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On Certain Decompositions of Solvable Lie Algebras

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Abstract. The author has previously shown that solvable Lie A-algebras and complemented solvable Lie algebras decompose as a vector space direct sum of abelian subalgebras, and their ideals relate nicely to this decomposition. However, neither of these classes is contained in the other. The object of this paper is to find a larger class of algebras, containing each of these classes, in which these same results hold.
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1. Introduction

Throughout L will denote a finite-dimensional solvable Lie algebra over a field F. Then L is an A-*algebra* if all of its nilpotent subalgebras are abelian. We say that L is *complemented* if, for any subalgebra S of L, there is a subalgebra T such that $S \cap T = 0$ and $\langle S, T \rangle = L$. In [4] and [5] it was shown that the algebras in each of these classes decompose as a vector space direct sum of abelian subalgebras, and their ideals relate nicely to this decomposition.

More precisely, it was shown that they split over each term in their derived series. This leads to a decomposition of L as $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ where A_i is an abelian subalgebra of L and $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ for each $0 \leq i \leq n$. It is shown that the ideals of L relate nicely to this decomposition: if K is an ideal of L then $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \ldots \dot{+} (K \cap A_0)$. However, [5, Examples 3.8 and 3.9] showed that neither of these classes was contained in the other. It is natural to ask, therefore, if we can find a larger class of algebras, containing each of these classes, in which these same results hold. We show that the class $q\mathcal{A}$ defined in Section 3 satisfies this.

We define the *nilpotent residual*, L^∞, of L to be the smallest ideal of L such that L/L^∞ is nilpotent. Clearly this is the intersection of the terms of the lower central series for L. Then the *lower nilpotent series* for L is the sequence of ideals L_i of L defined by $L_0 = L$, $L_{i+1} = (L_i)^\infty$ for $i \geq 0$. The *derived series* for L is the sequence of ideals $L^{(i)}$ of L defined by $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ for

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$i \geq 0$; we will also write L^2 for $L^{(1)}$. If $L^{(n)} = 0$ but $L^{(n-1)} \neq 0$ we say that that L has *derived length* n.

We shall denote the nilradical of L by $N(L)$ and the centre of L by $Z(L)$. If A is a subalgebra of L and B is an ideal of A, the *centraliser* of A/B in L is the set $Z_L(A/B) = \{x \in L : [x, A] \subseteq B\}$. The *Frattini subalgebra* of L, $\phi(L)$, is the intersection of the maximal subalgebras of L. When L is solvable this is always an ideal of L, by [2, Lemma 3.4]. Algebra direct sums will be denoted by $\oplus$, whereas direct sums of the vector space structure alone will be denoted by $\dot{+}$.

2. Splitting over the derived series

Let $\mathcal{S}_n$ denote the class of solvable Lie algebras that split over $L^{(n)}$. A class $\mathcal{H}$ of finite-dimensional solvable Lie algebras is called a *homomorph* if it contains, along with an algebra L, all epimorphic images of L.

Lemma 2.1. *$\mathcal{S}_n$ is a homomorph.*

Proof. This is straightforward. ■

Then we have the following properties of $\mathcal{S}_n$.

Theorem 2.2. *Let L be a solvable Lie algebra. Consider the following properties.*

(i) $L \in \mathcal{S}_n$;

(ii) $[L^{(n)}, L^{(n-1)}] = L^{(n)}$; *and*

(iii) the Cartan subalgebras of $L^{(n-1)}/L^{(n+1)}$ *are precisely the complements in* $L^{(n-1)}/L^{(n+1)}$ *of* $L^{(n)}/L^{(n+1)}$.

Then (i) ⇒ (ii) ⇒ (iii). Moreover, if $L^{(n+1)} = 0$ *then all three statements are equivalent.*

Proof. (i) ⇒ (ii): Suppose that $L = L^{(n)} \dot{+} B$ for some subalgebra B of L. Then $L^{(n-1)} = L^{(n)} \dot{+} B^{(n-1)}$, so

$$[L^{(n-1)}, L^{(n)}] = L^{(n+1)} + [B^{(n-1)}, L^{(n)}] = [L^{(n)} + B^{(n-1)}, L^{(n)} + B^{(n-1)}] = L^{(n)},$$

since $B^{(n)} \subseteq L^{(n)} \cap B = 0$.

(ii) ⇒ (iii): From (i) we have that $(L^{(n-1)})^{\infty} = L^{(n)}$, so the result follows from [6, Theorem 4.4.1.1].

So suppose now that $L^{(n+1)} = 0$ and that (iii) holds. Let C be a Cartan subalgebra of $L^{(n-1)}$ and let $L = \mathcal{L}_0 \dot{+} \mathcal{L}_1$ be the Fitting decomposition of L relative to ad C. Then $\mathcal{L}_1 = \cap_{k=1}^{\infty} L(\text{ad}C)^k \subseteq L^{(n)}$ which is abelian, and so $\mathcal{L}_1$ is an ideal of L. Also $L^{(n-1)} = \mathcal{L}_1 \dot{+} \mathcal{L}_0 \cap L^{(n-1)} = \mathcal{L}_1 \dot{+} C$. Now C is abelian, by (iii). But $C \cong L^{(n-1)}/\mathcal{L}_1$, so $L^{(n)} \subseteq \mathcal{L}_1$, whence $\mathcal{L}_1 = L^{(n)}$ and $L = L^{(n)} \dot{+} \mathcal{L}_0$. ■

Let $\mathcal{S}_\infty$ denote the class of solvable Lie algebras that split over each term in their derived series.

Lemma 2.3. *$\mathcal{S}_\infty$ is a homomorph.*

Proof. This is straightforward. ∎

We have the following characterisation of $\mathcal{S}_\infty$.

Theorem 2.4. *Let L be a solvable Lie algebra of derived length $n+1$. Then the following are equivalent:*

(i) $L \in \mathcal{S}_\infty$*;*

(ii) $[L^{(i)}, L^{(i-1)}] = L^{(i)}$ *for every* $1 \leq i \leq n$*;*

(iii) the Cartan subalgebras of $L^{(i-1)}/L^{(i+1)}$ *are precisely the subalgebras that are complementary to* $L^{(i)}/L^{(i+1)}$ *for* $1 \leq i \leq n$*;*

(iv) the lower nilpotent series for L *coincides with the derived series for* L*; and*

(v) every factor algebra in the lower nilpotent series for L *is abelian.*

Proof. (i) ⇒ (ii): This follows from Theorem 2.2.

(ii) ⇒ (iii): This also follows from Theorem 2.2.

(iii) ⇒ (i): It follows from (iii) and Theorem 2.2 that L splits over $L^{(n)}$. So we have that $L = L^{(n)} \dot{+} B$ where B is a subalgebra of L. Clearly $B^{(n)} = 0$ and $B \cong L/L^{(n)}$, and so, by the same argument, B splits over $B^{(n-1)}$, say $B = B^{(n-1)} \dot{+} D$. But then $L = L^{(n)} \dot{+} (B^{(n-1)} \dot{+} D) = L^{(n-1)} \dot{+} D$. Continuing in this way gives the desired result.

(ii) ⇒ (iv): We require that $L_i = L^{(i)}$ for each $i \geq 0$. We use induction. Clearly $L_0 = L^{(0)} = L$. Suppose that $L_k = L^{(k)}$ for some $k \geq 0$. Then $L^{(k)}/L^{(k+1)}$ is abelian and so $L_{k+1} \subseteq L^{(k+1)}$. Moreover, $L^{(k+1)} = [L^{(k)}, L^{(k+1)}] \subseteq [L_k, L_k] = L_k^2$, whence $L^{(k+1)} = [L^{(k)}, L^{(k+1)}] \subseteq [L_k, L_k^2] = L_k^3$ and a straightforward induction proof shows that $L^{(k+1)} \subseteq (L_k)^\infty = L_{k+1}$, and (iv) follows.

(iv) ⇒ (v): This is clear.

(v) ⇒ (iv): We proceed as above. We require that $L_i = L^{(i)}$ for each $i \geq 0$. We use induction. Clearly $L_0 = L^{(0)} = L$. Suppose that $L_k = L^{(k)}$ for some $k \geq 0$. Then $L^{(k)}/L^{(k+1)}$ is abelian and so $L_{k+1} \subseteq L^{(k+1)}$. Moreover, $L^{(k)}/L_{k+1} = L_k/L_{k+1}$ is abelian and so $L^{(k+1)} \subseteq L_{k+1}$, and (iv) follows.

(iv) ⇒ (ii): Suppose that (iv) holds. Then

$$[L^{(i)}, L^{(i-1)}] = [L_i, L_{i-1}] = [(L_{i-1})^\infty, L_{i-1}] = (L_{i-1})^\infty = L_i = L^{(i)}.$$ ∎

This yields the next result, which is analogous to [4, Corollary 3.2] and [5, Corollary 3.3].

Corollary 2.5. *Let $L \in \mathcal{S}_\infty$ be a solvable Lie algebra of derived length $n+1$. Then the following hold:*

(i) $L = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_0$ *where A_i is an abelian subalgebra of L for each $0 \leq i \leq n$;*

(ii) $L^{(i)} = A_n \dot{+} A_{n-1} \dot{+} \ldots \dot{+} A_i$ *for each $0 \leq i \leq n$; and*

(iii) $[A_i, A_{i+1}] \subseteq A_{i+1}$ *and so $A_i + A_{i+1}$ is a subalgebra.*

Proof. By Theorem 2.4 there is a subalgebra B_n of L such that $L = L^{(n)} \dot{+} B_n$. Put $A_n = L^{(n)}$. Similarly $B_n \cong L/L^{(n)}$ satisfies the conditions of Theorem 2.4, by Lemma 2.1, so $B_n = A_{n-1} \dot{+} B_{n-1}$ where $A_{n-1} = B_n^{(n-1)}$. Continuing in this way we get (i). A straightforward induction proof shows (ii). Finally,

$$[A_i, A_{i+1}] \subseteq [B_{i+1}, B_{i+2}^{(i+1)}] \subseteq [B_{i+2}, B_{i+2}^{(i+1)}] \subseteq B_{i+2}^{(i+1)} = A_{i+1},$$

which establishes (iii). ■

Our final result in this section is a generalisation of [4, Theorem 3.3].

Lemma 2.6. *If $L \in \mathcal{S}_\infty$ then $Z(L^{(i)}) \cap L^{(i+1)} = 0$.*

Proof. Clearly we can assume that $L^{(i+1)} \neq 0$. We have that $L^{(i)} = L^{(i+1)} \dot{+} A_i$ from Corollary 2.5. Let $x = a_n + \ldots + a_{i+1} \in Z(L^{(i)}) \cap L^{(i+1)}$ where $a_j \in A_j$ for $i+1 \leq j \leq n$. Then $0 = [x, A_{i+1} + A_i]$, so $[a_{i+1}, A_{i+1} + A_i] \subseteq (A_{i+1} + A_i) \cap L^{(i+2)} = 0$. It follows that $a_{i+1} \in Z_{A_{i+1}+A_i}(A_i) = A_i$ since A_i is complementary to $A_{i+1} \cong L^{(i+1)}/L^{(i+2)}$ in $A_{i+1} + A_i \cong L^{(i)}/L^{(i+2)}$ and so is a Cartan subalgebra of $A_{i+1} + A_i$, by Theorem 2.4. Hence $a_{i+1} = 0$. But now $0 = [x, A_{i+2} + A_{i+1}]$, so $[a_{i+2}, A_{i+2} + A_{i+1}] \subseteq (A_{i+2} + A_{i+1}) \cap L^{(i+3)} = 0$, giving $a_{i+2} = 0$ as before. Continuing in this way shows that $x = 0$. ■

3. Quasi-A-algebras

A solvable Lie algebra L will be called a *quasi-A-algebra* if $N(L/B)$ is abelian for every ideal B of L. We denote the class of all such algebras by $q\mathcal{A}$. Clearly A-algebras are quasi-A-algebras. The following lemma shows that so are solvable complemented Lie algebras.

Lemma 3.1. *Let L be a solvable complemented Lie algebra. Then $L \in q\mathcal{A}$.*

Proof. Since L is complemented, all factor algebras of L are ϕ-free, by [5, Theorem 2.1 (iii)], and so have abelian nilradical by [3, Theorem 7.4]. ■

Lemma 3.2. *$q\mathcal{A}$ is a homomorph.*

Proof. This is clear from the definition. ■

Lemma 3.3. $q\mathcal{A} \subseteq \mathcal{S}_\infty$.

Proof. Let $L \in q\mathcal{A}$ and let L_i be the i^{th} term of the lower nilpotent series for L. Then L_i/L_{i+1} is a nilpotent ideal of L/L_{i+1} and so is abelian. It follows from Theorem 2.4 that $L \in \mathcal{S}_\infty$. ■

The following result can be proved in the same way as [5, Lemma 3.5]. It is also a generalisation of [4, Lemma 3.4].

Lemma 3.4. *Let $L \in q\mathcal{A}$ have derived length $\leq n+1$, and suppose that $L = B \dot{+} C$ where $B = L^{(n)}$ and C is a subalgebra of L. If D is an ideal of L then $D = (B \cap D) \dot{+} (C \cap D)$.*

The next result generalises [4, Theorem 3.5] and [5, Theorem 3.6].

Theorem 3.5. *Let $L \in q\mathcal{A}$ be a solvable Lie algebra of derived length $\leq n+1$, let K be an ideal of L, and A be a minimal ideal of L. Put $N_K/K = N(L/K)$, the nilradical of L/K, and $N = N_{\{0\}}$. Then the following hold:*

(i) $K = (K \cap A_n) \dot{+} (K \cap A_{n-1}) \dot{+} \ldots \dot{+} (K \cap A_0)$ *for all ideals K of L;*

(ii) $N_K = A_n \oplus (N_K \cap A_{n-1}) \oplus \ldots \oplus (N_K \cap A_0)$*;*

(iii) $Z_{L^{(i)}}((L^{(i)} + K)/K) = N_K \cap A_i$ *for each $0 \leq i \leq n$; and*

(iv) $A \subseteq N \cap A_i$ *for some $0 \leq i \leq n$.*

Proof. The proofs of (i), (ii) and (iv) are the same as those in [5, Theorem 3.6 (i), (ii) and (iv)].
(iii) Since the hypothesis is factor algebra closed we can assume that $K = 0$; that is, it suffices to show that $Z(L^{(i)}) = N \cap A_i$. This follows as in [5, Theorem 3.6 (iii)]. ■

The following is a straightforward consequence of the above result.

Corollary 3.6. *Let $L \in \mathcal{S}_\infty$ be a solvable Lie algebra of derived length $\leq n+1$. Then $L \in q\mathcal{A}$ if and only if the following hold for every ideal K of L:*

(i) $N_K = A_n \oplus (N_K \cap A_{n-1}) \oplus \ldots \oplus (N_K \cap A_0)$*; and*

(ii) $Z_{L^{(i)}}((L^{(i)} + K)/K) = N_K \cap A_i$ *for each $0 \leq i \leq n$,*

where $N_K/K = N(L/K)$, the nilradical of L/K.

4. Completely solvable algebras

A Lie algebra L is called *completely solvable* if L^2 is nilpotent. Over a field of characteristic zero every solvable Lie algebra is completely solvable.

Lemma 4.1. *Let $L \in \mathcal{S}_n$, where $n \geq 2$, be a completely solvable Lie algebra. Then $L^{(n)} = 0$.*

Proof. We have $L = L^{(n)} \dot{+} B$ for some subalgebra B of L. Since L^2 is nilpotent and $n \geq 2$, $L^{(n)} \subseteq L^{(2)} \subseteq \phi(L)$, by [3, Corollary 4.2 and section 5]. Hence $L = \phi(L) + B = B$ and $L^{(n)} = 0$. ∎

Clearly then completely solvable Lie algebras $L \in \mathcal{S}_2$ are metabelian (that is, $L^{(2)} = 0$) so we get stronger versions of Theorems 2.4 and 3.5.

Theorem 4.2. *Let L be a completely solvable Lie algebra. Then the following are equivalent:*

(i) $L \in \mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}_\infty$;

(ii) $L^{(2)} = 0$ *and* $[L, L^2] = L^2$; *and*

(iii) $L^{(2)} = 0$ *and the Cartan subalgebras of L are precisely the subalgebras that are complementary to L^2.*

Proof. This is straightforward. ∎

Note that we can't replace $\mathcal{S}_2$ by $\mathcal{S}_n$ for any $n \neq 2$ in the above result, as the following example shows.

Example 4.3. Let L be the four-dimensional Lie algebra spanned by e_1, e_2, e_3, e_4 where $[e_1, e_2] = e_3$, $[e_1, e_3] = e_2$ and $[e_2, e_3] = e_4$. Then $L^2 = Fe_2 + Fe_3 + Fe_4$ is nilpotent, $L^{(2)} = Fe_4$ and $L \in \mathcal{S}_n$ for $n \neq 2$, but $L \notin \mathcal{S}_2$. Moreover, $L^{(2)} = Z(L)$ and $Fe_1 + Fe_4$ is a Cartan subalgebra of L.

Theorem 4.4. *Let $L \in q\mathcal{A}$ be a completely solvable Lie algebra with $L = L^2 \dot{+} B$ where B is a subalgebra of L, let K be an ideal of L and let A be a minimal ideal of L. Put $N_K/K = N(L/K)$, the nilradical of L/K. Then*

(i) $K = K \cap L^2 \dot{+} K \cap B$;

(ii) $N_K = L^2 \oplus Z_B(L/K)$; *and*

(iii) $A \subseteq L^2$ *and* $[A, L] = A$, *or* $A \subseteq B$ *and* $A \subseteq Z(L)$ *(in which case* $\dim A = 1$*).*

Proof. This follows easily from Theorem 3.5. ∎

Corollary 4.5. *Let $L \in \mathcal{S}_\infty$ be a completely solvable Lie algebra with $L = L^2 \dot{+} B$ where B is a subalgebra of L. Then $L \in q\mathcal{A}$ if and only if $N_K = L^2 \oplus Z_B(L/K)$ for all ideals K of L, where $N_K/K = N(L/K)$, the nilradical of L/K.*

Theorem 4.6. *Let L be a completely solvable Lie algebra. Then $L \in q\mathcal{A}$ if and only if L is an A-algebra.*

Proof. Suppose first that $L \in q\mathcal{A}$ and let U be a maximal nilpotent subalgebra of L. Let $L = \mathcal{L}_0 \dot{+} \mathcal{L}_1$ be the Fitting decomposition of L relative to $\operatorname{ad} U$. Then $\mathcal{L}_1 \subseteq L^2$, so $\mathcal{L}_1$ is an abelian ideal of L and $\mathcal{L}_0 \cong L/\mathcal{L}_1 \in q\mathcal{A}$. Thus we can work within $\mathcal{L}_0$; that is, we may assume that $\mathcal{L}_0 = L$.

For $x = u + v$ where $u \in U$ and $v \in L^2$, we have $(\operatorname{ad} x) \mid_{L^2} = (\operatorname{ad} u) \mid_{L^2}$, since L^2 is abelian. It follows that $\operatorname{ad} x$ is nilpotent for all $x \in U + L^2$. Thus, $U + L^2$ is a nilpotent ideal of L. Since $L \in q\mathcal{A}$, $U + L^2$ is abelian.

The converse is clear. ∎

The above result does not hold for all solvable Lie algebras, as complemented Lie algebras belong to $q\mathcal{A}$ and these are not all A-algebras (see, [5, Example 3.9]).

The classes $\mathcal{S}_\infty$ and $q\mathcal{A}$ are different even in the case of completely solvable Lie algebras, as the following example shows.

Example 4.7. Let $L = Fx_1 + Fx_2 + Fx_3 + Fx_4$ with $[x_4, x_1] = x_1$, $[x_4, x_2] = x_2$ and $[x_3, x_1] = x_2$, other products being zero. Then $L^2 = Fx_1 + Fx_2$, so $L \in \mathcal{S}_\infty$. However, $N = L^2 + Fx_3$, which is not abelian, and so $L \notin q\mathcal{A}$. Notice that $\phi(L) = Fx_2$.

The two classes do coincide for ϕ-free algebras as the next result shows. The *abelian socle*, $\operatorname{Asoc} L$, of L is the sum of the minimal abelian ideals of L.

Theorem 4.8. *Let L be a completely solvable Lie algebra. Then*

(i) L is ϕ-free only if $L \in q\mathcal{A}$; and

(ii) L is ϕ-free if and only if $L^2 \subseteq \operatorname{Asoc} L$ and $L \in \mathcal{S}_\infty$.

Proof. (i) Suppose that L is ϕ-free. Then L is complemented by [5, Theorem 2.2] and so $L \in q\mathcal{A}$ by Lemma 3.1.

(ii) Suppose first that L is ϕ-free. Then $L^2 \subseteq N = \operatorname{Asoc} L$, by [3, Theorem 7.4]. Also $L \in \mathcal{S}_\infty$ by [3, Lemma 7.2].

So suppose now that $L^2 \subseteq \operatorname{Asoc} L$ and $L \in \mathcal{S}_\infty$. Then L splits over $\operatorname{Asoc} L$ by Theorem 2.2. But now L is ϕ-free by [3, Theorem 7.3]. ∎

Part (i) of the above result does not hold if L is not completely solvable, as the following example shows.

Example 4.9. Let L be as in Example 4.7 above, considered over a field of characteristic p, let B be a faithful completely reducible L-module and put $X = B \dot{+} L$ where $B^2 = 0$ and L acts on B under the given L-module action. Then $B \subseteq \operatorname{Asoc} X$, since B is completely reducible as an X-module. Furthermore, $N(X) \subseteq C_X(B) = B$ since B is faithful. It follows that X splits over $\operatorname{Asoc} X = N(X) = B$, and hence that $\phi(X) = 0$, by [3, Theorem 7.3]. Moreover, $X^2 = B \dot{+} Fx_1 + Fx_2$ and $X^{(2)} = B$, so $X \in \mathcal{S}_\infty$. However, $X \notin q\mathcal{A}$ since $L \notin q\mathcal{A}$.

Notice further that if X is as in Example 4.9 then $X \in \mathcal{S}_\infty$ and (ii) and (iii) of Theorem 3.5 hold with $K = 0$, but $X \notin q\mathcal{A}$.

5. ϕ-free $q\mathcal{A}$-algebras

Following on from Theorem 4.8, in this section we seek a characterisation of ϕ-free $q\mathcal{A}$-algebras. A Lie algebra L is called *monolithic* if it has a unique minimal ideal W, the *monolith* of L. Then the following result can be proved in the same way as [4, Theorem 5.1].

Theorem 5.1. *Let $L \in q\mathcal{A}$ be a monolithic solvable Lie algebra of derived length $n+1$ with monolith W. Then, with the same notation as Corollary 2.5,*

(i) W is abelian;

(ii) $Z(L) = 0$ and $[L, W] = W$;

(iii) $N = A_n = L^{(n)}$;

(iv) $N = Z_L(W)$; and

(v) L is ϕ-free if and only if $W = N$.

Corollary 5.2. *Let $L \in q\mathcal{A}$ be a monolithic supersolvable Lie algebra. Then $L = L^2 \dot{+} Fx$ where $(\operatorname{ad} x)|_{L^2}$ is triangulable.*

Proof. We have $L = L^2 \dot{+} B$ for some abelian subalgebra B of L. Also $\dim W = 1$ and so $\dim L/L^2 = \dim L/N = \dim L/Z_L(W) \leq 1$. It follows that $L = L^2 \dot{+} Fx$, and $(\operatorname{ad} x)|_{L^2}$ is triangulable since L is supersolvable. ∎

Corollary 5.3. *Let $L \in q\mathcal{A}$ be a ϕ-free monolithic solvable Lie algebra of index $n+1$. Then $L = L^{(n)} \rtimes B$ is the semidirect product of $L^{(n)}$ and a solvable Lie algebra $B \in q\mathcal{A}$ of index n.*

Proof. Simply note that $L^{(n)} = Z_L(L^{(n)})$, by Theorem 5.1, and $L = L^{(n)} \dot{+} B$ where $B \cong L/Z_L(L^{(n)})$, which is isomorphic to an irreducible subalgebra of $gl(L^{(n)})$. ∎

The next result gives us a way of constructing a ϕ-free monolithic $q\mathcal{A}$ algebra of index $n+1$ from a $q\mathcal{A}$ algebra of index n.

Theorem 5.4. *Let $B \in q\mathcal{A}$ be a solvable Lie algebra and let A be an irreducible B-module. Then $L = A \rtimes B$ is a solvable Lie algebra with $L \in q\mathcal{A}$. Moreover, if A is faithful and B has index n, then L is ϕ-free, monolithic and has index $n+1$.*

Proof. Let K be an ideal of L, and let $N_K/K = N(L/K)$. Suppose first that $A \subseteq K$. Then $N_K/K \cong (N_K/A)/(K/A)$ which yields that $(N_K)^2 \subseteq K$ since $L/A \in q\mathcal{A}$.

So suppose now that $A \not\subseteq K$. Then $[N_K, A] = A$ implies that $A \subseteq (N_K)^r$ for every $r \geq 2$, and hence that $A \subseteq K$, a contradiction. It follows that

$[N_K, A] = 0$, from the irreducibility of A. Clearly $(N_K \cap B)^2 \subseteq K \cap B$, since $B \in q\mathcal{A}$. Hence $(N_K)^2 = (A + N_K \cap B)^2 \subseteq K$. We therefore have that $L \in q\mathcal{A}$.

Next assume that A is faithful and B has index n. Clearly $L^{(i)} = A + B^{(i)}$ for $1 \leq i \leq n$, so L has index $n+1$. If K is an ideal of L, then $A \cap K = 0$ or $A \cap K = A$ from the irreducibility of A. The former implies that $[A, K] = 0$, so $K \subseteq Z_L(A) = A$, a contradiction. It follows that $A \subseteq K$ and A is the monolith of L. If $\phi(L)$ is non-zero then $A \subseteq \phi(L)$ which yields $L = \phi(L) + B = B$, a contradiction. Hence L is ϕ-free. ■

Our final result gives the characterisation of ϕ-free $q\mathcal{A}$ algebras that we are seeking.

Theorem 5.5. *Let L be a solvable Lie algebra of with nilradical N. Then the following are equivalent:*

(i) $L \in q\mathcal{A}$ and is ϕ-free; and

(ii) N is abelian and $L = N \rtimes B$, where $B \in q\mathcal{A}$ and N is a completely reducible B-module.

Furthermore, L has index $n+1$ if B has index n and N has a faithful irreducible submodule.

Proof. (i) $\Rightarrow$ (ii): Assume (i). Since L is ϕ-free, $N = \operatorname{Asoc} L$ and $L = N \dot{+} B$, where B is a subalgebra of L, by [3, Theorems 7.3 and 7.4]. Moreover, $Z_L(N) = N$, so $B \cong L/Z_L(N)$ is isomorphic to a subalgebra of $gl(N)$ and N is faithful. Also, N is completely reducible, since $N = \operatorname{Asoc} L$. Finally, if A is a faithful irreducible submodule of N, then $L^{(i)} \supseteq A + B^{(i)}$ for $1 \leq i \leq n$, and so L has index one greater than that of B.
(ii) $\Rightarrow$ (i): So now suppose that (ii) holds. Since N is abelian and completely reducible, $N = \operatorname{Asoc} L$ and L is ϕ-free, by [3, Theorem 7.3].

Let $\operatorname{Asoc} L = A_1 \oplus \ldots \oplus A_r$, where A_i is a minimal ideal of L and put $L_0 = B$, $L_i = A_1 \oplus \ldots \oplus A_i + B$ for $1 \leq i \leq r$. Then A_i is an irreducible (L_{i-1})-module for $1 \leq i \leq r$. It follows from Theorem 5.4 and a simple induction argument that $L \in q\mathcal{A}$.

If B has index n and N has a faithful irreducible submodule, then L has index $n+1$ as above. ■

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