

Basic Relative Invariants of Homogeneous Cones

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Abstract. The purpose of this paper is to give an explicit expression to the basic relative invariants of a homogeneous cone in terms of the polynomials introduced by Vinberg in 1963. We present a closed formula that exhibits how we obtain the basic relative invariants at one time. We also factorize the determinant of the right multiplication operators of the corresponding clan V by giving an explicit expression to the exponents of the basic relative invariants in terms of the data of V .

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Introduction

Homogeneous convex cones in Euclidean vector spaces containing no entire lines (homogeneous cones for short in what follows) correspond in a one-to-one way, up to isomorphisms, to certain non-associative algebras called clans with unit element by Vinberg [8]. Homogeneous cones provide many examples of non-reductive prehomogeneous vector spaces (see [6] for prehomogeneous vector spaces), and as such the associated basic relative invariants are indispensable objects. Here the groups that we consider for the relative invariance are the split solvable Lie groups acting simply transitively on homogeneous cones. In this paper, we study the basic relative invariants of a homogeneous cone Ω by focusing on an inductive structure of the corresponding clan V . During the inductive arguments, there arise naturally representations of clans in the sense of Ishi [4]. The crucial step is to assign an orbit \mathcal{O} in the closure $\overline{\Omega}$ of Ω to every representation φ . This is done by using some of the results of Graczyk and Ishi [1], and Ishi [2] (see below for details).

Let us describe the contents of this paper in more detail. Let V be a clan of rank r with unit element e_V . The product in V is denoted by Δ . Fix an inner product $\langle \cdot | \cdot \rangle$ of V and a complete system c_1, \dots, c_r of orthogonal primitive idempotents. Then we have the corresponding normal decomposition of V written

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as $V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj}$. The space $\mathfrak{h} := \{L_x; x \in V\}$ of left multiplication operators L_x is a split solvable Lie algebra, and we denote by H the corresponding connected and simply connected Lie group. The H -orbit in V of e_V is a homogeneous cone Ω on which H acts simply transitively. The transpose of the left multiplication operators with respect to $\langle \cdot | \cdot \rangle$ defines another clan structure ∇ in V , which we call the dual clan of V . In order to describe the inductive structure of V mentioned above, we put $E = \bigoplus_{k>1} V_{k1}$ and $W = \bigoplus_{1 < j \leq k \leq r} V_{kj}$. Then

$$V = \mathbb{R}c_1 \oplus E \oplus W. \tag{0.1}$$

General elements x of V are denoted by $\lambda c_1 + \xi + w$ ($\lambda \in \mathbb{R}$, $\xi \in E$, $w \in W$) without any comments. We note that W is a subclan in both structures (V, Δ) and (V, ∇) . When we consider the clan (W, Δ) , we usually write simply W for it in the following. We also note $E \nabla W \subset E$, and this enables us to consider the representation (φ, E) of (W, ∇) defined by $\varphi(w)\xi = \xi \nabla w$ (Proposition 2.1). Let Q be the bilinear map associated with φ (see (1.10)), and $\underline{\varphi}$ the lower triangular part of φ (see (1.8)). Then, by putting $y = \mu c_1 + \eta + v$, the product Δ in V is described as

$$x \Delta y = (\lambda\mu)c_1 + (\mu\xi + \frac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi, \eta) + w \Delta v).$$

We know by Ishi and Nomura [5] that the irreducible factors of the determinant of the right multiplication operators are the basic relative invariants. This is a key to studying the basic relative invariants by the above inductive structure. We calculate $\text{Det } R_x$ ($x \in V$) in Proposition 3.2 for the right multiplication operators R_x of V to obtain

$$\text{Det } R_x = \lambda^{1+\dim E - \dim W} \text{Det}(R_{\lambda w - \frac{1}{2}Q[\xi]}^W) \quad (x \in V),$$

where R_w^W denotes the right multiplication operator of W and $Q[\xi] := Q(\xi, \xi)$. Then we see in Theorem 3.4 that the basic relative invariants $\Delta_1^V(x), \dots, \Delta_r^V(x)$ of V are described by the basic relative invariants $\Delta_2^W(w), \dots, \Delta_r^W(w)$ of W as

$$\Delta_1^V(x) = \lambda, \quad \Delta_j^V(x) = \lambda^{-\alpha_j} \Delta_j^W(\lambda w - \frac{1}{2}Q[\xi]) \quad (j = 2, \dots, r), \tag{0.2}$$

where $\alpha_2, \dots, \alpha_r$ are non-negative integers.

To determine the column vector $\alpha := {}^t(\alpha_2, \dots, \alpha_r)$ with α_j in (0.2), we need to look at representations of clans more closely. In this paper we do this in a general setting. Let (φ, E) be a selfadjoint representation of the dual clan (V, ∇) , anew. By using some of the results on Riesz measures in Graczyk and Ishi [1], and on Gindikin-Riesz distributions in Ishi [2], we can assign, to φ , an H -orbit in the closure $\overline{\Omega}$ of Ω . On the other hand, Ishi [2, Theorem 3.5] tells us that the H -orbits in $\overline{\Omega}$ are described as Hc_ε , where $c_\varepsilon := \varepsilon_1 c_1 + \dots + \varepsilon_r c_r$ with $\varepsilon = {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$. In this way, we attach ε to φ , and call φ an ε -representation.

Applying the above procedure to the representation (φ, E) of (W, ∇) arising from the decomposition (0.1), we take $\varepsilon = {}^t(\varepsilon_2, \dots, \varepsilon_r) \in \{0, 1\}^{r-1}$ so that φ

is an ε -representation. Then Theorem 5.1 shows that α is described by using the multiplier matrix σ_W of W (see (1.4) for definition of the multiplier matrix) as

$$\alpha = \sigma_W(\mathbf{1} - \varepsilon),$$

where $\mathbf{1} = {}^t(1, \dots, 1)$. Moreover the multiplier matrix σ_V of V is written as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W \varepsilon & \sigma_W \end{pmatrix}. \tag{0.3}$$

The formula (0.3) enables us to determine σ_V inductively. For any $k = 1, 2, \dots, r - 1$, let $V^{[k]}$ and $E^{[k]}$ be the subspaces of V respectively defined by

$$V^{[k]} := \bigoplus_{k < l \leq m \leq r} V_{ml}, \quad E^{[k]} := \bigoplus_{m > k} V_{mk}.$$

Then $V^{[k]}$ are subclans of (V, ∇) and $E^{[k]} \nabla V^{[k]} \subset E^{[k]}$. By the latter property, we have representations $\mathcal{R}^{[k]}$ defined by $\mathcal{R}^{[k]}(x^k)\xi^k = \xi^k \nabla x^k$ of $V^{[k]}$ on $E^{[k]}$. We put $\varepsilon^{[k]} := \varepsilon(\mathcal{R}^{[k]}) \in \{0, 1\}^{r-k}$, and define an $r \times r$ matrix \mathcal{E}_k by

$$\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[k]} & I_{r-k} \end{pmatrix} \quad (k = 1, \dots, r - 1).$$

Then σ_V is described in Theorem 6.1 as

$$\sigma_V = \mathcal{E}_{r-1} \mathcal{E}_{r-2} \cdots \mathcal{E}_1.$$

Now that the multiplier matrix σ_V is determined, we are able to have an explicit expression of the basic relative invariants $\Delta_j^V(x)$. Introducing the polynomials $D_1(x), \dots, D_r(x)$ defined by Vinberg [8, the formula (25), p. 385] (see also Ishi [3]), we obtain in Theorem 6.2

$$\Delta_1^V(x) = D_1(x), \quad \Delta_j^V(x) = \frac{D_j(x)}{\prod_{i < j} D_i(x)^{-\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj}}} \quad (j = 2, \dots, r).$$

Our final objective is to give an expression to the positive integers n_1, \dots, n_r appearing in the formula

$$\text{Det } R_x = \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad (x \in V).$$

The row vector $\underline{n} := (n_1, \dots, n_r)$, called the basic index of V in this paper, is expressed as (Theorem 6.5)

$$\underline{n} = \underline{m} \sigma_V^{-1},$$

where $\underline{m} = (m_1, \dots, m_r)$ is a row vector the entries of which are positive integers coming from the normal decomposition of V (see (6.6) for definition).

We organize this paper as follows. Section 1 collects definitions and facts about clans and homogeneous cones. In Section 2, we study an inductive structure of a clan V . Section 3 is devoted to expressing the basic relative invariants of V with non-negative integers α_j as in (0.2). In Section 4, we introduce ε -representations. The non-negative integers α_j in (0.2) are determined in Section 5. In the last section, Section 6, we present the multiplier matrix σ_V explicitly, and consequently we obtain an explicit expression of the basic relative invariants of V by the polynomials introduced by Vinberg. We also give an explicit expression to the basic index.

1. Preliminaries

Let V be a finite-dimensional real vector space with a bilinear product Δ . For $x \in V$, we denote by L_x the left multiplication operator $L_x y = x \Delta y$ ($y \in V$). The pair (V, Δ) (or simply V) is called a *clan* if the following three conditions are satisfied:

- (C1) (V, Δ) is left-symmetric: $L_x L_y - L_y L_x = L_{x \Delta y - y \Delta x}$ for all $x, y \in V$,
- (C2) there exists $s \in V^*$ such that $s(x \Delta y)$ defines an inner product in V ,
- (C3) for each $x \in V$, the operator L_x has only real eigenvalues.

Linear forms s with the property (C2) are said to be *admissible*. We denote by R_x the right multiplication operator $R_x y = y \Delta x$ ($y \in V$). In this paper, we always assume that a clan has a unit element.

Let V be a clan with unit element e_V . The rank of V is denoted by r . This means that there is a complete system of orthogonal primitive idempotents c_1, \dots, c_r with $c_1 + \dots + c_r = e_V$ such that we have the following decomposition of V :

$$V = \bigoplus_{1 \leq j \leq k \leq r} V_{kj}, \tag{1.1}$$

where $V_{jj} = \mathbb{R}c_j$ ($j = 1, \dots, r$) and

$$V_{kj} := \{x \in V; L_{c_i} x = \frac{1}{2}(\delta_{ij} + \delta_{ik})x, R_{c_i} x = \delta_{ij}x \ (i = 1, \dots, r)\} \quad (j < k).$$

The decomposition (1.1) is called the *normal decomposition* of V associated with c_1, \dots, c_r . The multiplication rules are

$$\begin{aligned} V_{ji} \Delta V_{lk} &= \{0\} \quad (\text{if } i \neq k, l), & V_{kj} \Delta V_{ji} &\subset V_{ki}, \\ V_{ji} \Delta V_{ki} &\subset V_{jk} \text{ or } V_{kj} & (\text{according to } j \geq k \text{ or } j \leq k). \end{aligned} \tag{1.2}$$

By (C1) and (C3), the space $\mathfrak{h} := \{L_x; x \in V\}$ of left multiplication operators forms a split solvable Lie algebra. We note here that \mathfrak{h} is linearly isomorphic to V . Let $H := \exp \mathfrak{h}$ be the connected and simply connected Lie group corresponding to \mathfrak{h} . We denote by Ω the H -orbit in V through e_V . We know that Ω is a proper open convex cone in V , and H acts on Ω simply transitively. For every $\varepsilon := {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$, we put $c_\varepsilon := \varepsilon_1 c_1 + \dots + \varepsilon_r c_r$. Then we have $c_\varepsilon \in \bar{\Omega}$, the closure of Ω , and let $\mathcal{O}_\varepsilon := Hc_\varepsilon$ be the H -orbit of c_ε . Note that $\mathcal{O}_1 = \Omega$, where $\mathbf{1} = {}^t(1, \dots, 1)$. By Ishi [2, Theorem 3.5], the H -orbit decomposition of $\bar{\Omega}$ is described as

$$\bar{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon.$$

By introducing the lexicographic order among the subspaces V_{kj} in (1.1), we see that every L_x ($x \in V$) is simultaneously represented by a lower triangular matrix. Then for each $h \in H$, there exist unique $h_{jj} > 0$ ($j = 1, \dots, r$) and $v_{kj} \in V_{kj}$ ($1 \leq j < k \leq r$) such that by setting $T_{jj} := (2 \log h_{jj})L_{c_j}$ and $L_j := \sum_{k>j} L_{v_{kj}}$, we have

$$h = (\exp T_{11})(\exp L_1)(\exp T_{22}) \cdots (\exp L_{r-1})(\exp T_{rr}). \tag{1.3}$$

This expression of h is useful for our later inductive arguments.

A function f on Ω is said to be *relatively invariant* under the action of H if there exists a one-dimensional representation χ of H with which we have $f(hx) = \chi(h)f(x)$ for all $h \in H$ and $x \in \Omega$. To each such χ , there corresponds an r -tuple $\underline{\tau} := (\tau_1, \dots, \tau_r) \in \mathbb{R}^r$ so that

$$\chi(h) = (h_{11})^{2\tau_1} \cdots (h_{rr})^{2\tau_r} \quad (\text{for } h \text{ as in (1.3)}).$$

We call $\underline{\tau}$ the *multiplier* of a relatively invariant function f and write $\chi = \chi_{\underline{\tau}}$.

Theorem 1.1 (Ishi [3]). *There exist irreducible relatively H -invariant polynomial functions $\Delta_1, \dots, \Delta_r$ by which any relatively H -invariant polynomial function p on V is written as*

$$p(x) = (\text{const}) \cdot \Delta_1(x)^{n_1} \cdots \Delta_r(x)^{n_r} \quad ((n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r).$$

Moreover Ω is described as

$$\Omega = \{x \in V; \Delta_1(x) > 0, \dots, \Delta_r(x) > 0\}.$$

The polynomials $\Delta_1(x), \dots, \Delta_r(x)$ are called the *basic relative invariants* of the cone Ω . They are also called the basic relative invariants of the clan V in this paper. We assume that the numbering of the basic relative invariants is given by the procedure of Ishi [3] according to c_1, \dots, c_r . For $j = 1, \dots, r$, let $\underline{\sigma}_j = (\sigma_{j1}, \dots, \sigma_{jr})$ be the multiplier of the basic relative invariant $\Delta_j(x)$, and we place them in an $r \times r$ matrix as

$$\sigma_V := \begin{pmatrix} \underline{\sigma}_1 \\ \vdots \\ \underline{\sigma}_r \end{pmatrix} = (\sigma_{jk}). \tag{1.4}$$

In this paper, we call σ_V the *multiplier matrix* of the clan V . We note that by the procedure of Ishi [3], the matrix σ_V is lower triangular with all $\sigma_{jk} \in \mathbb{Z}_{\geq 0}$ and $\sigma_{jj} = 1$ ($j = 1, \dots, r$). In particular, σ_V is invertible. We also note that the basic relative invariants are homogeneous polynomials, and we have by definition

$$\deg \Delta_j = \sigma_{j1} + \cdots + \sigma_{jj} \quad (j = 1, \dots, r). \tag{1.5}$$

Now we assume that the inner product $\langle \cdot | \cdot \rangle$ of V is given by an admissible linear form s_0 . Let us define a bilinear product ∇ in V through

$$\langle x \nabla y | z \rangle = \langle y | x \Delta z \rangle \quad (x, y, z \in V). \tag{1.6}$$

Then it turns out that the product ∇ defines a clan structure in V . The clan (V, ∇) is called the *dual clan* of (V, Δ) . The linear form s_0 is also an admissible linear form for (V, ∇) . In fact, we have $s_0(x \nabla y) = \langle x \nabla y | e_V \rangle = \langle y | x \rangle$. Moreover it is easy to see from (1.6) that e_V is also a unit element of (V, ∇) . The cone corresponding to (V, ∇) is the dual cone Ω^* of Ω with respect to the inner product $\langle \cdot | \cdot \rangle$, where

$$\Omega^* := \{x \in V; \langle x | y \rangle > 0 \text{ for all } y \in \overline{\Omega} \setminus \{0\}\}.$$

Let $L_x^\nabla: V \ni y \mapsto x \nabla y$ be the left multiplication operator by $x \in V$ of (V, ∇) .

Proposition 1.2. *The following relationships hold between Δ and ∇ .*

- (1) For $x, y \in V$, we have $x \Delta y + x \nabla y = y \Delta x + y \nabla x$.
- (2) For $i = 1, \dots, r$, one has $L_{c_i}^\nabla = L_{c_i}$.

Proof. (1) For any $z \in V$, we have by (C1)

$$\begin{aligned} \langle x \Delta y - y \Delta x | z \rangle &= s_0((x \Delta y - y \Delta x) \Delta z) = s_0(x \Delta (y \Delta z) - y \Delta (x \Delta z)) \\ &= \langle x | y \Delta z \rangle - \langle y | x \Delta z \rangle = \langle y \nabla x - x \nabla y | z \rangle. \end{aligned}$$

Hence, the assertion is proved.

(2) Suppose $x_{kj} \in V_{kj}$ ($j \leq k$). Then for any $y \in V$, we have

$$\langle c_i \nabla x_{kj} | y \rangle = \langle x_{kj} | c_i \Delta y \rangle = \langle x_{kj} | c_i \Delta y_{kj} \rangle = \frac{1}{2}(\delta_{ij} + \delta_{ik}) \langle x_{kj} | y \rangle,$$

where y_{kj} is the V_{kj} -component of y . Thus we get $L_{c_i}^\nabla x_{kj} = \frac{1}{2}(\delta_{ij} + \delta_{ik})x_{kj} = L_{c_i}x_{kj}$ for any $x_{kj} \in V_{kj}$. This shows $L_{c_i}^\nabla = L_{c_i}$. ■

Proposition 1.2 (2) shows that c_1, \dots, c_r form also a complete system of orthogonal primitive idempotents of the dual clan (V, ∇) . We denote by R_x^∇ the right multiplication operator of (V, ∇) by $x \in V$. By (1) and (2) of Proposition 1.2, we get $R_{c_i}^\nabla x_{kj} = \delta_{ik}x_{kj}$ for any $x_{kj} \in V_{kj}$ ($j \leq k$) and $i = 1, \dots, r$. Thus we have

$$V_{kj} = \left\{ x \in V; L_{c_i}^\nabla x = \frac{1}{2}(\delta_{ij} + \delta_{ik})x, R_{c_i}^\nabla x = \delta_{ik}x \ (i = 1, \dots, r) \right\}.$$

This implies that the decomposition (1.1) also serves as a normal decomposition of (V, ∇) relative to c_1, \dots, c_r with the multiplication rules

$$\begin{aligned} V_{ji} \nabla V_{lk} &= \{0\} \quad (\text{if } j \neq k, l), \quad V_{ji} \nabla V_{kj} \subset V_{ki}, \\ V_{ki} \nabla V_{kj} &\subset V_{ji} \text{ or } V_{ij} \quad (\text{according to } i \leq j \text{ or } i \geq j). \end{aligned} \tag{1.7}$$

For later sections, we give here the definition of a representation of the clan (V, ∇) . Let E be a real Euclidean vector space with inner product $\langle \cdot | \cdot \rangle_E$. We denote by $\mathcal{L}(E)$ the vector space of all linear operators on E . For a linear map $\varphi: V \rightarrow \mathcal{L}(E)$, let $\underline{\varphi}$ and $\overline{\varphi}$ be the ‘‘lower triangular part’’ and the ‘‘upper triangular part’’ of φ associated with c_1, \dots, c_r respectively given by

$$\begin{aligned} \underline{\varphi}(x) &:= \frac{1}{2} \sum_{j=1}^r x_{jj} \varphi(c_j) + \sum_{j < k} \varphi(c_k) \varphi(x_{kj}) \varphi(c_j), \\ \overline{\varphi}(x) &:= \frac{1}{2} \sum_{j=1}^r x_{jj} \varphi(c_j) + \sum_{j < k} \varphi(c_j) \varphi(x_{kj}) \varphi(c_k), \end{aligned} \tag{1.8}$$

where we write $x \in V$ as $x = \sum x_{jj}c_j + \sum_{j < k} x_{kj}$ according to (1.1). A linear map $\varphi: V \rightarrow \mathcal{L}(E)$ is called a *selfadjoint representation* of the clan (V, ∇) if $\varphi(x)$ is a selfadjoint operator for every $x \in V$ and if the following condition is satisfied:

$$\varphi(x \nabla y) = \overline{\varphi}(x) \varphi(y) + \varphi(y) \underline{\varphi}(x) \quad (x, y \in V). \tag{1.9}$$

We always require that $\varphi(e_V)$ is the identity operator. In this paper, we only consider selfadjoint representations, and often drop the adjective selfadjoint for simplicity. Associated with φ , we define a symmetric bilinear map $Q: E \times E \rightarrow V$ through

$$\langle \varphi(x)\xi | \eta \rangle_E = \langle Q(\xi, \eta) | x \rangle \quad (\xi, \eta \in E, x \in V). \tag{1.10}$$

From now on, we put $Q[\xi] := Q(\xi, \xi)$ and $Q[E] := \{Q[\xi]; \xi \in E\}$.

2. Inductive structure of a clan

Vinberg [8, Chapter II, Section 4] tells us that any clan has an inductive structure. In this section, we present the inductive structure explicitly by using a representation. Let V be a clan of rank r with unit element e_V . We keep to the notation used in Section 1. By the normal decomposition (1.1), we put

$$E = \bigoplus_{k>1} V_{k1}, \quad W = \bigoplus_{1<j\leq k\leq r} V_{kj}.$$

Then, we have

$$V = \mathbb{R}c_1 \oplus E \oplus W. \tag{2.1}$$

We denote general elements x of V by

$$\lambda c_1 + \xi + w \quad (\lambda \in \mathbb{R}, \xi \in E, w \in W) \tag{2.2}$$

without any comments. Inner products of E and W are taken respectively as the restrictions of the inner product $\langle \cdot | \cdot \rangle$ of V . The multiplication rules (1.2) and (1.7) tell us that W is a subclan in both structures (V, Δ) and (V, ∇) with unit element $e_W := e_V - c_1$. When we consider the clan (W, Δ) , we usually write simply W for it in what follows. We have $E \nabla W \subset E$ again by (1.7), and this enables us to consider the linear map $\varphi: W \rightarrow \mathcal{L}(E)$ defined by

$$\varphi(w)\xi = \xi \nabla w \quad (w \in W, \xi \in E). \tag{2.3}$$

Proposition 2.1. *The pair (φ, E) is a selfadjoint representation of (W, ∇) .*

Proof. Let $\xi \in E$ and $w \in W$. The multiplication rules (1.2) yield $\xi \Delta w = 0$. Thus we have by Proposition 1.2 (1)

$$\varphi(w)\xi = \xi \nabla w = w \Delta \xi + w \nabla \xi = (L_w + L_w^\nabla)\xi. \tag{2.4}$$

Since $L_w^\nabla = (L_w)^*$ by (1.6), we see that $\varphi(w)$ is selfadjoint. Furthermore for all $w, v \in W$ and $\xi \in E$, we have by (C1) and (2.4)

$$\begin{aligned} \varphi(w \nabla v)\xi &= w \nabla (\xi \nabla v) + (\xi \nabla w - w \nabla \xi) \nabla v \\ &= w \nabla (\xi \nabla v) + (w \Delta \xi) \nabla v. \end{aligned} \tag{2.5}$$

Taking the lower and the upper triangular part of $\varphi(x)$ in (2.4), we see that $\underline{\varphi}(w) = L_w$ and $\overline{\varphi}(w) = L_w^\nabla$. Then the last term of (2.5) is equal to

$$(\overline{\varphi}(w)\varphi(v) + \varphi(v)\underline{\varphi}(w))\xi.$$

Since $\varphi(e_W)$ is obviously the identity operator, the pair (φ, E) is now a selfadjoint representation of the clan (W, ∇) . ■

Let Q be the symmetric bilinear map associated with φ .

Proposition 2.2. *The product Δ in V is described as*

$$x \Delta y = (\lambda\mu)c_1 + (\mu\xi + \frac{1}{2}\lambda\eta + \underline{\varphi}(x)\eta) + (Q(\xi, \eta) + w \Delta v),$$

where we set $y = \mu c_1 + \eta + v \in V$ as we do for x in (2.2).

Proof. By definition of E , we have $c_1 \Delta \eta = \frac{1}{2}\eta$ and $\xi \Delta c_1 = \xi$. Next, the multiplication rules (1.2) imply that

$$c_1 \Delta W = W \Delta c_1 = E \Delta W = \{0\}.$$

By the proof of Proposition 2.1, we obtain $w \Delta \eta = \underline{\varphi}(w)\eta$. Moreover, we have for all $w \in W$

$$\langle Q(\xi, \eta) | w \rangle = \langle \varphi(w)\xi | \eta \rangle = \langle \xi \nabla w | \eta \rangle = \langle w | \xi \Delta \eta \rangle.$$

Thus Q is described as

$$Q(\xi, \eta) = \xi \Delta \eta \quad (\xi, \eta \in E). \quad (2.6)$$

Since the product Δ is bilinear, the proof is now completed. \blacksquare

We now describe the action of H on V with respect to the decomposition (2.1). Let $\mathfrak{h}_W := \{L_w; w \in W\}$ be the Lie algebra of left multiplication operators of W and $H_W := \exp \mathfrak{h}_W$ the corresponding Lie subgroup of H . Let $h \in H$ and we write h as in (1.3). Then putting

$$\xi_h = v_{21} + \cdots + v_{r1} \in E, \quad h_W = \exp T_{22} \exp L_2 \cdots \exp T_{rr} \in H_W,$$

we have $h = (\exp T_{11})(\exp L_{\xi_h})h_W$, where $T_{11} = (2 \log h_{11})L_{c_1}$ ($h_{11} > 0$).

Lemma 2.3. *Let $y = \mu c_1 + \eta + v \in V$. Then*

$$hy = \mu(h_{11})^2 c_1 + h_{11}(\mu\xi_h + h_W\eta) + \left(\frac{1}{2}\mu Q[\xi_h] + Q(\xi_h, h_W\eta) + h_W v \right).$$

Proof. We first note that the multiplication rules (1.2) tell us that

$$h_W c_1 = c_1, \quad h_W \eta \in E, \quad h_W v \in W.$$

Next we have again by (1.2)

$$R_{c_1}|_W = \text{id}_W, \quad E \Delta E \subset W, \quad E \Delta W = \{0\},$$

so that recalling (2.6), we obtain

$$\begin{aligned} (\exp L_{\xi_h})c_1 &= c_1 + \xi_h \Delta c_1 + \frac{1}{2}\xi_h \Delta (\xi_h \Delta c_1) = c_1 + \xi_h + \frac{1}{2}Q[\xi_h], \\ (\exp L_{\xi_h})\eta &= \eta + Q(\xi_h, \eta), \quad (\exp L_{\xi_h})v = v. \end{aligned}$$

Finally,

$$(\exp T_{11})c_1 = (h_{11})^2c_1, \quad (\exp T_{11})\eta = h_{11}\eta, \quad (\exp T_{11})v = v.$$

These observations yield that

$$\begin{aligned} hc_1 &= \exp T_{11}(c_1 + \xi_h + \frac{1}{2}Q[\xi_h]) = (h_{11})^2c_1 + h_{11}\xi_h + \frac{1}{2}Q[\xi_h], \\ h\eta &= \exp T_{11}(h_W\eta + Q(\xi_h, h_W\eta)) = h_{11}h_W\eta + Q(\xi_h, h_W\eta), \\ hv &= h_Wv. \end{aligned}$$

Hence we arrive at the formula in the lemma. ■

3. Inductive description of basic relative invariants

In this section, we study the basic relative invariants by considering the right multiplication operators of clans. The reason for this is the following theorem.

Theorem 3.1 (Ishi–Nomura [5]). *The basic relative invariants are the irreducible factors of the determinant of the right multiplication operators.*

Let V be a clan of rank r and we decompose V as in (2.1). We keep to the notation of the previous sections. By Proposition 2.2, the right multiplication operators R_x by $x \in V$ of the clan V are described as

$$R_x = \begin{pmatrix} \lambda & 0 & 0 \\ \frac{1}{2}\xi & \lambda \text{id}_E & R_\xi \\ 0 & R_\xi & R_w^W \end{pmatrix},$$

where R_w^W is the right multiplication operator of W , and we note that $R_\xi(W) \subset E$ and $R_\xi(E) \subset W$. As in [7, Proposition 4.1], we have the following proposition.

Proposition 3.2. *For $x \in V$, one has*

$$\text{Det } R_x = \lambda^{1+\dim E - \dim W} \text{Det}(R_{\lambda w - \frac{1}{2}Q[\xi]}^W).$$

Let $\Delta_2^W(w), \dots, \Delta_r^W(w)$ be the basic relative invariants of W , where the labeling $2, \dots, r$ is in accordance with the complete system of orthogonal primitive idempotents c_2, \dots, c_r of W . Noting that the basic relative invariants are homogeneous polynomials, we denote by d_j the homogeneous degree of $\Delta_j^W(w)$. Theorem 3.1 and Proposition 3.2 tell us that the basic relative invariants of V are exhausted by the polynomial λ and the irreducible factors of $\Delta_j^W(\lambda w - \frac{1}{2}Q[\xi])$ ($j = 2, \dots, r$).

Proposition 3.3. *For each $j = 2, \dots, r$, the only possible factor of the polynomial $\Delta_j^W(\lambda w - \frac{1}{2}Q[\xi])$ is λ^{α_j} for some non-negative integer $\alpha_j \leq d_j$.*

Proof. We put

$$P_j(x) := \Delta_j^W(\lambda w - \frac{1}{2}Q[\xi]) \quad (j = 2, \dots, r).$$

Since $x = \lambda c_1 + \xi + w \mapsto \lambda w - \frac{1}{2}Q[\xi]$ is a quadratic map, the polynomial $P_j(x)$ is a homogeneous polynomial of degree $2d_j$. Moreover, in each monomial of $P_j(x)$, the degrees of λ and w are the same. In particular, since $\Delta_j^W(\lambda w) = \lambda^{d_j} \Delta_j^W(w)$, the degree of the polynomial $P_j(x)$ with respect to λ is d_j , and the coefficient of λ^{d_j} is the irreducible polynomial $\Delta_j^W(w)$. Hence $P_j(x)$ is written as

$$P_j(x) = \lambda^{d_j} \Delta_j^W(w) + \lambda^{d_j-1} p_j^{(1)}(w, \xi) + \cdots + p_j^{(d_j)}(w, \xi),$$

where each $p_j^{(k)}(w, \xi)$ is a polynomial of w and ξ of degree $d_j + k$, and its degree with respect to w is strictly lower than d_j . Since $\Delta_j^W(w)$ is irreducible, $P_j(x)$ is factorized as

$$P_j(x) = \lambda^{\alpha_j} (\lambda^{d_j-\alpha_j} \Delta_j^W(w) + \lambda^{d_j-\alpha_j-1} p_j^{(1)}(w, \xi) + \cdots + p_j^{(d_j-\alpha_j)}(w, \xi)), \quad (3.1)$$

where α_j is the maximal non-negative integer such that $p_j^{(d_j-\alpha_j)} \neq 0$. Thus $\lambda^{-\alpha_j} P_j(x)$ is irreducible. The proof is now completed. ■

Propositions 3.2 and 3.3 immediately give the following theorem.

Theorem 3.4. *The basic relative invariants $\Delta_j^V(x)$ ($j = 1, \dots, r$) of V are given by*

$$\Delta_j^V(x) = \begin{cases} \lambda & (j = 1), \\ \lambda^{-\alpha_j} \Delta_j^W(\lambda w - \frac{1}{2}Q[\xi]) & (j = 2, \dots, r). \end{cases} \quad (3.2)$$

4. ε -representations

In order to determine the non-negative integers α_j in Theorem 3.4, we need to investigate the relationship between representations of clans and the range of the corresponding quadratic maps. Let V be a clan of rank r and (φ, E) a representation of (V, ∇) . We maintain the notation in Section 1. We begin with the following lemma.

Lemma 4.1. *For $x \in V$ and $\xi, \eta \in E$, one has*

$$(\exp L_x)Q(\xi, \eta) = Q((\exp \varphi(x))\xi, (\exp \varphi(x))\eta).$$

Proof. Recalling the definition of $\varphi(x)$ in (1.8), we have $\varphi(x)^* = \overline{\varphi(x)}$. Thus for any $y \in V$ we have by (1.9)

$$\begin{aligned} \langle L_x Q(\xi, \eta) | y \rangle &= \langle Q(\xi, \eta) | x \nabla y \rangle = \langle \varphi(x \nabla y) \xi | \eta \rangle_E \\ &= \langle \overline{\varphi(x)} \varphi(y) \xi + \varphi(y) \varphi(x) \xi | \eta \rangle_E \\ &= \langle \varphi(y) \xi | \varphi(x) \eta \rangle_E + \langle \varphi(y) \varphi(x) \xi | \eta \rangle_E \\ &= \langle Q(\xi, \varphi(x) \eta) + Q(\varphi(x) \xi, \eta) | y \rangle. \end{aligned}$$

This equation yields

$$L_x Q(\xi, \eta) = Q(\varphi(x) \xi, \eta) + Q(\xi, \varphi(x) \eta) \quad (x \in V, \xi, \eta \in E).$$

Thus we obtain the lemma. ■

Corollary 4.2. Q is Ω -positive, that is, $Q[\xi] \in \overline{\Omega} \setminus \{0\}$ for all $\xi \in E \setminus \{0\}$.

Proof. For each $x \in \Omega^*$, we take $x_0 \in V$ such that $(\exp L_{x_0}^\nabla)e_V = x$. Since $L_{x_0}^\nabla = (L_{x_0})^*$, we have by Lemma 4.1 and (1.10)

$$\langle Q[\xi] | x \rangle = \langle (\exp L_{x_0})Q[\xi] | e_V \rangle = \langle Q[(\exp \varphi(x_0))\xi] | e_V \rangle = \|(\exp \varphi(x_0))\xi\|_E^2 \geq 0,$$

where $\|\xi\|_E^2 := \langle \xi | \xi \rangle_E$ is the norm of E . Hence $Q[\xi] \in \overline{\Omega}$. Moreover, since $\exp \varphi(x_0)$ is invertible, we see that $Q[\xi] = 0$ if and only if $\xi = 0$. Thus the corollary is proved. ■

Let (φ, E) be any representation of (V, ∇) and Q the corresponding bilinear map. The Riesz measure μ_Q associated with the quadratic map $Q[\xi]$ is, by definition, the image of the Lebesgue measure $d\xi$ on E by $Q[\xi]$ (cf. Graczyk and Ishi [1]). In other words, we have for any measurable function f on V

$$\int_V f(x)\mu_Q(dx) = \int_E f(Q[\xi])d\xi.$$

Since Q is Ω -positive by Corollary 4.2, the Riesz measure μ_Q is supported by $\overline{\Omega}$. We now show that to φ we can assign an H -orbit \mathcal{O}_ε in $\overline{\Omega}$ ($\varepsilon \in \{0, 1\}^r$). In this case, we say that φ is an ε -representation.

Now Lemma 4.1 and (1.10) yield that for any $x \in V$ and $h \in H$

$$\text{Det } \varphi(h^*x) = \text{Det}(h^*\varphi(x)h) = (\text{Det } h)^2 \text{Det } \varphi(x).$$

Thus $\text{Det } \varphi(x)$ is a relatively H -invariant polynomial and its multiplier $\mathbf{l} = {}^t(l_1, \dots, l_r) \in \mathbb{Z}_{\geq 0}^r$ satisfies

$$\text{Det } \varphi(\lambda_1 c_1 + \dots + \lambda_r c_r) = (\lambda_1)^{l_1} \dots (\lambda_r)^{l_r} \quad (\lambda_1, \dots, \lambda_r \in \mathbb{R})$$

with $l_j = \dim \varphi(c_j)E$ ($j = 1, \dots, r$). Let $\mathcal{R}_\mathbf{s}$ ($\mathbf{s} \in \mathbb{R}^r$) be the Gindikin–Riesz distribution defined in Ishi [2] (in that paper, it is simply called the Riesz distribution). Then, by [1, (3.29)], we have

$$\mu_Q = \pi^{\dim E/2} \mathcal{R}_{\mathbf{l}/2}. \tag{4.1}$$

Since μ_Q is a positive measure, so is $\mathcal{R}_{\mathbf{l}/2}$. Let Ξ be the Gindikin–Wallach set, which is the set of \mathbf{s} for $\mathcal{R}_\mathbf{s}$ to be a positive measure (see Ishi [2, Theorem 6.2]). Thus by (4.1), we obtain $\mathbf{l}/2 \in \Xi$. Putting $d_{kj} := \dim V_{kj}$ for $1 \leq j < k \leq r$, we define $\mathbf{l}^{(j)} \in \mathbb{R}^r$ ($j = 1, \dots, r$) inductively by $\mathbf{l}^{(1)} := \mathbf{l}$ and, for $i = 1, \dots, r - 1$,

$$\mathbf{l}^{(i+1)} := \begin{cases} \mathbf{l}^{(i)} - {}^t(0, \dots, 0, d_{i+1,i}, \dots, d_{ri}) & \text{if } l_i^{(i)} > 0, \\ \mathbf{l}^{(i)} & \text{if } l_i^{(i)} \leq 0. \end{cases}$$

Further we define $\varepsilon(\varphi) = {}^t(\varepsilon_1, \dots, \varepsilon_r) \in \{0, 1\}^r$ by

$$\varepsilon_i = \begin{cases} 1 & \text{if } l_i^{(i)} > 0, \\ 0 & \text{if } l_i^{(i)} \leq 0 \end{cases} \quad (i = 1, \dots, r).$$

Then by [2, p. 183], $\mathcal{R}_{\mathbf{l}/2}$ is a measure on $\mathcal{O}_{\varepsilon(\varphi)}$. Using Proposition 3.10 and Theorem 3.13 of [1], we see that the support of $\mathcal{R}_{\mathbf{l}/2}$ is equal to $\overline{\mathcal{O}_{\varepsilon(\varphi)}}$, the closure of $\mathcal{O}_{\varepsilon(\varphi)}$. These observations together with (4.1) give the following proposition.

Proposition 4.3. φ is an $\varepsilon(\varphi)$ -representation.

5. Calculation of α_j

In this section, we determine the non-negative integers $\alpha_2, \dots, \alpha_r$ that appeared in Theorem 3.4. To do so, we return to the representation (φ, E) of (W, ∇) defined by (2.3). In particular, the orbits which we consider are the H_W -orbits in W . Let $e := e_W$ be the unit element of W . Let Ω_W be the H_W -orbit of e in W . Then Ω_W is the homogeneous cone corresponding to the clan (W, Δ) . Let us put $\varepsilon = \varepsilon(\varphi) = {}^t(\varepsilon_2, \dots, \varepsilon_r) \in \{0, 1\}^{r-1}$ so that φ is an ε -representation. The corresponding quadratic map $Q[\xi]$ satisfies $Q[E] = \overline{\mathcal{O}_\varepsilon}$, where \mathcal{O}_ε is the H_W -orbit through $c_\varepsilon := \varepsilon_2 c_2 + \dots + \varepsilon_r c_r$ in the closure $\overline{\Omega_W}$ of Ω_W .

Let us consider the polynomials $\Delta_j^W(\lambda w - w_\varepsilon)$ ($\lambda \in \mathbb{R}, w \in W, w_\varepsilon \in \overline{\mathcal{O}_\varepsilon}$). If $w \in \Omega_W$, then putting $w = he$ with $h \in H_W$, we have by the relative invariance

$$\Delta_j^W(\lambda w - w_\varepsilon) = \Delta_j^W(w) \Delta_j^W(\lambda e - h^{-1} w_\varepsilon). \tag{5.1}$$

For $j = 2, \dots, r$, we put

$$\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon) := \Delta_j^W(\lambda e - w_\varepsilon) = \lambda^{d_j} + \lambda^{d_j-1} q_j^{(1)}(w_\varepsilon) + \dots + q_j^{(d_j)}(w_\varepsilon), \tag{5.2}$$

where $q_j^{(k)}$ ($k = 1, \dots, d_j$) is a polynomial function on $\overline{\mathcal{O}_\varepsilon}$ of degree k . By the coefficient comparison of (5.1) with (3.1) relative to λ , the polynomials $q_j^{(k)}(w_\varepsilon)$ are the zero-polynomials on $\overline{\mathcal{O}_\varepsilon}$ for $k = d_j - \alpha_j + 1, \dots, d_j$, and the polynomial $q_j^{(d_j - \alpha_j)}(w_\varepsilon)$ is non-zero. In particular, $\lambda^{-\alpha_j} \tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon)$ is an irreducible polynomial. By (5.1), we see that $\lambda^{-\alpha_j} \Delta_j^W(\lambda w - w_\varepsilon)$ is also irreducible.

We put

$$\boldsymbol{\alpha} := \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_r \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{r-1}, \quad \mathbf{d} := \begin{pmatrix} d_2 \\ \vdots \\ d_r \end{pmatrix} \in \mathbb{Z}_{\geq 0}^{r-1}. \tag{5.3}$$

Let $\sigma_V = (\sigma_{jk})_{1 \leq j, k \leq r}$ be the multiplier matrix of V . We note here that by comparison of the degrees in (3.2) the multiplier matrix σ_V is described as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \mathbf{d} - \boldsymbol{\alpha} & \sigma_W \end{pmatrix}, \tag{5.4}$$

where σ_W is the multiplier matrix of W . Thus σ_W is equal to the $(r-1) \times (r-1)$ matrix $\sigma_W = (\sigma_{jk})_{2 \leq j \leq k \leq r}$.

Theorem 5.1. *It holds that $\boldsymbol{\alpha} = \sigma_W(\mathbf{1} - \boldsymbol{\varepsilon})$ with $\boldsymbol{\varepsilon} = {}^t(\varepsilon_2, \dots, \varepsilon_r)$ as above. In other words,*

$$\alpha_j = \sum_{k=2}^r \sigma_{jk} (1 - \varepsilon_k) \quad (j = 2, \dots, r). \tag{5.5}$$

Moreover, the multiplier matrix σ_V of V is written as

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W \boldsymbol{\varepsilon} & \sigma_W \end{pmatrix}. \tag{5.6}$$

Proof. We shall prove the theorem by induction on the rank r of V . We consider the polynomial $\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon)$ ($\lambda \in \mathbb{R}$, $w_\varepsilon \in \overline{\mathcal{O}_\varepsilon}$) defined in (5.2). First, we assume that $r = 2$. In this case, we have $W = \mathbb{R}c_2$ and $E = V_{21}$. For the case $\varepsilon_2 = 0$, we have $\mathcal{O}_\varepsilon = \{0\}$ and hence $\tilde{\mathcal{P}}_2^\varepsilon(\lambda, 0) = \lambda$. If $\varepsilon_2 = 1$, then $\tilde{\mathcal{P}}_2^\varepsilon(\lambda, w_\varepsilon) = \lambda - w_\varepsilon$ does not have the factor λ . Thus in both cases we have $\alpha_2 = 1 - \varepsilon_2$. Next we assume that $r \geq 3$, and that the theorem is true for clans of rank $r - 1$. Let us put $W' = \bigoplus_{2 < j \leq k \leq r} V_{kj}$ and $E' = \bigoplus_{k > 2} V_{k2}$. Then

$$W = \mathbb{R}c_2 \oplus E' \oplus W'. \tag{5.7}$$

Applying Proposition 2.1 to this decomposition, we see that the following linear map $\varphi': W' \rightarrow \mathcal{L}(E')$ is a representation of (W', ∇) :

$$\varphi'(w')\xi' := \xi' \nabla w' \quad (w' \in W', \xi' \in E').$$

We write general elements $w \in W$ as $w = w_{22}c_2 + \xi' + w'$ ($w_{22} \in \mathbb{R}$, $\xi' \in E'$, $w' \in W'$) without any comments. We put $e' := e - c_2$, which is the unit element of W' . Let $\Delta_3^{W'}(w'), \dots, \Delta_r^{W'}(w')$ be the basic relative invariants of W' . The equation (5.4) for σ_W tells us that the multiplier matrix $\sigma_{W'}$ of W' is equal to the $(r - 2) \times (r - 2)$ matrix $\sigma_{W'} = (\sigma_{jk})_{3 \leq j, k \leq r}$. We consider the Lie algebra $\mathfrak{h}' = \{L_{w'}; w' \in W'\}$ of the left multiplication operators of W' and the corresponding Lie group $H' = \exp \mathfrak{h}'$. Let $\Omega' := H'e'$ be the homogeneous cone associated with (W', Δ) . For each $\delta = {}^t(\delta_3, \dots, \delta_r) \in \{0, 1\}^{r-2}$, we put $c'_\delta := \delta_3c_3 + \dots + \delta_r c_r$. Then $c'_\delta \in \overline{\Omega'}$ and let $\mathcal{O}'_\delta := H'c'_\delta \subset \overline{\Omega'}$. Moreover, for $j = 3, \dots, r$, let $\alpha'_j: \{0, 1\}^{r-2} \rightarrow \mathbb{Z}_{\geq 0}$ and polynomials $\mathcal{P}_j^\delta(\lambda, y_\delta)$ ($\lambda \in \mathbb{R}$, $y_\delta \in \overline{\mathcal{O}'_\delta}$) be

$$\alpha'_j(\delta) := \sum_{k=3}^r \sigma_{jk}(1 - \delta_k), \quad \mathcal{P}_j^\delta(\lambda, y_\delta) := \Delta_j^{W'}(\lambda e' - y_\delta) \quad (j = 3, \dots, r). \tag{5.8}$$

By the induction hypothesis, there exist irreducible polynomials $\mathcal{F}_j^\delta(\lambda, y_\delta)$ ($j = 3, \dots, r$) such that

$$\mathcal{P}_j^\delta(\lambda, y_\delta) = \lambda^{\alpha'_j(\delta)} \mathcal{F}_j^\delta(\lambda, y_\delta) \quad (\lambda \in \mathbb{R}, y_\delta \in \overline{\mathcal{O}'_\delta}). \tag{5.9}$$

In order to know what power of λ is factored out from $\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon)$, it is clearly sufficient by continuity that we argue by restricting the variable w_ε to \mathcal{O}_ε . Thus we assume $w_\varepsilon \in \mathcal{O}_\varepsilon$ and take $h \in H_W$ such that $w_\varepsilon = hc_\varepsilon$. We note that h is decomposed as $h = (\exp T_{22})(\exp \xi'_h)h'$, where $T_{22} = (2 \log h_{22})L_{c_2}$ with some $h_{22} > 0$, $\xi'_h \in E'$ and $h' \in H'$. Let Q' be the symmetric bilinear map associated with φ' and we put

$$\varepsilon' := \varepsilon(\varphi') = \begin{pmatrix} \varepsilon'_3 \\ \vdots \\ \varepsilon'_r \end{pmatrix} \in \{0, 1\}^{r-2},$$

so that φ' is an ε' -representation. By Theorem 3.4 applied to W with the decomposition (5.7) and by the induction hypothesis, we have

$$\Delta_2^W(w) = w_{22}, \quad \Delta_j^W(w) = (w_{22})^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}(w_{22}w' - \frac{1}{2}Q'[\xi']) \quad (j \geq 3). \tag{5.10}$$

Let us put $\widehat{\varepsilon} = {}^t(\varepsilon_3, \dots, \varepsilon_r) \in \{0, 1\}^{r-2}$. Applying Lemma 2.3 to the clan W with (5.7) for $y = c_\varepsilon$, we obtain

$$w_\varepsilon = hc_\varepsilon = \varepsilon_2(h_{22})^2c_2 + \varepsilon_2h_{22}\xi'_h + \left(h'c'_\varepsilon + \frac{\varepsilon_2}{2}Q'[\xi'_h]\right).$$

Putting $y_{\widehat{\varepsilon}} = h'c'_\varepsilon$, we have

$$\lambda e - w_\varepsilon = (\lambda - \varepsilon_2(h_{22})^2)c_2 - \varepsilon_2h_{22}\xi'_h + \left(\lambda e' - y_{\widehat{\varepsilon}} - \frac{\varepsilon_2}{2}Q'[\xi'_h]\right). \tag{5.11}$$

(i) The case $j = 2$. In this case, we have $\widetilde{\mathcal{P}}_2^\varepsilon(\lambda, w_\varepsilon) = \lambda - \varepsilon_2(h_{22})^2$. If $\varepsilon_2 = 0$ then $\widetilde{\mathcal{P}}_2^\varepsilon(\lambda, w_\varepsilon) = \lambda$, and if $\varepsilon_2 = 1$ then $\widetilde{\mathcal{P}}_2^\varepsilon(\lambda, w_\varepsilon)$ does not have the factor λ . Hence in both cases we have $\alpha_2 = 1 - \varepsilon_2$. Since $\sigma_{2k} = \delta_{2k}$ ($k = 2, \dots, r$), we obtain (5.5) for α_2 .

(ii) The case $j = 3, \dots, r$. (a) We first assume that $\varepsilon_2 = 0$. In this case, (5.11) reduces to $\lambda e - w_\varepsilon = \lambda c_2 + (\lambda e' - y_{\widehat{\varepsilon}})$. Let d'_j be the homogeneous degree of $\Delta_j^{W'}(w')$ ($j = 3, \dots, r$). Using (5.10) and (5.9), we obtain

$$\begin{aligned} \Delta_j^W(\lambda c_2 + (\lambda e' - y_{\widehat{\varepsilon}})) &= \lambda^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}(\lambda(\lambda e' - y_{\widehat{\varepsilon}})) = \lambda^{-\alpha'_j(\varepsilon') + d'_j} \mathcal{P}_j^{\widehat{\varepsilon}}(\lambda, y_{\widehat{\varepsilon}}) \\ &= \lambda^{-\alpha'_j(\varepsilon') + d'_j + \alpha'_j(\widehat{\varepsilon})} \mathcal{F}_j^{\widehat{\varepsilon}}(\lambda, y_{\widehat{\varepsilon}}). \end{aligned}$$

Here the induction hypothesis for (5.6) says $\sigma_{j2} = \sum_{k=3}^r \sigma_{jk}\varepsilon'_k$. By using (1.5) for d'_j , we can rewrite σ_{j2} as $\sigma_{j2} = -\alpha'_j(\varepsilon') + d'_j$. Thus we get

$$\alpha_j = \sigma_{j2} + \alpha'_j(\widehat{\varepsilon}) = \sum_{k=2}^r \sigma_{jk}(1 - \varepsilon_k) \quad (j = 3, \dots, r).$$

(b) Next let us consider the case $\varepsilon_2 = 1$. We assume that λ is in a small open neighborhood U_0 of 0 and h_{22} in a small open neighborhood U_1 of 1, so that putting $a_\lambda := -(\lambda - (h_{22})^2)$, we have $a_\lambda > 0$. Then by (5.10) and (5.11)

$$\begin{aligned} \Delta_j^W(\lambda e - w_\varepsilon) &= (-a_\lambda)^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}\left((-a_\lambda)(\lambda e' - y_{\widehat{\varepsilon}} - \frac{1}{2}Q'[\xi'_h]) - \frac{1}{2}Q'[h_{22}\xi'_h]\right) \\ &= (-a_\lambda)^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}\left(-\lambda(a_\lambda e' + \frac{1}{2}Q'[\xi'_h]) + a_\lambda y_{\widehat{\varepsilon}}\right). \end{aligned}$$

Since $a_\lambda > 0$ and $Q'[\xi'_h] \in \overline{\mathcal{O}'_\varepsilon} \subset \overline{\Omega'}$, we have $a_\lambda e' + \frac{1}{2}Q'[\xi'_h] \in \Omega'$ for any $\lambda \in U_0$ and $h_{22} \in U_1$. Thus for each such λ and h_{22} , there exists a unique $g_\lambda \in H'$ so that $g_\lambda e' = a_\lambda e' + \frac{1}{2}Q'[\xi'_h]$. The one-dimensional representation associated with $\Delta_j^{W'}$ being $\chi_{\sigma'_j}$, we have $\chi_{\sigma'_j}(g_\lambda) = \Delta_j^{W'}(a_\lambda e' + \frac{1}{2}Q'[\xi'_h])$. Using the relative H' -invariance of $\Delta_j^{W'}$ and (5.9), we obtain

$$\begin{aligned} \Delta_j^W(\lambda e - w_\varepsilon) &= (-a_\lambda)^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}(-\lambda g_\lambda e' + a_\lambda y_{\widehat{\varepsilon}}) \\ &= (-a_\lambda)^{-\alpha'_j(\varepsilon')} \chi_{\sigma'_j}(g_\lambda) \Delta_j^{W'}(-\lambda e' + a_\lambda g_\lambda^{-1} y_{\widehat{\varepsilon}}) \\ &= (-1)^{2d'_j} (-a_\lambda)^{-\alpha'_j(\varepsilon')} \Delta_j^{W'}\left((-a_\lambda)e' - \frac{1}{2}Q'[\xi'_h]\right) \Delta_j^{W'}(\lambda e' - a_\lambda g_\lambda^{-1} y_{\widehat{\varepsilon}}) \\ &= \lambda^{\alpha'_j(\widehat{\varepsilon})} \mathcal{F}_j^{\varepsilon'}(-a_\lambda, \frac{1}{2}Q'[\xi'_h]) \mathcal{F}_j^{\widehat{\varepsilon}}(\lambda, a_\lambda g_\lambda^{-1} y_{\widehat{\varepsilon}}). \end{aligned}$$

To continue, we introduce a rational function $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ defined by

$$\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon) := \lambda^{-\alpha'_j(\hat{\varepsilon})} \tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon) = \mathcal{F}_j^{\varepsilon'}(-a_\lambda, \frac{1}{2}Q'[\xi'_h]) \mathcal{F}_j^{\hat{\varepsilon}}(\lambda, a_\lambda g_\lambda^{-1} y_{\hat{\varepsilon}}).$$

We shall show that $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ is actually an irreducible polynomial. Since $\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon)$ is a polynomial, it is sufficient to prove the existence of a non-zero limit of $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ as $\lambda \rightarrow 0$. Since both of $\mathcal{F}_j^{\varepsilon'}$ and $\mathcal{F}_j^{\hat{\varepsilon}}$ are polynomial functions, and since the map $g: U_0 \ni \lambda \mapsto g_\lambda^{-1} \in H'$ is continuous as well as $\lambda \mapsto g_\lambda$, we obtain

$$\lim_{\lambda \rightarrow 0} \tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon) = \mathcal{F}_j^{\varepsilon'}(-a_0, \frac{1}{2}Q'[\xi'_h]) \mathcal{F}_j^{\hat{\varepsilon}}(0, a_0 g_0^{-1} y_{\hat{\varepsilon}}).$$

In order to see that this limit is non-zero, we put $h_{22} = 1$ and $\xi'_h = 0$. Then we have $a_0 = 1$ and $g_0 e' = e'$, that is, $g_0 \in H'$ is the identity operator. By (5.9) and (5.8), we have

$$\mathcal{F}_j^{\varepsilon'}(-1, 0) = (-1)^{-\alpha'_j(\varepsilon')} \mathcal{P}_j^{\varepsilon'}(-1, 0) = (-1)^{d'_j - \alpha'_j(\varepsilon')}.$$

On the other hand, since $\mathcal{F}_j^{\hat{\varepsilon}}(\lambda, y_{\hat{\varepsilon}})$ does not have the factor of λ , we can take $z_{\hat{\varepsilon}} \in \mathcal{O}'_{\hat{\varepsilon}}$ such that $\mathcal{F}_j^{\hat{\varepsilon}}(0, z_{\hat{\varepsilon}}) \neq 0$. Thus we obtain

$$\tilde{\mathcal{F}}_j^\varepsilon(0, c_2 + z_{\hat{\varepsilon}}) = \mathcal{F}_j^{\varepsilon'}(-1, 0) \mathcal{F}_j^{\hat{\varepsilon}}(0, z_{\hat{\varepsilon}}) = (-1)^{d'_j - \alpha'_j(\varepsilon')} \mathcal{F}_j^{\hat{\varepsilon}}(0, z_{\hat{\varepsilon}}) \neq 0.$$

Hence $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ does not have the factor of λ . Since U_0 and U_1 are open sets and since we now know that $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ is a polynomial, the function $\tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ is extended to $\mathbb{R} \times \mathcal{O}_\varepsilon$ and does not have the factor of λ . Therefore $\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon) = \lambda^{\alpha'_j(\hat{\varepsilon})} \tilde{\mathcal{F}}_j^\varepsilon(\lambda, w_\varepsilon)$ is an irreducible factorization. This shows

$$\alpha_j = \alpha'_j(\hat{\varepsilon}) = \sum_{k=2}^r \sigma_{jk} (1 - \varepsilon_k) \quad (j = 3, \dots, r).$$

It remains to show (5.6). Since $\boldsymbol{\alpha} = \sigma_W(\mathbf{1} - \boldsymbol{\varepsilon})$ and $\boldsymbol{d} = \sigma_W \mathbf{1}$ by (1.5) for Δ_j^W , we have $\boldsymbol{d} - \boldsymbol{\alpha} = \sigma_W \boldsymbol{\varepsilon}$. Hence (5.6) follows from (5.4). ■

Remark 5.2. If we put $w_\varepsilon := \sum_{k=2}^r \varepsilon_k \lambda_k c_k$ ($\lambda_k > 0$) in (5.2), then we have

$$\tilde{\mathcal{P}}_j^\varepsilon(\lambda, w_\varepsilon) = \prod_{k=2}^r (\lambda - \varepsilon_k \lambda_k)^{\sigma_{jk}} = \lambda^{\sum_{\varepsilon_k=0} \sigma_{jk}} \prod_{\varepsilon_k=1} (\lambda - \lambda_k)^{\sigma_{jk}}.$$

But this only implies that $\alpha_j \leq \sum_{\varepsilon_k=0} \sigma_{jk} = \sum_{k=2}^r \sigma_{jk} (1 - \varepsilon_k)$, since, in general, the restriction to a lower dimensional set of an irreducible polynomial need not be irreducible.

6. Main theorems

In this section, we determine the multiplier matrix σ_V of V . For $k = 1, 2, \dots, r-1$, let $V^{[k]}$ and $E^{[k]}$ be the subspaces of V defined respectively by

$$V^{[k]} := \bigoplus_{k < l \leq m \leq r} V_{ml}, \quad E^{[k]} := \bigoplus_{m > k} V_{mk}.$$

We note here that $V^{[1]}$ and $E^{[1]}$ are W and E in (2.1) which we have worked with in the previous sections. Then analogously to that situation, we see that $V^{[k]}$ is a subclan of (V, ∇) and $E^{[k]} \nabla V^{[k]} \subset E^{[k]}$. The latter property allows us to define $\mathcal{R}^{[k]}: V^{[k]} \rightarrow \mathcal{L}(E^{[k]})$ by

$$\mathcal{R}^{[k]}(x^k)\xi^k = \xi^k \nabla x^k \quad (x^k \in V^{[k]}, \xi^k \in E^{[k]} \text{ and } k = 1, 2, \dots, r - 1).$$

Then similarly to Proposition 2.1, each $(\mathcal{R}^{[k]}, E^{[k]})$ is a representation of $(V^{[k]}, \nabla)$. Thus we put $\varepsilon^{[k]} := \varepsilon(\mathcal{R}^{[k]}) \in \{0, 1\}^{r-k}$, and consider the $r \times r$ matrix \mathcal{E}_k defined by

$$\mathcal{E}_k := \begin{pmatrix} I_{k-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[k]} & I_{r-k} \end{pmatrix} \quad (k = 1, \dots, r - 1).$$

Theorem 6.1. *The multiplier matrix σ_V of the clan V is given by*

$$\sigma_V = \mathcal{E}_{r-1}\mathcal{E}_{r-2} \cdots \mathcal{E}_1.$$

Proof. We shall prove the theorem by induction on the rank r of V . Let us decompose V as in (2.1), and let φ be the representation of (W, ∇) in (2.3). By the induction hypothesis, the multiplier matrix σ_W of W is described as

$$\sigma_W = \mathcal{E}'_{r-1}\mathcal{E}'_{r-2} \cdots \mathcal{E}'_2, \quad \mathcal{E}'_k = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \varepsilon^{[k]} & I_{r-k} \end{pmatrix} \quad (k = 2, 3, \dots, r - 1). \quad (6.1)$$

Let us put $\varepsilon = \varepsilon(\varphi)$. Applying Theorem 5.1 to V , we have by (6.1)

$$\sigma_V = \begin{pmatrix} 1 & 0 \\ \sigma_W \varepsilon & \sigma_W \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sigma_W \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & I_{r-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_{r-1} \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \varepsilon & I_{r-1} \end{pmatrix}.$$

By noting $\mathcal{E}_k = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{E}'_k \end{pmatrix}$ ($k = 2, 3, \dots, r - 1$), the proof is completed. ■

We are now able to describe the basic relative invariants $\Delta_1^V(x), \dots, \Delta_r^V(x)$ of V . To do so, we introduce the polynomials $D_j(x)$ ($j = 1, \dots, r$) defined by Vinberg [8, the formula (25), p. 385] (see also Ishi [3, p. 158]). Let $\|\cdot\|$ be the norm on V defined by $\|x\|^2 := \langle x|x \rangle$ ($x \in V$). Given $x \in V$, we introduce elements

$$x^{(j)} = \sum_{k=j}^r x_{kk}^{(j)} c_k + \sum_{m>k \geq j} X_{mk}^{(j)} \in V^{[j-1]} \quad (j = 1, \dots, r; V^{[0]} := V)$$

inductively by $x^{(1)} := x$ and, for $i = 1, \dots, r - 1$,

$$\begin{cases} x_{kk}^{(i+1)} := x_{ii}^{(i)} x_{kk}^{(i)} - \frac{1}{2s_0(c_k)} \|X_{ki}^{(i)}\|^2 & (i < k \leq r), \\ X_{mk}^{(i+1)} := x_{ii}^{(i)} X_{mk}^{(i)} - X_{mi}^{(i)} \Delta X_{ki}^{(i)} & (i < k < m \leq r). \end{cases}$$

Then the polynomials $D_j(x)$ are defined by

$$D_j(x) := x_{jj}^{(j)} \in \mathbb{R} \quad (j = 1, \dots, r).$$

The polynomials $D_j(x)$ appear in the solution $h \in H$ of the equation $he_V = x$ for a given $x \in \Omega$. In fact, Vinberg [8, Chapter III, Section 3] tells us that the numbers $h_{jj} > 0$ for T_{jj} in (1.3) are given by

$$h_{11}^2 = D_1(x), \quad h_{jj}^2 = D_1(x)^{-1} \cdots D_{j-1}(x)^{-1} D_j(x) \quad (j = 2, \dots, r).$$

Using these h_{jj} , we have $\Delta_j^V(x) = (h_{11})^{2\sigma_{j1}} \cdots (h_{jj})^{2\sigma_{jj}}$ by definition of σ_{jk} . From these observations, we obtain the following theorem, which tells us explicitly by what powers of $D_1(x), \dots, D_{j-1}(x)$ we have to divide $D_j(x)$ in order to obtain the irreducible factor $\Delta_j^V(x)$.

Theorem 6.2. *The basic relative invariants $\Delta_j^V(x)$ of V are written as*

$$\Delta_1^V(x) = D_1(x), \quad \Delta_j^V(x) = \frac{D_j(x)}{\prod_{i < j} D_i(x)^{-\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj}}} \quad (j = 2, \dots, r). \quad (6.2)$$

Remark 6.3. Let us verify that the power of $D_i(x)$ in the denominator in (6.2) is non-negative. Indeed, thanks to (5.4), it is sufficient to consider the case $i = 1$. By (5.6), we have $\sigma_{j1} = \sum_{k=2}^j \sigma_{jk} \varepsilon_k$, so that

$$-\sigma_{j1} + \sigma_{j2} + \cdots + \sigma_{jj} = \sum_{k=2}^j \sigma_{jk} (-\varepsilon_k + 1) \geq 0 \quad (j = 2, \dots, r).$$

Example 6.4. (1) Let Ω be an irreducible symmetric cone of rank $r \geq 3$. The corresponding clan is $V = \text{Herm}(r, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) with the product Δ defined by $x \Delta y = \underline{xy} + y(\underline{x})^*$. In this case, the polynomials $D_j(x)$ are calculated as

$$D_j(x) = \begin{cases} \det^{[j]}x & (j = 1, 2), \\ (\det^{[1]}x)^{2^{j-3}} (\det^{[2]}x)^{2^{j-4}} \cdots (\det^{[j-2]}x) (\det^{[j]}x) & (j \geq 3), \end{cases} \quad (6.3)$$

where $\det^{[k]}x$ ($k = 1, \dots, r$) are the left upper corner principal minors of x . Since it is clear from a glance at (6.3) that the basic relative invariants $\Delta_j(x)$ of Ω are equal to $\det^{[j]}(x)$ ($j = 1, \dots, r$), we can verify the formula (6.2). Put $d = \dim_{\mathbb{R}} \mathbb{K}$. Since we have $\dim V_{kj} = d$ ($j < k$), a straightforward computation yields $\varepsilon^{[k]} = {}^t(1, 0, \dots, 0) \in \{0, 1\}^{r-k}$ ($k = 1, \dots, r - 1$). Thus the multiplier matrix σ of V is given by $\sigma = \begin{pmatrix} 1 & & 0 \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix}$. Then $-\sigma_{ji} + \sigma_{j,i+1} + \cdots + \sigma_{jj} = j - i - 1$, and we have for $j \geq 2$

$$\Delta_j(x) = D_j(x) (D_1(x)^{j-2} D_2(x)^{j-3} \cdots D_{j-2}(x)^1)^{-1} = \det^{[j]}x.$$

(2) We fix a positive integer p . Let Ω^0 be the homogeneous cone defined by

$$\Omega^0 := \left\{ X = \begin{pmatrix} \lambda I_p & \xi^* \\ \xi & x \end{pmatrix}; \lambda \in \mathbb{R}, \xi \in \text{Mat}(r \times p, \mathbb{K}), x \in \text{Herm}(r, \mathbb{K}) \right\}.$$

X is positive-definite

In this case, Theorem 5.1 gives a quicker computation now that we have the previous example. The corresponding clan V has the decomposition (2.1) with

$$E = \text{Mat}(r \times p, \mathbb{K}), \quad W = \text{Herm}(r, \mathbb{K}).$$

The representation φ of (2.3) is given by $\varphi(x)\xi = x\xi$ ($x \in V, \xi \in E$), where the product on the right hand side is the ordinary matrix multiplication. We have

$$\varepsilon(\varphi) = \begin{cases} {}^t(\underbrace{1, \dots, 1}_p, 0, \dots, 0) & (1 \leq p < r), \\ {}^t(1, \dots, 1) & (p \geq r) \end{cases}$$

by definition, and φ is an $\varepsilon(\varphi)$ -representation. By the preceding example, we know σ_W . Thus Theorem 5.1 yields

$$\alpha = \sigma_W(\mathbf{1} - \varepsilon(\varphi)) = \begin{cases} {}^t(\underbrace{0, \dots, 0}_p, 1, 2, \dots, r - p) & (1 \leq p < r), \\ {}^t(0, \dots, 0) & (p \geq r), \end{cases}$$

and hence we obtain the formula for the basic relative invariants of V by Theorem 3.4. This α is computed in the previous paper [7] in a direct way. We note that the representation φ is regular if and only if $p \geq r$.

We conclude this paper by considering the actual factorization formula for $\text{Det } R_x$ ($x \in V$). More specifically, we will determine the positive integers n_1, \dots, n_r with which we have by Theorem 3.1

$$\text{Det } R_x = \Delta_1^V(x)^{n_1} \cdots \Delta_r^V(x)^{n_r} \quad (x \in V). \tag{6.4}$$

We set $\underline{n} := (n_1, \dots, n_r)$ in the form of row vector. We call \underline{n} the *basic index* of V , and are now going to express \underline{n} in terms of $d_{kj} = \dim V_{kj}$ ($j < k$).

Considering the degrees in (6.4), we have

$$\dim V = n_1 \deg \Delta_1^V + \cdots + n_r \deg \Delta_r^V. \tag{6.5}$$

Let

$$m_k := \sum_{l \geq k} \dim V_{lk} \quad (k = 1, \dots, r), \tag{6.6}$$

and we put them also in the form of row vector as $\underline{m} := (m_1, \dots, m_r)$. We note that $m_k = 1 + \dim E^{[k]}$ for $k = 1, 2, \dots, r - 1$, and $m_r = 1$.

Theorem 6.5. *One has $\underline{n} = \underline{m}\sigma_V^{-1}$.*

Proof. We shall prove the theorem by induction on the rank r of V . Let us decompose V as in (2.1). We denote by Δ_j^W ($j = 2, \dots, r$) the basic relative invariants of (W, Δ) , and by \underline{n}' the basic index of W . By (6.5) applied to W , we have $\dim W = \underline{n}'\mathbf{d}$, where we recall (5.3) for \mathbf{d} . Since $W = \bigoplus_{2 \leq j \leq k \leq r} V_{kj}$ is the normal decomposition of W , we have $m'_k := \sum_{l \geq k} \dim V_{lk} = m_k$ for $k = 2, \dots, r$. Let (φ, E) be the representation of (W, ∇) in (2.3) and Q the symmetric bilinear map associated with φ . Now Proposition 3.2 together with (3.2) gives

$$\begin{aligned} \text{Det } R_x &= \lambda^{1+\dim E - \dim W} \Delta_2^W(\lambda w - \frac{1}{2}Q[\xi])^{n'_2} \cdots \Delta_r^W(\lambda w - \frac{1}{2}Q[\xi])^{n'_r} \\ &= \Delta_1^V(x)^{1+\dim E - \underline{n}'(\mathbf{d} - \alpha)} \Delta_2^V(x)^{n'_2} \cdots \Delta_r^V(x)^{n'_r}. \end{aligned}$$

This tells us that

$$\underline{n} = (1 + \dim E - \underline{n}'(\mathbf{d} - \boldsymbol{\alpha}), \underline{n}') = (m_1 - \underline{n}'(\mathbf{d} - \boldsymbol{\alpha}), \underline{n}'). \quad (6.7)$$

Then by the induction hypothesis $\underline{n}'\sigma_W = \underline{m}'$, we obtain by (5.4) and (6.7)

$$\underline{n}\sigma_V = (m_1 - \underline{n}'(\mathbf{d} - \boldsymbol{\alpha}), \underline{n}') \begin{pmatrix} 1 & 0 \\ \mathbf{d} - \boldsymbol{\alpha} & \sigma_W \end{pmatrix} = (m_1, \underline{m}') = \underline{m}.$$

The proof is now completed. ■

Remark 6.6. The equation (6.7) tells us that $n_1 = 1 + \dim E - \underline{n}'(\mathbf{d} - \boldsymbol{\alpha})$. Let us verify here that we actually have $n_1 \geq 1$. We first note that Ishi [2, Lemma 3.3 (ii)] implies that for $\mathcal{O}_\varepsilon = H_W c_\varepsilon$ we have $\dim \overline{\mathcal{O}_\varepsilon} = \underline{m}'\varepsilon$. Then the fact that $Q[E] = \overline{\mathcal{O}_\varepsilon}$ together with Theorem 6.5 applied to W gives

$$\dim E \geq \dim \overline{\mathcal{O}_\varepsilon} = \underline{m}'\varepsilon = \underline{n}'\sigma_W\varepsilon = \underline{n}'(\mathbf{d} - \boldsymbol{\alpha}),$$

where the last equality follows from (5.6) and (5.4). Hence we obtain $n_1 \geq 1$.

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