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Radial Operators on Lie Supergroups and Characters of Representation

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Abstract. Foundational material on radial operators on complex Lie supergroups is presented. In particular, a local version of Berezin's recursion formula for describing the radial parts of fundamental operators in general linear and ortho-symplectic cases is proved. Results which are suitable for applications for computing characters which are only defined on proper subdomains or covering spaces thereof are established.
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We study the radial part of differential operators on Lie supergroups and in particular of the invariant Laplace-Casimir operators in the complex analytic setting. Arguments of these operators are radial functions, e.g. characters of Lie supergroup representations. Our original motivation for our work here came from applications which we now outline.

Applying Howe-duality in a Fock space context one observes that certain integrals of physical importance can be interpreted as numerical parts of characters of holomorphic semigroup representations of semigroups in Lie supergroups which appear as Howe partners of classical groups. As a result the integrals can be explicitly computed. This is carried out for unitary groups in [2] and orthogonal and symplectic groups in [4]. A unified approach, which, as that in [4] relies on the results of the present paper, is presented in [5]. The characters are holomorphic superfunction of a parameter which is varying in a covering space of a domain in the base complex reductive group of the Lie supergroup at hand. Such covering spaces, which arise, e.g., due to the involvement of the metaplectic representation, can be regarded as the domains of definition of the semigroup representations. They contain pieces of maximal tori so that at least in a local sense one has the appropriate notions of radial functions and operators. In fact the restrictions χ of the character to this torus piece is the function, i.e., the integral, which is of

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interest. The character property implies that the integrals are eigenfunctions of the radial parts of Laplace-Casimir operators. In fact, in the cases considered in [2, 4] the eigenvalues are zero! Thus in those cases every element D of the center universal enveloping algebra yields a differential equation which is of the form $\dot{D}(\chi) = 0$ where $\dot{D}$ is the associated radial part of D.

The above is conceptually pleasing, but to complete the task of obtaining an explicit formula for the correlation functions χ one needs more concrete information on the radial parts $\dot{D}$ of the Laplace-Casimir operators. Although all of the necessary information is contained in the fundamental work of F. Berezin, gleaning it from ([1]) and adapting it to the local setting indicated above requires a serious effort. Thus our work developed into the thesis project [6], the second author's paper [7] and our work here.

Let us now briefly summarize this paper. After fixing notation and defining radial functions and operators (sec. 1 and 2) we consider radial operators, primarily in the cases of $\mathfrak{gl}$ and $\mathfrak{osp}$ which are of interest for the above mentioned Fock space applications (sec. 3). The first general goal is to describe the radial part of a Laplace-Casimir operator D by $\dot{D} = J^{-1}P_D J$ where P_D is a constant coefficient polynomial differential operator on the given maximal torus of the Lie group G associated to $\mathfrak{g}_0$. A number of assumptions are needed for this, in particular that the function J should be an eigenfunction of the second order Laplace-Casimir operator which is defined in the usual way by an invariant nondegenerate supersymmetric bilinear form. This function appears as the square-root of the superdeterminant of a Jacobian of a coordinate chart which identifies a neighborhood $\mathcal{A}$ of a regular point in a maximal torus with a product $\mathcal{A} \times \mathcal{B}$ where $\mathcal{B}$ is a subsupermanifold which is defined as a local orbit of the Lie supergroup by conjugation. The assumptions required for this are satisfied in the two cases of interest mentioned above. Denoting by W the Weyl group at hand, we evaluate the radial part of an operator D given by an W-invariant polynomial P on the character χ of a highest weight representation and obtain its eigenvalue from $T(P) := P_D$ in section 4. This yields a better understanding of the dependence of $T(P)$ on P.

In both cases of interest there is an important infinite series $\{F_\ell\}$ of elements of the center of the universal enveloping algebra, which defines a series of $\{D_\ell\}$ Laplace-Casimir operators for which the constant coefficient operators P_{D_ℓ}, can be described via a certain recursive procedure (sec. 5). For example, in the case of $\mathfrak{osp}$, if we use the standard basis for the standard Cartan algebra, then the polynomials F_ℓ are defined by $F_\ell = \sum \varphi_i^{2\ell} + (-1)^\ell \sum \phi_j^{2\ell}$. Following Berezin we write $\dot{D}_\ell = J^{-1}T(F_\ell)J$. The goal is then to understand the map $F_\ell \mapsto T(F_\ell)$. Identifying the polynomials F_ℓ with the constant coefficient operators which they define, the main result of Berezin is that $T(F_\ell) = F_\ell + Q_\ell$ where Q_ℓ is a polynomial in $F_1, \ldots, F_{\ell-1}$. The proof of this result is discussed here in substantial detail.

The last paragraph is devoted to local versions of the formula mentioned above and its applications in [4] and [5]. First, we show that the global results of Berezin apply to give the same results on the local product neighborhoods $\mathcal{A} \times \mathcal{B}$. These then lift to the covering spaces mentioned at the outset to give global results there by applying the identity principle.

A different prove of the local version of $\dot{D} = J^{-1} P_D J$ on product neighborhoods $\mathcal{A} \times \mathcal{B}$ using methods which are much closer to those used by Helgason [3] in the classical case and Kostant's operator point of view on supermanifolds [8] in a complex version is given in the second authors thesis [6], Chapter 4. Of course this is only valid under the same conditions as Berezin's global result. As is shown in section 6, Berezin's global result implies the local result. Vice versa, application of the identity principle shows that the local result implies Berezin's global result. Thus the two results are equivalent.

1. Basic definitions

Here we give a short summary of Berezin's construction of Lie supergroups [1] and recall the basic objects for the study of radial differential operators on Lie supergroups.

Lie supergroups. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and G_0 be a Lie group with $\mathrm{Lie}(G_0) = \mathfrak{g}_0$. The sheaf of superfunctions $\mathcal{F}$ on the Lie supergroup corresponding to the pair $(G_0, \mathfrak{g})$ will be identified with $\mathcal{O}_{G_0} \otimes \Lambda \mathfrak{g}_1^*$. The central idea in Berezin's construction is the analysis of the Lie algebra

$$\widetilde{\mathfrak{g}} = (\mathfrak{g}_0 \otimes (\Lambda \mathfrak{g}_1^*)_0) \oplus (\mathfrak{g}_1 \otimes (\Lambda \mathfrak{g}_1^*)_1) \ .$$

Using the exponential series he defines in two steps coordinates (called Grassmann canonical coordinates) on the Lie groups $\widetilde{G}_0$ and $\widetilde{G}$ corresponding to $\mathfrak{g}_0 \otimes (\Lambda \mathfrak{g}_1^*)_0$, resp. $\widetilde{\mathfrak{g}}$ and a continuation of functions (called Grassmann analytic continuation) $\mathcal{O}_{G_0} \otimes \Lambda \mathfrak{g}_1^* \hookrightarrow \mathcal{O}_{\widetilde{G}_0} \otimes \Lambda \mathfrak{g}_1^* \hookrightarrow \mathcal{O}_{\widetilde{G}} \otimes \Lambda \mathfrak{g}_1^*$. The Lie algebra $\widetilde{\mathfrak{g}}$ is represented in the usual way by right- and left-invariant derivations on $\mathcal{O}_{\widetilde{G}}$ and hence on $\mathcal{O}_{\widetilde{G}} \otimes \Lambda \mathfrak{g}_1^*$. Introducing an auxiliary Grassmann variable in order to embed $\mathfrak{g}_1$ into $\widetilde{\mathfrak{g}}$ the mentioned derivations naturally restrict to superderivations on $\mathcal{O}_{G_0} \otimes \Lambda \mathfrak{g}_1^*$ and yield Lie superalgebra representations of $\mathfrak{g}$ on the sheaf of superfunctions $\mathcal{F}$ denoted by $X \mapsto R_X$, resp. L_X.

The morphisms for multiplication, inverse and neutral on the Lie supergroup can be defined via the group structure on $\widetilde{G}$. There are various other methods of constructing a Lie supergroup associated to $\mathfrak{g}$, (see [8] and [7] for constructions using Lie-Hopf algebras and [9] for the dual construction at the level of the structure sheaf). But it turns out that, even in the more delicate holomorphic setting, these are all equivalent ([11]). For further discussion of this matter, e.g., for a comparison of the various definitions and a detailed overview on Berezin's construction, see [6].

Universal enveloping algebra. The *universal enveloping algebra* of a Lie superalgebra $\mathfrak{g}$ is the quotient $U(\mathfrak{g})$ of the full tensor algebra $T(\mathfrak{g})$ by the ideal generated by

$$(X \otimes Y - (-1)^{|X||Y|} Y \otimes X) - [X, Y]$$

where X, Y are homogeneous elements of $\mathfrak{g}$ regarded as elements of $T(\mathfrak{g})$. If $X_1 \otimes X_2 \cdots \otimes X_k$ is a monomial in $T(\mathfrak{g})$, then its image in $U(\mathfrak{g})$ is equipped with the sign $(-1)^{|X_1|+\ldots+|X_k|}$. This defines a $\mathbb{Z}_2$-grading on $U(\mathfrak{g})$ for which the induced bracket defines a Lie superalgebra structure. Here we shall primarily be concerned with the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. This is the subalgebra of $U(\mathfrak{g})$ consisting of those elements X for which $\mathrm{ad}(X)(Y) := [X, Y] = 0$ for all $Y \in U(\mathfrak{g})$.

Laplace-Casimir operators. Recall that a representation of a Lie superalgebra $\mathfrak{g}$ is an (even) superalgebra morphism $\rho : \mathfrak{g} \to \mathrm{End}(V)$ to the superalgebra of linear (not necessarily even) maps of a graded vector space $V = V_0 \oplus V_1$. Such a representation extends to a representation $\rho : U(\mathfrak{g}) \to \mathrm{End}(V)$. The examples of main importance here are the representations L and R of $\mathfrak{g}$ on the structure sheaf $\mathcal{F}$ of superfunctions of the associated Lie supergroup. In particular, we consider the representation $X \mapsto L_X$ and extend it to a representation of $U(\mathfrak{g})$ by differential operators. The *Laplace-Casimir operators* are those in the image of the center $Z(\mathfrak{g})$.

2. Radial functions

A superfunction f on a Lie supergroup over a connected Lie group G_0 is said to be *radial* if it is annihilated by the representation $X \to L_X + R_X$ of $\mathfrak{g}$. Let $\mathcal{R}$ denote the sheaf of radial holomorphic functions on G_0. Observe that if D is a Laplace-Casimir operator, then $D|\mathcal{R} : \mathcal{R} \to \mathcal{R}$. We regard $\mathcal{R}$ as the natural domain of definition of these operators.

Note that in the classical case if f is globally defined on G_0, then the condition $L_X f + R_X f = 0$ for all $X \in \mathfrak{g}_0$ just means that f is conjugation invariant, i.e., $f(g_0 g g_0^{-1}) = f(g)$ for all $g_0 \in G_0$. If G_0 is reductive, which we assume from now on, we will see that globally defined conjugation invariant superfunctions are completely determined by their (numerical) restrictions to any given maximal torus H. We fix such a maximal torus and let $\mathfrak{h}$ denote its Lie algebra in $\mathfrak{g}_0$. Let us also assume, as will be the case in all applications, that $\mathfrak{h}$ is a Cartan algebra of the full Lie superalgebra $\mathfrak{g}$, i.e. $\mathfrak{g}_1$ decomposes into one-dimensional root spaces of non-trivial roots.

Restriction theorem at the Lie superalgebra level. At the infinitesimal level we are interested in understanding the image of the restriction map from the space $S(\mathfrak{g}^*)^{\mathfrak{g}}$ of $\mathrm{ad}_{\mathfrak{g}}$-invariant (super) polynomials on $\mathfrak{g}$ to the space $S(\mathfrak{h}^*)$. We view an element $P \in S(\mathfrak{g}^*)$ as a (holomorphic) polynomial map $P : \mathfrak{g}_0 \to \wedge \mathfrak{g}_1^*$. If U is a sufficiently small neighborhood of $0 \in \mathfrak{g}_0$ which is identified by the exponential map with a neighborhood V of Id in G_0, then $P \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ if and only if the resulting function on V is radial.

Since it has been assumed that $\mathfrak{h}$ is an even subspace of $\mathfrak{g}$, polynomials in $S(\mathfrak{h}^*)$ are just standard (numerically valued) polynomials. Since $P \in S(\mathfrak{g}^*)^{\mathfrak{g}}$ is invariant by the adjoint representation of G_0, it is immediate that $R(P)$ is invariant under the Weyl group $W = W(\mathfrak{g}_0, \mathfrak{h}_0)$. Thus we regard R as a map $R : S(\mathfrak{g}^*)^{\mathfrak{g}} \to S(\mathfrak{h}^*)^W$.

A great deal is known about the restriction morphism R. In particular, it is always injective. In our cases of interest, the basic results are proved in [1] and [10]. Let us quote Berezin's Theorem 3.1.

Theorem 2.1. *Let G_0 be reductive and assume that $\mathfrak{h}$ is a Cartan algebra of $\mathfrak{g}$ which is contained in $\mathfrak{g}_0$. Assume further that $\mathfrak{g}$ is endowed with a nondegenerate invariant scalar product and that its odd root spaces are 1-dimensional. Then a W-invariant polynomial P is in the image of R if and only if for every odd root β (with dual root β^*) it follows that*

$$\frac{d}{dt}\Big|_{t=0} P(h+t\beta^*) = \beta(h)Q(h) \tag{1}$$

where $Q=Q(h)$ is a polynomial on $\mathfrak{h}$.

We say that the *extendible polynomials* are exactly those which are W-invariant and satisfy the *divisibility condition* (1). Since our work here is aimed at understanding properties of radial functions and operators in the cases of $\mathfrak{gl}$ and $\mathfrak{osp}$, it should be emphasized that the conditions of Berezin's theorem are fulfilled in those cases.

For the statement of the version of Berizin's extension theorem for holomorphic functions we say that a holomorphic superfunction on $\mathfrak{g}_0$ is radial if and only if it is annihilated by all $\mathrm{ad}_{\mathfrak{g}}$-derivations. The divisibility condition for holomorphic functions is the same as that for polynomials.

Theorem 2.2. *Under the assumptions of Theorem 2.1 it follows that a W-invariant function $f \in \mathcal{O}(\mathfrak{h})$ can be (uniquely) extended to a radial holomorphic superfunction on $\mathfrak{g}_0$ if and only if it satisfies the divisibility condition.*

Sketch of Proof. For the *necessity*, i.e., that the divisibility condition is really needed for extension, one replaces the polynomial P in Berezin's proof by the convergent power series representation of the given function f at $0 \in \mathfrak{h}$. For the *sufficiency* Berezin uses the fact that the given polynomial P can be extended to a unique Ad_{G_0}-invariant polynomial on $\mathfrak{g}_0$ and then proceeds by using generalities valid for superfunctions as well as polynomials. By classical results W-invariant holomorphic functions on $\mathfrak{h}$ (possibly with values in $\Lambda\mathfrak{g}_1^*$) extend to $Ad_{\mathfrak{g}_0}$-invariant holomorphic functions on $\mathfrak{g}_0$. So the invariance under $Ad_{\mathfrak{g}}$ is just an additional condition not interfering with convergence. Hence Berezin's proof can be carried out in the holomorphic case. ∎

Restriction theorem at the group level. Now let us turn to the Lie supergroup associated to $\mathfrak{g}$ equipped with its sheaf $\mathcal{F}$ of holomorphic superfunctions, i.e., the sheaf of germs of holomorphic maps with values in $\wedge\mathfrak{g}_1^*$. We assume that $\mathfrak{g}$ satisfies the assumptions of Theorem 2.1 and let H be the maximal (complex) torus in G_0 associated to $\mathfrak{h}$. In this situation we wish to determine, e.g., the image of the restriction map $R: \mathcal{R}(G_0) \to \mathcal{O}(H)^W$ from the globally defined holomorphic radial functions on G_0 in the algebra of W-invariant holomorphic functions on H.

For this the divisibility condition must be transferred to the group level: A holomorphic function $f \in \mathcal{O}(H)$ is said to satisfy the divisibility condition if and only if its pull-back $f \circ \exp$ satisfies the divisibility condition on $\mathfrak{h}$. The following is an immediate consequence of the results in the previous paragraph.

Proposition 2.3. *A holomorphic function $f \in \mathcal{O}(H)^W$ satisfies the divisibility condition if and only if its lift $f \circ \exp$ is the restriction of a uniquely determined radial holomorphic superfunction on $\mathfrak{g}$.*

The extension theorem at the group level is stated as expected.

Theorem 2.4. *Let $\mathfrak{g}$ be a Lie superalgebra which satisfies the conditions of Theorem 2.1 and let G_0 be a base of an associated Lie supergroup. Fix a Cartan algebra $\mathfrak{h}$ in $\mathfrak{g}$ and let $H = \exp(\mathfrak{h})$. Then a W-invariant holomorphic function $f \in \mathcal{O}(H)$ is the restriction of a radial holomorphic superfunction on G_0 if and only if it satisfies the divisibility condition.*

Proof. To prove the sufficiency of the divisibility condition we let f be a W-invariant (numerical) holomorphic function on H which satisfies the divisibility condition and $\hat{f} = f \circ \exp$. Since $\hat{f}$ satisfies the divisibility condition on $\mathfrak{h}$, it is the restriction of a (unique) holomorphic radial superfunction $E(\hat{f})$ on $\mathfrak{g}_0$. This means that $\hat{f}$ is the numerical part of $E(\hat{f})|\mathfrak{h}$. The uniqueness of the extension $E(\hat{f})$ implies that $E(\hat{f})|\mathfrak{h}$ is periodic with respect to the discrete additive subgroup of $\mathfrak{h}$ which is the kernel of $\exp : \mathfrak{h} \to H$. Thus there is a $\wedge\mathfrak{g}_1^*$-valued holomorphic function f_s on H with $E(\hat{f})|\mathfrak{h} = f_s \circ \exp$.

Observe that since $E(\hat{f})$ is invariant by conjugation by elements of the normalizer of H, it follows that its restriction to $\mathfrak{h}$ is W-invariant and thus f_s is W-invariant. Classical invariant theory then implies that f_s is the restriction of a unique conjugation invariant superfunction $E(f)$ on G_0. The function $E(f) \circ exp$ is an Ad_{G_0}-invariant holomorphic superfunction which agrees with $E(\hat{f})$ on $\mathfrak{h}$. Thus $E(f) \circ exp$ *is* the radial extension of $\hat{f}$. In particular, if U is a neighborhood of $0 \in \mathfrak{g}_0$ such that $\exp : U \to V$ is biholomorphic, then the fact that the operators $L_X + R_X$ annihilate $E(\hat{f})|U$ for all $X \in \mathfrak{g}$ implies that they annihilate $E(f)|V$ for all $X \in \mathfrak{g}$. Hence the identity principle implies that $(L_X + R_X)E(f) = 0$ for all $X \in \mathfrak{g}$ and consequently $E(f)$ is the desired radial extension of f.

For the necessity of the divisibility condition, we just reverse the argument: If $E(f)$ is the radial extension of f, then $E(f) \circ \exp|U$ is annihilated by the operators $L_X + R_X$ and the identity principle implies that $E(f) \circ \exp$ is the radial extension of $\hat{f}$. Consequently $\hat{f}$ satisfies the divisibility condition which by definition is the divisibility condition for f. ∎

3. Jacobian formula

Recall that we have regarded the Laplace-Casimir operators as being differential operators $D : \mathcal{R} \to \mathcal{R}$ on the sheaf of radial holomorphic superfunctions on the Lie supergroup associated to a Lie superalgebra $\mathfrak{g}$. Here we restrict our considerations to the setting of Theorem 2.1 so that the algebra of global radial functions is decribed by the divisibility condition along a given maximal torus H in G_0. Note that divisibility at the group level means that for every odd root β the directional derivative $X_\beta(f)$ is divisible by $(r_\beta - 1)$ where r_β is the character associated to β.

Denote by $\mathcal{D}_H$ the image in $\mathcal{O}(H)^W$ of the restriction map R and $E : \mathcal{D}_H \to \mathcal{R}(G_0)$ the extension map which is R^{-1}. The associated *radial part* of a Laplace-Casimir operator D on H is defined as $\dot{D} := RDE : \mathcal{D}_H \to \mathcal{D}_H$. Our goal in this paragraph is to describe a basic result of Berezin which (at least for the Lie superalgebras and operators of main interest for the applications in [2] and [4]) shows that the study of the radial operators $\dot{D}$ can be reduced to analyzing certain constant coefficient polynomial differential operators on H. This is proved by introducing a sort of change of variables along H so that generically along H one has a local product decomposition in the H-direction and the transversal direction of the supergroup action by conjugation. As a consequence, a Jacobian J appears and therefore we refer to the result as the Jacobian formula.

The Jacobian J is defined as follows as a meromorphic function on H: For $\zeta = \exp(t) \in H$

$$J(\zeta) := \frac{\prod_{\alpha \in \Delta_0^+} 2\sin\frac{\alpha(t)}{2}}{\prod_{\beta \in \Delta_1^+} 2\sin\frac{\beta(t)}{2}}$$

where $\Delta^+ = \Delta_0^+ \cup \Delta_1^+$ is a system of even and odd positive roots. Under certain restrictive conditions the Jacobian formula states that given a Laplace-Casimir operator D there is a uniquely defined polynomial operator with constant coefficients P_D on H so that $\dot{D} = J^{-1} P_D J$. It should be remarked that in the classical case of Lie groups the analogous formula (without the odd roots in the Jacobian) holds in great generality. In the Lie supergroup setting we state it in the cases of $\mathfrak{g} = \mathfrak{gl}(m,n)$ and $\mathfrak{osp}(2m,2n)$. In [1] the latter Lie superalgebra is denoted by $C(m,n)$. From now on $\mathfrak{g}$ is restricted to be one of these (complex) Lie superalgebras equipped with the nondegenerate bilinear form $(X,Y) = \mathrm{STr}(XY)$.

In the case of $\mathfrak{gl}(m,n)$ we choose $\mathfrak{h}$ to be the Cartan algebra of diagonal matrices with coordinates

$$h = \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix}.$$

One checks that the polynomial function

$$F_k = \mathrm{STr}(h^k) = \sum \varphi_i^k - \sum \psi_j^k$$

satisfies the divisibility condition and therefore is extendible to an ad-invariant element of $S(\mathfrak{g}^*)$, i.e., to an element of $Z(\mathfrak{g})$.

In the case of $\mathfrak{osp}(2m,2n)$ we recall that $\mathfrak{g}_0 = \mathfrak{so}_{2m} \oplus \mathfrak{sp}_{2n}$ and as above choose $\mathfrak{h}$ to be in diagonal form with coordinates

$$h_{\mathfrak{so}} = \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix}$$

and

$$h_{\mathfrak{sp}} = \begin{pmatrix} \psi & 0 \\ 0 & -\psi \end{pmatrix}$$

with the full Cartan algebra given by $h = \mathrm{Diag}(h_{\mathfrak{so}}, h_{\mathfrak{sp}})$. In this case one defines the extendible polynomials

$$F_k := \frac{1}{2}\mathrm{STr}(h^{2k}) = \sum \varphi_i^{2k} - \sum \psi_j^{2k}\,.$$

In the case of $\mathfrak{gl}(m,n)$ the algebra of extendible polynomials consists of polynomials with constant coefficients in the F_k. In the case of $\mathfrak{osp}(2m,2n)$ one requires one additional generator which is most conveniently chosen as $L = \varphi_1 \cdot \ldots \cdot \varphi_m R$ where R is the product of the odd positive root functions. For the sake of brevity of notation we let

$$\tilde{F}_k = F_k(\frac{1}{i}\frac{\partial}{\partial t})$$

be the constant coefficient differential operator defined by F_k. A simplified version of Berezin's Jacobian theorem (see Theorem 3.2 on p. 302 of [1]) can be stated as follows.

Theorem 3.1. *Let $\mathfrak{g}$ be either $\mathfrak{gl}(m,n)$ or $\mathfrak{osp}(2m,2n)$ and P be an extendible polynomial on $\mathfrak{h}$ which defines the radial differential operator $\dot{D}_P$ by $\dot{D}_P(f) = RD_PE(f)$ for $f \in \mathcal{D}_H$ a W-invariant function satisfying the divisibility condition. Then there exists a uniquely determined polynomial function $T(P)$ on $\mathfrak{h}$ with associated constant coefficient differential operator denoted by $T(P)(\frac{1}{i}\frac{\partial}{\partial t})$ so that*

$$\dot{D}_P = J^{-1}T(P)(\frac{1}{i}\frac{\partial}{\partial t})J\,.$$

Furthermore, if $P = F_k$, then $T(P)$ is of degree k with top degree term F_k.

It should be underlined that the partial derivative operators

$$\frac{\partial}{\partial t} = (\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{m+n}})$$

are defined by the coordinates (φ, ψ) of $\mathfrak{h}$ which were introduced above.

Now we turn to understanding the mapping $P \mapsto T(P)$. This will be discussed for the polynomials $P = F_k$ in both cases $\mathfrak{gl}(m,n)$ and $\mathfrak{osp}(2m,2n)$.

4. Finite-dimensional representations

Here we explain how to compute the polynomials $T(P)$ in terms of the eigenvalues of the radial operators $\dot{D}_P$ on characters of *finite-dimensional* representations. We restrict to the cases $\mathfrak{g} = \mathfrak{gl}(m,n), \mathfrak{osp}(2m,2n)$, but most of the discussion applies in a much more general setting, e.g., where $\mathfrak{g}_0$ is semisimple. For more details see Chapter 3.10 (p.307-311) of [1].

Character formula. Consider a finite-dimensional irreducible representation ρ of a complex Lie supergroup associated to one of the Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m,n), \mathfrak{osp}(2m,2n)$. This is by definition a homomorphism of the Lie supergroup associated to $\mathfrak{g}$ to that associated to the Lie superalgebra $\mathfrak{gl}(V)$. Such is defined by a holomorphic mapping $G_0 \to \mathrm{GL}(V)$ which lifts to the sheaf level as a mapping $\mathcal{F}_{\mathrm{GL}(V)} \to \mathcal{F}_{G_0}$ which preserves the defining Lie supergroup triples (see [1], p.248). Taking a basis of homogeneous elements of V one interprets ρ as a holomorphic map of G_0 to matrices whose entries are superfunctions. The character of such a representation is defined by $\chi(g) := \mathrm{STr}(\rho(g))$. It is a radial superfunction on G_0 and we consider its restriction χ to a Cartan subgroup H. It is an eigenfunction of every radial operator $\dot{D}$. In other words there is a homomorphism λ defined on the space of radial differential operators with values in $\mathbb{C}$ so that $\dot{D}(\chi) = \lambda(\dot{D})\chi$.

Now apply the Jacobian formula, $\dot{D}_P = J^{-1}T(P)(\frac{1}{i}\frac{\partial}{\partial t})J$, define $\tilde{\chi} = J\chi$ and observe that

$$T(P)(\frac{1}{i}\frac{\partial}{\partial t})\tilde{\chi} = \lambda(\dot{D}_P)\tilde{\chi}\,.$$

In other words, the eigenvalue homomorphism for the radial operator $\dot{D}_P$ on the character χ is the same as the eigenvalue homomorphism for the constant coefficient operator $T(P)(\frac{1}{i}\frac{\partial}{\partial t})$ on the function $\tilde{\chi}$. This simple remark leads to an exact description of $T(P)$ in terms of eigenvalues of irreducible representations.

For this note that J is defined on H so that $\tilde{\chi}$ can be expanded in a Fourier series

$$\tilde{\chi}(t) = \sum_{k \in \mathbb{Z}^{m+n}} a_k e^{i\langle k,t\rangle}\,.$$

Applying $T(P)(\frac{1}{i}\frac{\partial}{\partial t})$ to both sides one shows that if $a_k \neq 0$, then

$$T(P)(k) = \lambda(\dot{D}_P) \tag{2}$$

for *every* extendible polynomial P. Letting P range over all such polynomials one proves the following fact.

Proposition 4.1. *The set of lattice elements k such that $a_k \neq 0$ is a W-orbit $W.k_0$.*

It should be noted that since $T(P)$ is itself W-invariant the lack of uniqueness of the lattice element k_0 is minimal.

Now χ is W-invariant. Furthermore, for σ in the Weyl group it follows that $\sigma(J) = \varepsilon(\sigma)J$ where $\varepsilon(\sigma) = \det(\sigma) = \pm 1$. Hence, up to a multiplicative constant

$$\chi(t) = J^{-1}(t) \sum_{\sigma \in W} \varepsilon(\sigma) e^{i\langle k_0, \sigma(t)\rangle}$$

for any fixed k_0 in the support of $\tilde{\chi}$.

Now order the weight lattice so that the roots α and β which occur in the above product formula for J are positive and write

$$\chi(t) = \sum c_j e^{i\langle m_j, t\rangle}$$

where the m_j are the weights of the representation ρ with Λ being the highest weight which occurs. Compare this expression for $\chi(t)$ with that above to obtain

$$\sum c_j e^{i\langle m_j, t\rangle} \prod (e^{i\frac{\alpha(t)}{2}} - e^{-i\frac{\alpha(t)}{2}}) = \sum \varepsilon(\sigma) e^{i\langle k_0, \sigma(t)\rangle} \prod (e^{i\frac{\beta(t)}{2}} - e^{-i\frac{\beta(t)}{2}}) \,.$$

Equating the highest order terms on each side yields

$$\Lambda + \frac{1}{2}\sum \alpha = k_0 + \frac{1}{2}\sum \beta$$

where k_0 is the highest of the elements in the W-orbit $W.k_0$. Turning this around, we see that

$$k_0 = \Lambda + \delta$$

where

$$\delta = \frac{1}{2}(\sum \alpha - \sum \beta) \,.$$

Theorem 4.2. *Let ρ be a finite-dimensional representation of a Lie supergroup associated to one of the Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m,n), \mathfrak{osp}(2m,2n)$. Let P be an extendible polynomial on $\mathfrak{h}$ and $T(P)$ be the polynomial which is defined by*

$$\dot{D}_P = J^{-1} T(P)(\frac{1}{i}\frac{\partial}{\partial t}) J \,.$$

If χ is the character of ρ with the homomorphism λ defined by

$$\dot{D}_P = \lambda(\dot{D}_P)\chi \,,$$

then

$$\lambda(\dot{D}_P) = T(P)(\Lambda + \delta)$$

where Λ is the highest weight of ρ.

Proof. This follows immediately from (2) and the fact that our choice of $k = k_0$ in the W-orbit is $k_0 = \Lambda + \delta$. ∎

5. Generating functions

Here we fix $\mathfrak{g}$ as one of the Lie superalgebras $\mathfrak{gl}(m,n)$ or $\mathfrak{osp}(2m,2n)$ and let $\dot{D}_\ell$ be the radial operator defined by the particular extendible polynomial F_ℓ. Using the Fourier series development of characters of representation, it was shown above that the value $\lambda(\dot{D}_\ell)$ of the eigenvalue homomorphism on $\dot{D}_\ell$ for the character of an irreducible representation ρ of highest weight Λ of the associated Lie supergroup is the value of the polynomial $T(P)$ on lattice point $k_0 = \Lambda + \delta$. Letting ρ range through all such representations, we see that $T(P)$ is the unique polynomial with this property. If we think of such a point k_0 as a weight, then it is in $\mathfrak{h}^*$; so we reformulate the result as follows: There is a uniquely determined polynomial function R_ℓ on $\mathfrak{h}^*$ with $R_\ell(\Lambda+\delta) = \lambda(\dot{D}_\ell)$ on every irreducible representation of highest weight Λ.

Associated to the sequence $\{R_\ell\}$ of polynomials one has the *generating function*

$$S(z) := \sum z^\ell R_\ell(x)$$

which is computed in closed form in [1] (see Lemma 4.3, pages 327-329, for the case of $\mathfrak{gl}(m,n)$ and Lemma 4.4, pages 335-341, for $\mathfrak{osp}(2m,2n)$). The resulting formulas for the polynomials R_ℓ are derived after the proofs of these lemmas. Using the identification $R_\ell = T(F_\ell)$, one has the following consequence which we formulate simultaneously for both $\mathfrak{gl}(m,n)$ and $\mathfrak{osp}(2m,2n)$.

Theorem 5.1. *If $\dot{D}_\ell$ is the radial differential operator defined by the extendible polynomial F_ℓ with*

$$\dot{D}_\ell = J^{-1} T(F_\ell)\left(\frac{1}{i}\frac{\partial}{\partial t}\right) J\,, \tag{3}$$

then $T(F_\ell) \in \mathbb{C}[F_1,\dots,F_\ell]$. Moreover $T(F_\ell) = F_\ell + Q_\ell$ where $Q_\ell \in \mathbb{C}[F_1,\dots,F_{\ell-1}]$ is a polynomial of lower degree.

6. An application

In [2] and [4] characters χ of representations of Lie supergroups on certain infinite-dimensional spaces play an important role. In [2] the complex Lie superalgebra at hand is $\mathfrak{gl}(m,n)$ and in [4] it is $\mathfrak{osp}(2n,2n)$. In these situations one would hope to apply the above results on radial operators. However, this can not be directly done, because the characters are defined by supertrace and only converge on certain open domains $\mathcal{H}$ in G_0 or on finite covering spaces $\widehat{\mathcal{H}}$ of such domains. On the other hand, Laplace-Casimir operators are local and can therefore be applied to such characters and in the settings of [2] and [4] the characters χ which appear are annihilated by Laplace-Casimir operators D_ℓ defined by the F_ℓ.

In the domains $\mathcal{H}$ or the covering spaces $\widehat{\mathcal{H}}$ there are closed connected complex submanifolds T^+ which are either open subsets of a Cartan algebra H or lifts of such into the covering space. Now the radial operators $\dot{D}_\ell$ are differential operators which are apriori defined on the space $\mathcal{D}_H$ of globally defined extendible W-invariant holomorphic functions and on that space we know how to compute them using the righthand side of (3). The restrictions of the characters χ to T^+, which are by definition the numerical parts of $\chi|T^+$, are by definition extendible as radial superfunctions, but they are only defined on T^+ and not on H. Nevertheless we wish to show that they are annihilated by the operators which are described by the righthand side of (3). For this we prove a local version of (3) and obtain the desired result on T^+ by applying the identity principle.

Local formula for $\dot{\mathbf{D}}_\ell$. We refer to a point in H as being *superregular* if it is regular in the sense of Lie theory and is not contained in any of the odd root hypersurfaces $\{r_\beta = 1\}$. Every superregular point x has a basis of open neighborhoods V in H which are relatively compact in the set of superregular points in H with the property that $\sigma(V) \cap V = \emptyset$ for every $\sigma \in W \setminus \{\mathrm{Id}\}$. Given such a V we thicken it as follows to an open neighborhood U in G_0. Let Δ be a polydisk in $\mathfrak{g}_0$ which is transversal to $\mathfrak{h}$ and define $U = \{\exp(\xi).x; \xi \in \Delta, x \in V\}$. We choose Δ small enough so that $U \cong \Delta \times V$. For $x \in V$ fixed we think of $\exp(\Delta).x$ as a local orbit of G_0.

Proposition 6.1. *Every superregular point x in H has a neighborhood basis of open sets V and U as above so that the restriction map $R : \mathcal{R}(U) \to \mathcal{O}(V)$ is an isomorphism.*

Proof. Since holomorphic maps $U \to \wedge\mathfrak{g}_1^*$ which are invariant by the local conjugation-action of G_0 are completely determined by their restrictions to V, it follows that R is injective. Surjectivity is proved by the following approximation argument.

First, in order to take care of W-invariance we consider the quotient $\pi : H \to Z = H/W$. The restriction $\pi|V$ maps V biholomorphically onto a domain $\tilde{V}$. A basic theorem of complex analysis states that we may choose $\tilde{V}$ (and accordingly V) so that the restriction map $\mathcal{O}(Z) \to \mathcal{O}(\tilde{V})$ has dense image.

Now let r be the product of the odd root functions on H and $\tilde{r}$ be the associated function on Z. Define $\tilde{f}$ be the function on $\tilde{V}$ associated to a given holomorphic function f on V. Let $\tilde{f}_n$ be a sequence of holomorphic functions on Z which converge to $\tilde{r}^{-2}\tilde{f}$ in $\mathcal{O}(\tilde{V})$. It follows that $\tilde{h}_n := \tilde{r}^2\tilde{f}_n$ converges to $\tilde{f}$. The point of this construction is that the sequence $\{h_n\}$ of lifts defined by $h_n := \pi^*(h_n)$ converge to f on V. In addition these are W-invariant and have the divisibility property. Thus we have the sequence $\{E(h_n)\}$ of radial extensions. Now the extension $h_n|V \to E(h_n)|V$ is such that the convergence of $h_n|V$ implies the convergence of $E(h_n)|V$ as a sequence of $\wedge\mathfrak{g}_1^*$-valued holomorphic maps. Consequently the maps $E(h_n)|U$ which are constant along the local G_0-orbits defined by Δ also converge. If a sequence of holomorphic functions converges, then so does any induced sequence of derivatives. Thus the limit $E(f)$ of the sequence $\{E(h_n)|U\}$ is a radial holomorphic function whose (numerical) restriction to V is the given function f. ∎

Having localized to the open sets $V = U \cap H$ of superregular elements of H and proved the above extension result, given a Laplace-Casimir operator D we define its radial part on $U \cap H$ in the same way as in the global case: $\dot{D}_{U\cap H}(f) := RDE(f)$. Since the extension result was proved by taking limits of globally defined extendible functions and the global operator $\dot{D}$ is continuous, it follows that $\dot{D}_{U\cap H}$ is just the restriction of $\dot{D}$ to $U \cap H$. Thus we have the following local version of Theorem 3.1.

Theorem 6.2. *Under the assumptions of Theorem 3.1, let D be a Laplace-Casimir operator defined by an extendible polynomial P. If U is as above, then the domain of definition of the radial operator $\dot{D}_{U\cap H}$ is the full algebra of holomorphic functions $\mathcal{O}(U \cap H)$ and*

$$\dot{D}_{U\cap H} = J^{-1}T(P)(\frac{1}{i}\frac{\partial}{\partial t})J\,.$$

As a result we have the local version of Theorem 5.1

Theorem 6.3. *Under the assumptions of Theorem 5.1, for U as above it follows that*

$$\dot{D}_{U\cap H} = J^{-1}(\tilde{F}_\ell + Q_\ell(\tilde{F}_1, \dots \tilde{F}_{\ell-1}))J$$

where Q_ℓ is a polynomial operator of lower degree than $\tilde{F}_\ell$.

Now let us return to the settings of [2] and [4] where we have an open piece T^+ of the Cartan algebra H contained as a closed submanifold of a domain $\mathcal{H}$ in G_0 or a finite-to-one covering space $\widehat{\mathcal{H}}$ of such a domain. The Weyl group W acts on these domains so that the above arguments apply: If $x \in T^+$ is superregular, then we setup U as above and prove the following result.

Theorem 6.4. *Let D_ℓ be the Laplace-Casimir operator defined on $\mathcal{H}$ or $\widehat{\mathcal{H}}$ by the extendible polynomial F_ℓ. Then a holomorphic superfunction on such a domain is annihilated by D_ℓ if and only if its (numerical) restriction to T^+ is annihilated by $J^{-1}(\tilde{F}_\ell + Q_\ell(\tilde{F}_1, \dots \tilde{F}_{\ell-1}))J$.*

Proof. Since D_ℓ acts on the full sheaf of radial functions, we may regard it as acting on the restriction of given holomorphic superfunction to U. Thus the restriction of those to $U\cap H$ is annihilated by the associated radial operator $\dot{D}_{U\cap H}$. By Theorem 6.3 the operator $J^{-1}(F_\ell + Q_\ell(F_1, \dots F_{\ell-1}))J$ annihilates this restriction on $U \cap H$ and the desired result follows by the identity principle. ■

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