : [0, 1] \to \Sigma \subset G \times M$ there exists a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, \gamma_G(0), \mathrm{id})$, for $\gamma_G := \pi_G \circ \gamma : [0, 1] \to G$, such that $\tilde{\psi}(U \times V)$ contains $\gamma([0, 1])$.

The infinitesimal action λ is called univalent if all leaves $\Sigma \subset G \times M$ are univalent.

Remark 5.10.   By Remark 4.6, a leaf Σ is univalent if for every path $\gamma : [0, 1] \to \Sigma$ there exists a flat chart ψ with respect to $(\mathcal{D}, U, V, \gamma_G(0), \rho)$ for some ρ or equivalently, after shrinking, for any ρ, with $\gamma([0, 1]) \subset \tilde{\psi}(U \times V)$.

Remark 5.11.   In the definition of a univalent leaf Σ it is enough for the defining property to hold true for closed paths $\gamma : [0, 1] \to \Sigma$ because for any path γ' the composition $(\gamma')^{-1} \cdot \gamma'$ of γ' and $(\gamma')^{-1}$, $(\gamma')^{-1}(t) = \gamma'(1-t)$, is a closed path with $\gamma'([0, 1]) = ((\gamma')^{-1} \cdot \gamma')([0, 1])$.

Remark 5.12.   If λ is univalent, then so is the induced infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$.

In [Pal57], an infinitesimal action (in the classical case) is called univalent if the restriction of the projection $\pi_G : G \times M \to G$ to an arbitrary leaf $\Sigma \subset G \times M$ is injective. The above defined notion of univalent infinitesimal actions on supermanifolds extends this definition:

Proposition 5.13.   *In the case of classical manifolds $\mathcal{G} = G$ and $\mathcal{M} = M$, an infinitesimal action is univalent if and only if the projection $\pi_G|_{\Sigma} : \Sigma \to G$ is injective for each leaf $\Sigma \subset G \times M$.*

Proof.   Let $\Sigma \subset G \times M$ be any leaf, $x = (g, p)$, $y = (g, q) \in \Sigma$, and $\gamma : [0, 1] \to \Sigma$ a path from x to y. Since λ is univalent, there is a flat chart $\psi : U \times V \to G \times M$ with respect to $(\mathcal{D}, U, V, g, \mathrm{id})$ with $\gamma([0, 1]) \subset \psi(U \times \{p\})$. We have $(g, q) = \gamma(1) = \psi(g, p) = (g, p)$ because $\pi_G \circ \psi = \pi_G$.

Assume now that $\pi_G|_{\Sigma}$ is injective for each leaf $\Sigma \subset G \times M$. Let $\Sigma \subset G \times M$ be a leaf and $\gamma : [0, 1] \to \Sigma$ a path. Using the compactness of $\gamma([0, 1])$ there are $0 = t_0 < \ldots < t_k = 1$ and flat charts $\psi_i : U_i \times V_i \to G \times M$, $i = 0, \ldots, k-1$, with

respect to $(\mathcal{D}, U_i, V_i, \gamma_G(t_i), \mathrm{id})$ such that the intersection $U_i \cap U_j$ is connected for all i, j and $\gamma([t_i, t_{i+1}]) \subset \psi(U_i \times V_i)$.

Set $\psi_0' = \psi_0$ and, after possibly shrinking V_0, inductively define flat charts $\psi_i' : U_i \times V_0 \to G \times M$ for $i \geq 1$ by composing ψ_i and a local diffeomorphism of the form $(\mathrm{id} \times \rho_i)$ such that ψ_i' and ψ_{i+1}' coincide on $(U_i \cap U_{i+1}) \times V_0$ for $i = 0, \ldots, k-2$ and $\gamma([t_i, t_{i+1}]) \subset \psi_i'(U_i \times V_0)$ holds. We have $\psi_0'(U_0 \times \{p\}) \subset \Sigma_{(g,p)}$ for $g := \gamma_G(0)$ and any $p \in V_0$, and then by induction $\psi_i'(U_i \times \{p\}) \subset \Sigma_{(g,p)}$ for any i. Therefore, $\psi_i' = \psi_j'$ on $(U_i \cap U_j) \times V_0$ since $\pi_G|_{\Sigma_{(g,p)}}$ is injective and $\pi_G \circ \psi = \pi_G$ for any flat chart ψ. Consequently, we can define a flat chart $\psi : U \times V \to G \times M$, $U := \bigcup_{i=0}^{k-1} U_i$, $V := V_0$, with respect to $(\mathcal{D}, U, V, g = \gamma_G(0), \mathrm{id})$ by setting $\psi|_{U_i \times V_0} = \psi_i'$. The map ψ satisfies $\gamma([0,1]) \subset \psi(U \times V)$ by construction. $\blacksquare$

Proposition 5.14. *The infinitesimal action λ is univalent if and only if for any two flat charts $\psi_i : \mathcal{U}_i \times \mathcal{V}_i \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i)$, $i = 1, 2$, and $\rho_1 = \rho_2$ on $V_1 \cap V_2$ we have $\psi_1 = \psi_2$ on their common domain of definition.*

Remark that by Proposition 4.10 any two flat charts coincide on their common domain of definition $(U_1 \cap U_2) \times (V_1 \cap V_2)$ if $U_1 \cap U_2$ is connected.

Proof. Let λ be univalent and ψ_i flat charts with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i)$ and $\rho_1 = \rho_2$ on $V_1 \cap V_2$. Let $(h, p) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. Since ψ_1 and ψ_2 are flat charts, $\tilde{\psi}_1(U_1 \times \{p\})$ and $\tilde{\psi}_2(U_2 \times \{p\})$ are both contained in the leaf $\Sigma = \Sigma_{(g, \tilde{\rho}_1(p))} = \Sigma_{(g, \tilde{\rho}_2(p))}$. The univalence of the reduced infinitesimal action $\tilde{\lambda}$ implies that $\pi_G|_\Sigma : \Sigma \to G$ is injective and thus $\tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, p)$. Let $\gamma : [0, 1] \to U_1 \cup U_2$ be a closed path with $\gamma(0) = \gamma(1) = g$, $\gamma([0, \frac{1}{2}]) \subset U_1$, $\gamma([\frac{1}{2}, 1]) \subset U_2$ and $\gamma(\frac{1}{2}) = h$. Then

$$\gamma' : [0, 1] \to \Sigma, \ \gamma'(t) = \begin{cases} \tilde{\psi}_1(\gamma(t), p), & t \leq \frac{1}{2} \\ \tilde{\psi}_2(\gamma(t), p), & t > \frac{1}{2} \end{cases}$$

is a closed path. As λ is univalent, there is a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, g, \mathrm{id})$ with $\gamma'([0, 1]) = \tilde{\psi}_1(\gamma([0, \frac{1}{2}]) \times \{p\}) \cup \tilde{\psi}_2(\gamma([\frac{1}{2}, 1]) \times \{p\}) \subset \tilde{\psi}(U \times V)$. Then, after possibly shrinking V_1, $\psi \circ (\mathrm{id} \times \rho_1)$ is a flat chart with respect to $(\mathcal{D}, U, V_1, g, \rho_1)$. By Proposition 4.10 the flat charts $\psi \circ (\mathrm{id} \times \rho_1)$ and ψ_1 coincide on a neighbourhood of $\gamma([0, \frac{1}{2}]) \times \{p\}$. Moreover, $\psi \circ (\mathrm{id} \times \rho_1)$ and ψ_2 coincide on a neighbourhood of $\gamma([\frac{1}{2}, 1]) \times \{p\}$ since $\rho_1 = \rho_2$ on $V_1 \cap V_2$. In particular, we get $\psi_1 = \psi \circ (\mathrm{id} \times \rho_1) = \psi_2$ near $(h, p) = (\gamma(\frac{1}{2}), p)$.

Suppose now that any two flat charts ψ_i with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i)$, $i = 1, 2$, with $\rho_1 = \rho_2$ already coincide on their common domain of definition. Assume that there is a leaf $\Sigma \subset G \times M$ which is not univalent and let $\gamma : [0, 1] \to \Sigma$ be a path for which there is no flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\gamma([0, 1]) \subset \tilde{\psi}(U \times V)$. Define I to be the set of points $t \in [0, 1]$ such that there exists a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\gamma([0, t]) \subset \tilde{\psi}(U \times V)$. The set I is open and $I \neq [0, 1]$ by assumption. Let s be the minimum of $[0, 1] \setminus I$. There is a flat chart $\psi_1 : \mathcal{U}_1 \times \mathcal{V}_1 \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_1, V_1, \pi_G(\gamma(s)), \mathrm{id})$ with $\gamma(s) \in \tilde{\psi}_1(U_1 \times V_1)$. By the choice of s there is $t \in [0, s)$ with $\gamma([t, s]) \subset \tilde{\psi}_1(U_1 \times V_1)$ such that there is a flat chart $\psi_2 : \mathcal{U}_2 \times \mathcal{V}_2 \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U_2, V_2, \pi_G(\gamma(0)), \mathrm{id})$

with $\gamma([0,t]) \in \tilde{\psi}_2(U_2 \times V_2)$. Then $h = \pi_G(\gamma(t)) \in U_1 \cap U_2$ and after possibly shrinking V_2 there exists a diffeomorphism ρ such that $\psi_1 \circ (\mathrm{id} \times \rho)$ is a flat chart with respect to $(\mathcal{D}, U_1, V_2, h, \rho')$ and ψ_2 with respect to $(\mathcal{D}, U_2, V_2, h, \rho')$ for some $\rho' : \mathcal{V}_2 \to \mathcal{M}$. Therefore, $\psi_1 \circ (\mathrm{id} \times \rho)$ and ψ_2 agree on their common domain of definition and they define a flat chart $\psi : (\mathcal{U}_1 \cup \mathcal{U}_2) \times V_2 \to \mathcal{G} \times \mathcal{M}$ with $\gamma([0,s]) \subset \tilde{\psi}((U_1 \cup U_2) \times V_2)$, contradicting the definition of s. ∎

The preceding proposition allows us to glue together flat charts in the case of a univalent infinitesimal action. This also implies the next corollary.

Corollary 5.15. *The infinitesimal action λ is univalent if and only if for any compact subset Σ' of a leaf $\Sigma \subset G \times M$ there exists a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\Sigma' \subset \tilde{\psi}(U \times V)$.*

In the following the structure of the distribution $\mathcal{D}$ associated to λ is investigated further. Let $\Sigma \subset G \times M$ be a leaf, $(g,p) \in \Sigma$ and $\gamma : [0,1] \to \Sigma$ a closed path with $\gamma(0) = \gamma(1) = (g,p)$ and let $\gamma_G = \pi_G \circ \gamma$. We now want to associate a germ of a local diffeormorphism of $\mathcal{M}$ around p measuring the "holonomy" along the path γ.

To do so, let $0 = t_0 < \ldots < t_k = 1$ be a partition of $[0,1]$ such that there are flat charts $\psi_i : \mathcal{U}_i \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$, $i = 0, \ldots, k-1$, with $\gamma([t_i, t_{i+1}]) \subset \tilde{\psi}_i(U_i \times \{p\})$, such that ψ_i and ψ_{i+1} coincide on their common domain of definition and such that ψ_0 is a flat chart with respect to $(\mathcal{D}, U_0, V, g, \mathrm{id})$. We have $\tilde{\psi}_i(U_i \times \{p\}) \subset \Sigma$ for any i. By Lemma 4.9 there is a diffeomorphism $\rho : \mathcal{V} \to \mathcal{M}$ onto its image such that ψ_{k-1} is a flat chart with respect to $(\mathcal{D}, U_{k-1}, V, g = \gamma_G(1), \rho)$. Since $(g,p) = \gamma(1) \in \tilde{\psi}_{k-1}(U_{k-1} \times \{p\})$, we have $(g, \tilde{\rho}(p)) = \tilde{\psi}_{k-1}(g,p) = (g,p)$ and thus $\tilde{\rho}(p) = p$.

Define $\Phi(\gamma)$ to be the germ of the local diffeomorphism ρ in p. The local uniqueness of flat chart implies that $\Phi(\gamma)$ does not depend on the actual choice of the flat charts $\psi_i : \mathcal{U}_i \times \mathcal{V}_i \to \mathcal{G} \times \mathcal{M}$. Let $\mathrm{Diff}_p(\mathcal{M})$ denote the set of germs of local diffeomorphisms $\chi : \mathcal{V}_1 \to \mathcal{V}_2$ in $p \in M$, where $\mathcal{V}_i = (V_i, \mathcal{O}_\mathcal{M})$, $i = 1, 2$, are open subsupermanifolds of $\mathcal{M}$ with $p \in V_i$. Then $\Phi(\gamma)$ is an element of $\mathrm{Diff}_p(\mathcal{M})$ for each closed path $\gamma : [0,1] \to \Sigma$ with $\gamma(0) = \gamma(1) = (g,p)$.

In the case complex supermanifolds and holomorphic maps $\mathrm{Diff}_p(\mathcal{M})$ should be replaced by $\mathrm{Hol}_p(\mathcal{M})$, the set of germs of local biholomorphisms $\chi : \mathcal{V}_1 \to \mathcal{V}_2$.

Proposition 5.16. *The germ $\Phi(\gamma)$ only depends on the homotopy class $[\gamma]$ of the closed path γ with $\gamma(0) = \gamma(1) = (g,p)$. Therefore, the assignment $[\gamma] \mapsto \Phi([\gamma]) = \Phi(\gamma)$ defines a maps*

$$\Phi = \Phi_\Sigma = \Phi_{\Sigma,(g,p)} : \pi_1(\Sigma, (g,p)) \to \mathrm{Diff}_p(\mathcal{M}).$$

Proof. Let $\gamma_s : [0,1] \to \Sigma$, $s \in [0,1]$, be a continuous family of closed paths with $\gamma_s(0) = \gamma_s(1) = (g,p)$ and $\gamma_0 = \gamma$. Let $s_0 \in [0,1]$, $0 = t_0 < \ldots < t_k = 1$ and $\psi_i : \mathcal{U}_i \times \mathcal{V}_i \to \mathcal{G} \times \mathcal{M}$, $i = 0, \ldots, k-1$, flat charts with $\gamma_{s_0}([t_i, t_{i+1}]) \subset \tilde{\psi}_i(U_i \times V_i)$ and such that ψ_0 is a flat chart with respect to $(\mathcal{D}, U_0, V_0, g, \mathrm{id})$ and ψ_i and ψ_{i+1} coincide on their common domain of definition. Then ψ_{k-1} is a flat chart with

respect to $(\mathcal{D}, U_{k-1}, V_{k-1}, g, \rho)$ for some local diffeomorphism ρ around p and $\Phi(\gamma_{s_0})$ is the germ of ρ in p. Since all intervals $[t_i, t_{i+1}]$ are compact there exists an open neighbourhood $J \subseteq [0,1]$ of s_0 such that $\gamma_s([t_i, t_{i+1}]) \subset \tilde{\psi}_i(U_i \times V_i)$ for $i = 0, \dots, k-1$ and all $s \in J$. Therefore, $\Phi(\gamma_{s_0}) = \Phi(\gamma_s)$ for all $s \in J$ and the set $\{s \in [0,1] \,|\, \Phi(\gamma_s) = \Phi(\gamma)\}$ is open and closed. Since $\gamma = \gamma_0$ we get $\Phi(\gamma_s) = \Phi(\gamma)$ for all $s \in [0,1]$. ∎

Remark 5.17. The definition of Φ implies that it is a group homomorphism: If γ is a constant path, then $\Phi([\gamma])$ is the germ of the identity $\mathrm{id} : \mathcal{M} \to \mathcal{M}$ and we have $\Phi([\gamma_1] \cdot [\gamma_2]) = \Phi([\gamma_1]) \circ \Phi([\gamma_2])$ for the composition $\gamma_1 \cdot \gamma_2$ of two closed paths γ_1 and γ_2.

Remark 5.18. By Lemma 4.7 and Remark 4.11 the morphism Φ does not depend on whether its construction is done with respect to the distribution $\mathcal{D}$ on $\mathcal{G} \times \mathcal{M}$ associated to λ or to the distribution $\mathcal{D}_0$ on $G \times \mathcal{M}$ associated to the infinitesimal action $\lambda_0 = \lambda|_{\mathfrak{g}_0}$.

Remark 5.19. Due to the connectedness of the leaves Σ the map $\Phi_{\Sigma,(g,p)}$ is trivial for some $(g,p) \in \Sigma$ if and only if $\Phi_{\Sigma,(h,q)}$ is trivial for all $(h,p) \in \Sigma$.

The triviality of the map $\Phi = \Phi_\Sigma = \Phi_{\Sigma,(g,p)}$ can be viewed as a sort of absence of holonomy for the leaf Σ.

Example 5.20. Let $\mathcal{G} = S^1$, with coordinate ϕ, and $\mathcal{M} = \mathbb{R}^{0|2}$, with coordinates θ_1, θ_2. Let $X = \theta_1 \frac{\partial}{\partial \theta_2}$ and consider the infinitesimal S^1-action $\lambda : \mathrm{Lie}(S^1) \cong \mathbb{R} \to \mathrm{Vec}(\mathcal{M})$, $\lambda(t) = tX$. The unique leaf of the distribution $\mathcal{D}$ on $S^1 \times \mathcal{M}$, spanned by $\frac{\partial}{\partial \phi} + X$, is $\Sigma = S^1 \times \{0\} = S^1 \times M$. Let $\phi_0 \in S^1$ and $r : \Omega \to \mathbb{R}$ be a local inverse around $1 \in \Omega \subset S^1$ of $\mathbb{R} \to S^1$, $t \mapsto e^{it}$. Then

$$\psi^*(\phi, \theta_1, \theta_2) = (\phi, \rho^*(\theta_1), \rho^*(\theta_2)) + (0, 0, r(\phi\phi_0^{-1})\rho^*(\theta_1))$$

defines the pullback of a flat chart ψ with respect to $(\mathcal{D}, U, V, \phi_0, \rho)$ for $V = \{0\} = M$, $U \subset S^1$ with $\phi_0 \in U$ and $U\phi_0^{-1} \subseteq \Omega$ and a diffeormorphism $\rho : \mathbb{R}^{0|2} \to \mathbb{R}^{0|2}$. For any $\phi_0' \in U$ the map ψ is also a flat chart with respect to $(\mathcal{D}, U, V, \phi_0', \rho')$ for $\rho' : \mathbb{R}^{0|2} \to \mathbb{R}^{0|2}$ with pullback

$$(\rho')^*(\theta_1, \theta_2) = \rho^*(\theta_1, \theta_2) + (0, r(\phi_0'\phi_0^{-1})\rho^*(\theta_1))$$

$$= \rho^*(\theta_1, \theta_2) + \left(0, \left(\int_{\phi_0}^{\phi_0'} 1 d\phi\right)\rho^*(\theta_1)\right),$$

where the integral might be taken along any path in Ω. In particular $(\rho')^*(\theta_1) = \rho^*(\theta_1)$.

A calculation shows that the map $\Phi : \pi_1(\Sigma, (\phi_0, 0)) \to \mathrm{Diff}_0(\mathbb{R}^{0|2}) = \mathrm{Diff}(\mathbb{R}^{0|2})$ is given by

$$\Phi([\gamma])^*(\theta_1, \theta_2) = \mathrm{id}^*(\theta_1, \theta_2) + \left(0, \left(\int_\gamma 1 d\phi\right)\theta_1\right) = \left(\theta_1, \theta_2 + \left(\int_\gamma 1 d\phi\right)\theta_1\right)$$

for any $\gamma : [0,1] \to S^1 \cong \Sigma$, $\gamma(0) = \gamma(1) = \phi_0$. Thus, identifying $\pi_1(\Sigma, (\phi_0, 0))$ and $\mathbb{Z}$, we get

$$\Phi : \mathbb{Z} \to \mathrm{Diff}(\mathbb{R}^{0|2}), \ \Phi(k)^*(\theta_1, \theta_2) = (\theta_1, \theta_2 + 2\pi k\theta_1).$$

We now establish an equivalence between univalent infinitesimal actions and infinitesimal actions with univalent reduced infinitesimal action and leaves without holonomy.

Proposition 5.21.   *The infinitesimal action λ is univalent if and only if the reduced infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ is univalent and for all leaves Σ the map Φ_Σ is trivial.*

Proof.   If the infinitesimal action λ is univalent, then the reduced infinitesimal action $\tilde{\lambda}$ is univalent as a direct consequence. Moreover, for any closed path $\gamma : [0,1] \to \Sigma$ there is a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\gamma([0,1]) \subset \tilde{\psi}(U \times V)$ and thus $\Phi([\gamma]) = \mathrm{id}$.

Now, suppose that $\tilde{\lambda}$ is univalent and Φ_Σ is trivial for all leaves $\Sigma \subset G \times M$. We need to show that any two flat charts $\psi_i : \mathcal{U}_i \times \mathcal{V}_i \to \mathcal{G} \times \mathcal{M}$, $i = 1, 2$, with respect to $(\mathcal{D}, U_i, V_i, g, \rho_i)$ with $\rho_1 = \rho_2$ coincide on their common domain of definition (cf. Proposition 5.14). By replacing ψ_i by $\psi_i \circ (\mathrm{id} \times \rho_i^{-1})$ it is enough to show $\psi_1 = \psi_2$ in the case of $\rho_1 = \rho_2 = \mathrm{id}$.

Let $(h, p) \in (U_1 \times V_1) \cap (U_1 \times V_1) = (U_1 \cap U_2) \times (V_1 \cap V_2)$ be arbitrary. We have $\tilde{\psi}_i(h, p) \in \tilde{\psi}_i(U_i \times \{p\}) \subset \Sigma_{(g,p)}$ and thus $\tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, p)$ since $\pi_G|_{\Sigma_{(g,p)}}$ is injective. After shrinking V_2 there is a diffeomorphism $\rho : \mathcal{V}_2 \to \mathcal{M}$ such that $\psi_2 \circ (\mathrm{id} \times \rho)$ is defined and coincides with ψ_1 near (h, p). Note that $\tilde{\rho}(p) = p$ since $\tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, p)$. The composition $\psi_2 \circ (\mathrm{id} \times \rho)$ is a flat chart with respect to $(\mathcal{D}, U_2, V_2, g, \rho)$. Let $\alpha : [0,1] \to \Sigma_{(g,p)}$ be a closed path with $\alpha(0) = \alpha(1) = (g, p)$, $\alpha(\frac{1}{2}) = \tilde{\psi}_1(h, p) = \tilde{\psi}_2(h, \tilde{\rho}(p)) = \tilde{\psi}_2(h, p)$ and $\alpha([0, \frac{1}{2}]) \subset \tilde{\psi}_1(U_1 \times \{p\})$ and $\alpha([\frac{1}{2}, 1]) \subset \tilde{\psi}_2(U_2 \times \{p\}) = \tilde{\psi}_2((\mathrm{id} \times \tilde{\rho})(U_2 \times \{p\}))$. By the definition of $\Phi = \Phi_{\Sigma_{(g,p)}}$ we have $\Phi([\alpha]) = \rho$. The triviality of Φ then gives $\rho = \mathrm{id}$ so that ψ_1 and ψ_2 agree near (h, p).   ∎

Corollary 5.22.   *The infinitesimal action λ is univalent if and only if its restriction $\lambda_0 = \lambda|_{\mathfrak{g}_0}$ is univalent.*

Proof.   The statement follows from the preceding proposition and the observation formulated in Remark 5.18.   ∎

The action on $\mathcal{G} \times \mathcal{M}$ from the right and invariant functions   In this paragraph, the action $R : \mathcal{G} \times (\mathcal{G} \times \mathcal{M}) \to \mathcal{G} \times \mathcal{M}$ of the Lie supergroup $\mathcal{G}$ on the product $\mathcal{G} \times \mathcal{M}$ from the right, which is in the classical case given by $(g, (h, x)) \mapsto (hg^{-1}, x)$, is introduced. Its behaviour with respect to the distribution $\mathcal{D}$ associated to $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ and in particular $\mathcal{D}$-invariant functions is studied.

Definition 5.23.   Let $\mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}$ be the sheaf of $\mathcal{D}$-invariant functions on $\mathcal{G} \times \mathcal{M}$,

i.e.

$$\mathcal{O}_{\mathcal{G}\times\mathcal{M}}^{\mathcal{D}}(\Omega) = \{f \in \mathcal{O}_{\mathcal{G}\times\mathcal{M}}(\Omega)| \, \mathcal{D} \cdot f = 0\}$$

for any open subset $\Omega \subset G \times M$.

Remark 5.24. For any $\mathcal{D}$-invariant function f on $\mathcal{G} \times \mathcal{M}$, the underlying function $\tilde{f}$ is constant along the leaves, and in the classical case every such function is $\mathcal{D}$-invariant.

Moreover, if $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ is a flat chart and f is $\mathcal{D}$-invariant, then $\psi^*(f)$ is $\mathcal{D}_{\mathcal{G}}$-invariant since $\psi_*(\mathcal{D}_{\mathcal{G}}) = \mathcal{D}$. The $\mathcal{D}_{\mathcal{G}}$-invariant functions on $\mathcal{U} \times \mathcal{V} \subseteq \mathcal{G} \times \mathcal{M}$ are of the form $f_{\mathcal{M}} = 1 \otimes f_{\mathcal{M}}$ for $f_{\mathcal{M}} \in \mathcal{O}_{\mathcal{M}}(V)$ so that $\psi^*(f) = 1 \otimes f_{\mathcal{M}}$ for an appropriate $f_{\mathcal{M}}$.

We have the following identity principle for $\mathcal{D}$-invariant functions on $\mathcal{G}\times\mathcal{M}$:

Lemma 5.25. *Let $W \subseteq G \times M$ be open, $\Sigma \subset G \times M$ be a leaf with $\Sigma \subset W$ and let $f_1, f_2 \in \mathcal{O}_{\mathcal{G}\times\mathcal{M}}^{\mathcal{D}}(W)$. If $f_1 = f_2$ on an open neighbourhood of some $x \in \Sigma$, then f_1 and f_2 coincide on an open neighbourhood of the leaf Σ.*

Proof. Define $\Sigma' \subseteq \Sigma$ to be the subset of points $y \in \Sigma$ such f_1 and f_2 are equal on some open neighbourhood of y. The set Σ' is open and contains x by assumption.

For any flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\tilde{\psi}(U \times V) \subset W$ we have $\psi^*(f_i) = 1 \otimes f_{\mathcal{M},i}$ for appropriate $f_{\mathcal{M},i}$ by Remark 5.24. Therefore, $f_1 = f_2$ near $\tilde{\psi}(U \times \{q\})$ if and only if $f_{\mathcal{M},1} = f_{\mathcal{M},2}$ near $q \in M$. It follows that the set Σ' is also closed and thus equal to Σ. ∎

Definition 5.26. Let $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the multiplication of $\mathcal{G}$, $\iota : \mathcal{G} \to \mathcal{G}$ the inversion and $\tau : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$ the morphism which interchanges the two components. Then define the action of $\mathcal{G}$ on itself from the right as $r : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $r = \mu \circ \tau \circ (\iota \times \text{id})$, whose underlying action is given by $(g,h) \mapsto hg^{-1}$. Define now a $\mathcal{G}$-action on $\mathcal{G} \times \mathcal{M}$ by

$$R : \mathcal{G} \times (\mathcal{G} \times \mathcal{M}) \to \mathcal{G} \times \mathcal{M}, \, R = r \times \text{id}.$$

Lemma 5.27. *For every right-invariant vector field X on $\mathcal{G}$ we have*

$$(\text{id}^* \otimes (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X))) \circ R^* = R^* \circ (X \otimes \text{id}^* + \text{id}^* \otimes \lambda(X)).$$

Proof. The right-invariance of X is equivalent to $\mu^* \circ X = (X \otimes \text{id}^*) \circ \mu^*$ and a short calculation yields $(\text{id}^* \otimes X) \circ r^* = r^* \circ X$, which directly implies the desired equality. ∎

Corollary 5.28. *We have*

$$R^*(\mathcal{O}_{\mathcal{G}\times\mathcal{M}}^{\mathcal{D}}) \subset \tilde{R}_*(\mathcal{O}_{\mathcal{G}\times\mathcal{G}\times\mathcal{M}}^{1\otimes\mathcal{D}}),$$

where $1 \otimes \mathcal{D}$ is the distribution on $\mathcal{G} \times (\mathcal{G} \times \mathcal{M})$ spanned by vector fields of the form $\text{id}^* \otimes Y$ for vector fields Y belonging to $\mathcal{D}$ and $\mathcal{O}_{\mathcal{G}\times\mathcal{G}\times\mathcal{M}}^{1\otimes\mathcal{D}}$ denotes the sheaf of $1 \otimes \mathcal{D}$-invariant functions.

Proof.　Using the preceding lemma, we have

$$(\mathrm{id}^* \otimes (X \otimes \mathrm{id}^* + \mathrm{id}^* \otimes \lambda(X)))(R^*(f)) = R^* \circ (X \otimes \mathrm{id}^* + \mathrm{id}^* \otimes \lambda(X))(f) = R^*(0) = 0$$

for any $\mathcal{D}$-invariant function f on $\mathcal{G} \times \mathcal{M}$ and $X \in \mathfrak{g}$. Thus, $R^*(f)$ is $\mathbb{1} \otimes \mathcal{D}$-invariant. ∎

Definition 5.29.　For $g \in G$, let $r_g : \mathcal{G} \to \mathcal{G}$ denote the composition of the action r and the inclusion $\mathcal{G} \hookrightarrow \{g\} \times \mathcal{G} \subset \mathcal{G} \times \mathcal{G}$, and define

$$R_g : \mathcal{G} \times \mathcal{M} \to \mathcal{G} \times \mathcal{M}, \; R_g = (r_g \times \mathrm{id}).$$

Since r is an action, r_g and R_g are diffeomorphisms, $(r_g)^{-1} = r_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$.

Lemma 5.30.　*For each $g \in G$ the map $R_g : \mathcal{G} \times \mathcal{M} \to \mathcal{G} \times \mathcal{M}$ satisfies*

(i) $(R_g)_*(\mathcal{D}_\mathcal{G}) = \mathcal{D}_\mathcal{G}$, and

(ii) $(R_g)_*(\mathcal{D}) = \mathcal{D}$.

Proof.　Property (i) can be directly obtained from the definition of R_g. Property (ii) follows from the fact that $(r_g)_*(X) = X$ for every right-invariant vector field X, and therefore $(R_g)_*(X + \lambda(X)) = (r_g)_*(X) + \lambda(X) = X + \lambda(X)$ so that $(R_g)_*(\mathcal{D}) = \mathcal{D}$. ∎

The composition of flat charts and maps of the form R_g exhibits a special behaviour as specified in the following lemma.

Lemma 5.31.　*Let $g \in G$ and let $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ be a flat chart with respect to $(\mathcal{D}, U, V, h, \rho)$. Then the composition*

$$\psi' = R_{g^{-1}} \circ \psi \circ (R_g|_{Ug \times V}) : \mathcal{U}g \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$$

is a flat chart with respect to $(\mathcal{D}, Ug, V, hg, \rho)$ where $Ug = \{ug | u \in U\}$ and $\mathcal{U}g = (Ug, \mathcal{O}_\mathcal{G}|_{Ug})$.

Proof.　We have $(\psi')_*(\mathcal{D}_\mathcal{G}) = \mathcal{D}$ using $(R_g)_*(\mathcal{D}_\mathcal{G}) = \mathcal{D}_\mathcal{G}$ and $(R_g)_*(\mathcal{D}) = \mathcal{D}$. Moreover, direct calculations show $\pi_\mathcal{G} \circ \psi' = \pi_\mathcal{G}$ and $\psi' \circ \iota_{hg} = \iota_{hg} \circ \rho$. ∎

Corollary 5.32.　*The underlying classical Lie group G acts on the space of leaves $\Sigma \subset G \times M$ by $(g, \Sigma) \mapsto \tilde{R}_g(\Sigma)$. For $(h, p) \in G \times M$ we have $\tilde{R}_g(\Sigma_{(h,p)}) = \Sigma_{(hg^{-1}, p)}$.*

Proof.　Since R_g preserves the distribution $\mathcal{D}$, $\tilde{R}_g$ maps leaves diffeomorphically onto leaves. We have $\tilde{R}_g(\Sigma_{(h,p)}) = \Sigma_{(hg^{-1}, p)}$ because $(hg^{-1}, p) = \tilde{R}_g(h, p) \in \tilde{R}_g(\Sigma_{(h,p)})$. ∎

Remark 5.33. If there exists an action $\varphi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ with infinitesimal action λ, then $\tilde{R}_g(\Sigma_{(e,p)}) = \Sigma_{(g^{-1},p)} = \Sigma_{(e,\tilde{\varphi}(g,p))}$ for any $p \in M$:
Let $\psi = (\mathrm{id} \times \varphi) \circ (\mathrm{diag} \times \mathrm{id}) : \mathcal{G} \times \mathcal{M} \to \mathcal{G} \times \mathcal{M}$. The map ψ is a flat chart (cf. Lemma 4.12) and $\tilde{\psi}(G \times \{p\}) = \Sigma_{(e,p)}$. We have $(e, \tilde{\varphi}(g,p)) = \tilde{R}_g(g, \tilde{\varphi}(g,p) = \tilde{R}_g(\tilde{\psi}(g,p)) \in \tilde{R}_g(\tilde{\psi}(G \times \{p\}))$ and thus $\tilde{R}_g(\Sigma_{(e,p)}) = \Sigma_{(g^{-1},p)} = \Sigma_{(e,\tilde{\varphi}(g,p))}$.

Proposition 5.34. *The infinitesimal action* $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ *is univalent if and only if every leaf of the form* $\Sigma_{(e,p)}$ *for* $p \in M$ *is univalent.*

Proof. If λ is univalent, then all leaves are univalent, in particular each leaf of the form $\Sigma_{(e,p)}$. Assume now that all leaves $\Sigma_{(e,p)}$ are univalent. Let Σ be an arbitrary leaf and let $(g,p) \in \Sigma$. We have $\Sigma_{(e,p)} = \tilde{R}_g(\Sigma_{(g,p)}) = \tilde{R}_g(\Sigma)$. If $\Omega \subset \Sigma = \Sigma_{(g,p)}$ is a relatively compact subset, then $\tilde{R}_g(\Omega) \subset \Sigma_{(e,p)}$ is relatively compact and the univalence of $\Sigma_{(e,p)}$ yields the existence of a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with $\tilde{R}_g(\Omega) \subset \tilde{\psi}(U \times V)$. By Lemma 5.31 the map

$$R_{g^{-1}} \circ \psi \circ R_g : \mathcal{U}g \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$$

is a flat chart and

$$\Omega = \tilde{R}_{g^{-1}}(\tilde{R}_g(\Omega)) \subset \tilde{R}_{g^{-1}}(\tilde{\psi}(U \times V)) = (\tilde{R}_{g^{-1}} \circ \tilde{\psi} \circ \tilde{R}_g)(Ug \times V). \qquad \blacksquare$$

Globalizations of infinitesimal actions on supermanifolds We now study conditions for the existence of globalizations. The main result is the following:

Theorem 5.35. *Let* $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ *be an infinitesimal action. Then the following statements are equivalent:*

(i) *The infinitesimal action* λ *is globalizable.*

(ii) *The restricted infinitesimal action* $\lambda_0 = \lambda|_{\mathfrak{g}_0}$ *is globalizable.*

(iii) *The infinitesimal action* λ *is univalent.*

(iv) *The reduced infinitesimal action* $\tilde{\lambda}$ *is univalent, i.e.* $\pi_G|_\Sigma$ *is injective for an arbitrary leaf* Σ*, and all leaves* $\Sigma \subset G \times M$ *are "holonomy free", i.e. the morphism* Φ_Σ *is trivial.*

(v) *There exists a local action* $\varphi : \mathcal{W} \to \mathcal{M}$ *with induced infinitesimal action* λ *whose domain of definition is maximally balanced.*

The equivalence of (*iii*) and (*iv*) is the content of Proposition 5.21 and once the equivalence of (*i*) and (*iii*) is established the equivalence of (*i*) and (*ii*) is a consequence of Corollary 5.22. The other equivalences shall be proven in the following: The implication (*iii*) $\Rightarrow$ (*i*) is the content of Proposition 5.44, (*i*) $\Rightarrow$ (*v*) follows from Proposition 5.47, and (*v*) $\Rightarrow$ (*iii*) is proven in Proposition 5.48.

Remark 5.36 (see [Pal57], Chapter III, Theorem IV and Theorem V)**.** In the classical case, Palais shows that for a univalent infinitesimal action the space of

leaves $\Sigma \subset G \times M$ of the distribution $\mathcal{D}$, which is denoted by $M^* = (G \times M)/\sim$, carries in a natural way the structure of a possibly non-Hausdorff manifold, the action of G on $G \times M$ by $g \cdot (h, p) = (hg^{-1}, p)$ induces an action on the quotient space M^* and $M \to M^*$, $p \mapsto \Sigma_{(e,p)}$ is an injective embedding.

A similar construction is also important in the case of supermanifolds.

Definition 5.37. Let λ be an infinitesimal action of $\mathcal{G}$ on $\mathcal{M}$ and $\mathcal{D}$ the associated distribution on $\mathcal{G} \times \mathcal{M}$. Define

$$M^* = (G \times M)/\sim$$

to be the space of leaves $\Sigma \subset G \times M$ and denote by $\tilde{\pi} : G \times M \to M^*$, $\tilde{\pi}(g, p) = \Sigma_{(g,p)}$, the projection. Now endow M^* with the quotient topology and define the sheaf $\mathcal{O}_{\mathcal{M}^*}$ of $\mathbb{Z}_2$-graded algebras on M^* by setting

$$\mathcal{O}_{\mathcal{M}^*} = \tilde{\pi}_*(\mathcal{O}^{\mathcal{D}}_{\mathcal{G} \times \mathcal{M}}),$$

i.e. $\mathcal{O}_{\mathcal{M}^*}(\Omega) = \mathcal{O}^{\mathcal{D}}_{\mathcal{G} \times \mathcal{M}}(\tilde{\pi}^{-1}(\Omega))$ for any open subset $\Omega \subset M^*$, where $\mathcal{O}^{\mathcal{D}}_{\mathcal{G} \times \mathcal{M}}$ denotes again the sheaf of $\mathcal{D}$-invariant functions on $\mathcal{G} \times \mathcal{M}$. Define the ringed space

$$\mathcal{M}^* = (M^*, \mathcal{O}_{\mathcal{M}^*})$$

and let $\pi = (\pi^*, \tilde{\pi}) : \mathcal{G} \times \mathcal{M} \to \mathcal{M}^*$, where $\pi^* : \mathcal{O}_{\mathcal{M}^*} \to \tilde{\pi}_*(\mathcal{O}_{\mathcal{G} \times \mathcal{M}})$ is given by the canonical inclusion $\mathcal{O}_{\mathcal{M}^*}(\Omega) = \mathcal{O}^{\mathcal{D}}_{\mathcal{G} \times \mathcal{M}}(\tilde{\pi}^{-1}(\Omega)) \hookrightarrow \mathcal{O}_{\mathcal{G} \times \mathcal{M}}(\tilde{\pi}^{-1}(\Omega))$ for any open subset $\Omega \subseteq M^*$.

Definition 5.38. For any open subset $V \subseteq M$, $\mathcal{V} = (V, \mathcal{O}_{\mathcal{M}}|_V)$, and $g \in G$ we define a morphism of ringed spaces

$$\iota_{\mathcal{V},g} : \mathcal{V} \to \mathcal{M}^* \text{ by } \iota_{\mathcal{V},g} = \pi \circ \iota_g,$$

where $\iota_g : \mathcal{V} \hookrightarrow \{g\} \times \mathcal{V} \subset \mathcal{G} \times \mathcal{M}$ denotes again the inclusion. The reduced map of $\iota_{\mathcal{V},g}$ is given by $p \mapsto \Sigma_{(g,p)}$. For $g = e$ and $\mathcal{V} = \mathcal{M}$, let

$$\iota_{\mathcal{M}} = \iota_{\mathcal{M},e} : \mathcal{M} \to \mathcal{M}^*, \ \iota_{\mathcal{M}} = \pi \circ \iota_e.$$

Remark 5.39. Let $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ be a flat chart with respect to $(\mathcal{D}, U, V, g, \mathrm{id})$ and let $\pi_{\mathcal{M}} = \pi_{\mathcal{M}}|_{U \times V} : \mathcal{U} \times \mathcal{V} \to \mathcal{V}$ denote the projection onto the second component. As $\iota_g : \mathcal{V} \to \mathcal{U} \times \mathcal{V}$ is a section of $\pi_{\mathcal{M}}$ and $\psi \circ \iota_g = \iota_g$, the diagram

$$
\begin{array}{ccc}
\mathcal{U} \times \mathcal{V} & \xrightarrow{\psi} & \mathcal{G} \times \mathcal{M} \\
\downarrow{\scriptstyle \pi_{\mathcal{M}}} & & \downarrow{\scriptstyle \pi} \\
\mathcal{V} & \xrightarrow{\iota_{\mathcal{V},g}} & \mathcal{M}^*
\end{array}
$$

is commutative.

We will see later on that $\mathcal{M}^*$ carries the structure of a supermanifold and $\iota_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^*$ is an open embedding if and only if the infinitesimal action λ is globalizable. In this case $\mathcal{M}^*$ is itself a globalization of λ and the $\mathcal{G}$-action on $\mathcal{M}^*$ is induced by the $\mathcal{G}$-action R on $\mathcal{G} \times \mathcal{M}$.

Remark 5.40. The topological space M^* only depends on the underlying infinitesimal action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ since $\widetilde{\mathcal{D}} = \mathcal{D}_{\tilde{\lambda}}$ (cf.Lemma 5.7). Hence, the map $\tilde{\pi}$ is an open map as in the classical case (cf. [Pal57], Chapter I, Theorem III, or [HI97], § 2, Proposition 2).

The topological space M^* fulfills the second axiom of countability because $\tilde{\pi}$ is an open quotient map and $G \times M$ a manifold.

Lemma 5.41. *The space $\mathcal{M}^* = (M^*, \mathcal{O}_{\mathcal{M}^*})$ is a locally ringed space.*

Proof. For $f \in \mathcal{O}_{\mathcal{M}^*}(\Omega) = \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(\tilde{\pi}^{-1}(\Omega))$ with $\Sigma \in \Omega \subseteq M^*$ denote by $[f]_\Sigma$ the germ of f in the stalk $(\mathcal{O}_{\mathcal{M}^*})_\Sigma$. We can define the ideal

$$\mathfrak{m}_\Sigma = \{[f]_\Sigma \mid \tilde{f}(\Sigma) = 0\} \lhd (\mathcal{O}_{\mathcal{M}^*})_\Sigma$$

for any leaf $\Sigma \in M^*$. Assume that $\mathfrak{m}_\Sigma$ is not a maximal ideal in the stalk $(\mathcal{O}_{\mathcal{M}^*})_\Sigma$. Then there is a proper ideal $I \lhd (\mathcal{O}_{\mathcal{M}^*})_\Sigma$ which is not contained in $\mathfrak{m}_\Sigma$. Let $[f]_\Sigma \in I \setminus \mathfrak{m}_\Sigma$. Then we have $\tilde{f}(\Sigma) \neq 0$ and the continuity of $\tilde{f}$ gives the existence of an open neighbourhood Ω of $\Sigma \in M^*$ such that f is defined on $\tilde{\pi}^{-1}(\Omega)$, i.e.

$$f \in \mathcal{O}_{\mathcal{M}^*}(\Omega) = \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(\tilde{\pi}^{-1}(\Omega)) \subset \mathcal{O}_{\mathcal{G} \times \mathcal{M}}(\tilde{\pi}^{-1}(\Omega)),$$

and $\tilde{f}(x) \neq 0$ for all $x \in \tilde{\pi}^{-1}(\Omega)$. Consequently, there exists $g \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}(\tilde{\pi}^{-1}(\Omega))$ with $gf = fg = 1$. The function g is also $\mathcal{D}$-invariant since

$$0 = Y(1) = Y(gf) = Y(g)f + (-1)^{|g||Y|}gY(f) = Y(g)f + 0 = Y(g)f$$

for all vector fields Y belonging to $\mathcal{D}$ and thus $Y(g) = 0$. Therefore, g defines an element in $\mathcal{O}_{\mathcal{M}^*}(\Omega)$ and $[g]_\Sigma[f]_\Sigma = [gf]_\Sigma = 1 \in I$ which contradicts the assumption $I \neq (\mathcal{O}_{\mathcal{M}^*})_\Sigma$. ∎

Lemma 5.42. *The action of $\mathcal{G}$ on $\mathcal{G} \times \mathcal{M}$ from the right $R : \mathcal{G} \times (\mathcal{G} \times \mathcal{M}) \to \mathcal{G} \times \mathcal{M}$ induces an action χ of $\mathcal{G}$ on the space $\mathcal{M}^*$, i.e. a morphism of ringed spaces satisfying the usual action properties, such that*

$$
\begin{array}{ccc}
\mathcal{G} \times (\mathcal{G} \times \mathcal{M}) & \xrightarrow{\ R\ } & \mathcal{G} \times \mathcal{M} \\
{\scriptstyle \mathrm{id} \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\
\mathcal{G} \times \mathcal{M}^* & \xrightarrow[\ \chi\ ]{} & \mathcal{M}^*
\end{array}
$$

commutes. In particular, if $\mathcal{M}^*$ is a supermanifold, then χ is an action of the Lie supergroup $\mathcal{G}$ on $\mathcal{M}^*$.

Proof. The underlying action $\tilde{\chi}$ of G on M^* is given by

$$\tilde{\chi} : G \times M^* \to M^*, \ (g, \Sigma_{(h,p)}) \mapsto \tilde{R}_g(\Sigma_{(h,p)}) = \Sigma_{(hg^{-1},p)}.$$

and continuous since $\tilde{\pi} \circ \tilde{R} = \tilde{\chi} \circ (\mathrm{id}_G \times \tilde{\pi})$.

If f is any $\mathcal{D}$-invariant function on $\mathcal{G} \times \mathcal{M}$, then $R^*(f)$ is $(\mathbb{1} \otimes \mathcal{D})$-invariant (see Corollary 5.28). Therefore,

$$(\pi \circ R)^*(f) \in (\mathcal{O}_{\mathcal{G} \times \mathcal{G} \times \mathcal{M}}^{\mathbb{1} \otimes \mathcal{D}})((\tilde{\pi} \circ \tilde{R})^{-1}(\Omega)) \cong (\mathcal{O}_{\mathcal{G}} \hat{\otimes} \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}})((\tilde{\pi} \circ \tilde{R})^{-1}(\Omega))$$

for any $f \in \mathcal{O}_{\mathcal{M}^*}(\Omega) = \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(\tilde{\pi}^{-1}(\Omega))$, $\Omega \subseteq M^*$, and $(\pi \circ R)^*$ induces a morphism $\chi^* : \mathcal{O}_{\mathcal{M}^*} \to \tilde{\chi}_*(\mathcal{O}_{\mathcal{G} \times \mathcal{M}^*})$ with $R^* \circ \pi^* = (\mathrm{id} \times \pi)^* \circ \chi^*$. The map χ defines an action of $\mathcal{G}$ on $\mathcal{M}^*$ since R is an action and χ inherits the respective properties. ∎

The infinitesimal action

$$\lambda_\chi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M}^*), \ \lambda_\chi(X) = (X(e) \otimes \mathrm{id}^*) \circ \chi^*$$

on the ringed space $\mathcal{M}^*$ extends the infinitesimal action λ on $\mathcal{M}$ in the following sense:

Lemma 5.43. *For any $X \in \mathfrak{g}$ we have*

$$\lambda(X) \circ \iota_{\mathcal{M}}{}^* = \iota_{\mathcal{M}}{}^* \circ \lambda_\chi(X).$$

Proof. Let $\Omega \subseteq M^*$ be open and $f \in \mathcal{O}_{\mathcal{M}^*}(\Omega)$. Then $\pi^*(f)$ is $\mathcal{D}$-invariant on $\tilde{\pi}^{-1}(\Omega) \subset G \times M$ and thus $(\mathrm{id}^* \otimes \lambda(X))(\pi^*(f)) = -(X \otimes \mathrm{id}^*)(\pi^*(f))$ for $X \in \mathfrak{g}$. Consequently, we have

$$(\lambda(X) \circ \iota_{\mathcal{M}}{}^*)(f) = (\lambda(X) \circ \iota_e{}^*)(\pi^*(f)) = \iota_e{}^*((\mathrm{id}^* \otimes \lambda(X))(\pi^*(f)))$$
$$= -((X(e) \otimes \mathrm{id}^*)(\pi^*(f)).$$

A calculation using the identities $\pi \circ R = \chi \circ (\mathrm{id} \times \pi)$ and $R \circ (\mathrm{id} \times \iota_e) = (\iota \times \mathrm{id})$ gives $\iota_{\mathcal{M}}{}^* \circ \lambda_\chi(X) = ((-X(e)) \otimes \mathrm{id}^*) \circ \pi^*$ so that $\lambda(X) \circ \iota_{\mathcal{M}}{}^* = \iota_{\mathcal{M}}{}^* \circ \lambda_\chi(X)$. ∎

Proposition 5.44. *Let λ be univalent. Then $\mathcal{M}^*$ is a supermanifold, with a possibly non-Hausdorff underlying manifold M^*.*

Moreover, the morphism $\iota_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}^*$ is an open embedding, the infinitesimal action λ_χ induced by χ extends the infinitesimal action of λ on $\mathcal{M} \cong (\tilde{\iota}_{\mathcal{M}}(M), \mathcal{O}_{\mathcal{M}^*}|_{\tilde{\iota}_{\mathcal{M}}(M)})$, and we have $G \cdot \tilde{\iota}_{\mathcal{M}}(M) = \tilde{\chi}(G \times \tilde{\iota}_{\mathcal{M}}(M)) = M^*$.

Thus $\mathcal{M}^*$ is a globalization of λ and the infinitesimal action λ is globalizable.

The proof of the proposition makes use of the next lemma.

Lemma 5.45. *Let λ be univalent and $W \subseteq G \times M$ an open connected subset. For any $f \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(W)$ there exists a unique extension*

$$\hat{f} \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(\tilde{\pi}^{-1}(\tilde{\pi}(W))) \ \text{with} \ \hat{f}|_W = f.$$

Proof. Note that the open set

$$\tilde{\pi}^{-1}(\tilde{\pi}(W)) = \bigcup_{\substack{\Sigma \text{ leaf} \\ \Sigma \cap W \neq \emptyset}} \Sigma$$

is again connected and by Lemma 5.25 a $\mathcal{D}$-invariant extension $\hat{f}$ of f is unique.

Let Σ be a leaf with $\Sigma \cap W \neq \emptyset$, i.e. $\Sigma \in \tilde{\pi}(W)$, and let $\Sigma' \subset \Sigma$ be relatively compact. Since λ is univalent, there exists a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with

$\Sigma' \subset \tilde{\psi}(U \times V)$ and $\tilde{\psi}(U' \times V) \subset W$ for some open subset $U' \subseteq U$. The function $(\psi|_{\tilde{\psi}^{-1}(W)})^*(f)$ is $\mathcal{D}_{\mathcal{G}}$-invariant and thus of the form $(\psi|_{\tilde{\psi}^{-1}(W)})^*(f) = 1 \otimes f_{\mathcal{M}}$ on $U' \times V \subseteq \tilde{\psi}^{-1}(W)$ for some $f_{\mathcal{M}} \in \mathcal{O}_{\mathcal{M}}(V)$. Now, $1 \otimes f_{\mathcal{M}}$ is already defined on $U \times V$ and we define $\hat{f}$ on $\tilde{\psi}(U \times V)$ by

$$\hat{f}|_{\tilde{\psi}(U \times V)} = (\psi^{-1})^*(1 \otimes f_{\mathcal{M}}).$$

This yields a well-defined function $\hat{f}$ on $\tilde{\pi}^{-1}(\tilde{\pi}(W))$ due to the uniqueness of flat charts following from the univalence of λ (see Proposition 5.14). Moreover, $\hat{f}$ is $\mathcal{D}$-invariant by construction and $\hat{f}|_W = f$. ∎

Proof of Proposition 5.44.   We prove that in the case of a univalent infinitesimal action the morphism $\iota_{\mathcal{V},g} : \mathcal{V} \to \mathcal{M}^*$ defines a chart for $\mathcal{M}^*$ if there is a flat chart $\psi : \mathcal{U} \times \mathcal{V} \to \mathcal{G} \times \mathcal{M}$ with respect to $(\mathcal{D}, U, V, g, \mathrm{id})$. Due to the local existence of flat charts this implies that $\mathcal{M}^*$ is a supermanifold.

By Proposition 5.21, the restriction $\tilde{\pi}_{\mathcal{G}}|_\Sigma : \Sigma \to G$ of the projection $\tilde{\pi}_{\mathcal{G}} : G \times M \to G$ is injective for all leaves $\Sigma \subset G \times M$. Therefore, we have $\tilde{\iota}_{\mathcal{V},g}(p) = \Sigma_{(g,p)} = \Sigma_{(g,q)} = \tilde{\iota}_{\mathcal{V},g}(q)$ if and only if $p = q$. The map $\tilde{\iota}_{\mathcal{V},g}$ is open because for any open subset $V' \subseteq V$ the set $\tilde{\iota}_{\mathcal{V},g}(V') = \tilde{\iota}_{\mathcal{V},g}(\tilde{\pi}_{\mathcal{M}}(U \times V')) = \tilde{\pi}(\tilde{\psi}(U \times V))$ is open in M^* using that ψ is a local diffeomorphism and $\tilde{\pi}$ an open map. Consequently, $\tilde{\iota}_{\mathcal{V},g}$ is a homeomorphism onto its image.

To show that $\iota_{\mathcal{V},g}^*$ is injective, let $\Omega \subseteq M^*$ be open and $f_1, f_2 \in \mathcal{O}_{\mathcal{M}^*}(\Omega)$. If $\iota_{\mathcal{V},g}^*(f_1) = \iota_{\mathcal{V},g}^*(f_2)$, then also

$$\psi^*(\pi^*(f_1)) = \pi_{\mathcal{M}}^*(\iota_{\mathcal{V},g}^*(f_1)) = \pi_{\mathcal{M}}^*(\iota_{\mathcal{V},g}^*(f_2)) = \psi^*(\pi^*(f_2)).$$

Since $\pi^* : \mathcal{O}_{\mathcal{M}^*} = \tilde{\pi}_*(\mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}) \hookrightarrow \tilde{\pi}_*(\mathcal{O}_{\mathcal{G} \times \mathcal{M}})$ is the canonical inclusion, $\pi^*(f_i)$ and f_i can be identified. If $\psi^*(f_1) = \psi^*(f_2)$, then $f_1 = f_2$ on $\tilde{\psi}(\tilde{\psi}^{-1}(\tilde{\pi}^{-1}(\Omega)))$ and thus $f_1 = f_2$ by Lemma 5.25.

For any $f_{\mathcal{V}} \in \mathcal{O}_{\mathcal{M}}|_V(\tilde{\iota}_{\mathcal{V},g}^{-1}(\Omega))$ we have
$$1 \otimes f_{\mathcal{V}} = \pi_{\mathcal{M}}^*(f_{\mathcal{V}}) \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}_{\mathcal{G}}}(U \times \tilde{\iota}_{\mathcal{V},g}^{-1}(\Omega)) = \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}_{\mathcal{G}}}(\tilde{\psi}^{-1}(\tilde{\pi}^{-1}(\Omega))).$$
Thus $(\psi^{-1})^*(1 \otimes f_{\mathcal{V}})$ is a $\mathcal{D}$-invariant function on $\tilde{\psi}(U \times V) \cap \tilde{\pi}^{-1}(\Omega) = \tilde{\psi}(\tilde{\psi}^{-1}(\tilde{\pi}^{-1}(\Omega)))$. By Lemma 5.45 there is a $\mathcal{D}$-invariant extension $\hat{f} \in \mathcal{O}_{\mathcal{G} \times \mathcal{M}}^{\mathcal{D}}(\tilde{\pi}^{-1}(\Omega)) = \mathcal{O}_{\mathcal{M}^*}(\Omega)$ of $(\psi^{-1})^*(1 \otimes f_{\mathcal{V}})$ and we have $\iota_{\mathcal{V},g}^*(\hat{f}) = f_{\mathcal{V}}$ since $\psi^*(\pi^*(\hat{f})) = \psi^*((\psi^{-1})^*(1 \otimes f_{\mathcal{V}})) = 1 \otimes f_{\mathcal{V}}$. Consequently, $\iota_{\mathcal{V},g}^*$ is also surjective and thus $\iota_{\mathcal{V},g}$ is a chart for $\mathcal{M}^*$.

The morphism $\iota_{\mathcal{M}} = \iota_{\mathcal{M},e}$ is an open embedding since the univalence of λ implies that $\tilde{\iota}_{\mathcal{M}}$ is injective with the same argument as for $\tilde{\iota}_{\mathcal{V},g}$ and locally $\iota_{\mathcal{M}}$ is of the form $\iota_{\mathcal{V},e}$ such that there exists a flat chart ψ with respect to $(\mathcal{D}, U, V, e, \mathrm{id})$.

By Lemma 5.43, the infinitesimal action λ_χ induced by χ extends the infinitesimal action of λ on $\mathcal{M} \cong (\tilde{\iota}_{\mathcal{M}}(M), \mathcal{O}_{\mathcal{M}^*}|_{\tilde{\iota}_{\mathcal{M}}(M)})$. Since

$$g \cdot \Sigma_{(e,p)} = \tilde{\chi}(g, \Sigma_{(e,p)}) = \tilde{R}_g(\Sigma_{(e,p)}) = \Sigma_{(g^{-1},p)}$$

for any $p \in M$ and $\tilde{\iota}_{\mathcal{M}}(M)$ consists exactly of those leaves which intersect $\{e\} \times M$, we have $G \cdot \tilde{\iota}_{\mathcal{M}}(M) = \tilde{\chi}(G \times \tilde{\iota}_{\mathcal{M}}(M)) = M^*$. ∎

Lemma 5.46. *Let $\varphi : \mathcal{W} \to \mathcal{M}$ a local action with induced infinitesimal action λ and maximally balanced domain of definition. Then we have $\tilde{\psi}(W_p \times \{p\}) = \Sigma_{(e,p)}$ for any $p \in M$ and $\psi = (\mathrm{id} \times \varphi) \circ (\mathrm{diag} \times \mathrm{id}) : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$, which is locally a flat chart.*

Proof. The lemma follows from the analogous classical result (see [Pal57], Chapter II, Theorem VI) since $\varphi : \mathcal{W} \to \mathcal{M}$ has a maximally balanced domain of definition if and only if the domain of definition W of the reduced local action is maximally balanced. ∎

Proposition 5.47. *Let $\varphi' : \mathcal{G} \times \mathcal{M}' \to \mathcal{M}'$ be a globalization of λ. Then there is a local action $\varphi : \mathcal{W} \to \mathcal{M}$ with maximally balanced domain of definition and infinitesimal action λ.*

Moreover, $\varphi : \mathcal{W} \to \mathcal{M}$ is the unique maximal local action with infinitesimal action λ. Any two local actions $\varphi_i : \mathcal{W}_i \to \mathcal{M}$, $i = 1, 2$, with infinitesimal action λ coincide on their common domain of definition and define a local action $\chi : \mathcal{W}_1 \cup \mathcal{W}_2 \to \mathcal{M}$ with $\chi|_{W_1} = \varphi_1$ and $\chi|_{W_2} = \varphi_2$.

Proof. The set $(\tilde{\varphi}')^{-1}(M) \cap (G \times M)$ is open in $G \times M$ and contains $\{e\} \times M$. Let W_p be the connected component of e in $\{g \in G \mid \tilde{\varphi}'(g, p) \in M\}$ for $p \in M$ and define

$$W = \bigcup_{p \in M} W_p \times \{p\}.$$

The set $W \subseteq G \times M$ is open and the largest domain of definition of a local action included in $(\tilde{\varphi}')^{-1}(M) \cap (G \times M)$ (cf. [Pal57], Chapter II, Theorem I). Let $\mathcal{W} = (W, \mathcal{O}_{\mathcal{W}})$ and define $\varphi = \varphi'|_W : \mathcal{W} \to \mathcal{M}$. The map φ is a local action of $\mathcal{G}$ on $\mathcal{M}$. A direct calculation shows that the domain of definition $\mathcal{W}$ of φ is maximally balanced. By the preceding lemma we have $\tilde{\psi}(W_p \times \{p\}) = \Sigma_{(e,p)}$ for any $p \in M$, where $\psi = (\mathrm{id} \times \varphi) \circ (\mathrm{diag} \times \mathrm{id}) : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$.

Let $\chi : \mathcal{W}_\chi \to \mathcal{M}$ be any local action with induced infinitesimal action λ and set $\psi_\chi = (\mathrm{id} \times \chi) \circ (\mathrm{diag} \times \mathrm{id}) : \mathcal{W}_\chi \to \mathcal{G} \times \mathcal{M}$. For any $p \in M$, $\tilde{\psi}_\chi(W_{\chi,p} \times \{p\})$ is contained in the leaf $\Sigma_{(e,p)}$. Since $\Sigma_{(e,p)} = \tilde{\psi}(W_p \times \{p\})$, we get

$$W_{\chi,p} = \pi_G(\tilde{\psi}_\chi(W_{\chi,p} \times \{p\})) \subseteq \pi_G(\Sigma_{(e,p)}) = \pi_G(\tilde{\psi}(W_p \times \{p\})) = W_p,$$

which implies $W_\chi \subseteq W$. The uniqueness of local actions with a given domain of definition (see Corollary 4.13) yields $\varphi = \chi$ on W_χ. ∎

Proposition 5.48. *Let $\varphi : \mathcal{W} \to \mathcal{M}$ be a local action with maximally balanced domain of definition. Then its induced infinitesimal action is univalent.*

Proof. Let $\psi = (\mathrm{id} \times \varphi) \circ (\mathrm{diag} \times \mathrm{id}) : \mathcal{W} \to \mathcal{G} \times \mathcal{M}$ be the locally flat chart associated to the local action φ. By Lemma 5.46 we have $\tilde{\psi}(W_p \times \{p\}) = \Sigma_{(e,p)}$ for any $p \in M$.

Let $\Omega \subset \Sigma_{(e,p)}$ be a relatively compact connected subset. By Lemma 4.12 there are subsets $U \subset G$ and $V \subset M$, $p \in V$, such that $\psi|_{U \times V}$ is a flat chart and $\Omega \subset \tilde{\psi}(U \times V)$. Consequently, $\Sigma_{(e,p)}$ is univalent for any $p \in M$ and hence λ is univalent by Proposition 5.34 ∎

Example 5.49. Let $\mathcal{M} = (\mathbb{C} \setminus \{0\}) \times \mathbb{C}^{0|2}$, with coordinates z, θ_1, θ_2, and let $\alpha : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ be a holomorphic function. Consider the even holomorphic vector field

$$X_\alpha = (1 + \alpha(z)\theta_1\theta_2)\frac{\partial}{\partial z}$$

on $\mathcal{M}$. We now examine for which α the infinitesimal $\mathbb{C}$-action λ_α, $\lambda_\alpha(t) = tX_\alpha$, generated by X_α is globalizable.

Let $\mathcal{D}_\alpha$ be the distribution spanned by $\frac{\partial}{\partial t} + X_\alpha$ on $\mathbb{C} \times \mathcal{M}$. The leaves $\Sigma \subset \mathbb{C} \times M$ of $\mathcal{D}_\alpha$ are of the form $\Sigma = \Sigma_{(t,z)} = \{(t + s, z + s)|\, s \in \mathbb{C} \setminus \{-z\}\}$ for $(t, z) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\})$. Each leaf Σ is therefore biholomorphic to $\mathbb{C} \setminus \{0\}$.

The reduced vector field $\tilde{X}_\alpha = \frac{\partial}{\partial z}$ always generates a globalizable infinitesimal action and a globalization of $M = \mathbb{C} \setminus \{0\}$ is $M^* = \mathbb{C}$ with the usual addition as $\mathbb{C}$-action. If λ_α is globalizable, then the globalization $\mathcal{M}_\alpha^* = (M^*, \tilde{\pi}_* \mathcal{O}_{\mathbb{C} \times \mathcal{M}}^{\mathcal{D}_\alpha})$ is a complex supermanifold of dimension $(1|2)$. Every complex supermanifold $\mathcal{N}$ with underlying manifold $\mathbb{C}$ is split since $\mathbb{C}$ is Stein (see [Oni98], Theorem 3.4). Moreover, $\mathcal{N}$ is isomorphic to $\mathbb{C}^{1|n}$ for some $n \in \mathbb{N}$ since all holomorphic vector bundles on $\mathbb{C}$ are trivial. Therefore, $\mathcal{M}_\alpha^*$ is isomorphic to $\mathbb{C}^{1|2}$ if it is a supermanifold.

An element $f = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2 \in \mathcal{O}_{\mathbb{C} \times \mathcal{M}}(\mathbb{C} \times M)$ is $\mathcal{D}_\alpha$-invariant if and only if there exist holomorphic functions $g_i : \mathbb{C} \to \mathbb{C}$, $i = 1, 2, 3$, with

$$f_i(t, z) = g_i(z - t)$$

and f_{12} is locally of the form

$$f_{12}(t, z) = -A(z)\left(\frac{\partial}{\partial z}f_0\right)(t, z) + g_{12}(z - t),$$

where A is a primitive of α and g_{12} a holomorphic function. There are two different cases:

(i) If α has a global primitive $A : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ we have

$$\mathcal{O}_{\mathcal{M}_\alpha^*}(M^*)$$
$$= \left\{ g_0 + g_1\theta_1 + g_2\theta_2 + \left(g_{12} - A\frac{\partial}{\partial z}g_0\right)\theta_1\theta_2 \,\middle|\, g_0, g_1, g_2, g_{12} \text{ holomorphic}\right\}$$
$$\cong \mathcal{O}_{\mathbb{C}^{1|2}}(\mathbb{C}).$$

and it turns out that λ_α is globalizable with globalization $\mathcal{M}_\alpha^* \cong \mathbb{C}^{1|2}$, where $\mathbb{C}$ acts on $\mathbb{C}^{1|2}$ by the extension of the usual addition on $\mathbb{C}$, i.e. $(z, \theta_1, \theta_2) \mapsto (z + t, \theta_1, \theta_2)$. Moreover, the open embedding $\iota_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}_\alpha^*$ can be realised as

$$\iota_{\mathcal{M}} : \mathbb{C} \setminus \{0\} \times \mathbb{C}^{0|2} \hookrightarrow \mathbb{C}^{1|2}, \; \iota_{\mathcal{M}}^*(z, \theta_1, \theta_2) = (z - A(z)\theta_1\theta_2, \theta_1, \theta_2).$$

(ii) In the case where α does not have a global primitive A on $\mathbb{C} \setminus \{0\}$, we get

$$\mathcal{O}_{\mathcal{M}_\alpha^*}(M^*) = \left\{\lambda + g_1\theta_1 + g_2\theta_2 + g_{12}\theta_1\theta_2 \,\middle|\, \lambda \in \mathbb{C}, \, g_1, g_2, g_{12} \text{ holomorphic}\right\},$$

which is not isomorphic to $\mathcal{O}_{\mathbb{C}^{1|2}}(\mathbb{C})$. Therefore, the ringed space $\mathcal{M}_\alpha^*$ is not a supermanifold and λ_α is not globalizable.

For any α, a flat chart ψ with respect to $(\mathcal{D}_\alpha, U, V, t_0, \rho)$ is given by the pullback

$$\psi^*(t, z, \theta_1, \theta_2)$$
$$= \left(t, \rho^*(z, \theta_1, \theta_2)\right) + \left(0, (t - t_0) + \left(A(\tilde{\rho}(z) + (t - t_0)) - A(\tilde{\rho}(z))\right)\rho^*(\theta_1\theta_2), 0, 0\right),$$

where $U \subseteq \mathbb{C}$ and $V \subseteq \mathbb{C} \setminus \{0\}$ need to be open subsets such that there exists a primitive A of α on $U + \tilde{\rho}(V) = \{t + z \mid t \in U, z \in \tilde{\rho}(V)\} \subset \mathbb{C} \setminus \{0\}$.

We shall now explicitly describe the morphism $\Phi = \Phi_{\Sigma,(s,z)} : \pi_1(\Sigma, (s, z)) \to$ $\mathrm{Hol}_z(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{0|2})$ for a leaf $\Sigma \subset \mathbb{C} \times M$ with $(s, z) \in \Sigma$. First remark that if ψ is a flat chart with respect to $(\mathcal{D}_\alpha, U, V, t_0, \rho)$ and $t'_0 \in U$, then ψ is also a flat chart with respect to $(\mathcal{D}_\alpha, U, V, t'_0, \rho')$ for

$$(\rho')^*(z, \theta_1, \theta_2)$$
$$= \rho^*(z, \theta_1, \theta_2) + \left((t'_0 - t_0) + \left(A(\tilde{\rho}(z) + (t'_0 - t_0)) - A(\tilde{\rho}(z))\right)\rho^*(\theta_1\theta_2), 0, 0\right)$$
$$= \rho^*(z, \theta_1, \theta_2) + \left(\int_{t_0}^{t'_0} 1 dt + \left(\int_{t_0}^{t'_0} \alpha(\tilde{\rho}(z) + (t - t_0))dt\right)\rho^*(\theta_1\theta_2), 0, 0\right),$$

where the integrals do not depend on the path if contained in U since α has a primitive on $U + \tilde{\rho}(V)$. If ψ' is another flat chart with respect to $(\mathcal{D}, U', V', t'_0, \rho')$ with $V \cap V' \neq \emptyset$ and $t''_0 \in U'$, then ψ' is also a flat chart with respect to $(\mathcal{D}, U', V', t''_0, \rho'')$ for

$$(\rho'')^*(z, \theta_1, \theta_2)$$
$$= \rho^*(z, \theta_1, \theta_2) + \left(\int_{t_0}^{t''_0} 1 dt + \left(\int_{t_0}^{t''_0} \alpha(\tilde{\rho}(z) + (t - t_0))dt\right)\rho^*(\theta_1\theta_2), 0, 0\right),$$

where the integrals need to be taken along appropriate paths. For any closed path $\gamma : [0, 1] \to \Sigma$, $\gamma(r) = (\gamma_1(r), \gamma_2(r))$, $\gamma(0) = \gamma(1) = (s, z)$, we consequently get

$$\Phi([\gamma])^*(z, \theta_1, \theta_2) = \left(z + \left(\int_{\gamma_1} \alpha_{s,z}dt\right)\theta_1\theta_2, \theta_1, \theta_2\right)$$

for $\alpha_{s,z}(t) = \alpha((z-s)+t)$. Thus, Φ is trivial if and only if α has a global primitive on $\mathbb{C} \setminus \{0\}$. Using Theorem 5.35 this shows again that λ_α is globalizable precisely if α has a global primitive.

In the special case of $\alpha(z) = z^{-1}$, we have for example

$$\Phi : \mathbb{Z} \to \mathrm{Hol}_p(\mathbb{C} \setminus \{0\} \times \mathbb{C}^{0|2}), \ \Phi(k)^*(z, \theta_1, \theta_2) = (z + 2\pi i k \theta_1\theta_2, \theta_1, \theta_2)$$

for any leaf Σ, identifying the fundamental group of $\Sigma \cong \mathbb{C} \setminus \{0\}$ with $\mathbb{Z}$.

6. Actions of simply-connected Lie supergroups

In this section a few consequences of the characterization of globalizable infinitesimal actions of simply-connected Lie supergroups, i.e. Lie supergroups $\mathcal{G}$ whose underlying Lie group G is simply-connected, are given.

If G is simply-connected and acts on the classical manifold M, the leaves $\Sigma \subset G \times M$ of the distribution associated to the infinitesimal action are all isomorphic to G and thus simply-connected. This yields consequences for the existence of globalizations of infinitesimal actions of simply-connected Lie supergroups since there are no holonomy phenomena, i.e. the morphisms $\Phi_\Sigma : \pi_1(\Sigma) \to \mathrm{Diff}_p(\mathcal{M})$ are all trivial, if the reduced infinitesimal action is global.

Definition 6.1. An infinitesimal action $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ is called global if there is a $\mathcal{G}$-action on $\mathcal{M}$ which induces λ.

Remark 6.2 (cf. [Pal57], Chapter II, Section 4)**.** Let G be a Lie group acting on a manifold M. Then the leaves $\Sigma \subset G \times M$ of the distribution $\mathcal{D}_{\tilde{\lambda}}$ associated to the infinitesimal action $\tilde{\lambda}$ induced by the G-action are all isomorphic to G.

As a consequence we obtain the following theorem.

Theorem 6.3. *Let $\mathcal{G}$ be a simply-connected Lie supergroup and $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ an infinitesimal action of $\mathcal{G}$ such that its reduced action $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ is global. Then the infinitesimal action λ is globalizable, and $\mathcal{M}$ is the unique globalization.*

Proof. Let $\mathcal{D}$ denote again the distribution on $\mathcal{G} \times \mathcal{M}$ associated to λ. By Lemma 5.7 we have $\widetilde{\mathcal{D}} = \mathcal{D}_{\tilde{\lambda}}$ and by definition the leaves of $\mathcal{D}$ are the leaves of $\mathcal{D}_{\tilde{\lambda}}$. By the preceding remark all leaves $\Sigma \subset G \times M$ are isomorphic to G and consequently simply-connected. Therefore, the morphisms $\Phi : \pi_1(\Sigma, (g, p)) \cong \{1\} \to \mathrm{Diff}_p(\mathcal{M})$ are all trivial. Since $\tilde{\lambda}$ is global and thus in particular globalizable, $\tilde{\lambda}$ is univalent. This implies that λ is globalizable using the equivalent characterizations of globalizability formulated in Theorem 5.35. Let $\mathcal{M}'$ be a globalization of λ and $\iota_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}'$ the open embedding. Then M' is a globalization of $\tilde{\lambda}$ and because M' is the unique globalization of $\tilde{\lambda}$ (cf. [Pal57], Chapter III, Theorem XII,(4)) we have $M = M'$ and $\tilde{\iota}_{\mathcal{M}} = \mathrm{id}_M$. Since $\iota_{\mathcal{M}}$ is an open embedding, this implies $\iota_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$ and $\mathcal{M}' = \mathcal{M}$. ∎

If the assumption on the simply-connectedness of $\mathcal{G}$ is dropped in the above theorem, there exist counterexamples to the statement, see e.g. Example 5.20. Also, as e.g. illustrated in Example 5.49, it is not enough for $\tilde{\lambda}$ to be globalizable. We really need that $\tilde{\lambda}$ is global.

Definition 6.4. Let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ an infinitesimal action. The set of points $p \in M$ such that there exists an even vector field $X \in \mathfrak{g}_0$ with $\lambda(X)(p) \neq 0$ is called the support of λ.

Remark 6.5. The definition of the support of an infinitesimal action λ implies that the support of λ coincides with the support of the underlying infinitesimal action $\tilde{\lambda}$.

In the classical case, we have the following two theorems on actions of

simply-connected Lie groups.

Theorem 6.6 (see [Pal57], Chapter III, Theorem XVIII). *Let G be a simply-connected Lie group and $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ an infinitesimal action of G on a manifold M. If the support of $\tilde{\lambda}$ is relatively compact in M, then $\tilde{\lambda}$ is global.*

In particular, any infinitesimal action of a simply-connected Lie group G on a compact manifold M is global.

Theorem 6.7 (see [Pal57], Chapter IV, Theorem III). *Let $\tilde{\lambda} : \mathfrak{g}_0 \to \mathrm{Vec}(M)$ be an infinitesimal action of a simply-connected Lie group G on M. Suppose there exists a set of generators $\{X_i\}_{i \in I}$, $X_i \in \mathfrak{g}_0$, of the Lie algebra $\mathfrak{g}_0$ such that the flow of each vector field $\tilde{\lambda}(X_i)$ is global. Then the infinitesimal action $\tilde{\lambda}$ is global.*

Applying Theorem 6.3, these results in the classical case can be directly carried over to the case of infinitesimal actions of simply-connected Lie supergroups on supermanifolds.

Corollary 6.8. *Let $\mathcal{G}$ be a simply-connected Lie supergroup and $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ an infinitesimal action whose support is relatively compact in M. Then the infinitesimal action λ is global.*

In particular, any infinitesimal action of a simply-connected Lie supergroup on a supermanifold with compact underlying manifold is global.

Corollary 6.9. *Let $\lambda : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ be an infinitesimal action of a simply-connected Lie supergroup $\mathcal{G}$ such that there exists a set of generators $\{X_i\}_{i \in I}$, $X_i \in \mathfrak{g}_0$, of $\mathfrak{g}_0$ such that each vector field $\tilde{\lambda}(X_i)$ has a global flow. Then the infinitesimal action λ is global.*

A slightly weaker version of this corollary, in a formulation for DeWitt supermanifolds, has been proven, in a different way, in [Tuy13]. The assumption there is that all even vector fields have global flows, and not only a set of generators.

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