Journal of Lie Theory
Volume **24** (2014) 957–967
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Limits of Contraction Groups and the Tits Core

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Communicated by K.-H. Neeb

Abstract. The *Tits core* $G^\dagger$ of a totally disconnected locally compact group G is defined as the abstract subgroup generated by the closures of the contraction groups of all its elements. We show that a dense subgroup is normalised by the Tits core if and only if it contains it. It follows that every dense subnormal subgroup contains the Tits core. In particular, if G is topologically simple, then the Tits core is abstractly simple, and when $G^\dagger$ is non-trivial, it is the smallest dense normal subgroup. The proofs are based on the fact, of independent interest, that the map which associates to an element the closure of its contraction group is continuous.

Mathematics Subject Classification 2010: 22D05, 20E32.
Key Words and Phrases: Totally disconnected locally compact group, simple group, contraction group, Chabauty topology.

1. Introduction

In his seminal paper [8], J. Tits uses his notion of BN-pairs to show the following result: *if* $\mathbf{G}$ *is a simple algebraic group over a field* k*, then every subgroup of* $\mathbf{G}(k)$ *normalised by* $\mathbf{G}(k)^\dagger$ *either is central, or contains* $\mathbf{G}(k)^\dagger$. The group $\mathbf{G}(k)^\dagger$ is defined to be the subgroup of $\mathbf{G}(k)$ generated by the unipotent radicals of the k-defined parabolic subgroups. Tits deduces in particular that the quotient of $\mathbf{G}(k)^\dagger$ by its centre is simple as an abstract group.

In the present paper, we associate a subgroup $G^\dagger$ to an arbitrary locally compact group G and call it the **Tits core** of G. It is defined as the subgroup of G generated by the closures of the contraction groups of all elements of G. Recall

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[*] F.R.S.-FNRS research associate, supported in part by the European Research Council (grant #278469).

[†] Supported in part by Australian Research Council Discovery Projects DP120100996.

[‡] Supported in part by Australian Research Council Discovery Projects DP0984342.

ISSN 0949–5932 / $2.50 © Heldermann Verlag

that the *contraction group* of an element $g \in G$ is given by

$$\mathrm{con}(g) = \{u \in G \mid g^n u g^{-n} \to 1 \text{ as } n \to +\infty\}.$$

Thus by definition we have

$$G^\dagger = \langle \overline{\mathrm{con}(g)} \mid g \in G \rangle.$$

The Tits core is clearly normal, and even topologically characteristic, in G, but it need not be closed *a priori.*

In case G is the group of k-rational points of a simple algebraic group $\mathbf{G}$ over a local field k, the contraction group of each element coincides with the unipotent radical of some k-defined parabolic subgroup, and is in particular closed (see [7, Lemma 2.4]). It follows that $G^\dagger = \mathbf{G}(k)^\dagger$ in this case; this justifies our choice of terminology for $G^\dagger$.

The main results of this paper may be viewed as analogues of the aforementioned theorem by Tits in the case of totally disconnected locally compact groups. For the sake of brevity, we shall write *t.d.l.c.* for *totally disconnected locally compact.*

Theorem 1.1. *Let G be a t.d.l.c. group and let D be a dense subgroup of G. If $G^\dagger$ normalises D, then $G^\dagger \leq D$.*

The following consequence is immediate by induction on the length of a subnormal chain:

Corollary 1.2. *Let G be a t.d.l.c. group. Then every dense subnormal subgroup of G contains the Tits core $G^\dagger$.*

We shall now specialise the above results to the case where the ambient group G is *topologically simple*, i.e. the only closed normal subgroups of G are the identity subgroup and the whole group. Any non-trivial normal subgroup of G is then clearly dense. More generally, since a closed subgroup has a closed normaliser, one sees by induction on the length of a subnormal chain that every non-trivial subnormal subgroup of G is dense. Consequently, if G is topologically simple, then Corollary 1.2 ensures that $G^\dagger$ is contained in every non-trivial subnormal subgroup. This yields the following.

Corollary 1.3. *Let G be a topologically simple t.d.l.c. group. If the Tits core $G^\dagger$ is non-trivial, then it is the smallest non-trivial subnormal subgroup of G.*

Notice that the very existence of a smallest normal subgroup in the abstract group underlying a topologically simple group has nothing obvious *a priori.* As in the aforementioned situation studied by Tits, we obtain a result of abstract simplicity, which is immediate from Corollary 1.3.

Corollary 1.4. *Let G be a t.d.l.c. group. If G is topologically simple, then the Tits core $G^\dagger$ is either trivial or abstractly simple.*

The proof of Theorem 1.1 is short, but it elaborates on tools developed in earlier papers (e.g. [1, 10, 13]) such as the *contraction group* and *tidy subgroups*, designed to study individual group elements in order to obtain global information. This global information offers a new approach to such vital and general questions as to whether compactly generated, topologically simple groups must be abstractly simple. A sign of the potential of this approach is that the results above have recently been used by T. Marquis, [6], in his proof that irreducible complete Kac–Moody groups over finite fields are abstractly simple, thereby settling the main remaining open case of a question of J. Tits going back to [9]. The results of the present paper also play a role in the study of compactly generated topologically simple t.d.l.c. groups undertaken in [2], and are notably used to prove that if such a group admits an open subgroup decomposing as a non-trivial direct product, then it is abstractly simple (see Theorem L in [2]).

An essential ingredient in proving Theorem 1.1 is an analysis of the behaviour of the contraction groups associated to a converging sequence of elements in G. In order to describe our results in this direction, we recall that the *nub* of the element g in a t.d.l.c. group G is defined by

$$\mathrm{nub}(g) = \overline{\mathrm{con}(g)} \cap \overline{\mathrm{con}(g^{-1})}.$$

The nub is a compact subgroup normalised by g; it coincides with the unique maximal compact subgroup of G normalised by g and on which g acts ergodically (see [13] and Theorem 2.2 below). A key property of the nub is that $\mathrm{nub}(g)$ is trivial if and only if $\mathrm{con}(g)$ is closed (see [1, Theorem 3.32] and [5]).

Theorem 1.5. *Let G be a t.d.l.c. group. For any $g \in G$, there is a filter basis of identity neighbourhoods $\mathcal{U}$, consisting of open compact subgroups, enjoying the following property. For all $U \in \mathcal{U}$ and $h \in gU$, there exist $r, t \in U$ such that*

$$\mathrm{con}(h) = t\mathrm{con}(g)t^{-1}; \ \mathrm{nub}(h) = r\mathrm{nub}(g)r^{-1}.$$

Denoting by $\mathcal{SUB}(G)$ the compact space of closed subgroups of G endowed with the Chabauty topology (see [3] or [4]), we obtain the following consequence.

Corollary 1.6. *Let G be a t.d.l.c. group. Then the maps*

$$G \to \mathcal{SUB}(G) : g \mapsto \overline{\mathrm{con}(g)} \quad \textit{and} \quad G \to \mathcal{SUB}(G) : g \mapsto \mathrm{nub}(g)$$

are both continuous.

Consideration of the map $g \mapsto \mathrm{con}(g)$ for the group $\mathbf{R}^* \ltimes \mathbf{R}$ shows that this corollary may fail when the group is not totally disconnected.

Acknowledgements. We thank the referee for his/her detailed comments, and for pointing out an inaccuracy in an earlier version of this article.

2. Preliminaries

Relatively little is known about the overall topological dynamics of a t.d.l.c. group acting on itself. By contrast, there is a well-developed theory of the dynamics of $\mathbf{Z}$ acting on a t.d.l.c. group by continuous automorphisms. We recall some basic concepts and results in this area which will be needed in the sequel; the main references are [1], [10] and [13]. Notice that the results from [1] are stated under the hypothesis that the ambient groups are metrisable; this hypothesis has been removed by Jaworski [5]; we shall therefore freely refer to the results of [1] without any further comment about metrisability.

For the rest of this section, fix a t.d.l.c. group G. We only allow automorphisms that preserve the topology of G.

Given an automorphism f of G, the *contraction group* $\mathrm{con}(f)$ of f on G is given by

$$\mathrm{con}(f) = \{u \in G \mid f^n(u) \to 1 \text{ as } n \to +\infty\}.$$

One sees that $\mathrm{con}(f)$ is a subgroup of G, although not necessarily a closed subgroup.

Given a subgroup U of G, define the following subgroups:

$$U_- := \bigcap_{i \leq 0} f^i(U); \quad U_+ := \bigcap_{i \geq 0} f^i(U); \quad U_0 := \bigcap_{i \in \mathbf{Z}} f^i(U);$$

$$U_{--} := \bigcup_{j \leq 0} f^j(U_-); \quad U_{++} := \bigcup_{j \geq 0} f^j(U_+).$$

Say U is *tidy above for* f if $U = U_+U_-$. Say U is *tidy below for* f if U_{--} and U_{++} are closed subgroups of G. A *tidy subgroup for* f is an open compact subgroup of G that is tidy both above and below for f.

Theorem 2.1. *Let U be an open compact subgroup of G.*

(i) *There is some integer k such that $\bigcap_{i=0}^k f^i(U)$ is tidy above for f. Consequently U_+U_- contains an open subgroup of U.*

(ii) *There is an open compact subgroup of G that is tidy for f.*

(iii) *We have $U_{--} = \mathrm{con}(f)U_0$ (and similarly $U_{++} = \mathrm{con}(f^{-1})U_0$). In particular, $\mathrm{con}(f) = \mathrm{con}(f^{-1}) = 1$ if and only if the set of open compact subgroups of G that are f-invariant forms a base of neighbourhoods of the identity.*

(iv) *We have $U_- = (\mathrm{con}(f) \cap U_-)U_0$ and $U_+ = (\mathrm{con}(f^{-1}) \cap U_+)U_0$.*

(v) *If in addition U is tidy for f, then we have $U_- = (\mathrm{con}(f) \cap U)U_0$ and $U_+ = (\mathrm{con}(f^{-1}) \cap U)U_0$.*

Proof. For (i) and (ii), see [10] and [11]. The first assertion of part (iii) is [1, Proposition 3.16]. The second follows from the first, combined with (i). Part (iv) follows from (iii) since $U_0 \leq U_- \leq U_{--}$, and similarly for U_+. Finally, for (v) we observe that by [10, Lemma 3] the group U_{--} is closed if and only if $U_{--} \cap U = U_-$. Thus if U is tidy for f, then (iii) yields $U_- = U_{--} \cap U = (\mathrm{con}(f) \cap U)U_0$ since $U_0 \leq U$. The proof for U_+ is similar. ■

A more recent development in the dynamical theory of automorphisms of t.d.l.c. groups is the *nub* of an automorphism. As usual, we will define the nub of a group element to be the nub of the corresponding inner automorphism. Before presenting several characterisations of the nub, we introduce one more subgroup associated with an automorphism f of a t.d.l.c. group G, namely the *parabolic group* $\mathrm{par}(f)$. It is defined to consist of those elements $x \in G$ such that the set $\{f^n(x) \mid n \in \mathbf{N}\}$ is relatively compact. By [10, Proposition 3], the parabolic subgroup $\mathrm{par}(f)$ is always closed.

Theorem 2.2. *Let G be a t.d.l.c. group and let f be an automorphism of G. Then there is an f-invariant compact subgroup $\mathrm{nub}(f)$ of G, which is equal to each of the following sets:*

(i) *the intersection of all tidy subgroups for f;*

(ii) *the closure of the set $\mathrm{bco}(f) := \mathrm{con}(f) \cap \mathrm{par}(f^{-1})$;*

(iii) *the intersection $\overline{\mathrm{con}(f)} \cap \mathrm{par}(f^{-1})$;*

(iv) *the intersection $\overline{\mathrm{con}(f)} \cap \overline{\mathrm{con}(f^{-1})}$;*

(v) *the intersection $\bigcap_V \overline{\mathrm{rbco}(f,V)}$, taken over all compact open subgroups V of G, where*

$$\mathrm{rbco}(f,V) := \{x \in \mathrm{par}(f^{-1}) \mid \exists N \; \forall n \geq N \; f^n(x) \in V\};$$

(vi) *the largest compact, f-stable subgroup of G having no relatively open f-stable subgroups;*

(vii) *the largest compact, f-stable subgroup of G on which f acts ergodically.*

Proof. Assertions (i), (ii), (v), (vi), (vii) are contained in [13] Theorem 4.12. For assertion (iii) and (iv), see respectively Lemma 3.29 and Corollary 3.27 in [1]. ■

3. Anisotropic groups

A t.d.l.c. group G with trivial Tits core will be called **anisotropic**; equivalently G is anisotropic if all its elements have trivial contraction group. This is certainly the case if G is discrete, or compact, or if every element of G is contained in a compact subgroup. Results from [1] and [10] imply that if a t.d.l.c. group G is anisotropic, then all its closed subgroups are unimodular (but not conversely). It should be emphasised that non-discrete topologically simple groups may also be anisotropic: indeed, the examples of topologically simple t.d.l.c. groups constructed in [12, §3] are all unions of ascending chains of compact open subgroups, hence anisotropic. Notice however that such groups are not compactly generated; it is in fact an interesting open problem to determine whether a non-compact compactly generated t.d.l.c. group without infinite discrete quotient can be anisotropic.

In any case, the Tits core is contained in the kernel of all anisotropic quotients of G. More precisely, results by Baumgartner–Willis [1] imply the following characterisation.

Proposition 3.1. *Let G be a t.d.l.c. group and N be a closed normal subgroup. Then G/N is anisotropic if and only if $G^\dagger \leq N$. In particular $\overline{G^\dagger}$ is the smallest closed normal subgroup of G affording an anisotropic quotient.*

Proof. It is clear that the image of $G^\dagger$ in G/N is contained in $(G/N)^\dagger$. Thus if G/N is anisotropic, then $G^\dagger \leq N$.

For each $g \in G$, we denote by $\mathrm{con}(g/N)$ the inverse image in G of the contraction group of gN in G/N. By [1, Theorem 3.8], we have $\mathrm{con}(g/N) = \mathrm{con}(g)N$. Therefore, if $\mathrm{con}(g/N)/N$ is non-trivial, then $\mathrm{con}(g)$ is not contained in N. In particular, if G/N is not anisotropic, then $G^\dagger$ is not contained in N. ■

4. Limits of contraction groups

The goal in this section is to prove Theorem 1.5 from the introduction. This will be achieved after a series of technical preparations.

We start with a lemma which strengthens [10, Lemma 14]. The original version is used there in the proof that, if U is tidy for g then it is also tidy for every element in gU and that the scale is constant on gU.

Lemma 4.1. *Let $g \in G$ and let U be an open compact subgroup of G that is tidy above for g. Then for every $u \in U$ there is $t \in U_+ \cap \mathrm{con}(g^{-1})$ such that, for every $k \geq 0$,*

$$t^{-1}(gu)^k t g^{-k} \in U.$$

Proof. It will be shown by induction that for every $n \geq 0$ there is $t_n \in U_+$ with

$$t_n^{-1}(gu)^k t_n = b_{n,k} g^k, \text{ with } b_{n,k} \in U, \text{ for every } k \in \{0, 1, \ldots, n\}. \tag{1}$$

The base case of the induction, when $n = 0$, is certainly true. Suppose that (1) has been established for n. The element t_{n+1} will be constructed as $t_{n+1} = t_n y$, where $y \in g^{-n} U_+ g^n$.

Notice first that, for any choice of $y \in g^{-n} U_+ g^n$ and all $k \in \{0, 1, \ldots, n\}$, we have

$$t_{n+1}^{-1}(gu)^k t_{n+1} = y^{-1} b_{n,k} g^k y = b_{n+1,k} g^k$$

where $b_{n+1,k} = y^{-1} b_{n,k} g^k y g^{-k}$ belongs to U. Now, to find suitable y, consider

$$t_{n+1}^{-1}(gu)^{n+1} t_{n+1} = y^{-1} t_n^{-1}(gu)(gu)^n t_n y = y^{-1} t_n^{-1} g (u t_n b_{n,n}) g^n y.$$

Since $u t_n b_{n,n}$ belongs to U which is tidy above for g, we have $u t_n b_{n,n} = w_- w_+$ with $w_\pm \in U_\pm$. Put $y = g^{-n} w_+^{-1} g^n$. Then

$$t_{n+1}^{-1}(gu)^{n+1} t_{n+1} = t_{n+1}^{-1} g w_- g^n = b_{n+1,n+1} g^{n+1}$$

with $b_{n+1,n+1} = y^{-1} t_n^{-1} g w_- g^{-1}$ in U as required.

Since U_+ is compact, the sequence $(t_n)_{n \in \mathbf{N}}$ has a subnet converging to an element $s \in U_+$. We see that

$$s^{-1}(gu)^k s g^{-k} \in U \quad \forall k \geq 0$$

for this choice of s. Moreover, note that we can freely replace s with sv given some $v \in U_0$, since U_0 is normalised by $\langle g \rangle$ and hence $[g^l, v] \in U$ for all $l \in \mathbf{Z}$. Now by Theorem 2.1(iv), we have

$$U_+ = (U_+ \cap \mathrm{con}(g^{-1})) U_0,$$

hence by a suitable choice of v, we obtain $t = sv \in U_+ \cap \mathrm{con}(g^{-1})$ with the desired property. ■

Corollary 4.2. *Let g, u and t be as in Lemma 4.1. Then $t\,\mathrm{con}(g) t^{-1} = \mathrm{con}(gu)$.*

Proof. For $k \geq 0$, there is $b_k \in U$ such that $(gu)^k t = t b_k g^k$. Therefore, for each $c \in \mathrm{con}(g)$, we have

$$(gu)^k (tct^{-1})(gu)^{-k} = tb_k(g^k c g^{-k})(tb_k)^{-1}.$$

The right side converges to the identity as $k \to \infty$ because tb_k belongs to U, which has a neighbourhood base of the identity comprising normal subgroups. Hence $t\mathrm{con}(g)t^{-1} \leq \mathrm{con}(gu)$. Similarly, given $c \in \mathrm{con}(gu)$, then, for $k \geq 0$,

$$g^k(t^{-1}ct)g^{-k} = b_k^{-1}t^{-1}((gu)^k c(gu)^{-k})tb_k$$

showing that $t^{-1}ct \in \mathrm{con}(g)$, so $\mathrm{con}(g) \geq t^{-1}\mathrm{con}(gu)t$ and hence $t\mathrm{con}(g)t^{-1} = \mathrm{con}(gu)$. ■

Lemma 4.3. *Let $g \in G$ and let U be an open compact subgroup of G that is tidy above for g. Then for every $u \in U \cap g^{-1}Ug$ there is $r \in U$ such that, for every $k \in \mathbf{Z}$,*

$$r^{-1}(gu)^k r = b_k g^k \text{ with } b_k \in U. \tag{2}$$

Proof. Applying Lemma 4.1 yields $t \in U_+$ such that (2) holds for $k \geq 0$ with $r = t$. Applying the lemma a second time replacing g by g^{-1} and u by $gu^{-1}g^{-1}$, which belongs to U, yields $s \in U_-$ such that (2) holds for $k \leq 0$ with $r = s$.

Since $s^{-1}t$ belongs to U, it is equal to v_+v_- with $v_\pm \in U_\pm$. Put $r = tv_-^{-1} = sv_+$. Then for $k \geq 0$

$$r^{-1}(gu)^k r = v_- t^{-1}(gu)^k t v_-^{-1} = v_- b_k g^k v_-^{-1} = b_k' g^k$$

with $b_k' = v_- b_k g^k v_-^{-1} g^{-k}$, which belongs to U, and for $k \leq 0$

$$r^{-1}(gu)^k r = v_+^{-1} s^{-1}(gu)^k s v_+ = v_+^{-1} b_k g^k v_+ = b_k' g^k$$

with $b_k' = v_+^{-1} b_k g^k v_+ g^{-k}$ which belongs to U. ■

Corollary 4.4. *Let g, u and r be as in Lemma 4.3. Then $r\mathrm{nub}(g)r^{-1} = \mathrm{nub}(gu)$.*

Proof. By Theorem 2.2 (ii) we have $\mathrm{nub}(g) = \overline{\mathrm{bco}(g)}$. It follows from Lemma 4.3 that $r\mathrm{bco}(g)r^{-1} = \mathrm{bco}(gu)$, whence the claim. ■

Proof. [Proof of Theorem 1.5] Fix $g \in G$ and let $\mathcal{U}$ be the set of open compact subgroups of G that are tidy above for g. The tidying procedure (Theorem 2.1(i)) ensures that $\mathcal{U}$ is a filter basis of identity neighbourhoods. Given $U \in \mathcal{U}$ and $u \in U$, Corollary 4.2 yields an element $t \in U_+$ such that $t\mathrm{con}(g)t^{-1} = \mathrm{con}(gu)$. Similarly, Corollary 4.4 yields an element $r \in U$ such that $r\mathrm{nub}(g)r^{-1} = \mathrm{nub}(gu)$. ■

Proof. [Proof of Corollary 1.6] Let $\varphi\colon G \to \mathcal{SUB}(G) : g \mapsto \overline{\mathrm{con}(g)}$. If φ were not continuous at some $g \in G$, there would be a neighbourhood V of $\varphi(g)$ in $\mathcal{SUB}(G)$ such that $\varphi^{-1}(V)$ is not a neighbourhood of g. Denoting by $\mathcal{U}$ the filter basis of identity neighbourhoods afforded by Theorem 1.5, we infer that for each $U \in \mathcal{U}$, there exists $h_U \in gU$ such that $\varphi(h_U) \notin V$. Now, Theorem 1.5 ensures that the net $(\varphi(h_U))_{U\in\mathcal{U}}$ converges to $\varphi(g)$ along the filter $\mathcal{U}$, a contradiction.

The proof of continuity of the map $G \to \mathcal{SUB}(G) : g \mapsto \mathrm{nub}(g)$ is similar. ∎

5. Normal closures

Given a subset A of a group G, we define the **normal closure** of A in G to be the smallest (abstract) normal subgroup of G containing A. The following result establishes a connection between the normal closure of an element f and the topological closure of its contraction group; it generalizes [13, Prop. 7.1].

Proposition 5.1. *Let G be a t.d.l.c. group and let A be an abstract subgroup of G. Given any $g \in A$, if $\overline{\mathrm{con}(g)}$ normalises A, then $\overline{\mathrm{con}(g)} \leq A$. In particular, any (abstract) normal subgroup of G containing g also contains $\overline{\mathrm{con}(g)}$.*

Proof. The closed subgroup $\overline{\mathrm{con}(g)}$ is normalised by g. We may therefore find a compact subgroup $U \leq \overline{\mathrm{con}(g)}$ which is open in $\overline{\mathrm{con}(g)}$ and tidy for g. Then U_{--} is a closed subgroup of $\overline{\mathrm{con}(g)}$ which contains $\mathrm{con}(g)$ by Theorem 2.1(iii), so that $\overline{\mathrm{con}(g)} = U_{--}$. Since $\mathrm{con}(g) \cap U$ is dense in U because U is open, it then follows from Theorem 2.1(v) that $U_{-} = U$. In particular $gUg^{-1} \leq U$.

Let $u \in U \cap \mathrm{con}(g)$. We define inductively a sequence $(v_n) \subset U \cap \mathrm{con}(g)$ by setting $v_0 = u$ and $v_n = v_{n-1}g^n u g^{-n}$ for all $n > 0$. The fact that $u \in \mathrm{con}(g)$ implies that (v_n) converges. Its limit v belongs to U since U is compact. By construction, we have $v = ugvg^{-1}$, so that $u = [v, g]$. Therefore, denoting the commutator map $G \to G : x \mapsto [x, g]$ by ψ, we deduce that $\psi(U)$ contains $U \cap \mathrm{con}(g)$. Since ψ is continuous and U is compact, it follows that $\psi(U)$ is closed. Therefore $\psi(U) = U$ since $U \cap \mathrm{con}(g)$ is dense in U.

By hypothesis, we have $g \in A$ and $\overline{\mathrm{con}(g)} \leq \mathrm{N}_G(A)$, so that $[x, g] \in A$ for any $x \in \overline{\mathrm{con}(g)}$. The previous paragraph therefore implies that $U \leq A$. We conclude that

$$\overline{\mathrm{con}(g)} = U_{--} = \bigcup_{j\leq 0} g^j U g^{-j} \leq A,$$

as desired. ∎

By combining Corollary 4.2 and Proposition 5.1, we arrive at a proof of Theorem 1.1.

Proof. [Proof of Theorem 1.1] Let $g \in G$. Since D is dense in G, there is a net (g_n) in D that converges to g. By hypothesis, we have $\overline{\mathrm{con}(g_n)} \leq \overline{G^\dagger} \leq \mathrm{N}_G(D)$ for all n. Applying Proposition 5.1 with $A = D$, we infer that $\overline{\mathrm{con}(g_n)}$ is contained in D for all n. Moreover, by Corollary 4.2, for some n there is $t \in \mathrm{con}(g^{-1})$ such that $t\mathrm{con}(g)t^{-1} = \mathrm{con}(g_n)$, and consequently $\overline{t\mathrm{con}(g)t^{-1}} = \overline{\mathrm{con}(g_n)}$. Since $t \in \mathrm{con}(g^{-1}) \leq \mathrm{N}_G(D)$ by assumption, it follows that $\overline{\mathrm{con}(g)} \leq D$. As g was arbitrary, we conclude that $G^\dagger \leq D$. ■

We point out the following refinement of Corollary 1.4.

Corollary 5.2. *Let G be a t.d.l.c. group. Then the following are equivalent:*

(i) *G is topologically simple, and* $\mathrm{con}(g) \neq 1$ *for some $g \in G$;*

(ii) *$G^\dagger$ is abstractly simple and dense in G, and the centre of G is trivial.*

Proof. Suppose that G is topologically simple and that $\mathrm{con}(g) \neq 1$ for some $g \in G$. Evidently G is non-abelian, so $\mathrm{Z}(G)$ is a proper closed normal subgroup of G, hence $\mathrm{Z}(G) = 1$. Moreover, $G^\dagger$ is a non-trivial normal subgroup of G, and in fact $G^\dagger$ must be dense in G by topological simplicity. Let N be a non-trivial normal subgroup of $G^\dagger$. Then $\mathrm{N}_G(\overline{N}) \geq \overline{(G^\dagger)} = G$, since the normaliser of any closed subgroup of G is closed; thus N is a dense subnormal subgroup of G, and thus $N = G^\dagger$ by Corollary 1.2. Hence $G^\dagger$ is abstractly simple.

Conversely, suppose that $G^\dagger$ is abstractly simple and dense in G, and that $\mathrm{Z}(G) = 1$. Then in particular $G^\dagger$ is non-trivial, so $\mathrm{con}(g) \neq 1$ for some $g \in G$. Moreover, given a closed normal subgroup N of G, then either $N \cap G^\dagger = 1$ or $N \geq G^\dagger$, and in the latter case $N = G$. But if $N \cap G^\dagger = 1$, then $G^\dagger \leq \mathrm{C}_G(N)$, so in fact N is central in G, using the fact that the centraliser of a subgroup is closed, and hence N is trivial by the assumption that $\mathrm{Z}(G) = 1$. Hence G is topologically simple. ■

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Received November 20, 2013
and in final form February 24, 2014