

## Existence of Lattices on General $H$ -Type Groups

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Communicated by S. Gindikin

**Abstract.** Let  $\mathcal{N}$  be a two step nilpotent Lie algebra endowed with non-degenerate scalar product  $\langle \cdot, \cdot \rangle$  and let  $\mathcal{N} = V \oplus_{\perp} Z$ , where  $Z$  is the center of the Lie algebra and  $V$  its orthogonal complement with respect to the scalar product. We prove that if  $(V, \langle \cdot, \cdot \rangle_V)$  is the Clifford module for the Clifford algebra  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$  such that the homomorphism  $J: \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \rightarrow \text{End}(V)$  is skew symmetric with respect to the scalar product  $\langle \cdot, \cdot \rangle_V$ , or in other words the Lie algebra  $\mathcal{N}$  satisfies conditions of general  $H$ -type Lie algebras [6, 13], then there is a basis with respect to which the structural constants of the Lie algebra  $\mathcal{N}$  are all  $\pm 1$  or 0.

*Mathematics Subject Classification 2000:* 17B30, 22E25.

*Key Words and Phrases:* Clifford module, nilpotent two step algebra, lattice, general  $H$ -type algebras.

### 1. Introduction and definitions

We denote by  $\langle \cdot, \cdot \rangle_V$  a real valued symmetric non-degenerate bi-linear form defined on a real vector space  $V$  and call it a scalar product. If the form is positive definite, we denote it by  $(\cdot, \cdot)$  and call an inner product. We use the notation  $[\cdot, \cdot]$  for commutator, or in other words for a skew symmetric bi-linear vector valued form. The  $H$ -type Lie algebras were introduced by A. Kaplan in [18] and were widely studied, see, for instance [5, 9, 11, 19, 21]. In works [6, 8, 13] a generalisation of classical  $H$ -type Lie algebras were introduced and studied.

**Definition 1.1.** A 2-step nilpotent Lie algebra  $\mathcal{N}$  endowed with a scalar product  $\langle \cdot, \cdot \rangle$  is called a general  $H$ -type algebra, if

1.  $\mathcal{N} = V \oplus_{\perp} Z$ , where  $Z$  is the center of the Lie algebra  $\mathcal{N}$ , which is non-degenerate subspace of the scalar product vector space  $(\mathcal{N}, \langle \cdot, \cdot \rangle)$ , and  $V$

its orthogonal complement,

2. the skew symmetric map  $J: Z \rightarrow \text{End}(V)$  defined by

$$\langle J_z u, v \rangle = \langle z, [u, v] \rangle \tag{1}$$

satisfies the orthogonality condition

$$\langle J_z u, J_z v \rangle = \langle z, z \rangle \langle u, v \rangle, \quad \text{for all } z \in Z, \quad u, v \in V. \tag{2}$$

Conditions 1 and 2 imply

$$J_z^2 = -\langle z, z \rangle \text{Id}_V \tag{3}$$

see, for example, [6, 13, 23].

Due to (3) the space  $V$  becomes a  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ -module, where  $\langle \cdot, \cdot \rangle_Z$  is the restriction of the scalar product  $\langle \cdot, \cdot \rangle$  onto the center  $Z$ . So, from the definition we see that any general  $H$ -type algebra  $\mathcal{N}$  defines a  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ -module  $V$ . Moreover, the module  $V$  is endowed with the scalar product  $\langle \cdot, \cdot \rangle_V$ , obtained by the restriction of  $\langle \cdot, \cdot \rangle$  on  $V$ , such that the representations  $J_z$  are skew symmetric with respect to the scalar product  $\langle \cdot, \cdot \rangle_V$  for any  $z \in Z$ .

If, from the other side, we assume that  $V$  is a Clifford module for some Clifford algebra  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ , and  $V$  carried a scalar product  $\langle \cdot, \cdot \rangle_V$  such that (2) holds, then  $J$  is skew symmetric with respect to  $\langle \cdot, \cdot \rangle_V$ :

$$\langle J_z u, v \rangle_V = -\langle u, J_z v \rangle_V. \tag{4}$$

Therefore, one can define the Lie bracket  $[\cdot, \cdot]: V \times V \rightarrow Z$  by using (1) with the scalar product  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V + \langle \cdot, \cdot \rangle_Z$  and show that the Lie algebra  $\mathcal{N} = (V \oplus_{\perp} Z, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$  is a general  $H$ -type algebra, see [6, 9, 13, 19].

In general, among the conditions (2), (3), and (4) any two of them imply the third one. We say that a  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ -module  $V$  is an *admissible module*, if there is a scalar product  $\langle \cdot, \cdot \rangle_V$  defined on  $V$  such that the representations  $J_z: \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \rightarrow \text{End}(V)$ , satisfy the skew symmetry condition (4) for any  $z \in Z$ . The scalar product  $\langle \cdot, \cdot \rangle_V$  will be called an *admissible scalar product*.

In the case of a positive definite scalar product  $(\cdot, \cdot)_Z$  for every irreducible Clifford module  $V$  of the Clifford algebra  $\text{Cl}(Z, (\cdot, \cdot)_Z)$  it is always possible to find an inner product  $(\cdot, \cdot)_V$  on  $V$  such that, the representations  $J_z$  satisfy the orthogonality condition (2). The Lie algebra  $\mathcal{N} = V \oplus_{\perp} Z$ , where the Lie bracket is defined in (1) by making use of skew symmetric maps  $J_z$  and the inner product on  $\mathcal{N}$  is the sum of inner products on  $V$  and  $Z$ , is the  $H$ -type algebra introduced by A. Kaplan in [18]. We call such an algebra a *classical  $H$ -type Lie algebra*.

Let  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$  be a Clifford algebra generated by an indefinite scalar product  $\langle \cdot, \cdot \rangle_Z$ . It was shown in [6] that given a  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ -module  $V$  there always exists a scalar product  $\langle \cdot, \cdot \rangle_V$  on  $V$  (or on  $V \oplus V$ ), such that the map  $J_z: Z \rightarrow \text{End}(V)$  (or a modified map  $\tilde{J}_z: Z \rightarrow \text{End}(V \oplus V)$ ) satisfies (2), or equivalently (4), for an arbitrary  $z \in Z$ . As a consequence, we obtain that for any Clifford algebra  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$  there exists an admissible module  $(V, \langle \cdot, \cdot \rangle_V)$  (or  $(V \oplus V, \langle \cdot, \cdot \rangle_{V \oplus V})$ ). Moreover, the admissible module  $(V, \langle \cdot, \cdot \rangle_V)$  (or  $(V \oplus V, \langle \cdot, \cdot \rangle_{V \oplus V})$ ) will be necessarily a neutral space, see Proposition 2.1 and [6]. The corresponding 2-step nilpotent Lie algebra satisfies Definition 1.1 and is called general  $H$ -type algebra.

Now, let  $N$  be a simply connected, nilpotent Lie group and  $\Gamma$  its discrete subgroup such that the quotient space  $\Gamma \backslash N$  is compact. Then the group  $\Gamma$  is called *lattice* and quotient  $\Gamma \backslash N$  is called a *compact nilmanifold*, see, for instance, [12, 20]. Nilmanifolds, as a generalization of higher dimensional tori, play important role in study of the sub-Riemannian geometry, the Riemannian geometry with singularities, hypoelliptic operators, and spectral properties of differential operators of the Grushin type, see for example [2, 3, 4, 15, 17, 28]. Compact sub-Riemannian manifolds coming from simple Lie groups were studied in [1, 7]. According to the Mal'cev criterium [25] a nilpotent Lie group  $N$  admits a lattice if and only if the corresponding Lie algebra of  $N$  has a basis with rational structure constants. Not all, even 2-step, nilpotent Lie algebras admit such a basis. In the work [10] it was shown that classical  $H$ -type Lie algebras  $\mathcal{N}$  have integer structure constants, or more precisely, there is a basis  $\{v_1, \dots, v_m\}$  of  $V$  and a basis  $\{z_1, \dots, z_n\}$  of  $Z$ , such that  $[v_\alpha, v_\beta] = \sum_{k=1}^n A_{\alpha\beta}^k z_k$ , where the numbers  $A_{\alpha\beta}^k$  equal 0, or  $\pm 1$ . So it is be natural to ask whether general  $H$ -type algebras have such a basis too. In the present work we show the following statement.

**Theorem 1.2.** *Let  $(\mathcal{N} = V \oplus_\perp Z, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$  be a general  $H$ -type algebra. Then there is an orthonormal basis  $\{v_1, \dots, v_m, z_1, \dots, z_n\}$  of  $V \oplus_\perp Z$  such that  $[v_\alpha, v_\beta] = \sum_{k=1}^n A_{\alpha\beta}^k z_k$ , where the coefficients  $A_{\alpha\beta}^k$  are equal to  $\pm 1$ , or 0.*

Denote by  $\langle \cdot, \cdot \rangle_V$  and  $\langle \cdot, \cdot \rangle_Z$  the restrictions of the scalar product to the subspaces  $V$  and  $Z$ , and assume that the scalar product space  $(V, \langle \cdot, \cdot \rangle_V)$  and thus  $(Z, \langle \cdot, \cdot \rangle_Z)$  are nondegenerate. Let  $\nu_\alpha^V = \langle v_\alpha, v_\alpha \rangle_V$ ,  $\alpha = 1, \dots, m$  and  $\nu_k^Z = \langle z_k, z_k \rangle_Z$ ,  $k = 1, \dots, n$  be the corresponding indices. Let  $J_{z_k}: \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \rightarrow \text{End}(V)$  be representations of orthonormal generators  $z_1, \dots, z_n$  of  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ . We write

$$[v_\alpha, v_\beta] = \sum_{k=1}^n A_{\alpha\beta}^k z_k \quad \text{and} \quad J_{z_k} v_\alpha = \sum_{\beta=1}^m B_{\alpha\beta}^k v_\beta. \tag{5}$$

Then, as a consequence of (1) and (5), we obtain

$$\langle J_{z_k} v_\alpha, v_\beta \rangle_V = \langle z_k, [v_\alpha, v_\beta] \rangle_Z \implies \nu_\beta^V B_{\alpha\beta}^k = \nu_k^Z A_{\alpha\beta}^k. \tag{6}$$

Therefore the result of Theorem 1.2 can be reformulated as follows.

**Theorem 1.3.** *Given a scalar product space  $(Z, \langle \cdot, \cdot \rangle_Z)$  with an orthonormal basis  $z_1, \dots, z_n$  there is an admissible Clifford  $\text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ -module  $(V, \langle \cdot, \cdot \rangle_V)$  of minimal dimension with representations  $J: \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z) \rightarrow \text{End}(V)$  and an orthonormal basis  $v_1, \dots, v_m$  on  $(V, \langle \cdot, \cdot \rangle_V)$ , such that  $J_z$  satisfies (2), (3) and moreover,*

$$\langle J_{z_k} v_\alpha, v_\beta \rangle_V = \pm 1, \text{ or } 0, \text{ for all } k = 1, \dots, n, \alpha, \beta = 1, \dots, m. \tag{7}$$

In the following we always use the identification  $\text{Cl}_{r,s} \cong \text{Cl}(Z, \langle \cdot, \cdot \rangle_Z)$ , arising from the isomorphism  $(Z, \langle \cdot, \cdot \rangle_Z) \cong \mathbb{R}^{r,s}$ . Here  $\mathbb{R}^{r,s} = (\mathbb{R}^{r+s}, \mathbb{Q}_{r,z})$ , with  $\mathbb{Q}_{r,z}(x) = \sum_{i=1}^r x_i^2 - \sum_{i=r+1}^{r+s} x_i^2$ .

We call  $\text{Cl}_{r,s}$ -modules, satisfying Theorem 1.3 *admissible integral modules* and the corresponding orthonormal basis  $\{v_\alpha\}$  *integral basis*. The existence of admissible integral  $\text{Cl}_{r,0}$ -modules was shown in [10]. The admissible integral  $\text{Cl}_{r,0}$ -modules lead to the presents of a lattice on classical  $H$ -type groups. Notice also, that in the work [12], the existence of a rational structure constants on classical  $H$ -type algebras was shown by realizing its Lie algebra as a direct sum of the space  $\mathbb{R}^m$  and the center  $Z$ , given as a *Lie triple system* of  $\mathfrak{so}(m)$ .

*In the present work we construct explicitly an orthonormal basis of any minimal dimensional admissible  $\text{Cl}_{r,s}$ -module with respect to which the structure constants  $A_{\alpha\beta}^k$  defined in (5), or equivalently in (6), equal to  $\pm 1$  or 0.* There are several methods of construction of such an integral basis. To be able to use the Bott 8-periodicity it is necessary to construct 64 admissible integral modules of minimal dimension. We use the isomorphism  $\text{Cl}_{r,s+1} \cong \text{Cl}_{s,r+1}$  that preserves the integral basis, we take the tensor products with  $\text{Cl}_{4,4}$ -module or with  $\text{Cl}_{1,1}$ -module, we construct  $\text{Cl}_{r,1}$ -module from  $\text{Cl}_{r,0}$ -module and reduce the number of required modules to 12. To explain the main idea for the construction of the integral basis in the remaining cases we recall the terminology. A vector  $v \in V$  is called *spacelike* if  $\langle v, v \rangle_V > 0$  or  $v = 0$ ; *timelike* if  $\langle v, v \rangle_V < 0$ ; *null* if  $\langle v, v \rangle_V = 0$ . A linear map  $P: V \rightarrow V$  is called *involution* if  $P^2 = \text{Id}_V$  and *anti- $\text{involution}$*  if  $P^2 = -\text{Id}_V$ . We say that a bijective linear map  $T: V \rightarrow V$  is an *isometry* if  $\langle Tv, Tv \rangle_V = \langle v, v \rangle_V$ , and it is an *anti-isometry* if  $\langle Tv, Tv \rangle_V = -\langle v, v \rangle_V$  for all vectors  $v \in V$ . The principal method for the construction of integral bases starts by picking up a maximal number of mutually commuting isometric involutions together with “complementary” isometries or anti-isometries

satisfying some commutation relations with the original involutions. These choice of involutions and complementary operators give an orthogonal decomposition of the representation space for the Clifford algebra. Choosing a common spacelike eigenvector of the original involutions we construct an integral basis by means of Clifford multiplication. There are several differences in the construction of orthogonal decompositions of the representation spaces by those involutions and complementary operators. The purpose of the present work is, not only to show the existence of an integral structure for all  $\text{Cl}_{r,s}$ -modules, but also to present several possible methods for such kind of constructions, especially for the cases of low dimensions.

The structure of this paper is as follows: Section 2 is an auxiliary section where we collected the information about properties of admissible modules and auxiliary technical lemmas. In Section 3 we prove that the isomorphism between Clifford algebras  $\text{Cl}_{r,s+1}$  and  $\text{Cl}_{s,r+1}$  preserves the admissible integral modules. This isomorphism reduces significantly the number of the Clifford modules where we need to construct integral basis before we are able to apply the Bott periodicity. In Section 4 we construct a minimal admissible module of  $\text{Cl}_{r,1}$  with an integral basis basing on the existence of integral basis for the algebra  $\text{Cl}_{r,0}$ . Section 5 devoted to the construction of integral structures on admissible  $\text{Cl}_{r,s}$ -modules for  $0 \leq r, s \leq 8$  with  $r + s \leq 8$ . In Section 5 we actively develop a method of the simultaneous orthogonal decomposition of eigenspaces for a collection of mutually commuting isometric involutions. We also exploit results of Sections 3 and 4. In Section 6 we prove some theorems that allow to use the Bott periodicity of Clifford algebras for construction of integral bases. We also construct admissible modules with integral basis for  $\text{Cl}_{r+1,s+1}$  and  $\text{Cl}_{0,n+2}$  based on the admissible module of  $\text{Cl}_{1,1}$  and  $\text{Cl}_{2,0}$ . This method shows that the tensor product representation with some modification gives us an admissible module with integral basis, but it need not be a minimal. It remains to decompose this admissible module into minimal one's together with an integral basis. Section 7 deals with integral structures on admissible  $\text{Cl}_{r,s}$ -modules for  $r + s \geq 9$ . In the last Section 8 we make some observations about the presented constructions.

## 2. Properties of admissible $\text{Cl}_{r,s}$ -modules

We recall the basic properties of admissible  $\text{Cl}_{r,s}$ -modules when  $s > 0$ . We say that  $W$  is an admissible sub-module of an admissible module  $(V, \langle \cdot, \cdot \rangle_V)$  if  $W$  is a Clifford sub-module of  $V$  and the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $W$ , denoted by  $\langle \cdot, \cdot \rangle_W$ , is an admissible scalar product. There are decompositions of a given admissible  $\text{Cl}_{r,s}$ -module,  $s > 0$ , into non-admissible sub-modules. In the following proposition we give conditions that ensure a decomposition of an admissible Clifford module

into admissible sub-modules.

**Proposition 2.1.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible  $\text{Cl}_{r,s}$ -module with  $s > 0$  and  $J_{z_k}$ ,  $k = 1, \dots, r + s$ , representations of the orthonormal generators  $z_1, \dots, z_{r+s}$  of the Clifford algebra  $\text{Cl}_{r,s}$ .*

(1) *Then the scalar product space  $(V, \langle \cdot, \cdot \rangle_V)$  is neutral, i. e. the maximal dimension of subspaces where the restriction of  $\langle \cdot, \cdot \rangle_V$  is positive or negative definite coincide and, particularly, the dimension of  $V$  can be only even.*

(2) *If  $W$  is an admissible sub-module of a Clifford  $\text{Cl}_{r,s}$ -module, then  $W^\perp$  is also an admissible sub-module. Hence, we have the decomposition of an admissible  $\text{Cl}_{r,s}$ -module  $(V, \langle \cdot, \cdot \rangle_V)$  into admissible sub-modules.*

**Proof.** The proof of the first statement can be found in [6]. For the second statement we note that if  $W$  is an admissible sub-module of a Clifford  $\text{Cl}_{r,s}$ -module, then the action of each  $J_{z_k}$  leaves invariant the orthogonal complement  $W^\perp$ . Since the scalar product restricted to  $W^\perp$  is non-degenerate,  $W^\perp$  is also an admissible sub-module. ■

Further we show the existence of a non-trivial scalar product satisfying two conditions (consequently all three) among (2), (3), and (4) for Clifford  $\text{Cl}_{r,s}$ -modules with  $s > 0$ .

**Lemma 2.2.** *Let  $V$  be an irreducible  $\text{Cl}_{r,s}$ -module:  $J: \text{Cl}_{r,s} \rightarrow \text{End}(V)$ ,  $J_z^2 = -\langle z, z \rangle_{\mathbb{R}^{r,s}} \text{Id}_V$ , with a symmetric bilinear form  $\langle \cdot, \cdot \rangle_V: V \times V \rightarrow \mathbb{R}$  which satisfies  $\langle J_z v, w \rangle_V + \langle v, J_z w \rangle_V = 0$  for any  $z \in \mathbb{R}^{r,s}$  and any  $v, w \in V$ . Then the scalar product  $\langle \cdot, \cdot \rangle_V$  is non-degenerate or identically vanishes.*

**Proof.** Let  $N = \{v \in V \mid \langle v, w \rangle_V = 0 \text{ for any } w \in V\}$ . Then for any  $z \in \mathbb{R}^{r,s}$  and  $v \in N$  one obtain  $\langle J_z v, w \rangle_V + \langle v, J_z w \rangle_V = 0$ , and  $\langle v, J_z w \rangle_V = 0$ , for all  $w \in V$ . Hence  $\langle J_z v, w \rangle_V = 0$ , which shows that  $N$  is invariant under  $J$ . So if  $v \in V$  with  $\langle v, v \rangle_V \neq 0$  then  $N$  must be the trivial space  $\{0\}$  or entire  $V$  due to the irreducibility of the module  $V$ . ■

As was mentioned above, in the case of classical  $H$ -type algebras, there is an admissible inner product for any  $\text{Cl}_{r,0}$ -module, particularly, any irreducible module can be an admissible module with an inner product. However, for  $s > 0$  not all irreducible  $\text{Cl}_{r,s}$ -modules can be admissible modules. For instance, the Clifford algebra  $\text{Cl}_{0,2}$  is isomorphic to the algebra  $\mathbb{R}(2)$  of  $(2 \times 2)$  real matrices, and the irreducible module is 2 dimensional, whereas the admissible  $\text{Cl}_{0,2}$ -module of minimal dimension is isomorphic to  $\mathbb{R}^{2,2}$ . However the following property is still hold.

**Proposition 2.3.** (see [6]) *Let  $V$  be a  $\text{Cl}_{r,s}$ -module, then there is a scalar product on  $V$  or on  $V \oplus V$  with respect to which the resulting  $\text{Cl}_{r,s}$ -module is an admissible module. The representation on  $V \oplus V$  should be redefined in an obvious way.*

**Corollary 2.4.** *An irreducible  $\text{Cl}_{r,s}$ -module can be either an admissible module of minimal dimension or the double of the irreducible module is the admissible module of minimal dimension. As a result, a general  $H$ -type algebra constructed from the minimal dimensional admissible module is unique up to isomorphism.*

Any irreducible  $\text{Cl}_{r,s}$ -module  $V$  can be generated by a non-zero vector  $v \in V$  by the subsequent actions of  $J_z$ ,  $z \in \mathbb{R}^{r,s}$  (or Clifford multiplication by  $z$ ). Similarly, we have the following statement.

**Proposition 2.5.** *Any minimal dimensional admissible module is generated by a non-null vector.*

**Lemma 2.6.** *Let  $\text{Cl}_{r,s}$  be a Clifford algebra with orthonormal generators  $\{z_i\}$ ,  $i = 1, \dots, r + s$  and  $(V, \langle \cdot, \cdot \rangle_V)$  an admissible  $\text{Cl}_{r,s}$ -module with an orthonormal basis  $\{v_\alpha\}_{\alpha=1}^{\dim V}$ . Then  $J_{z_i}v_\alpha \neq \pm J_{z_j}v_\alpha$  for  $i \neq j$ .*

**Proof.** Let us assume that  $J_{z_i}v_\alpha = \pm J_{z_j}v_\alpha$ ,  $i \neq j$  and  $v_\alpha$  be an element of the orthonormal basis of  $V$ . The assumption gives  $J_{z_i \pm z_j}v_\alpha = 0$  and hence  $J_{z_i \pm z_j}^2v_\alpha = -\langle z_i \pm z_j, z_i \pm z_j \rangle_{\mathbb{R}^{r,s}}v_\alpha = 0$ , which implies that  $\langle z_i \pm z_j, z_i \pm z_j \rangle_{\mathbb{R}^{r,s}} = 0 = \langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} + \langle z_j, z_j \rangle_{\mathbb{R}^{r,s}}$ . So, one of  $z_i$  or  $z_j$  is spacelike, and the other is timelike. The assumption also implies that  $J_{z_i}J_{z_j}v_\alpha = \pm v_\alpha$ , which leads to

$$\langle \pm v_\alpha, \pm v_\alpha \rangle_V = \langle J_{z_i}J_{z_j}v_\alpha, J_{z_i}J_{z_j}v_\alpha \rangle_V = \langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} \langle z_j, z_j \rangle_{\mathbb{R}^{r,s}} \langle v_\alpha, v_\alpha \rangle_V = -\langle v_\alpha, v_\alpha \rangle_V.$$

This is a contradiction since  $v_\alpha$  is a non-null vector. ■

**Corollary 2.7.** *Under assumptions of Lemma 2.6 if we additionally assume that each operator  $J_{z_i}$  permutes the basis  $\{v_\alpha\}$  up to sign, that is  $J_{z_i}v_\alpha = \pm v_\beta$  for any  $i, \alpha$  and some  $\beta$ , then with the given orthonormal generators  $\{z_i\}$  of the Clifford algebra the basis  $\{v_\alpha, z_i\}$  of the general  $H$ -type Lie algebra  $V \oplus_{\perp} \mathbb{R}^{r,s}$  has structure constants  $A_{\alpha,\beta}^k$  equal  $\pm 1$  or  $0$ .*

In fact, in the present work we show that for any minimal dimensional admissible module  $(V, \langle \cdot, \cdot \rangle_V)$ , one can find a vector  $w \in V$  with  $\langle w, w \rangle_V = 1$  such that the  $2^{r+s}$ -vectors

$$w, J_1w, \dots, J_{r+s}w, J_1J_2w, \dots, J_{r+s-1}J_{r+s}w, \dots, J_{i_1}J_{i_2} \dots J_{i_k}w, \dots, J_1J_2 \dots J_{r+s}w,$$

satisfy the property that any two vectors  $J_{i_1} J_{i_2} \cdots J_{i_k} w$  and  $J_{j_1} J_{j_2} \cdots J_{j_\ell} w$  coincide up to sign or orthogonal and therefore we can select an orthonormal basis of  $V$  produced from  $w \in V$  by action of  $J_i = J_{z_i}$ . Moreover  $J_{z_i}, i = 1, \dots, r + s$ , act on the obtained basis as permutations up to sign. Hence we prove that any general H-type Lie algebra have an integral basis according Lemma 2.6 and Corollary 2.7.

Further we collect auxiliary technical lemmas, that will be used later. We say that a linear operator  $\Omega: V \rightarrow V$  on a scalar product space  $(V, \langle \cdot, \cdot \rangle_V)$  is *symmetric* if  $\langle \Omega v, w \rangle_V = \langle v, \Omega w \rangle_V$  for all  $v, w \in V$ .

**Lemma 2.8.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible module,  $\Omega: V \rightarrow V$  a symmetric anti-involution. Then for any  $w \in V$  with  $\langle w, w \rangle_V = 1$  there is  $\lambda \in \mathbb{R}$  such that  $\tilde{w} = w + \lambda \Omega(w)$  satisfies:  $\langle \tilde{w}, \Omega \tilde{w} \rangle_V = 0$ , and  $\langle \tilde{w}, \tilde{w} \rangle_V = 1$ .*

**Proof.**

Note  $\langle \Omega w, \Omega w \rangle_V = \langle \Omega^2 w, w \rangle_V = -\langle w, w \rangle_V$ . Let  $w \in V$  be such that  $\langle w, w \rangle_V = 1$  and assume  $\langle w, \Omega w \rangle_V = a$ . If  $a = 0$ , then we choose  $\lambda = 0$ . Thus, we can assume that  $\langle w, \Omega w \rangle_V = a \neq 0$ . Then, by solving the equation  $\langle \tilde{w}, \Omega \tilde{w} \rangle_V = -(a\lambda^2 + 2\lambda - a) = 0$ , we find  $\lambda = -\frac{1}{a} \pm \sqrt{1 + \frac{1}{a^2}}$  for  $a \neq 0$  and get  $\langle \tilde{w}, \tilde{w} \rangle_V = 2\lambda \frac{a^2+1}{a} \neq 0$ . If  $a > 0$  then we choose  $\lambda = -\frac{1}{a} + \sqrt{1 + \frac{1}{a^2}} > 0$  and if  $a < 0$  then we choose  $\lambda = -\frac{1}{a} - \sqrt{1 + \frac{1}{a^2}} < 0$ . This choice makes the product  $\langle \tilde{w}, \tilde{w} \rangle_V$  strictly positive. Normalizing,  $\tilde{w}$  we get  $\langle \tilde{w}, \tilde{w} \rangle = 1$ . ■

The next lemma is a generalisation of the previous one.

**Lemma 2.9.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible module,  $\Omega_1, \dots, \Omega_l$  symmetric linear operators on  $V$  such that 1)  $\Omega_k^2 = -\text{Id}_V, k = 1, \dots, l$ ; and 2)  $\Omega_k \Omega_j = -\Omega_j \Omega_k, k, j = 1, \dots, l$ . Then for any  $w \in V$  with  $\langle w, w \rangle_V = 1$  there is a vector  $\tilde{w} \in V$  satisfying:*

$$\langle \tilde{w}, \Omega_k \tilde{w} \rangle_V = 0, \quad \text{and} \quad \langle \tilde{w}, \tilde{w} \rangle_V = 1, \quad k = 1, \dots, l.$$

**Proof.** Notice that symmetry of operators and property 2) imply

$$\langle \Omega_k v, \Omega_j v \rangle_V = 0 \quad \text{for any} \quad v \in V. \tag{8}$$

We choose  $w \in V, \langle w, w \rangle_V = 1$  and apply Lemma 2.8 to  $w$  and  $\Omega_1$  and construct  $w_1 \in V$  such that

$$\langle w_1, w_1 \rangle_V = 1, \quad \langle w_1, \Omega_1 w_1 \rangle_V = 0. \tag{9}$$

Then we define  $w_2 = w_1 + \lambda_2 \Omega_2 w_1$  and find that for suitable  $\lambda_2 \in \mathbb{R}$  we have  $\langle w_2, w_2 \rangle_V = 1, \langle w_2, \Omega_2 w_2 \rangle_V = 0$ . Moreover,  $\langle w_2, \Omega_1 w_2 \rangle_V = 0$  by assumptions of

Lemma 2.9 and properties (8) and (9). Now, applying Lemma 2.8 we assume that the vector  $w_k = w_{k-1} + \lambda_k \Omega_k w_{k-1}$ ,  $2 < k < l$ , is chosen and satisfies  $\langle w_k, w_k \rangle_V = 1$ ,  $\langle \Omega_k w_k, w_k \rangle_V = 0$ , where it was shown that  $\langle w_{k-1}, w_{k-1} \rangle_V = 1$ ,  $\langle w_{k-1}, \Omega_j w_{k-1} \rangle_V = 0$ , for  $j = 1, \dots, k - 1$ . Then  $\langle w_k, \Omega_j w_k \rangle_V = 0$  for any  $j = 1, \dots, k - 1$  by (8) and assumption on operators  $\Omega_j$ . Denoting the last vector  $w_l$  by  $\tilde{w}$ , we finish the proof of Lemma 2.9. ■

**Corollary 2.10.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  and linear maps  $\Omega_1, \dots, \Omega_l$  satisfies the conditions of Lemma 2.9. Let  $P$  be an involution on  $V$  such that  $P\Omega_k = \Omega_k P$ ,  $k = 1, \dots, l$ . If  $w \in V$  satisfies  $Pw = w$  and  $\langle w, w \rangle_V = 1$ , then for the vector  $\tilde{w}$  constructed in Lemma 2.9 it holds:  $P\tilde{w} = \tilde{w}$ .*

**Proof.** Let  $w \in V$  satisfies  $Pw = w$  and  $\langle w, w \rangle_V = 1$ . Then for the vector  $w_1 = w + \lambda_1 \Omega_1 w$  we calculate  $Pw_1 = Pw + \lambda_1 P\Omega_1 w = w + \lambda_1 \Omega_1 Pw = w_1$ . Thus, we proceed further by induction and proof the Corollary. ■

One of the principal parts in our construction is a presence of an operator having orthogonal eigenspaces. The following lemma describes some of them.

**Lemma 2.11.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be a neutral scalar product space. Assume that an involution  $P: V \rightarrow V$  is either symmetric or isometric operator. We denote its eigenspaces  $E_+ = \{v \in V \mid Pv = v\}$ ,  $E_- = \{v \in V \mid Pv = -v\}$ . Then the decomposition  $P = E_+ \oplus E_-$  is orthogonal with respect to  $\langle \cdot, \cdot \rangle_V$ .*

**Proof.** Let  $w \in E_+$  and  $v \in E_-$  be arbitrary vectors. If involution  $P$  is symmetric then we argue as follows  $\langle w, v \rangle_V = \langle Pw, v \rangle_V = \langle w, Pv \rangle_V = -\langle w, v \rangle_V$ .

Let  $P$  be an isometry. Then  $\langle w, v \rangle_V = \langle Pw, Pv \rangle_V = -\langle w, v \rangle_V$ , where in the first equality we used the isometry property of  $P$  and in the second the definition of eigenvectors. ■

We fix the notation  $E_{\pm}$  for eigenspaces of an involution  $P$  corresponding to eigenvalues  $\pm 1$ .

**Lemma 2.12.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be a neutral scalar product space and  $P: V \rightarrow V$  be an isometric involution. Then we have the following cases.*

- 1) *If a linear map  $T: V \rightarrow V$  is an isometry such that  $PT = -TP$ , then the restriction of  $\langle \cdot, \cdot \rangle_V$  on each of  $E_{\pm}$  is non-degenerate neutral,*
- 2) *If a linear map  $T: V \rightarrow V$  is an anti-isometry such that  $PT = -TP$ , then the restriction of  $\langle \cdot, \cdot \rangle_V$  on each of  $E_{\pm}$  is non-degenerate neutral or sign definite,*

3) If a linear map  $T: V \rightarrow V$  is an anti-isometry such that  $PT = TP$ , then the restriction of  $\langle \cdot, \cdot \rangle_V$  on each of  $E_{\pm}$  is non-degenerate neutral.

**Proof.** To show the first statement we observe that  $T: E_+ \rightarrow E_-$  is an isometry. Since the eigenspaces  $E_{\pm}$  are orthogonal, the scalar product  $\langle \cdot, \cdot \rangle_V$  restricted to each of  $E_{\pm}$  is non-degenerate. If the scalar product restricted to  $E_+$  would be positive definite, then the scalar product restricted to  $E_-$  would be also positive definite, since the map  $T$  is an isometry that contradicts the assumption that space  $(V, \langle \cdot, \cdot \rangle_V)$  is neutral. The same arguments shows that the restriction to  $E_+$  could not be negative definite. So the scalar product restricted to  $E_+$  and therefore to  $E_-$  should be neutral.

In order to prove the second statement, we observe that since  $T: E_+ \rightarrow E_-$  is an anti-isometry, the restriction of  $\langle \cdot, \cdot \rangle_V$  to  $E_+$  can be sign definite and the restriction of  $\langle \cdot, \cdot \rangle_V$  to  $E_-$  will have opposite sign due to neutral nature of  $(V, \langle \cdot, \cdot \rangle_V)$ .

In the third case since the eigenspaces  $E_{\pm}$  are invariant under  $T$  but contains spacelike and timelike vectors, they are decomposed into subspaces of equal dimensions where the restriction of  $\langle \cdot, \cdot \rangle_V$  sign definite but of opposite signs. ■

Note that assumptions made in Lemma 2.12, item 2), do not guarantee the existence of  $w \in V$  with

$$Pw = w, \quad \langle w, w \rangle_V = 1. \tag{10}$$

The following lemma contains a benchmark example for our work, describing one of possible solutions of this problem.

**Lemma 2.13.** Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible  $Cl_{r,s}$ -module,  $z_1, \dots, z_{r+s}$  orthonormal generators of the Clifford algebra  $Cl_{r,s}$ , and  $J_{z_i} \in \text{End}(V)$ ,  $i = 1, \dots, r + s$  are representations for the Clifford algebra. Assume that the operator  $P = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}} J_{z_{i_4}}$ ,  $i_1 \neq i_2 \neq i_3 \neq i_4$ , is an isometric involution and  $T: V \rightarrow V$  is an anti-isometry such that  $PT = -TP$ . Then there is  $w \in V$  satisfying (10) or  $P$  can be modified to other isometric involution  $\hat{P}$  such that (10) holds for  $\hat{P}$ .

**Proof.** First we notice that operator  $P = J_{z_{i_1}} J_{z_{i_2}} J_{z_{i_3}} J_{z_{i_4}}$ ,  $i_1 \neq i_2 \neq i_3 \neq i_4$  is also symmetric. We apply Lemma 2.12, item 2). If the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E_+$  is positive definite or neutral, then we are done. If the restriction is negative definite, then we define the operator  $\hat{P} = J_{z_{i_2}} J_{z_{i_1}} J_{z_{i_3}} J_{z_{i_4}}$  and denote by  $\hat{E}_{\pm}$  its eigenspaces corresponding to eigenvalues  $\pm 1$ . Thus, if  $w \in E_+$ , then  $\hat{P}w = -Pw = -w$  and therefore  $w \in \hat{E}_-$ . Continuing to argue in the same way,

we conclude that  $E_+ = \hat{E}_-$  and  $E_- = \hat{E}_+$ . So, we change the operator  $P$  and its eigenspaces  $E_{\pm}$  to the operator  $\hat{P}$  and the corresponding eigenvectors  $\hat{E}_{\mp}$  to satisfy (10). ■

To ensure existence of  $w \in V$  satisfying (10) for a general isometric involution  $P$  we need to have one more operator commuting with  $P$ . More precisely we state the following generalisation of Lemma 2.12.

**Lemma 2.14.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be a neutral scalar product space. Let  $P$  be an isometric involution and assume that there are two anti-isometric operators  $T_i: V \rightarrow V$ ,  $i = 1, 2$ , such that  $T_1P = -PT_1$ , and  $T_2P = PT_2$ . Then the eigenspaces  $E_{\pm}$  of  $P$  are non-trivial and the scalar product  $\langle \cdot, \cdot \rangle_V$  restricted to each of  $E_{\pm}$  is non-degenerate and neutral.*

**Proof.** The presence of the operator  $T_1$  ensures that the restriction of  $\langle \cdot, \cdot \rangle_V$  to  $E_{\pm}$  is neutral or sign definite and non-degeneracy follows from Lemma 2.12. Actually the restriction of  $\langle \cdot, \cdot \rangle_V$  to  $E_+$  can not be sign definite. Indeed, since  $T_2$  preserves  $E_+$  and it is an anti-isometry, the space  $E_+$  contains both spacelike and timelike vectors forming subspaces of an equal dimension, see the proof of Proposition 2.1. The same arguments, applied to  $E_-$ , finish the proof. ■

**Definition 2.15.** Let  $P_1, \dots, P_m$  be isometric mutually commuting involutions defined on a neutral scalar product space  $(V, \langle \cdot, \cdot \rangle_V)$ . Then the collection  $T_1, \dots, T_m, T_{m+1}$  of linear operators on  $V$  is called complementary operators to the family  $P_1, \dots, P_m$  if

$$\begin{aligned} P_1T_1 = -T_1P_1, \quad P_1T_2 = T_2P_1, \quad \dots \quad P_1T_m = T_mP_1, \quad P_1T_{m+1} = T_{m+1}P_1, \\ P_2T_2 = -T_2P_2, \quad \dots \quad P_2T_m = T_mP_2, \quad P_2T_{m+1} = T_{m+1}P_2, \\ \vdots \\ P_mT_m = -T_mP_m, \quad P_mT_{m+1} = T_{m+1}P_m. \end{aligned}$$

**Remark 2.16.** In some of situations the operator  $T_{m+1}$  can be omitted, but we still call the system of operators  $T_1, \dots, T_m$  complementary to  $P_1, \dots, P_m$ .

Based on Lemmas 2.12 and 2.14 we construct an integral structure by giving an explicit simultaneous eigenspace decomposition of a given admissible module by a family of isometric involutions and their complementary operators. Simultaneously, we calculate the dimension of the minimal admissible modules for all cases.

### 3. Isomorphism preserving admissibility

There are several types of isomorphisms between Clifford algebras. Among them the periodicity with the period 8

$$\text{Cl}_{r+8,s} \cong \text{Cl}_{r+4,s+4} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16), \quad \text{Cl}_{r,s+8} \cong \text{Cl}_{r+4,s+4} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16)$$

are basic and used to construct an integral basis for all the cases  $\text{Cl}_{r,s}$  after we prove the existence of the integral basis for  $\text{Cl}_{r,s}$  of  $0 \leq r, s \leq 8$ , see Theorems 6.1, 6.2 and 6.3.

Not all isomorphisms of the Clifford algebras lead to the isometric admissible modules, for instance, the isomorphism  $\text{Cl}_{r,s+4} \cong \text{Cl}_{s,r+4}$  does not preserve the admissibility in general, since in particular, the isomorphism  $\text{Cl}_{0,4} \cong \text{Cl}_{4,0}$  does not directly give us a required scalar product from the positive one for  $\text{Cl}_{4,0}$ -module to a neutral one for  $\text{Cl}_{0,4}$ -module. In this section we show that the isomorphism  $\text{Cl}_{r,s+1} \cong \text{Cl}_{s,r+1}$  preserves the admissibility.

We recall an isomorphism between  $\text{Cl}_{r,s+1}$  and  $\text{Cl}_{s,r+1}$ . It is given as follows: let  $z_1, \dots, z_r, \zeta_1, \dots, \zeta_{s+1}$  be the orthonormal generators of  $\text{Cl}_{r,s+1}$  with the property  $\langle z_i, z_i \rangle_{\mathbb{R}^{r,s+1}} = 1$ ,  $\langle \zeta_j, \zeta_j \rangle_{\mathbb{R}^{r,s+1}} = -1$ . Likewise let  $a_1, \dots, a_s, b_1, \dots, b_{r+1}$  be orthonormal generators of  $\text{Cl}_{s,r+1}$  with  $\langle a_i, a_i \rangle_{\mathbb{R}^{s,r+1}} = 1$ ,  $\langle b_j, b_j \rangle_{\mathbb{R}^{s,r+1}} = -1$ . We define a correspondence  $\Phi: \mathbb{R}^{r,s+1} \rightarrow \text{Cl}_{s,r+1}$  by

$$\begin{aligned} z_1 &\longmapsto b_1 b_{r+1}, & \zeta_1 &\longmapsto a_1 b_{r+1}, \\ z_2 &\longmapsto b_2 b_{r+1}, & \dots & \dots, \\ \dots & \dots, & \zeta_s &\longmapsto a_s b_{r+1}, \\ z_r &\longmapsto b_r b_{r+1}, & \zeta_{s+1} &\longmapsto b_{r+1}. \end{aligned}$$

Then we have  $\Phi(z_i)^2 = -1$ ,  $\Phi(\zeta_j)^2 = 1$ . Moreover  $\{\Phi(z_i), \Phi(\zeta_j)\}_{i,j}$  anti commute and are linearly independent. Thus, one can extend the isomorphism to an isomorphism of the Clifford algebras  $\text{Cl}_{r,s+1}$  and  $\text{Cl}_{s,r+1}$ .

**Theorem 3.1.** *Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible  $\text{Cl}_{s,r+1}$ -module and  $J: \text{Cl}_{s,r+1} \rightarrow \text{End}(V)$  its representation. Then the Clifford module  $(V, \langle \cdot, \cdot \rangle_V)$  with the representation  $J \circ \Phi: \text{Cl}_{r,s+1} \rightarrow \text{End}(V)$  is admissible.*

**Proof.** The skew symmetry condition  $\langle J_{\Phi(z_i)}v, w \rangle_V + \langle v, J_{\Phi(z_i)}w \rangle_V = 0$  for any  $v, w \in V$  holds by the following

$$\begin{aligned} \langle J_{\Phi(z_i)}v, w \rangle_V &= \langle J_{b_i} J_{b_{r+1}} v, w \rangle_V = -\langle J_{b_{r+1}} v, J_{b_i} w \rangle_V \\ &= \langle v, J_{b_{r+1}} J_{b_i} w \rangle_V = -\langle v, J_{b_i} J_{b_{r+1}} w \rangle_V = -\langle v, J_{\Phi(z_i)} w \rangle_V \quad \text{for } i \leq r. \end{aligned}$$

The rest of the cases can be shown in a similar way. ■

**Corollary 3.2.** *If an integral basis for an admissible  $\text{Cl}_{s,r+1}$ -module satisfies the assumptions of Lemma 2.6 and Corollary 2.7, then it is also an integral basis of the admissible  $\text{Cl}_{r,s+1}$ -module.*

**Proof.** Indeed, let  $\{v_\alpha\}$  be an integral basis for  $\text{Cl}_{s,r+1}$ -module, then

$$\langle J_{\Phi(z_i)}v_\alpha, v_\beta \rangle_V = \langle J_{b_i}J_{b_{r+1}}v_\alpha, v_\beta \rangle_V = -\langle J_{b_{r+1}}v_\alpha, J_{b_i}v_\beta \rangle_V = \pm 1 \text{ or } 0,$$

since the vectors  $J_{b_{r+1}}v_\alpha$  and  $J_{b_i}v_\beta$  are also basis vectors up to sign by the assumption. ■

#### 4. Integral structure on admissible $\text{Cl}_{r,1}$ -modules

In this section we show the existence of an integral basis for  $\text{Cl}_{r,1}$ -modules based on the existence of those for  $\text{Cl}_{r,0}$ -modules. Let  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible  $\text{Cl}_{r,1}$ -module. Denote by  $J_i = J_{z_i}$ ,  $i = 1, \dots, r + 1$  the representations of orthonormal generators of the algebra  $\text{Cl}_{r,1}$  such that  $J_i^2 = -\text{Id}_V$ ,  $i = 1, \dots, r$ ,  $J_{r+1}^2 = \text{Id}_V$ . Let  $U$  be a subspace of  $V$  invariant under the action of  $J_i$ ,  $i = 1, \dots, r$ , and let  $U = \text{span}\{u_1, \dots, u_l\}$ , be a basis such that

- $\{u_\alpha\}_{\alpha=1}^l$  is an orthonormal basis of  $U$ ;
- maps  $J_i = J_{z_i}$ ,  $i = 1, \dots, r$ , permute the basis  $\{u_\alpha\}_{\alpha=1}^l$  up to sign;
- $\langle J_i u_\alpha, u_\beta \rangle_V = \pm 1$  or  $0$ .

Denote  $\tilde{u}_\alpha = J_{r+1}u_\alpha$ ,  $\alpha = 1, \dots, l$ , and the vector spaces  $\tilde{U} = \text{span}\{\tilde{u}_1, \dots, \tilde{u}_l\}$ , and  $W = \text{span}\{u_1, \dots, u_l, \tilde{u}_1, \dots, \tilde{u}_l\}$ .

**Theorem 4.1.** *In the notations above if the decomposition  $W = U \oplus \tilde{U}$  is orthogonal with respect to  $\langle \cdot, \cdot \rangle_V$ , then  $(W, \langle \cdot, \cdot \rangle_W)$ , where  $\langle \cdot, \cdot \rangle_W$  is the restriction of  $\langle \cdot, \cdot \rangle_V$  onto  $W$  is an admissible integral  $\text{Cl}_{r,1}$ -module. If  $U$  has minimal dimension, then  $W$  is a minimal admissible integral  $\text{Cl}_{r,1}$ -module.*

**Proof.** Observe that the restriction of  $\langle \cdot, \cdot \rangle_V$  to  $U$  will be positive definite or negative definite since  $\langle J_i u_\alpha, J_i u_\alpha \rangle_V = \langle z_i, z_i \rangle_{\mathbb{R}^{r,1}} \langle u_\alpha, u_\alpha \rangle_V = \langle u_\alpha, u_\alpha \rangle_V$ ,  $i = 1, \dots, r$ . Let us assume that it is positive definite. Then  $\tilde{U} = \text{span}\{\tilde{u}_1, \dots, \tilde{u}_l\}$  is isomorphic to  $U$  and the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $\tilde{U}$  is negative definite since  $\langle \tilde{u}_\alpha, \tilde{u}_\alpha \rangle_V = \langle J_{r+1}u_\alpha, J_{r+1}u_\alpha \rangle_V = \langle z_{r+1}, z_{r+1} \rangle_{\mathbb{R}^{r,1}} \langle u_\alpha, u_\alpha \rangle_V = -\langle u_\alpha, u_\alpha \rangle_V$ . We conclude that  $J_{r+1}: U \rightarrow \tilde{U}$  defines an anti-isometry with respect to restrictions of the scalar product  $\langle \cdot, \cdot \rangle_V$  onto spaces  $U$  and  $\tilde{U}$  and the space  $(W, \langle \cdot, \cdot \rangle_W)$ , where  $W = U \oplus \tilde{U}$  and  $\langle \cdot, \cdot \rangle_W$  is the restriction of  $\langle \cdot, \cdot \rangle_V$  onto  $W$  is neutral.

The space  $W$  is invariant under the action of  $J_i$ ,  $i = 1, \dots, r + 1$ , and all maps  $J_i$ , permute the basis  $\{v_\alpha, \tilde{v}_\alpha\}_{\alpha=1}^l$  up to sign. We conclude that the sub-module  $(W, \langle \cdot, \cdot \rangle_W)$  is admissible integral and has a minimal possible dimension if the space  $U$  has minimal dimension. Note that the orthogonality of the decomposition  $U \oplus \tilde{U}$  can be achieved by applying Lemmas 2.8 and 2.9.  $\blacksquare$

**Corollary 4.2.** *The dimension of the minimal admissible  $Cl_{r,1}$ -module is twice of the  $Cl_{r,0}$ -module.*

**Corollary 4.3.** *Combining Theorems 3.1 and 4.1 we conclude that Theorems 1.2 and 1.3 are true for  $Cl_{0,s}$ -modules,  $s > 0$  and the corresponding general  $H$ -type Lie algebras.*

**5. Integral structure on admissible  $Cl_{r,s}$ -modules with  $r + s \leq 8$**

We show the existence of an admissible integral  $Cl_{r,s}$ -module for  $r + s = 2, \dots, 8$  by direct construction. In these constructions we start from an admissible module  $(V, \langle \cdot, \cdot \rangle_V)$  and find an orthonormal set  $\{v_\alpha\}_{\alpha=1}^l$  such that the sub-module  $(W, \langle \cdot, \cdot \rangle_W)$ , where  $W = \text{span}\{v_1, \dots, v_l\}$  and the scalar product  $\langle \cdot, \cdot \rangle_W$  is the restriction of  $\langle \cdot, \cdot \rangle_V$  onto  $W$ , is an admissible integral and has the minimal possible dimension. In some cases  $W$  is not an irreducible sub-module. The cases not presented in Section 5 can be obtained by applying Theorems 3.1 and 4.1. Through the present section let  $z_i$ ,  $i = 1, \dots, r + s$  be orthonormal generators of the Clifford algebra  $Cl_{r,s}$  with  $\langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} = 1$  for  $i = 1, \dots, r$  and  $\langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} = -1$  for  $i = r + 1, \dots, s$ . The scalar product space  $(V, \langle \cdot, \cdot \rangle_V)$  be an admissible  $Cl_{r,s}$ -module, and  $J_i^2 := J_{z_i}^2 = -\langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} \text{Id}_V = -\text{Id}_V$  for  $i = 1, \dots, r$  and  $J_i^2 := J_{z_i}^2 = -\langle z_i, z_i \rangle_{\mathbb{R}^{r,s}} \text{Id}_V = \text{Id}_V$  for  $i = r + 1, \dots, s$ .

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{1,1}$ -MODULE. The existence of the integral basis follows from the isomorphism  $Cl_{r,s+1} = Cl_{0,2} \cong Cl_{1,1} = Cl_{s,r+1}$  by Theorem 3.1, but we prefer to give a direct construction, since we use this construction further. The Clifford algebra  $Cl_{1,1}$  is isomorphic to the space  $\mathbb{R}(2)$  of  $(2 \times 2)$ -matrices with real entries. Choose  $w \in V$  such that  $\langle w, w \rangle_V = 1$ . Then

$$v_1 = w, \quad v_2 = J_1 w, \quad v_3 = J_2 w, \quad v_4 = J_2 J_1 w \tag{11}$$

is an orthogonal basis satisfying  $\langle v_1, v_1 \rangle_V = \langle v_2, v_2 \rangle_V = -\langle v_3, v_3 \rangle_V = -\langle v_4, v_4 \rangle_V = 1$ . It is easy to see that the admissible module is integral. The admissible module is not irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{1,2}$ -MODULE. The Clifford algebra  $Cl_{1,2}$  is isomorphic to the space  $\mathbb{R}(2) \oplus \mathbb{R}(2)$ . Consider the isometric involution  $P = J_1 J_2 J_3$ . The operator  $P$  commutes with the anti-isometry  $T = J_2$ . Thus

the eigenspace  $E_+$  of  $P$  is a neutral space by Lemma 2.12, item 3). We pick up a vector  $w \in V$  such that  $Pw = w$  and  $\langle w, w \rangle_V = 1$ . It gives the linear dependent vectors  $J_1w = -J_2J_3w$ ,  $J_2w = J_1J_3w$ ,  $J_3w = J_1J_2w$ . In this case we choose the basis

$$v_1 = w, \quad v_2 = J_1w, \quad v_3 = J_2w, \quad v_4 = J_3w,$$

that is orthogonal with  $\langle v_1, v_1 \rangle_V = \langle v_2, v_2 \rangle_V = -\langle v_3, v_3 \rangle_V = -\langle v_4, v_4 \rangle_V = 1$ . The basis is integral due to the orthogonality of generators, skew symmetry of  $J_k$  and the condition  $J_1J_2J_3w = w$ . The admissible integral module is not irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $\text{Cl}_{2,2}$ -MODULE. The Clifford algebra  $\text{Cl}_{2,2}$  is isomorphic to the space  $\mathbb{R}(4)$ . We consider the isometric involution  $P = J_1J_2J_3J_4$  and the isometry  $T = J_1$ . Thus  $PT = -TP$ . We choose  $w \in V$  such that  $Pw = w$  and  $\langle w, w \rangle_V = 1$  by Lemma 2.12, part 1). The linear dependence relations listed in the case of  $\text{Cl}_{1,2}$ -module still hold up to sign. We write the basis

$$\begin{aligned} v_1 = w, & & v_2 = J_1w, & & v_3 = J_2w, & & v_4 = J_1J_2w, \\ v_5 = J_3w, & & v_6 = J_4w, & & v_7 = J_1J_3w, & & v_8 = J_1J_4w. \end{aligned}$$

It is orthonormal and  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, 2, 3, 4$ ,  $\langle v_\alpha, v_\alpha \rangle_V = -1$ ,  $\alpha = 5, 6, 7, 8$ . Since  $J_i$ ,  $i = 1, 2, 3, 4$ , permute the basis up to sign the sub-module  $W = \text{span}\{v_1, \dots, v_8\}$  is integral. The constructed module is not irreducible because the dimension of an irreducible module is 4 and with any choice of a non-null vector  $v \in V$  the five vectors  $v, J_1v, J_2v, J_3v, J_4v$  are already linear independent.

We give an alternative construction of an admissible  $\text{Cl}_{2,2}$ -module. As it was mentioned the  $\pm 1$ -eigenspaces  $E_\pm$  of  $P$  have equal dimension. Consider two operators  $\hat{J}_1 = J_1J_2$  and  $\hat{J}_2 = J_1J_3$ . Since they commute with  $P$  they leave invariant  $E_+$ . Moreover  $\hat{J}_1^2 = -\text{Id}_{E_+}$ ,  $\hat{J}_2^2 = \text{Id}_{E_+}$ ,  $\hat{J}_1\hat{J}_2 = -\hat{J}_2\hat{J}_1$ . Thus, the algebra generated by  $\hat{J}_1$  and  $\hat{J}_2$  in  $\text{End}(E_+)$  is isomorphic to the Clifford algebra  $\text{Cl}_{1,1}$  and the representation is admissible, since, for example,

$$\begin{aligned} \langle \hat{J}_1u, v \rangle_{E_+} = \langle J_1J_2u, v \rangle_{E_+} &= -\langle J_2u, J_1v \rangle_{E_+} = \langle u, J_2J_1v \rangle_{E_+} \\ &= -\langle u, J_1J_2v \rangle_{E_+} = -\langle u, \hat{J}_1v \rangle_{E_+}. \end{aligned}$$

The same arguments valid for  $E_-$ . Because  $\dim(E_+) = 4$  and  $E_+ \perp E_-$ , we have an integral structure on  $V = E_+ \oplus_\perp E_-$  inherited from that of  $\text{Cl}_{1,1}$ .

AN INTEGRAL BASIS ON ADMISSIBLE  $\text{Cl}_{3,2}$ -MODULE. The Clifford algebra  $\text{Cl}_{3,2}$  is isomorphic to the space  $\mathbb{C}(4)$ . We choose mutually commuting isometric involutions  $P_1 = J_2J_3J_4J_5$  and  $P_2 = J_1J_2J_3$  and complementary isometries  $T_1 = J_2$  and  $T_2 = J_2J_4$ , satisfying the relations

$$P_1T_1 = -T_1P_1, \quad P_1T_2 = T_2P_1 \quad P_2T_2 = -T_2P_2. \tag{12}$$

We pick up a vector  $w \in V$  such that  $P_1w = P_2w = w$  and  $\langle w, w \rangle_V = 1$ , which existence is guaranteed by Lemma 2.12, item 1). The orthonormal basis is

$$\begin{aligned} v_1 &= w, & v_2 &= J_1w, & v_3 &= J_2w, & v_4 &= J_3w, \\ v_5 &= J_4w, & v_6 &= J_5w, & v_7 &= J_2J_4w, & v_8 &= J_2J_5w, \end{aligned}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 4$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$ ,  $\alpha = 5, \dots, 8$ . In this case we have enough linear dependent relations, defined by  $P_1w = P_2w = w$ , and showing that action of  $J_i$ ,  $i = 1, \dots, 5$ , permutes basis vectors up to sign. The constructed module is admissible, integral, and irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{2,3}$ -MODULE. The Clifford algebra  $Cl_{2,3}$  is isomorphic to the space  $\mathbb{R}(4) \oplus \mathbb{R}(4)$ . We fix two mutually commuting isometric involutions  $P_1 = J_1J_2J_3J_4$  and  $P_2 = J_1J_4J_5$  and two complementary isometries  $T_1 = J_1$  and  $T_2 = J_1J_2$  which satisfy (12). The common eigenspaces of  $P_1$  and  $P_2$  are neutral spaces by Lemma 2.12, part 1). So we find a common eigenvector  $w \in E_{1+} \cap E_{2+} \subset V$  such that  $P_1w = P_2w = w$  with  $\langle w, w \rangle_V = 1$ . We have the simultaneous eigenspace decomposition of the representation space  $V$  by  $P_1$  and  $P_2$  with mutually orthogonal eigenvectors, see Table 1. It gives the

Table 1: Eigenspace decomposition:  $Cl_{2,3}$  case

Involution	Eigenvalue			
$P_1$	+1		-1	
$P_2$	+1	-1	+1	-1
Eigenvectors	$w, J_1J_4w$	$J_5w, J_1J_2w$	$J_1w, J_4w$	$J_2w, J_3w$

orthonormal basis

$$\begin{aligned} v_1 &= w, & v_2 &= J_1w, & v_3 &= J_2w, & v_4 &= J_1J_2w, \\ v_5 &= J_3w, & v_6 &= J_4w, & v_7 &= J_5w, & v_8 &= J_1J_4w, \end{aligned}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 4$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$ ,  $\alpha = 5, \dots, 8$ . As in previous cases we can show that  $J_i$ ,  $i = 1, \dots, 5$  permute basis vectors up to sign. The constructed module is admissible integral, but not irreducible because we know that with any choice of non-null vector  $v \in V$  the 6 vectors  $v, J_1v, J_2v, J_3v, J_4v, J_5v$  are linearly independent but the dimension of the irreducible module is 4.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{4,2}$ -MODULE. The Clifford algebra  $Cl_{4,2}$  is isomorphic to the space  $\mathbb{H}(4)$ . We fix two mutually commuting isometric involutions  $P_1 = J_1J_2J_3J_4$  and  $P_2 = J_1J_2J_5J_6$ . Then we have two complementary isometries  $T_1 = J_1$ ,  $T_2 = J_2J_3$  which satisfy relations (12). So, the simultaneous eigenspace  $E_{1+} \cap E_{2+}$  of  $P_1$  and  $P_2$  is neutral by Lemma 2.12, part 1). Hence we

find a common eigenvector  $w \in V$  satisfying  $P_1w = P_2w = w$  with  $\langle w, w \rangle_V = 1$ . It gives a simultaneous eigenspace decomposition of the representation space  $V$ , generated by Clifford multiplication of  $w$ , and which is presented in Table 2. We

Table 2: Eigenspace decomposition:  $Cl_{4,2}$  case

Involutions	Eigenvalues			
$P_1$	+1		-1	
$P_2$	+1	-1	+1	-1
Eigenvector	$w, J_1J_2w$ $J_1J_3J_5w, J_2J_3J_5w$	$J_5w, J_6w$ $J_1J_3w, J_1J_4w$	$J_3w, J_4w$ $J_1J_5w, J_1J_6w$	$J_1w, J_2w$ $J_3J_5w, J_4J_6w$

choose the orthonormal basis

$$\begin{aligned}
 v_1 &= w, & v_2 &= J_1w, & v_3 &= J_2w, & v_4 &= J_3w, \\
 v_5 &= J_4w, & v_6 &= J_1J_2w, & v_7 &= J_1J_3w, & v_8 &= J_1J_4w, \\
 v_9 &= J_5w, & v_{10} &= J_6w, & v_{11} &= J_1J_5w, & v_{12} &= J_1J_6w, \\
 v_{13} &= J_3J_5w, & v_{14} &= J_4J_6w, & v_{15} &= J_1J_3J_5w, & v_{16} &= J_2J_3J_5w
 \end{aligned}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 8$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$  for  $\alpha = 9, \dots, 16$ . The vectors can be made orthogonal if we apply Lemma 2.9 to operators  $J_1J_3J_5$  and  $J_2J_3J_5$ . The relations  $P_1w = P_2w = w$  show that all other relations will be made also orthogonal and that the operators  $J_j$ ,  $j = 1, \dots, 6$ , permute the basis up to sign. Hence we constructed a minimal admissible integral sub-module of  $Cl_{4,2}$ -module of the dimension 16, which is irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{3,3}$ -MODULE. The Clifford algebra  $Cl_{3,3}$  is isomorphic to the space  $\mathbb{R}(8)$ . We consider isometric mutually commuting involutions  $P_1 = J_1J_2J_4J_5$ ,  $P_2 = J_2J_3J_5J_6$ ,  $P_3 = J_1J_2J_3$ . The table of commutation relations with the generators  $J_i$  help to construct the complementary operators. By  $T(+ \rightarrow +)$  we denote the isometric and  $T(+ \rightarrow -)$  is denote the anti-isometric operator.

Involutions\Generators	$J_1$	$J_2$	$J_3$	$J_4$	$J_5$	$J_6$
$P_1 = J_1J_2J_4J_5$	$a$	$a$	$c$	$a$	$a$	$c$
$P_2 = J_2J_3J_5J_6$	$c$	$a$	$a$	$c$	$a$	$a$
$P_3 = J_1J_2J_3$	$c$	$c$	$c$	$a$	$a$	$a$

Involutions\Comp. op.	$J_1(+ \rightarrow +)$	$J_3(+ \rightarrow +)$	$J_1J_4(+ \rightarrow -)$
$P_1 = J_1J_2J_4J_5$	$a$	$c$	$c$
$P_2 = J_2J_3J_5J_6$		$a$	$c$
$P_3 = J_1J_2J_3$			$a$

The tables show that the common eigenspace  $E_{1+} \cap E_{2+}$  of operators  $P_1$  and  $P_2$  is neutral scalar product space by Lemma 2.12, item 1). If the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E_{1+} \cap E_{2+} \cap E_{3+}$  is negative definite, we apply procedure of Lemma 2.13 and change the operator  $P_3 = J_1 J_2 J_3$  to the operator  $\hat{P}_3 = J_2 J_1 J_3$ . So we can find a vector  $w \in E_{1+} \cap E_{2+} \cap E_{3+} \subset V$  with properties  $P_1 w = P_2 w = P_3 w = w$  and  $\langle w, w \rangle_V = 1$ . Then we obtain Table 3, expressing the simultaneous eigenspace decomposition by the involutions  $P_1, P_2$  and  $P_3$ . It allows us to choose the

Table 3: Eigenspace decomposition:  $Cl_{3,3}$  case

Involutions	Eigenvalues							
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvector	$w$	$J_1 J_4 w$	$J_3 w$	$J_6 w$	$J_1 w$	$J_4 w$	$J_2 w$	$J_5 w$

orthonormal basis (13).

$$\begin{aligned}
 v_1 &= w, & v_2 &= J_1 w, & v_3 &= J_2 w, & v_4 &= J_3 w, \\
 v_5 &= J_4 w, & v_6 &= J_5 w, & v_7 &= J_6 w, & v_8 &= J_1 J_4 w,
 \end{aligned}
 \tag{13}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 4$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$  for  $\alpha = 5, \dots, 8$ . Relations  $P_1 w = P_2 w = P_3 w = w$  leave only 8 linear independent vectors and shows that operators  $J_j$ ,  $j = 1, \dots, 6$  permute the basis up to sign. The minimal admissible integral  $Cl_{3,3}$ -module is of dimension 8 and it is irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{5,2}$ -MODULE. The Clifford algebra  $Cl_{5,2}$  is isomorphic to the space  $\mathbb{H}(4) \oplus \mathbb{H}(4)$ . The isometric involutions  $P_1 = J_1 J_2 J_3 J_4$ ,  $P_2 = J_1 J_2 J_6 J_7$ ,  $P_3 = J_5 J_6 J_7$  mutually commute and the complementary isometric operators are  $T_1 = J_1$ ,  $T_2 = J_2 J_3$ . Here we present the table of their commutation relations that can be obtained from the table analogous to (5).

Involutions \ Comp. op.	$J_1(+ \rightarrow +)$	$J_2 J_3(+ \rightarrow +)$
$P_1 = J_1 J_2 J_3 J_4$	$a$	$c$
$P_2 = J_1 J_2 J_6 J_7$		$a$
$P_3 = J_5 J_6 J_7$		

Since there is no a complementary isometric operator with the property that it commutes with  $P_1$  and  $P_2$  and anti-commutes with  $P_3$  we use the property that isometry  $P_3$  commutes with  $P_1$  and  $P_2$  and we change  $P_3$  to  $\hat{P}_3$  if it is necessary. We find a vector  $w \in V$  such that  $P_1 w = P_2 w = P_3 w = w$  and  $\langle w, w \rangle_V = 1$ . This relations give the orthonormal basis and show that  $J_j$  acts by permutation on this basis. The eigenspace decomposition is presented in Table 4. The basis is given

Table 4: Eigenspace decomposition:  $Cl_{5,2}$  case

Involutions	Eigenvalues							
	+1				-1			
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1		-1		+1		-1	
Eigenvector	$w, J_5w$ $J_1J_3J_6w, J_1J_3J_7w$		$J_6w, J_7w$ $J_1J_3w, J_1J_4w$		$J_3w, J_4w$ $J_1J_6w, J_1J_7w$		$J_1w, J_2w$ $J_3J_6w, J_3J_7w$	

in (14).

$$\begin{aligned}
 v_1 &= w, & v_2 &= J_1w, & v_3 &= J_2w, & v_4 &= J_3w, \\
 v_5 &= J_4w, & v_6 &= J_5w, & v_7 &= J_1J_3w, & v_8 &= J_1J_4w, \\
 v_9 &= J_5w, & v_{10} &= J_6w, & v_{11} &= J_1J_6w, & v_{12} &= J_1J_7w, \\
 v_{13} &= J_3J_6w, & v_{14} &= J_3J_7w, & v_{15} &= J_1J_3J_6w, & v_{16} &= J_1J_3J_7w
 \end{aligned}
 \tag{14}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 8$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$  for  $\alpha = 9, \dots, 16$ . Finally, since the vectors  $w$  and  $J_1J_3J_6w$  and  $J_1J_3J_7w$  need not be orthogonal, we apply Lemma 2.9 to change the vector  $w$  to  $\tilde{w}$ . We constructed an integral basis in the admissible sub-module, which is irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{4,3}$ -MODULE. The Clifford algebra  $Cl_{4,3}$  is isomorphic to the space  $\mathbb{C}(8)$ . We use the mutually commuting isometries  $P_1 = J_1J_2J_3J_4$ ,  $P_2 = J_1J_2J_5J_6$ ,  $P_3 = J_2J_3J_6J_7$ . We start from the common eigenvector  $w \in V$ :  $P_1w = P_2w = P_3w = w$ ,  $\langle w, w \rangle_V = 1$ . The existence of such a common eigenvector  $w$  is guaranteed by Lemma 2.12, part 1) and presence of three complementary isometries  $T_1 = J_1$ ,  $T_2 = J_2J_3$  and  $T_3 = J_1J_2$  with the commutation relations

Involutions \ Comp. op.	$J_1(+ \rightarrow +)$	$J_2J_3(+ \rightarrow +)$	$J_1J_2(+ \rightarrow +)$
$P_1 = J_1J_2J_3J_4$	$a$	$c$	$c$
$P_2 = J_1J_2J_5J_6$		$a$	$c$
$P_3 = J_2J_3J_6J_7$			$a$

Then we have simultaneous eigenspace decomposition of  $P_i$  showed in Table 5. The eigenvectors listed in Table 5 form a basis. Since  $w$  and  $J_1J_3J_6w$  are not necessarily orthogonal, we apply Lemma 2.8 to make them orthogonal. The relations  $P_1w = P_2w = P_3w = w$  show that  $J_i$ ,  $i = 1, \dots, 7$ , permute the basis vectors up to sign. The constructed  $Cl_{4,3}(w)$ -module is admissible, integral, and irreducible.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{3,4}$ -MODULE. The Clifford algebra  $Cl_{3,4}$  is isomorphic to the space  $\mathbb{R}(8) \oplus \mathbb{R}(8)$ . We fix mutually commuting isometric involutions  $P_1 = J_1J_2J_4J_5$ ,  $P_2 = J_2J_3J_5J_6$ ,  $P_3 = J_1J_2J_6J_7$ ,  $P_4 = J_3J_4J_5$ , and the complementary operators  $T_1 = J_1$ ,  $T_2 = J_3$  and  $T_3 = J_7$  having the following

Table 5: Eigenspace decomposition:  $Cl_{4,3}$  case

Involutions	Eigenvalues							
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvector	$w$	$J_7w$	$J_5w$	$J_6w$	$J_4w$	$J_3w$	$J_1w$	$J_2w$
	$J_1J_3J_6w$	$J_1J_2w$	$J_1J_4w$	$J_1J_3w$	$J_1J_5w$	$J_1J_6w$	$J_3J_6w$	$J_1J_7w$

commutation relations.

Involutions\Comp. op.	$J_1(+ \rightarrow +)$	$J_3(+ \rightarrow +)$	$J_7(+ \rightarrow -)$
$P_1 = J_1J_2J_4J_5$	$a$	$c$	$c$
$P_2 = J_2J_3J_5J_6$		$a$	$c$
$P_3 = J_1J_2J_6J_7$			$a$
$P_4 = J_3J_4J_5$			

Since there are no complementary isometric operator with the property that it commutes with  $P_1$  and  $P_2$  and anti-commutes with  $P_3$ , we only know that the simultaneous eigenspaces of  $P_1$  and  $P_2$  are neutral spaces by Lemma 2.12, item 1). The complementary anti-isometry  $T_3 = J_7$  and Lemma 2.12, part 2) guaranties that the space  $E = E_{1+} \cap E_{2+} \cap E_{3+}$  either neutral or sign definite. We consider both possibilities.

Let the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E$  be neutral, then we find  $w \in V$  such that

$$P_1w = P_2w = P_3w = P_4w = w \quad \text{and} \quad \langle w, w \rangle_V = 1. \tag{15}$$

In this case we can directly proceed further and construct an orthonormal basis.

Let the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E$  be negative definite, then we change  $P_3$  to  $\hat{P}_3 = J_2J_1J_6J_7$ , that will make the space  $\hat{E} = E_{1+} \cap E_{2+} \cap \hat{E}_{3+}$  positive definite space.

If the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E$  is positive definite, we do nothing.

Thus, from now on we can assume that the restriction of  $\langle \cdot, \cdot \rangle_V$  on  $E = E_{1+} \cap E_{2+} \cap E_{3+}$  is positive definite. Hence it can happen only two cases for  $w \in E$ :

- (1)  $\langle w + P_4w, w + P_4w \rangle_V = 0$ , that is  $P_4w = -w$ . We change the operator  $P_4 = J_3J_4J_5$  to the operator  $\hat{P}_4 = J_4J_3J_5$ , then the vector  $w$  is a common eigenvector of all four involutions  $P_i$  with the eigenvalue 1, i. e.  $w$  satisfies (15).
- (2)  $\langle w + P_4w, w + P_4w \rangle_V > 0$ . In this case we get the eigenvector  $\hat{w} = w + P_4w$  of  $P_4$  with the eigenvalue 1. Normalising the vector  $\hat{w}$  we obtain that  $\hat{w}$  satisfies (15).

So in both cases the admissible  $\text{Cl}_{3,4}$ -module is 8 dimensional and decomposed into 8 common eigenspaces as shown in Table 6. The orthogonal basis is

Table 6: Eigenspace decomposition:  $\text{Cl}_{3,4}$  case

Involutions	Eigenvalues							
	+1				-1			
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1	-1	+1	-1	+1	-1	+1	-1
$P_4$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvectors	$w$	$J_7w$	$J_3w$	$J_6w$	$J_4w$	$J_1w$	$J_5w$	$J_2w$

given in

$$\begin{aligned}
 v_1 &= w, & v_2 &= J_1w, & v_3 &= J_2w, & v_4 &= J_3w, \\
 v_5 &= J_4w, & v_6 &= J_5w, & v_7 &= J_6w, & v_8 &= J_7w,
 \end{aligned}
 \tag{16}$$

with  $\langle v_\alpha, v_\alpha \rangle_V = 1$ ,  $\alpha = 1, \dots, 4$  and  $\langle v_\alpha, v_\alpha \rangle_V = -1$  for  $\alpha = 5, \dots, 8$ . The relations (15) show that  $J_j$ ,  $j = 1, \dots, 7$ , permute the basis up to sign. The module is admissible integral, and irreducible.

AN INTEGRAL BASIS ON  $\text{Cl}_{6,2}$ -ADMISSIBLE MODULE. The Clifford algebra  $\text{Cl}_{6,2}$  is isomorphic to the space  $\mathbb{H}(8)$ . We consider the three mutually commuting isometric involutions  $P_1 = J_1J_2J_3J_4$ ,  $P_2 = J_1J_2J_5J_6$ , and  $P_3 = J_1J_2J_7J_8$ , and four complementary operators:  $T_1 = J_1$ ,  $T_2 = J_5$ ,  $T_3 = J_7$ , and  $T_4 = J_1J_3J_5J_7$ . Then we have commutation relations between involutions  $P_i$  and complementary operators  $T_j$ .

Involutions\Comp. op.	$J_1(+ \rightarrow +)$	$J_5(+ \rightarrow +)$	$J_7(+ \rightarrow -)$	$J_1J_3J_5J_7(+ \rightarrow -)$
$P_1 = J_1J_2J_3J_4$	$a$	$c$	$c$	$c$
$P_2 = J_1J_2J_5J_6$		$a$	$c$	$c$
$P_3 = J_1J_2J_7J_8$			$a$	$c$

These relations imply the common eigenspace  $E_{1+} \cap E_{2+}$  of the first two involutions is neutral space by Lemma 2.12, item 1). Then we use Lemma 2.14 and conclude that the common eigenspace  $E_{1+} \cap E_{2+} \cap E_{3+}$  of all three involutions  $P_i$  is a neutral space. So, we may find an element  $w$  such that  $P_1w = P_2w = P_3w = w$ , and  $\langle w, w \rangle_V = 1$ . The eigenspace decomposition presented in Table 7. By a direct calculations, we know that each 4 vectors belong to a common eigenspace, especially the vectors  $w$ ,  $J_1J_2w$ ,  $J_1J_3J_5J_7w$ , and  $J_1J_3J_5J_8w$  are in the comon eigenspace with eigenvalue of 1. First two are spacelike and orthonormal and also last two are timelike and orthonormal. Unfortunately, first two and last two need not be orthogonal, so we need to apply Lemma 2.9. Since the operators  $J_1J_3J_5J_7$  and  $J_1J_3J_5J_8$  are anti-involutions and anti-commute each other, they satisfy the

conditions of Lemma 2.9. Hence we can obtain a new common eigenvector  $\tilde{w}$ :  $P_1\tilde{w} = P_2\tilde{w} = P_3\tilde{w} = \tilde{w}$ , such that  $\langle \tilde{w}, J_1J_3J_5J_7\tilde{w} \rangle_V = \langle \tilde{w}, J_1J_3J_5J_8\tilde{w} \rangle_V = 0$ . The linear dependence relations arising from  $P_1\tilde{w} = P_2\tilde{w} = P_3\tilde{w} = \tilde{w}$  shows that all other vectors presented in Table 7 become orthogonal and moreover  $J_j$ ,  $j = 1, \dots, 8$ , permute them up to sign. The vectors listed in Table 7 form an integral basis of 32 dimensional admissible sub-module of  $\text{Cl}_{6,2}$ -module. The sub-module is irreducible.

Table 7: Eigenspace decomposition:  $\text{Cl}_{6,2}$  case

Involutions	Eigenvalues							
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvectors	$w$	$J_7w$	$J_5w$	$J_1J_3w$	$J_3w$	$J_1J_5w$	$J_1J_7w$	$J_1w$
	$J_1J_2w$	$J_8w$	$J_6w$	$J_1J_4w$	$J_4w$	$J_1J_6w$	$J_1J_8w$	$J_2w$
	$J_1J_3J_5J_7w$	$J_2J_3J_5w$	$J_2J_3J_7w$	$J_5J_7w$	$J_1J_5J_7w$	$J_3J_7w$	$J_3J_5w$	$J_3J_5J_7w$
	$J_1J_3J_5J_8w$	$J_2J_3J_6w$	$J_2J_3J_8w$	$J_5J_8w$	$J_1J_5J_8w$	$J_3J_8w$	$J_3J_6w$	$J_3J_5J_8w$

AN INTEGRAL BASIS ON ADMISSIBLE  $\text{Cl}_{5,3}$ -MODULE. The Clifford algebra  $\text{Cl}_{5,3}$  is isomorphic to the space  $\mathbb{H}(8)$ . We consider the three mutually commuting isometric involutions  $P_1 = J_1J_2J_3J_4$ ,  $P_2 = J_1J_2J_6J_7$ , and  $P_3 = J_2J_3J_7J_8$ . In this case, we choose four complementary operators:  $T_1 = J_1$ ,  $T_2 = J_1J_3$ ,  $T_3 = J_8$ , and  $T_4 = J_1J_3J_5J_7$ . Then we have commutation relations between involutions  $P_i$  and complementary operators  $T_j$ .

Involutions \ Comp. op.	$J_1(+ \rightarrow +)$	$J_1J_3(+ \rightarrow +)$	$J_8(+ \rightarrow -)$	$J_1J_3J_5J_7(+ \rightarrow -)$
$P_1 = J_1J_2J_3J_4$	$a$	$c$	$c$	$c$
$P_2 = J_1J_2J_6J_7$		$a$	$c$	$c$
$P_3 = J_2J_3J_7J_8$			$a$	$c$

The common eigenspace  $E_{1+} \cap E_{2+}$  of the first two involutions is neutral space by Lemma 2.12, part 1). Then we use Lemma 2.14 and conclude that the common eigenspace  $E_{1+} \cap E_{2+} \cap E_{3+}$  of all three involutions  $P_i$  is a neutral space. So, we may find an element  $w$  such that  $P_1w = P_2w = P_3w = w$  and  $\langle w, w \rangle_V = 1$ . The eigenspace decomposition presented in Table 8. We need to apply Lemma 2.9 to

Table 8: Eigenspace decomposition:  $\text{Cl}_{5,3}$  case

Involutions	Eigenvalues							
$P_1$	+1				-1			
$P_2$	+1		-1		+1		-1	
$P_3$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvectors	$w$	$J_8w$	$J_6w$	$J_7w$	$J_4w$	$J_3w$	$J_1w$	$J_2w$
	$J_5w$	$J_1J_2w$	$J_1J_4w$	$J_1J_3w$	$J_1J_6w$	$J_1J_7w$	$J_1J_5w$	$J_1J_8w$
	$J_1J_2J_8w$	$J_5J_8w$	$J_5J_6w$	$J_5J_7w$	$J_4J_5w$	$J_3J_5w$	$J_3J_7w$	$J_2J_5w$
	$J_1J_2J_5J_8w$	$J_1J_2J_5w$	$J_1J_4J_5w$	$J_1J_3J_5w$	$J_1J_5J_6w$	$J_1J_5J_7w$	$J_3J_5J_7w$	$J_1J_5J_8w$

operators  $\Omega_1 = J_1J_2J_8$  and  $\Omega_2 = J_1J_2J_5J_8$  to make the vectors in  $E_{1+} \cap E_{2+} \cap E_{3+}$

orthogonal. It also makes all other vectors orthogonal by relations  $P_1w = P_2w = P_3w = w$ . The same relations show that  $J_j, j = 1, \dots, 8$  permute the basis up to sign. It proves that basis listed in Table 8 is integral. The constructed sub-module is irreducible, since its dimension is 32.

AN INTEGRAL BASIS ON ADMISSIBLE  $Cl_{4,4}$ -MODULE. The Clifford algebra  $Cl_{4,4}$  is isomorphic to the space  $\mathbb{R}(16)$ . Choose mutually commuting isometric involutions  $P_1 = J_1J_2J_3J_4, P_2 = J_1J_2J_5J_6, P_3 = J_2J_3J_5J_7,$  and  $P_4 = J_1J_2J_7J_8,$  and four complementary operators  $T_1 = J_1, T_2 = J_1J_3, T_3 = J_1J_2,$  and  $T_4 = J_8.$  Here are the tables of the commutation relations complementary operators:

Involution\Comp. op.	$J_1(+ \rightarrow +)$	$J_1J_3(+ \rightarrow +)$	$J_1J_2(+ \rightarrow +)$	$J_8(+ \rightarrow -)$
$P_1 = J_1J_2J_3J_4$	$a$	$c$	$c$	$c$
$P_2 = J_1J_2J_5J_6$		$a$	$c$	$c$
$P_3 = J_2J_3J_5J_7$			$a$	$c$
$P_4 = J_1J_2J_7J_8$				$a$

We can choose a vector  $w \in V$  satisfying (15) by Lemma 2.12, part 1) and Lemma 2.13. Hence, by the relations (15) we have a simultaneous eigenspace decomposition of a subspace in  $V$  spanned by the 16 common eigenvectors that form the basis listed in Table 9. The relations (15) shows that  $J_j, j = 1, \dots, 8$

Table 9: Eigenspace decomposition:  $Cl_{4,4}$  case

Involutions	Eigenvalues							
$P_1$	+1							
$P_2$	+1				-1			
$P_3$	+1		-1		+1		-1	
$P_4$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvectors	$w$	$J_8w$	$J_1J_2w$	$J_7w$	$J_6w$	$J_1J_4w$	$J_5w$	$J_1J_3w$

Involutions	Eigenvalues							
$P_1$	-1							
$P_2$	+1				-1			
$P_3$	+1		-1		+1		-1	
$P_4$	+1	-1	+1	-1	+1	-1	+1	-1
Eigenvectors	$J_4w$	$J_1J_6w$	$J_3w$	$J_1J_5w$	$J_1J_8w$	$J_1w$	$J_1J_7w$	$J_2w$

permutes the basis up to sign. The 16 dimensional admissible module of  $Cl_{4,4}$  is minimal, irreducible, and integral.

### 6. Admissible modules obtained by tensor product

#### 6.1. Bott periodicity and admissible modules of dimensions $r + s > 8$ .

In this section we present theorems that allow to use the Bott periodicity

$$\text{Cl}_{r+8,s} \cong \text{Cl}_{r+4,s+4} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16), \quad \text{Cl}_{r,s+8} \cong \text{Cl}_{r+4,s+4} \cong \text{Cl}_{r,s} \otimes \mathbb{R}(16)$$

of Clifford algebras in order to prove that  $\text{Cl}_{r,s}$ -modules are integer for  $r + s > 8$ .

**Theorem 6.1.** *Let us assume that  $(V, \langle \cdot, \cdot \rangle_V)$  is an admissible integral  $\text{Cl}_{r,s}$ -module and  $(U, \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{0,8}$ -module, where the representations  $J_{y_j} \in \text{End}(U)$  permute the integral basis of  $U$  up to sign for all orthonormal generators  $y_j$  of the Clifford algebra  $\text{Cl}_{0,8}$ . Then the tensor product vector space  $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{r,s+8}$ -module.*

**Proof.** Let  $(z_1, \dots, z_r, \zeta_1, \dots, \zeta_s)$  be orthonormal generators of the Clifford algebra  $\text{Cl}_{r,s}$  with the quadratic form  $\mathbb{Q}_{r,s}(a) = \sum_{i=1}^r a_i^2 - \sum_{k=1}^s \alpha_k^2$  for  $a = \sum_{i=1}^r a_i z_i + \sum_{k=1}^s \alpha_k \zeta_k$ . Let  $\{y_1, \dots, y_8\}$  be orthonormal generators for  $\text{Cl}_{0,8}$  with quadratic form  $\mathbb{Q}_{0,8}(b) = -\sum_{j=1}^8 b_j^2$  for  $b = \sum_{j=1}^8 b_j y_j$  and, finally let  $(\tilde{z}_1, \dots, \tilde{z}_r, \tilde{\zeta}_1, \dots, \tilde{\zeta}_{s+8})$  be orthonormal generators for the Clifford algebra  $\text{Cl}_{r,s+8}$  with quadratic form  $\mathbb{Q}_{r,s+8}(c) = \sum_{i=1}^r c_i^2 - \sum_{k=1}^{s+8} \zeta_k^2$  for  $c = \sum_{i=1}^r c_i \tilde{z}_i + \sum_{k=1}^{s+8} \zeta_k \tilde{\zeta}_k$ .

We know that the minimal admissible  $\text{Cl}_{0,8}$ -module  $(U, \langle \cdot, \cdot \rangle_U)$  is isomorphic to  $\mathbb{R}^{8,8}$  with quadratic form  $\mathbb{Q}_{8,8}(u) = \sum_{i=1}^8 u_i^2 - \sum_{j=9}^{16} u_j^2$  for  $u = \sum_{i=1}^{16} u_i e_i$ , where  $e_i, i = 1, \dots, 16$  is the standard basis in  $\mathbb{R}^{8,8}$ . This module is also irreducible. Then the endomorphisms  $J_{y_j} \in \text{End}(\mathbb{R}^{8,8})$  are such that  $J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}, j = 1, \dots, 8, J_{y_i} J_{y_j} = -J_{y_j} J_{y_i}$  for  $i \neq j$ .

Now one needs to find  $E \in \text{End}(\mathbb{R}^{8,8})$  satisfying conditions

$$\begin{aligned} E J_{y_j} &= -J_{y_j} E, \quad j = 1, \dots, 8, \quad E^2 = \text{Id}_{\mathbb{R}^{8,8}}, \\ \langle E u, u' \rangle_{\mathbb{R}^{8,8}} &= \langle u, E u' \rangle_{\mathbb{R}^{8,8}} \quad \text{for } u, u' \in \mathbb{R}^{8,8}, \end{aligned} \tag{17}$$

where the scalar product is defined by the quadratic form  $\mathbb{Q}_{8,8}$ . Define  $E = \prod_{j=1}^8 J_{y_j}$  to be a volume form for the Clifford algebra  $\text{Cl}_{0,8}$ . Then it is easy to check that  $E$  satisfies conditions (17).

Denote  $\tilde{V} = V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}}$  and notice that the scalar product  $\langle \cdot, \cdot \rangle_{\tilde{V}}$  is non-degenerate. Set

$$\begin{aligned} \mathbb{R}^{r,s+8} \ni \tilde{z}_i &\mapsto \tilde{J}_{\tilde{z}_i} = J_{z_i} \otimes E \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad i = 1, \dots, r, \\ \mathbb{R}^{r,s+8} \ni \tilde{\zeta}_k &\mapsto \tilde{J}_{\tilde{\zeta}_k} = J_{\zeta_k} \otimes E \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad k = 1, \dots, s, \\ \mathbb{R}^{r,s+8} \ni \tilde{\zeta}_{s+j} &\mapsto \tilde{J}_{\tilde{\zeta}_{s+j}} = \text{Id}_V \otimes J_{y_j} \in \text{End}(V \otimes \mathbb{R}^{8,8}), \quad j = 1, \dots, 8, \end{aligned}$$

where  $J_{z_i}, J_{\zeta_k} \in \text{End}(V)$ ,  $i = 1, \dots, r$ ,  $k = 1, \dots, s$ , such that  $J_{z_i}^2 = -\text{Id}_V$ ,  $J_{\zeta_k}^2 = \text{Id}_V$ , and  $J_{y_j} \in \text{End}(\mathbb{R}^{8,8})$ ,  $j = 1, \dots, 8$ , such that  $J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}$ . Then, it is easy to see that  $\tilde{J}_{z_i}^2 = -\text{Id}_{\tilde{V}}$  for  $i = 1, \dots, r$ ,  $\tilde{J}_{\zeta_k}^2 = \text{Id}_{\tilde{V}}$  for  $k = 1, \dots, s + 8$ . The direct calculation shows that  $\tilde{J}_{z_i}$  and  $\tilde{J}_{\zeta_k}$  mutually anti-commute if the indices are different.

The next step is to verify that the scalar product  $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}}$  satisfies  $\langle \tilde{J}_{\tilde{z}} \tilde{v}, \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{v}, \tilde{J}_{\tilde{z}} \tilde{v}' \rangle_{\tilde{V}} = 0$ . We write  $\tilde{z} = a + b$ ,  $a \in \mathbb{R}^{r,s}$ ,  $b \in \mathbb{R}^{0,8}$ , and  $\tilde{v} = v \otimes u$ ,  $\tilde{v}' = v' \otimes u'$  for  $v, v' \in V$ ,  $u, u' \in \mathbb{R}^{8,8}$ . Then

$$\begin{aligned} & \langle \tilde{J}_{\tilde{z}} \tilde{v}, \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{v}, \tilde{J}_{\tilde{z}} \tilde{v}' \rangle_{\tilde{V}} = \langle \tilde{J}_a \tilde{v}, \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{J}_b \tilde{v}, \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{v}, \tilde{J}_a \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{v}, \tilde{J}_b \tilde{v}' \rangle_{\tilde{V}} \\ & = \langle J_a v \otimes Eu, v' \otimes u' \rangle_{\tilde{V}} + \langle v \otimes u, J_a v' \otimes Eu' \rangle_{\tilde{V}} \\ & + \langle v \otimes J_b u, v' \otimes u' \rangle_{\tilde{V}} + \langle v \otimes u, v' \otimes J_b u' \rangle_{\tilde{V}} \tag{18} \\ & = (J_a v, v')_V \langle Eu, u' \rangle_{\mathbb{R}^{8,8}} + (v, J_a v')_V \langle u, Eu' \rangle_{\mathbb{R}^{8,8}} \\ & + (v, v')_V \underbrace{\left( \langle J_b u, u' \rangle_{\mathbb{R}^{8,8}} + \langle u, J_b u' \rangle_{\mathbb{R}^{8,8}} \right)}_{=0} = 0. \end{aligned}$$

To show that the resulting  $\text{Cl}_{r,s+8}$ -module is integral we assume that both modules  $(V, \langle \cdot, \cdot \rangle_V)$ ,  $(\mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})$  are integral. Then, if  $\{v_\alpha\}_{\alpha=1}^{\dim V}$  and  $\{e_p\}_{p=1}^{16}$  are integral bases for  $V$  and  $\mathbb{R}^{8,8}$  respectively, we denote by  $\{\tilde{v}_n = v_\alpha \otimes e_p\}_{n=1}^{16 \dim V}$  the basis of  $\tilde{V}$ . We assumed that the maps  $J_{y_j}$ ,  $j = 1, \dots, 8$ , permute the basis  $\{e_p\}_{p=1}^{16}$  up to sign. Then the map  $E$  also permutes the basis  $\{e_p\}_{p=1}^{16}$ . We have  $\langle \tilde{J}_{z_i} \tilde{v}_n, \tilde{v}_m \rangle_{\tilde{V}} = \langle J_{z_i} v_\alpha \otimes E e_p, v_\beta \otimes e_q \rangle_{\tilde{V}} = \langle J_{z_i} v_\alpha, v_\beta \rangle_V \cdot \langle E e_p, e_q \rangle_{\mathbb{R}^{8,8}} = \pm 1$  or  $0$ , for all  $i = 1, \dots, r$  and  $\langle \tilde{J}_{\zeta_k} \tilde{v}_n, \tilde{v}_m \rangle_{\tilde{V}} = \pm 1$  or  $0$ , for  $k = 1, \dots, s$  and  $n, m = 1, \dots, 16 \dim V$ . Analogously

$$\langle \tilde{J}_{\zeta_{s+j}} \tilde{v}_n, \tilde{v}_m \rangle_{\tilde{V}} = \langle v_\alpha \otimes J_{y_j} e_p, v_\beta \otimes e_q \rangle_{\tilde{V}} = \langle v_\alpha, v_\beta \rangle_V \cdot \langle J_{y_j} e_p, e_q \rangle_{\mathbb{R}^{8,8}} = \pm 1 \text{ or } 0,$$

for all  $j = 1, \dots, 8$  and  $n, m = 1, \dots, 16 \dim V$ . ■

**Theorem 6.2.** *Let us assume that  $(V, \langle \cdot, \cdot \rangle_V)$  is an admissible integral  $\text{Cl}_{r,s}$ -module and  $(U, (\cdot, \cdot)_U)$  is an admissible integral  $\text{Cl}_{8,0}$ -module, where the representations  $J_{y_j} \in \text{End}(U)$  permute the integral basis of  $U$  up to sign for all orthonormal generators  $y_j$  of the Clifford algebra  $\text{Cl}_{8,0}$ . Then the scalar product vector space  $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{r+8,s}$ -module.*

**Proof.** Let  $(z_1, \dots, z_r, \zeta_1, \dots, \zeta_s)$  be orthonormal generators of the Clifford algebra  $\text{Cl}_{r,s}$ ,  $\{y_1, \dots, y_8\}$  orthonormal generators for  $\text{Cl}_{8,0}$ , and  $\tilde{z}_1, \dots, \tilde{z}_{r+8}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_s$  be orthonormal generators for  $\text{Cl}_{r+8,s}$ . The minimal admissible  $\text{Cl}_{8,0}$ -module  $(U, (\cdot, \cdot)_U)$  is isomorphic to  $\mathbb{R}^{16,0} = \mathbb{R}^{16}$  and it is irreducible. The endomorphisms  $J_{x_j} \in \text{End}(\mathbb{R}^{16})$  satisfy  $J_{x_j}^2 = -\text{Id}_{\mathbb{R}^{16}}$ ,  $j = 1, \dots, 8$ ,  $J_{x_i} J_{x_j} = -J_{x_j} J_{x_i}$  for

$i \neq j$ . Define  $\mathcal{E} = \prod_{j=1}^8 J_{y_j}$  to be a volume form for  $\text{Cl}_{8,0}$ . Then  $\mathcal{E}$  satisfies the conditions  $\mathcal{E}J_{x_j} = -J_{x_j}\mathcal{E}$ ,  $j = 1, \dots, 8$ ,  $\mathcal{E}^2 = \text{Id}$ , and  $(\mathcal{E}u, u')_{\mathbb{R}^{16}} = (u, \mathcal{E}u')_{\mathbb{R}^{16}}$  for all  $u, u' \in \mathbb{R}^{16}$ . Denote  $\tilde{V} = V \otimes \mathbb{R}^{16}$  and non-degenerate scalar product  $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{16}}$ . Set

$$\begin{aligned} \mathbb{R}^{r+8,s} \ni \tilde{z}_i &\mapsto \tilde{J}_{\tilde{z}_i} = J_{z_i} \otimes \mathcal{E} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad i = 1, \dots, r, \\ \mathbb{R}^{r+8,s} \ni \tilde{\zeta}_k &\mapsto \tilde{J}_{\tilde{\zeta}_k} = J_{\zeta_k} \otimes \mathcal{E} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad k = 1, \dots, s, \\ \mathbb{R}^{r+8,s} \ni \tilde{z}_{r+j} &\mapsto \tilde{J}_{\tilde{z}_{r+j}} = \text{Id}_V \otimes J_{x_j} \in \text{End}(V \otimes \mathbb{R}^{16}), \quad j = 1, \dots, 8, \end{aligned}$$

where  $J_{z_i}, J_{\zeta_k} \in \text{End}(V)$ ,  $i = 1, \dots, r$ ,  $k = 1, \dots, s$ , such that  $J_{z_i}^2 = -\text{Id}_V$ ,  $J_{\zeta_k}^2 = \text{Id}_V$ , and  $J_{x_j} \in \text{End}(\mathbb{R}^{16})$ ,  $j = 1, \dots, 8$ , such that  $J_{x_j}^2 = -\text{Id}_{\mathbb{R}^{16}}$ . Then, we finish the proof as in Theorem 6.1. ■

**Theorem 6.3.** *Let us assume that  $(V, \langle \cdot, \cdot \rangle_V)$  is an admissible integral  $\text{Cl}_{r,s}$ -module and  $(U, \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{4,4}$ -module, where the representations  $J_{y_j} \in \text{End}(U)$  permute the integral basis of  $U$  up to sign for all orthonormal generators  $y_j$  of the Clifford algebra  $\text{Cl}_{4,4}$ . Then the tensor product vector space  $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{r+4,s+4}$ -module.*

**Proof.** Let  $(z_1, \dots, z_r, \zeta_1, \dots, \zeta_s)$  be orthonormal generators of the Clifford algebra  $\text{Cl}_{r,s}$ , the collection  $\{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$  orthonormal generators for  $\text{Cl}_{4,4}$ , and let  $\tilde{z}_1, \dots, \tilde{z}_{r+4}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_{s+4}$  be orthonormal generators for the Clifford algebra  $\text{Cl}_{r+4,s+4}$ . The minimal admissible  $\text{Cl}_{4,4}$ -module  $(U, \langle \cdot, \cdot \rangle_U)$  is isomorphic to  $\mathbb{R}^{8,8}$  and it is irreducible. The endomorphisms  $J_{x_i}, J_{y_j} \in \text{End}(\mathbb{R}^{8,8})$  satisfies  $J_{x_i}^2 = -\text{Id}_{\mathbb{R}^{8,8}}$ ,  $J_{y_j}^2 = \text{Id}_{\mathbb{R}^{8,8}}$ ,  $j = 1, \dots, 4$ , and mutually anti-commute.

We define the endomorphism  $\mathcal{E}: \mathbb{R}^{8,8} \rightarrow \mathbb{R}^{8,8}$  by the volume form  $\mathcal{E} = \prod_{i=1}^4 J_{x_i} \prod_{j=1}^4 J_{y_j}$ . Then  $\mathcal{E}^2 = \text{Id}_{\mathbb{R}^{8,8}}$ ,  $\mathcal{E}J_{x_i} = -J_{x_i}\mathcal{E}$ ,  $\mathcal{E}J_{y_i} = -J_{y_i}\mathcal{E}$ ,  $i = 1, 2, 3, 4$ , and  $\langle \mathcal{E}u, u' \rangle_{\mathbb{R}^{8,8}} = \langle u, \mathcal{E}u' \rangle_{\mathbb{R}^{8,8}}$ . Denote  $\tilde{V} = V \otimes U \cong V \otimes \mathbb{R}^{8,8}$  and  $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}}$ . Set also

$$\begin{aligned} \tilde{J}_{\tilde{z}_k} &= J_{z_k} \otimes \mathcal{E}, \quad k = 1, \dots, r, & \tilde{J}_{\tilde{\zeta}_l} &= J_{\zeta_l} \otimes \mathcal{E}, \quad l = 1, \dots, s, \\ \tilde{J}_{\tilde{z}_{k+i}} &= \text{Id}_V \otimes J_{x_i}, \quad i = 1, 2, 3, 4, & \tilde{J}_{\tilde{\zeta}_{s+j}} &= \text{Id}_V \otimes J_{y_j} \quad j = 1, 2, 3, 4. \end{aligned}$$

Then it is easy to see that  $\tilde{J}_{\tilde{z}_i}^2 = -\text{Id}_{\tilde{V}}$ ,  $i = 1, \dots, r + 4$ ,  $\tilde{J}_{\tilde{\zeta}_j}^2 = \text{Id}_{\tilde{V}}$  for  $j = 1, \dots, s + 4$ . It can be shown as in Theorem 6.1 that all  $\tilde{J}_{\tilde{z}_i}, \tilde{J}_{\tilde{\zeta}_j}$  mutually anti-commute and the module  $\tilde{V} = V \otimes U \cong V \otimes \mathbb{R}^{8,8}$  is admissible integral  $\text{Cl}_{r+4,s+4}$ -module. ■

**Proposition 6.4.** *If the admissible integral  $\text{Cl}_{r,s}$ -module  $(V, \langle \cdot, \cdot \rangle_V)$  in Theorems 6.1, 6.2, and 6.3 is of minimal dimension then the resulting  $\text{Cl}_{r,s+8}$ -module*

$(V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})$ ,  $\text{Cl}_{r+8,s}$ -module  $(V \otimes \mathbb{R}^{16}, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{16}})$ , and  $\text{Cl}_{r+4,s+4}$ -module  $(V \otimes \mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})$  are minimal admissible integral modules.

**Proof.** The admissible integral  $\text{Cl}_{0,8}$ -module  $(\mathbb{R}^{8,8}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{8,8}})$  is of minimal dimension equals 16. Since admissible and irreducible modules has periodicity 8 the resulting  $\text{Cl}_{r,s+8}$ -module  $\tilde{V}$  will have minimal dimension equals  $16 \dim V$ . Similar arguments used in the cases of  $\text{Cl}_{r+8,s}$  and  $\text{Cl}_{r+4,s+4}$ -modules. ■

**6.2. Twisted tensor product.**

In this subsection, we give two methods of construction of an admissible module from a given admissible module of lower dimensions by making use of tensor product. We show two cases, that is a construction of an admissible module for  $\text{Cl}_{0,n+2}$  from that of  $\text{Cl}_{n,0}$  and  $\text{Cl}_{0,2}$ . Another one is the construction of an admissible module of  $\text{Cl}_{r+1,s+1}$  from that of  $\text{Cl}_{r,s}$  and  $\text{Cl}_{1,1}$ . Both methods also give us the integral structure from those of the lower dimensions. Note that these constructions not always give the minimal dimensional resulting module, even if the initial admissible modules  $\text{Cl}_{n,0}$  (or  $\text{Cl}_{r,s}$ ) and  $\text{Cl}_{0,2}$  (and  $\text{Cl}_{1,1}$ ) are of minimal dimensions. The resulting module can exceed the minimal dimension two or four times.

**Theorem 6.5.** *Let us assume that  $(V, \langle \cdot, \cdot \rangle_V)$  is an admissible integral  $\text{Cl}_{n,0}$ -module and  $(U, \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{0,2}$ -module. Then the scalar product vector space  $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{0,n+2}$ -module.*

**Proof.** Let  $(z_1, \dots, z_n)$  be orthonormal generators of the Clifford algebra  $\text{Cl}_{n,0}$ ,  $\{y_1, y_2\}$  orthonormal generators for  $\text{Cl}_{0,2}$ , and, finally let  $\tilde{z}_1, \dots, \tilde{z}_{n+2}$  be orthonormal generators of  $\text{Cl}_{0,n+2}$ . The map

$$\tilde{z}_i \mapsto z_i \otimes y_1 y_2, \quad \text{if } i = 1, \dots, n, \quad \tilde{z}_{n+1} \mapsto 1 \otimes y_1, \quad \tilde{z}_{n+2} \mapsto 1 \otimes y_2.$$

defines the isomorphism between the Clifford algebras  $\text{Cl}_{0,n+2}$  and  $\text{Cl}_{n,0} \otimes \text{Cl}_{0,2}$ . The minimal admissible  $\text{Cl}_{0,2}$ -module  $(U, \langle \cdot, \cdot \rangle_U)$  is isomorphic to  $\mathbb{R}^{2,2}$ . Then the endomorphisms  $J_{y_1}$  and  $J_{y_2}$  from  $\text{End}(\mathbb{R}^{2,2})$  are written in the standard basis  $\{e_i\}_{i=1}^4$  as follows

$$J_{y_1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad J_{y_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We have  $J_{y_1}^2 = \text{Id}$ ,  $J_{y_2}^2 = \text{Id}$ ,  $J_{y_1} J_{y_2} = -J_{y_2} J_{y_1}$ , and  $J_{y_1} J_{y_2} e_1 = e_2$ ,  $J_{y_1} e_1 = e_3$ ,  $J_{y_2} e_1 = e_4$ . We need to find  $\mathcal{F} \in \text{End}(\mathbb{R}^{2,2})$  satisfying conditions

$$\mathcal{F} J_{y_1} = -J_{y_1} \mathcal{F}, \quad \mathcal{F} J_{y_2} = -J_{y_2} \mathcal{F}, \quad \mathcal{F}^2 = -\text{Id}, \quad (19)$$

$$\langle \mathcal{F}u, u' \rangle_{\mathbb{R}^{2,2}} = \langle u, \mathcal{F}u' \rangle_{\mathbb{R}^{2,2}} \quad \text{for } u, u' \in \mathbb{R}^{2,2}. \tag{20}$$

Conditions (19) imply that the matrix for  $\mathcal{F}$  has the form

$$\begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & -a & -b \\ -d & c & b & -a \end{pmatrix} \quad \text{with} \quad \begin{aligned} a^2 - b^2 - c^2 - d^2 &= -1 \\ ab = 0, \quad bc = 0, \quad bd &= 0. \end{aligned}$$

Checking the condition (20) we find that  $b = 0$ . Denote  $\tilde{V} = V \otimes \mathbb{R}^{2,2}$  and  $\langle \cdot, \cdot \rangle_{\tilde{V}} = (\cdot, \cdot)_V \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$ . Set

$$\tilde{J}_{\tilde{z}_i} = J_{z_i} \otimes \mathcal{F}, \quad i = 1, \dots, n, \quad \tilde{J}_{\tilde{z}_{n+1}} = \text{Id}_V \otimes J_{y_1}, \quad \text{and} \quad \tilde{J}_{\tilde{z}_{n+2}} = \text{Id}_V \otimes J_{y_2},$$

where  $J_{z_i} \in \text{End}(V)$ ,  $i = 1, \dots, n$ , such that  $J_{z_i}^2 = -\text{Id}$ . Then, it is easy to see that  $\tilde{J}_{\tilde{z}_i}^2 = \text{Id}_{\tilde{V}}$  for  $i = 1, \dots, n+2$  and the representations anti-commute. To verify that the scalar product  $\langle \cdot, \cdot \rangle_{\tilde{V}} = (\cdot, \cdot)_V \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$  satisfies  $\langle \tilde{J}_{\tilde{z}} \tilde{v}, \tilde{v}' \rangle_{\tilde{V}} + \langle \tilde{v}, \tilde{J}_{\tilde{z}} \tilde{v}' \rangle_{\tilde{V}} = 0$ , we write  $\tilde{z} = a + b$ , where  $a \in \mathbb{R}^{n,0}$ ,  $b \in \mathbb{R}^{0,2}$ , and  $\tilde{v} = v \otimes u$ ,  $\tilde{v}' = v' \otimes u'$  for  $v, v' \in V$ ,  $u, u' \in \mathbb{R}^{2,2}$ . Then we argue as in (18).

To show that the resulting  $\text{Cl}_{0,n+2}$ -module is integral we assume that both modules  $(V, \langle \cdot, \cdot \rangle_V)$ ,  $(\mathbb{R}^{2,2}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}})$  are integral and choose special form of the map  $\mathcal{F}$ , for instance we set  $a = d = 0$  and  $c = 1$ . Then, if  $\{v_\alpha\}$  and  $\{e_p\}$  are integral bases for  $V$  and  $\mathbb{R}^{2,2}$  respectively, then for  $\tilde{v}_n = v_\alpha \otimes e_p$

$$\langle \tilde{J}_{\tilde{z}_i} \tilde{v}_n, \tilde{v}_m \rangle_{\tilde{V}} = \langle J_{z_i} v_\alpha \otimes \mathcal{F} e_p, v_\beta \otimes e_q \rangle_{\tilde{V}} = \langle J_{z_i} v_\alpha, v_\beta \rangle_V \cdot \langle \mathcal{F} e_p, e_q \rangle_{\mathbb{R}^{2,2}} = \pm 1 \text{ or } 0.$$

Note that  $\mathcal{F} \neq \pm J_{y_1} J_{y_2}$ . ■

**Theorem 6.6.** *Let us assume that  $(V, \langle \cdot, \cdot \rangle_V)$  is an admissible integral  $\text{Cl}_{r,s}$ -module and  $(U, \langle \cdot, \cdot \rangle_U)$  is the minimal dimensional admissible integral  $\text{Cl}_{1,1}$ -module, then  $(V \otimes U, \langle \cdot, \cdot \rangle_V \langle \cdot, \cdot \rangle_U)$  is an admissible integral  $\text{Cl}_{r+1,s+1}$ -module.*

**Proof.** Let  $(z_1, \dots, z_r, \zeta_1, \dots, \zeta_s)$  be orthonormal generators of the Clifford algebra  $\text{Cl}_{r,s}$ ,  $\{x, y\}$  orthonormal generators for  $\text{Cl}_{1,1}$ , and let  $(\tilde{z}_1, \dots, \tilde{z}_{r+1}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_{s+1})$  be orthonormal generators for the Clifford algebra  $\text{Cl}_{r+1,s+1}$ . It is known that there is the isomorphism between  $\text{Cl}_{r+1,s+1}$  and  $\text{Cl}_{r,s} \otimes \text{Cl}_{1,1}$  given by the following relation between the generators  $\tilde{z}_1, \dots, \tilde{z}_{r+1}, \tilde{\zeta}_1, \dots, \tilde{\zeta}_{s+1}$  of  $\text{Cl}_{r+1,s+1}$  and generators of  $\text{Cl}_{r,s} \otimes \text{Cl}_{1,1}$ :

$$\tilde{z}_i \cong z_i \otimes xy, \quad i = 1, \dots, r, \quad \tilde{\zeta}_j \cong \zeta_j \otimes xy, \quad j = 1, \dots, s, \quad \tilde{z}_{r+1} \cong 1 \otimes x, \quad \tilde{\zeta}_{s+1} \cong 1 \otimes y.$$

The admissible  $\text{Cl}_{1,1}$ -module  $(U, \langle \cdot, \cdot \rangle_U)$  is isomorphic to  $\mathbb{R}^{2,2}$ . Then the endomorphisms  $J_{y_1}$  and  $J_{y_2}$  from  $\text{End}(\mathbb{R}^{2,2})$  are written in the standard basis  $\{e_i\}_{i=1}^4$

as follows

$$J_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad J_y J_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then  $J_x^2 = -\text{Id}_{\mathbb{R}^{2,2}}$ ,  $J_y^2 = \text{Id}_{\mathbb{R}^{2,2}}$ ,  $J_x J_y = -J_y J_x$ , and  $J_x e_1 = e_2$ ,  $J_y e_1 = e_3$ ,  $J_y J_x e_1 = e_4$ . We need to find the endomorphism  $F: \mathbb{R}^{2,2} \rightarrow \mathbb{R}^{2,2}$  such that

$$F^2 = \text{Id}_{\mathbb{R}^{2,2}}, \quad F J_x = -J_x F, \quad F J_y = -J_y F, \quad \langle F u, u' \rangle_{\mathbb{R}^{2,2}} = \langle u, F u' \rangle_{\mathbb{R}^{2,2}}$$

for all  $u, u' \in \mathbb{R}^{2,2}$ . Checking these conditions, we find that the matrix for  $F$  has the form

$$\begin{pmatrix} a & b & c & d \\ b & -a & d & -c \\ -c & -d & -a & -b \\ -d & c & -b & a \end{pmatrix} \quad \text{with} \quad \begin{aligned} a^2 + b^2 - c^2 - d^2 &= 1 \\ bc &= ad. \end{aligned}$$

Denote  $\tilde{V} = V \otimes \mathbb{R}^{2,2}$  and the scalar product  $\langle \cdot, \cdot \rangle_{\tilde{V}} = \langle \cdot, \cdot \rangle_V \cdot \langle \cdot, \cdot \rangle_{\mathbb{R}^{2,2}}$ . Set also

$$\begin{aligned} \tilde{J}_{z_i} &= J_{z_i} \otimes F, & i &= 1, \dots, r, & \tilde{J}_{\zeta_j} &= J_{\zeta_j} \otimes F, & j &= 1, \dots, s, \\ \tilde{J}_{z_{r+1}} &= \text{Id}_V \otimes J_x, & & & \tilde{J}_{\zeta_{s+1}} &= \text{Id}_V \otimes J_y. \end{aligned}$$

We finish the proof as in Theorem 6.5, choosing the map  $F$  with the matrix having the entries  $a = 1$  and  $b = c = d = 0$ . Observe that  $J_x J_y \neq \pm F$  since if we choose  $d = \pm 1$ , then  $a$  or  $b$  must be different from zero. ■

### 7. Integral $\text{Cl}_{r,s}$ -modules with $r + s \geq 9$

To get the integral structure on  $\text{Cl}_{7,2}$ -module, we obtain the integral structure on  $\text{Cl}_{1,8}$ -module by Theorem 6.1 and then use Theorem 3.1 and Corollary 3.2. For  $\text{Cl}_{6,3}$ -module we exploit the isomorphisms  $\text{Cl}_{6,3} \cong \text{Cl}_{5,2} \otimes \text{Cl}_{1,1}$  and Theorem 6.6. Counting dimensions, we conclude that  $\text{Cl}_{6,3}$  is of minimal dimension. We use Theorem 6.3 and show the existence of integral basis due to the isomorphisms

$$\text{Cl}_{5,4} \cong \text{Cl}_{1,0} \otimes \text{Cl}_{4,4}, \quad \text{Cl}_{4,5} \cong \text{Cl}_{0,1} \otimes \text{Cl}_{4,4}.$$

For the rest of cases we use Theorems 3.1, 6.1, 6.2, and 6.3, and Corollary 3.2.

## 8. Final remarks

(1) The constructed admissible integral modules show that the corresponding general  $H$ -type Lie algebras admit integer structural constants. The natural question arises: how many different Lie algebras are behind the general  $H$ -type Lie algebras if we discard the presence of the scalar product? Having in hand the integral basis, it is easier to answer this question. Both of the Lie algebras based on spaces  $\mathbb{R}^{1,1} \oplus_{\perp} \mathbb{R}^{0,1}$  and  $\mathbb{R}^{2,0} \oplus_{\perp} \mathbb{R}^{1,0}$ , corresponding to  $\text{Cl}_{1,0}$ - and  $\text{Cl}_{0,1}$ -modules, are isomorphic to the three dimensional Heisenberg algebra, although the metrics are different. The 6-dimensional Lie algebras based on vector spaces  $\mathbb{R}^{4,0} \oplus_{\perp} \mathbb{R}^{2,0}$ ,  $\mathbb{R}^{2,2} \oplus_{\perp} \mathbb{R}^{0,2}$ , related to  $\text{Cl}_{2,0}$ - and  $\text{Cl}_{0,2}$ -modules, are also isomorphic, but not isomorphic to the Lie algebra based on the space  $\mathbb{R}^{2,2} \oplus_{\perp} \mathbb{R}^{1,1}$ , arising from  $\text{Cl}_{1,1}$ -module. This can be proved by making use of the above constructed integral basis. In general, the isomorphism  $\text{Cl}_{r,s+1} \cong \text{Cl}_{s,r+1}$  preserving the admissibility of modules, does not imply that the resulting general  $H$ -type algebras are isomorphic. The details including these and more general relations between  $\mathbb{R}^{k,k} \oplus_{\perp} \mathbb{R}^{r,s}$  is treated in a forthcoming paper, see also [26] and [27] for the classification of low dimensional nilpotent Lie algebras.

(3) Although we used the results in [10] for the classical cases for the sake of simplicity, applying our method for the construction of the integral basis on  $\text{Cl}_{r,0}$ -modules, it can be shown that our method not only gives the integral basis but also detects the possible signature of the scalar product on the admissible module. Remind that for  $\text{Cl}_{r,0}$ -modules the scalar product is positive definite, but for the rest of cases is neutral.

(4) The integral admissible  $\text{Cl}_{0,s}$ -modules could be also found by using Theorem 6.5. In this case the resulting  $\text{Cl}_{0,3}$ - and  $\text{Cl}_{0,5}$ -modules would be of minimal dimensions, but  $\text{Cl}_{0,s}$ -modules for  $r = 2, 4, 6, 7, 8$  would exceed the minimal dimension twice.

(5) The table that can be found in arXiv:1305.6814 shows the Clifford algebras, where the circled Clifford algebras has admissible modules of double dimension comparing with irreducible modules. It is also easy to see the symmetry of Clifford algebras with respect to the axis  $r - s = -1$  that allows to use Theorem 3.1.

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Received November 12, 2013  
and in final form February 27, 2014