Tulczyjew's Triplet for Lie Groups I: Trivializations and Reductions

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Abstract. All semidirect product and functorial trivializations of first order and iterated bundles over a Lie group are presented. For cotangent bundles, symplectic reduction is applied to obtain coadjoint orbit symplectic structure. All ingredients of Tulczyjew's triplet namely, iterated bundles, symplectomorphisms and special symplectic structures, for a Lie group are realized in trivializations. Symplectic reductions of iterated bundles by right invariance results in Tulczyjew's triplet for reduced manifolds. Results for left invariance are summarized in an appendix.
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1. Introduction

Tulczyjew's triplet consists of three iterated bundles

$$T^*T\mathcal{M} \xleftarrow{\quad \sigma_{\mathcal{M}} \quad} TT^*\mathcal{M} \xrightarrow{\quad \Omega^\flat_{T^*\mathcal{M}} \quad} T^*T^*\mathcal{M} \qquad (1)$$

over a smooth manifold $\mathcal{M}$ and two special symplectic structures on $TT^*\mathcal{M}$ each of which results in the same, namely, the Tulczyjew symplectic structure on $TT^*\mathcal{M}$. Special symplectic structures are induced from canonical symplectic spaces $T^*T^*\mathcal{M}$ and $T^*T\mathcal{M}$ by means of the symplectomorphisms $\sigma_{\mathcal{M}}$ and $\Omega^\flat_{T^*\mathcal{M}}$ which also carry Lagrangian submanifolds therein. When $\mathcal{M}$ is the configuration manifold of some mechanical system, Lagrangian and Hamiltonian formulations become Lagrangian submanifolds of Tulczyjew symplectic space $TT^*\mathcal{M}$ with respect to two different special symplectic structures. This geometrizes the Legendre transformation as a change of realization of a Lagrangian submanifold [59, 60, 61, 62, 63]. See Appendix A for a more detailed discussion on Tulczyjew's construction.

When the configuration manifold $\mathcal{M}$ is a Lie group G the kinematical symmetries reduce Lagrangian dynamics on TG to the Lie algebra $\mathfrak{g}$ of G with

Euler-Poincaré equations and, Hamiltonian dynamics on T^*G to the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ with Lie-Poisson equations [1, 3, 23, 39, 43]. In this case, Tulczyjew construction is expected to provide a Legendre transformation between reduced dynamics as well. The purpose of this series of papers is to give an explicit construction of Tulczyjew triplet for Lie groups and make relations between Lagrangian and Hamiltonian formulations precise at various levels of trivializations admitted by iterated bundles over G.

In the present work, we will focus on trivializations of bundles and representations of vector fields and one-forms on trivialized structures. Our treatment will be based on global trivializations of TG and T^*G as well as their group structures. The latter can be realized as semidirect product groups $G\circledS\mathfrak{g}$ and $G\circledS\mathfrak{g}^*$, respectively. The trivializations of iterated bundles can be achieved in several ways which are discussed in subsection 1.2. We shall adapt the one on which iterated bundles are trivialized as tangent and cotangent groups of semidirect products $G\circledS\mathfrak{g}$ and $G\circledS\mathfrak{g}^*$.

Our motivation comes from the plasma dynamics as described by Poisson-Vlasov equations [14, 25]. The configuration space is the group $Diff_{can}(T^*\mathcal{Q})$ of canonical diffeomorphisms of the particle phase space $T^*\mathcal{Q}$ on which the motion is generated by left actions [40]. The kinematical symmetries are generated by the right action and the well-known Lie Poisson formulation is obtained by reduction with this symmetry [47, 49, 50, 51]. Right invariance of Eulerian dynamics is also the case for incompressible fluid with configuration space being the group $Diff_{vol}(\mathcal{Q})$ of volume preserving diffeomorphisms [4, 42]. On the contrary, rigid body dynamics is left invariant [43, 54]. In this case, the configuration space is the group of rotations $SO(3)$ and, the reduction by left action results in the usual formulation of rigid body dynamics, that is the one with body or convected coordinates. Thus, we shall adapt and proceed by right actions in order to obtain a geometric set up for right invariant dynamics over G. In other words, the dynamics will not feel the trivializations performed for the purpose of Tulczyjew construction for Lie groups. Within the right invariant geometric set up obtained, we shall perform reductions of dynamics generated by the left action. Inevitably, our notation will have some differences from those familiar in the literature [1, 2, 3, 4, 23, 39, 43, 46]. For the sake of completeness, we shall present, in Appendix B, trivializations of Tulczyjew's triplet for left invariant dynamics as well, namely, the construction by left action and left trivialization.

In the literature, various analysis and generalizations of Tulczyjew's triplet can be found. To cite some examples, we refer to [8, 11, 17, 18, 19] for construction for field theories using jet bundles, to [7, 20, 21] for constructions in the framework of Lie algebroids, to [55] and [24] for $k-$cosymplectic and presymplectic structures, to [22] for double groups, and to the recent preprint [16] for the Marsden-Weinstein reductions of Tulczyjew triplets.

1.1. Notations.

G is a Lie group and, $\mathfrak{g} = Lie(G) \simeq T_eG$ is its Lie algebra. The dual of $\mathfrak{g}$ is $\mathfrak{g}^* = Lie^*(G) \simeq T_e^*G$. Throughout the paper, we will denote arbitrary elements of certain spaces by specific letters. We will adapt

$$g, h \in G, \quad \xi, \eta, \zeta \in \mathfrak{g}, \quad \mu, \nu, \lambda \in \mathfrak{g}^*. \tag{2}$$

A superscript R will stand for right invariance. For example,

$$\xi_g^R := T_e R_g\left(\xi\right) \in T_g G, \quad \mu_g^R := T_g^* R_{g^{-1}}\left(\mu\right) \in T_g^* G \tag{3}$$

are right invariant vector field and one-form on G over g translated from $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, respectively. Similarly, a superscript L will show the left invariance. More generally for an arbitrary manifold $\mathcal{M}$, we will use the following notation

$$u, v \in \mathcal{M}, \quad V_u, U_u \in T_u \mathcal{M}, \quad \alpha_u, \beta_u, \gamma_u \in T_u^* \mathcal{M} \tag{4}$$

to denote vectors and one-forms over specific points. This will also hold for a vector space W. For vectors and one-forms which are either right or left invariant, we adapt letters V, U and α, β, respectively. For example, $V_g \in T_g G$ and $\alpha_g \in T_g^* G$ will have arbitrary invariance properties. The diagonalization map

$$\mathfrak{D} : \mathcal{M} \to \mathcal{M} \times \mathcal{M} : v \to (v, v) \tag{5}$$

will be used in writing trivializations in a coordinate invariant form.

1.2. Trivializations.

The tangent TG and the cotangent T^*G bundles of a Lie group G carry canonical group multiplications. The requirement that the trivialization maps be Lie group isomorphisms, results with semi-direct product structures $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^*$ on the trivialized spaces, respectively. The decomposition of iterated bundles TTG, TT^*G, T^*TG and T^*T^*G, into the products of G, $\mathfrak{g}$ and $\mathfrak{g}^*$, can be achieved by replacing the first order bundles TG and T^*G by their semi direct product trivializations $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^*$, respectively. This reduces the problem to decompositions of $T\left(G \circledS \mathfrak{g}\right)$, $T\left(G \circledS \mathfrak{g}^*\right)$, $T^*\left(G \circledS \mathfrak{g}\right)$ and $T^*\left(G \circledS \mathfrak{g}^*\right)$. To write the iterated bundles in terms of products of algebra and its dual, the remaining step can be taken in several ways. The first way is to consider $T\left(G \circledS \mathfrak{g}\right)$, $T\left(G \circledS \mathfrak{g}^*\right)$, $T^*\left(G \circledS \mathfrak{g}\right)$ and $T^*\left(G \circledS \mathfrak{g}^*\right)$ as tangent and cotangent groups of semidirect products and express them as semidirect products of base group with its Lie algebra and dual of the algebra, respectively. This will result in trivializations of the first kind

$$
\begin{aligned}
T\left(G \circledS \mathfrak{g}\right) &\simeq \left(G \circledS \mathfrak{g}\right) \circledS Lie\left(G \circledS \mathfrak{g}\right) \\
&\simeq \left(G \circledS \mathfrak{g}\right) \circledS \left(\mathfrak{g} \circledS \mathfrak{g}\right) =: {}^1 TTG \\
T\left(G \circledS \mathfrak{g}^*\right) &\simeq \left(G \circledS \mathfrak{g}^*\right) \circledS Lie\left(G \circledS \mathfrak{g}^*\right) \\
&\simeq \left(G \circledS \mathfrak{g}^*\right) \circledS \left(\mathfrak{g} \circledS \mathfrak{g}^*\right) =: {}^1 TT^*G \\
T^*\left(G \circledS \mathfrak{g}\right) &\simeq \left(G \circledS \mathfrak{g}\right) \circledS Lie^*\left(G \circledS \mathfrak{g}\right) \\
&\simeq \left(G \circledS \mathfrak{g}\right) \circledS \left(\mathfrak{g}^* \times \mathfrak{g}^*\right) =: {}^1 T^*TG \\
T^*\left(G \circledS \mathfrak{g}^*\right) &\simeq \left(G \circledS \mathfrak{g}^*\right) \circledS Lie^*\left(G \circledS \mathfrak{g}^*\right) \\
&\simeq \left(G \circledS \mathfrak{g}^*\right) \circledS \left(\mathfrak{g}^* \times \mathfrak{g}\right) =: {}^1 T^*T^*G
\end{aligned}
\tag{6}
$$

which induce multi-semidirect product group structures on iterated bundles. The second way is first to distribute functors T and T^* to $G \circledS \mathfrak{g}$ and $G \circledS \mathfrak{g}^*$, obtain product of first order bundles and then trivialize each factor involving the product.

This gives trivializations of the second kind

$$
\begin{aligned}
T\left(G\circledS\mathfrak{g}\right) &\simeq TG\circledS T\mathfrak{g} \simeq \left(G\circledS\mathfrak{g}\right)\circledS\left(\mathfrak{g}\times\mathfrak{g}\right) =: {}^{2}TTG \\
T\left(G\circledS\mathfrak{g}^{*}\right) &\simeq TG\circledS T\mathfrak{g}^{*} \simeq \left(G\circledS\mathfrak{g}\right)\circledS\left(\mathfrak{g}^{*}\times\mathfrak{g}^{*}\right) =: {}^{2}TT^{*}G \\
T^{*}\left(G\circledS\mathfrak{g}\right) &\simeq T^{*}G\circledS T^{*}\mathfrak{g} \simeq \left(G\circledS\mathfrak{g}^{*}\right)\circledS\left(\mathfrak{g}\times\mathfrak{g}^{*}\right) =: {}^{2}T^{*}TG \\
T^{*}\left(G\circledS\mathfrak{g}^{*}\right) &\simeq T^{*}G\circledS T^{*}\mathfrak{g}^{*} \simeq \left(G\circledS\mathfrak{g}^{*}\right)\circledS\left(\mathfrak{g}^{*}\times\mathfrak{g}\right) =: {}^{2}T^{*}T^{*}G, \quad (7)
\end{aligned}
$$

where we use the identifications $TW = W \times W$ and $T^{*}W = W \times W^{*}$ for vector spaces $\mathfrak{g}$ and $\mathfrak{g}^{*}$ with the assumption of reflexivity. In trivializations of first and second kinds, all maps are Lie group morphisms. In trivializations of the second kind, the orders of fibrations are mixed up in an unusual way. For our purpose of investigating dynamical structures, we shall prefer studying trivializations of the first kind. However, for the sake of completeness, we shall also present results for trivializations of the second kind. Some other trivializations are also possible but they usually do not satisfy requirements of being Lie group morphisms. For example, two other immediate ways to make trivializations can be achieved by replacing semidirect products with direct products. For example, the third and the fourth kinds of trivializations of $T^{*}\left(G\circledS\mathfrak{g}\right)$ can be given by

$$
\begin{aligned}
T^{*}\left(G\circledS\mathfrak{g}\right) &\simeq \left(G\circledS\mathfrak{g}\right)\times Lie^{*}\left(G\circledS\mathfrak{g}\right) \\
&\simeq \left(G\circledS\mathfrak{g}\right)\times\left(\mathfrak{g}^{*}\times\mathfrak{g}^{*}\right) =: {}^{3}T^{*}TG \quad (8) \\
T^{*}\left(G\circledS\mathfrak{g}\right) &\simeq T^{*}G\circledS T^{*}\mathfrak{g} \simeq \left(G\times\mathfrak{g}^{*}\right)\circledS\left(\mathfrak{g}^{*}\times\mathfrak{g}^{*}\right) =: {}^{4}T^{*}TG. \quad (9)
\end{aligned}
$$

The trivializations in Eqs.(6), (7) and (9) are based on identifications of first order bundles with their semidirect product trivializations. Alternatively, the group structures on trivializations of TG and $T^{*}G$ can be given in direct product form $TG \simeq G\times\mathfrak{g}$ and $T^{*}G \simeq G\times\mathfrak{g}^{*}$. This reduces the problem of trivializations to decompositions of $T\left(G\times\mathfrak{g}\right)$, $T\left(G\times\mathfrak{g}^{*}\right)$, $T^{*}\left(G\times\mathfrak{g}\right)$, $T^{*}\left(G\times\mathfrak{g}^{*}\right)$ which can be achieved either by regarding this as a cotangent group or, by functorial way, with or without semidirect product structures. As an example, we choose $T^{*}\left(G\times\mathfrak{g}\right)$ and list yet another four trivializations of $T^{*}TG$ in addition to above four trivializations of $T^{*}\left(G\times\mathfrak{g}\right)$

$$
\begin{aligned}
T^{*}\left(G\times\mathfrak{g}\right) &\simeq \left(G\times\mathfrak{g}\right)\times Lie^{*}\left(G\times\mathfrak{g}\right) \\
&\simeq \left(G\times\mathfrak{g}\right)\times\left(\mathfrak{g}^{*}\times\mathfrak{g}^{*}\right) =: {}^{5}T^{*}TG \\
T^{*}\left(G\times\mathfrak{g}\right) &\simeq T^{*}G\times T^{*}\mathfrak{g} \simeq \left(G\times\mathfrak{g}^{*}\right)\times\left(\mathfrak{g}\times\mathfrak{g}^{*}\right) =: {}^{6}T^{*}TG \\
T^{*}\left(G\times\mathfrak{g}\right) &\simeq \left(G\times\mathfrak{g}\right)\circledS Lie^{*}\left(G\times\mathfrak{g}\right) \\
&\simeq \left(G\times\mathfrak{g}\right)\circledS\left(\mathfrak{g}^{*}\times\mathfrak{g}^{*}\right) =: {}^{7}T^{*}TG \\
T^{*}\left(G\times\mathfrak{g}\right) &\simeq T^{*}G\times T^{*}\mathfrak{g} \simeq \left(G\circledS\mathfrak{g}^{*}\right)\times\left(\mathfrak{g}\times\mathfrak{g}^{*}\right) =: {}^{8}T^{*}TG.
\end{aligned}
$$

The fifth and sixth approaches that takes $TG = G\times\mathfrak{g}$ and $T^{*}G = G\times\mathfrak{g}^{*}$ with direct product group structure and then trivialize iterated bundles with the same understanding have already been considered in [46] and [16].

1.3. Content of the work.

In the next section, we shall start by reviewing the right adjoint and coadjoint actions of a Lie group on its tangent and cotangent spaces. Then, semidirect

product structures and their representations on tangent and cotangent spaces will be studied. The left and right trivializations of tangent group TG and cotangent group T^*G will be established. Various actions and lifts of group operations to their tangent and cotangent spaces will be computed. It will be shown that, the trivialization of T^*G is a symplectic manifold with an exact symplectic two-form $\Omega_{G\circledS_R\mathfrak{g}^*} = d\theta_{G\circledS_R\mathfrak{g}^*}$. A symplectic reduction will be applied to trivialized symplectic manifold $G\circledS_R\mathfrak{g}^*$ and Kostant-Kirillov-Souriou symplectic form on the coadjoint orbit in $\mathfrak{g}^*$ will be rederived.

In section three, we shall start with trivializations of iterated tangent bundle TTG. A canonical involution on the trivialized space ^1TTG will be derived. Trivializations of cotangent bundles $T^*TG \simeq T^* (G\circledS_R\mathfrak{g})$ and $T^*T^*G \simeq T^* (G\circledS_R\mathfrak{g}^*)$ will be introduced. The symplectic structures on trivialized bundles $^1T^*TG$ and $^1T^*T^*G$ induced from those of T^*TG and T^*T^*G will be presented. The left group actions of G on $^1T^*TG$ and $^1T^*T^*G$ are Hamiltonian actions with ad-invariant momentum mappings. Marsden-Weinstein symplectic reduction theorem will be applied to both. Finally, we shall present trivializations of Tulczyjew's symplectic space $TT^*G \simeq T (G\circledS\mathfrak{g}^*)$. Over the trivialization $^1TT^*G$, we shall have two potential one-forms $^1\theta_1$ and $^1\theta_2$ whose exterior derivatives coincide and give Tulczyjew's symplectic two-form $\Omega_{^1TT^*G}$. A symplectic reduction will also be applied to Tulczyjew's symplectic manifold $^1TT^*G$. The reduced manifolds obtained by the reductions of $^1T^*TG$, $^1T^*T^*G$ and $^1TT^*G$ will all be expressible as products of the Lie algebra $\mathfrak{g}$, its dual $\mathfrak{g}^*$ and and the coadjoint orbit $\mathfrak{g}^*_\mu$ but with different orders. The reduced symplectic structures are the products of the canonical symplectic structure on $\mathfrak{g}^* \times \mathfrak{g}$, as a trivialized cotangent space, and Kostant-Kirillov-Souriou symplectic structure on coadjoint orbit.

In section four, trivialization of Tulczyjew's triplet for Lie groups will be established by the diagram

$$^1T^*TG \quad \xleftarrow{\quad^1\bar{\sigma}_G\quad} \quad ^1TT^*G \quad \xrightarrow{\quad^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\quad} \quad ^1T^*T^*G \ , \qquad (10)$$

where the symplectic diffeomorphisms $^1\bar{\sigma}_G$ and $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ will be defined through compositions with trivialization maps. This triplet will contain two special symplectic structures for $^1TT^*G$. The rest of this section will be devoted to the reduction of Tulczyjew's triplet by right invariance under G, and a triplet diagram consisting of reduced symplectic manifolds will be obtained.

Appendix A contains a brief summary of construction of Tulczyjew's triplet for an arbitrary manifold, general definitions of special symplectic structures, derivations on differential forms and, Morse families. In appendix B, we shall present results of almost all constructions of the main body for left representations and reductions. To each section, we shall place an introduction where we summarize contents in more technical terms.

2. The First Order Bundles

In the literature, adjoint and coadjoint representations of G on its Lie algebra $\mathfrak{g}$ and dual $\mathfrak{g}^*$ are usually defined in terms of left representations, see for example

[1, 3, 43, 39, 23]. We shall use right representations in the main body of this work. So, in the first subsection we shall review basic ingredients of right representations of G on its tangent and cotangent spaces. In the second subsection, we will particularly focus on the right representations of semidirect product groups. In the third subsection, we shall identify the tangent bundle TG with the semidirect product group $G⑤_R\mathfrak{g}$, calculate tangent and cotangent lifts of group multiplication on $G⑤_R\mathfrak{g}$ and representations of $G⑤_R\mathfrak{g}$ on Lie algebra $\mathfrak{g}⑤_R\mathfrak{g}$ and its dual $\mathfrak{g}^* \times \mathfrak{g}^*$. Here, subscript R in the semidirect product symbol ⑤ indicates that, the group structure on $G⑤_R\mathfrak{g}$ and Lie algebra structure on $\mathfrak{g}⑤_R\mathfrak{g}$ are derived from right representation of G on $\mathfrak{g}$. In the third subsection, we shall trivialize the cotangent group T^*G into $G⑤_R\mathfrak{g}^*$. Tangent and cotangent lifts, adjoint and coadjoint representations of $G⑤_R\mathfrak{g}^*$ will be presented. In order to have definitions of canonical and symplectic forms on $G⑤_R\mathfrak{g}$ which are invariant under trivialization, Jacobi-Lie bracket of right invariant vector fields on $G⑤_R\mathfrak{g}^*$ will be derived. An exact symplectic structure on $G⑤_R\mathfrak{g}^*$ which is symplectomorphic to the canonical symplectic structure on T^*G will be constructed. By construction, this structure will be invariant under dynamics generated by left actions (e.g. fluids and plasmas),

2.1. Representations.

Right representation of a Lie group G on a vector space W is a map $\rho : W \times G \to W$ satisfying the condition $\rho_h \circ \rho_g = \rho_{gh}$ for $g, h \in G$ [15]. Here, ρ_g is an automorphism on W obtained from ρ by fixing $g \in G$, that is, $\rho_g : W \to W$. The dual map is defined by

$$\rho_g^* : W^* \to W^*, \quad \left\langle \rho_g^* \left(\alpha_v \right), V_v \right\rangle = \left\langle \alpha_v, \rho_g \left(V_v \right) \right\rangle,$$

where $V_v \in T_v W = W$ and $\alpha_v \in T_v W^* \simeq W^*$. By fixing $v \in W$, we obtain the orbit map

$$\rho_v : G \to W : g \to \rho \left(g, v \right) \tag{11}$$

and the image of ρ_v is called orbit of v in W. Tangent and cotangent lifts of ρ_v are

$$T_g \rho_v : T_g G \to W, \ T_g^* \rho_v : W^* \to T_g^* G, \tag{12}$$

respectively. By taking $g = e$, the infinitesimal right action

$$T_e \rho : W \times \mathfrak{g} \to W$$

of $\mathfrak{g}$ on W is obtained. In other words, over the identity $e \in G$, we obtain the induced Lie algebra homomorphism $T_e \rho : \mathfrak{g} \to End(W)$ into the space of linear continuous maps of W. That is, fixing a Lie algebra element $\xi \in \mathfrak{g}$, we arrive at the tangent map and its dual

$$T_e \rho_\xi : W \to W, \quad T_e^* \rho_\xi : W^* \to W^*. \tag{13}$$

Since ρ is linear with respect to its second argument, the tangent map of ρ_g is

$$T_v \rho_g \left(V_v \right) = \rho_g \left(V_v \right) = \rho \left(g, V_v \right) = \rho_{V_v} \left(g \right), \tag{14}$$

We denote left and right multiplications on G by L_g and R_g, respectively. The inner automorphism

$$I_g^R = L_{g^{-1}} \circ R_g \tag{15}$$

satisfies $I_g^R \circ I_h^R = I_{hg}^R$ and is a right representation of G on G. The right adjoint action $Ad_g^R = T_e I_g^R$ of G on $\mathfrak{g}$ is defined as the tangent map of right inner automorphism at the identity $e \in G$. The infinitesimal adjoint representation

$$ad_\xi^R \eta = [\xi, \eta]_\mathfrak{g}^R \tag{16}$$

is defined as the derivative of Ad_g^R at identity and is a right action of $\mathfrak{g}$ on itself. Distribution of Ad_g^R on ad^R is given by

$$Ad_g^R [\xi, \eta]_\mathfrak{g}^R = \left[Ad_g^R \xi, Ad_g^R \eta \right]_\mathfrak{g}^R . \tag{17}$$

A right invariant vector field ξ^R on G can be obtained by right translation

$$\xi_g^R = T_e R_g \xi \tag{18}$$

of $\xi \in \mathfrak{g}$ for each $g \in G$. The identity

$$[\xi, \eta]_\mathfrak{g}^R = \left[\xi^R, \eta^R \right]_{JL} \tag{19}$$

gives the isomorphism between $\mathfrak{g}$ endowed with the bracket in Eq.(16) and, the space $\mathfrak{X}^R (G)$ of right invariant vector fields endowed with the Jacobi-Lie bracket. The coadjoint action Ad_g^{R*} of G on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a right representation and is the linear algebraic dual of $Ad_{g^{-1}}^R$, namely,

$$\left\langle Ad_g^{R*} \mu, \xi \right\rangle = \left\langle \mu, Ad_{g^{-1}}^R \xi \right\rangle \tag{20}$$

holds for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. The infinitesimal coadjoint action ad_ξ^{R*} of $\mathfrak{g}$ on $\mathfrak{g}^*$ is the linear algebraic dual of ad_ξ^R. Note that, the infinitesimal generator of the coadjoint action Ad_g^{R*} is minus the infinitesimal coadjoint action ad_ξ^{R*}, that is, if $g^t \subset G$ is a curve passing through the identity in the direction of $\xi \in \mathfrak{g}$, then

$$\left. \frac{d}{dt} \right|_{t=0} Ad_{g^t}^{R*} \mu = -ad_\xi^{R*} \mu. \tag{21}$$

The distribution law in Eq.(17) leads a commutation relation for ad_ξ^{R*} and Ad_g^{R*} given by

$$Ad_g^{R*} \circ ad_\xi^{R*} = ad_{Ad_g \xi}^{R*} \circ Ad_g^*$$

or in an alternative form

$$ad_\xi^{R*} \circ Ad_g^{R*} = Ad_g^{R*} \circ ad_{Ad_{g^{-1}} \xi}^{R*}. \tag{22}$$

Remark 2.1. The definition of inner automorphism in Eq.(15) differs from the one given, for example, in [1, 3, 23, 39, 43]. This definition of the inner automorphism is the most crucial for the present work, see for example, the representations of G on $\mathfrak{g}$ and $\mathfrak{g}^*$ in Eqs.(17) and (20), respectively, which are right representations.

2.2. Semidirect product structures.

For an arbitrary right representation $\rho : G \times W \to W$, define the group multiplication

$$L^R_{(g,u)}(h,v) = R^R_{(h,v)}(g,u) = (g,u)(h,v) = (gh, u + \rho(g^{-1}, v)) \qquad (23)$$

that endows $S = G \circledS W$ with a semidirect product group structure with the identity and inverse elements

$$(e,0), \quad (g,u)^{-1} = (g^{-1}, -\rho(g,u)), \qquad (24)$$

respectively. Since the group structure is defined by right representation of G on W, we shall denote this semidirect product by $S^R = G\circledS_R W$. We remark again that, our formulation of semidirect product structure may contain differences with those in [9, 27, 28, 38, 44, 45, 53], formulations in these references use left representation of the group over the vector space. The simple relation $\rho_g \to \rho_{g^{-1}}$ sends a left representation to a right one, hence, establishes a link between Eq.(23) with formulations left representations.

The tangent and cotangent spaces over the point $(g,u) \in S$ are

$$\begin{aligned}
T_{(g,u)}S &= \{(V_g, V_u) : V_g \in T_g G \text{ and } V_u \in T_u W = W\} \\
T^*_{(g,u)}S &= \{(\alpha_g, \alpha_u) : \alpha_g \in T^*G \text{ and } \alpha_u \in T^*_u W = W^*\}
\end{aligned} \qquad (25)$$

and, over the identity $(e,0)$, we shall use $(\xi, V_0) \in T_{(e,0)}S$ and $(\alpha_e, \alpha_0) \in T^*_{(e,0)}S$. The derivative of left and right actions give the respective left and right actions of S on $T_{(g,u)}S^R$

$$\begin{aligned}
T_{(h,v)}L^R_{(g,u)}(V_h, V_v) &= (T_h L_g V_h, \rho(g^{-1}, V_v)) \\
T_{(g,u)}R^R_{(h,v)}(V_g, V_u) &= (T_g R_h V_g, V_u + T_{g^{-1}}\rho_v(V_{g^{-1}})),
\end{aligned} \qquad (26)$$

where $T_g \rho_v$ is defined as in Eq.(12) and, $T L_g V_{g^{-1}} = -T R_{g^{-1}} V_g$. The inner automorphism

$$I^R_{(g,u)}(h,v) = (g,u)^{-1}(h,v)(g,u) = \left(I^R_g h, \rho_g\left(v - u + \rho_{h^{-1}}(u)\right)\right)$$

is a right representation of S^R on itself. The adjoint action of S^R on its Lie algebra $T_{(e,0)}S^R = \mathfrak{s}^R$ and the infinitesimal action of $\mathfrak{s}^R$ on itself are given by

$$\begin{aligned}
Ad^R_{(g,u)}(\xi, V_0) &= \left(Ad^R_g \xi, \rho_g V_0 - \rho_g \circ T_e \rho_u(\xi)\right) \\
[(\xi, V_0), (\eta, U_0)]^R_{\mathfrak{s}} &= \left([\xi, \eta]^R_{\mathfrak{g}}, T_e \rho_{U_0}(\xi) - T_e \rho_{V_0}(\eta)\right),
\end{aligned} \qquad (27)$$

where $\xi, \eta \in \mathfrak{g}$, $V_0, U_0 \in T_0 W = W$ and we used the identities in Eq.(14). Denoting $uv = u + \rho(g^{-1}, v)$ for brevity, the lifted left and right actions of S^R on T^*S^R are

$$\begin{aligned}
T^*_{(h,v)}L^R_{(g,u)}(\alpha_{gh}, \alpha_{uv}) &= (T^*_h L_g(\alpha_{gh}), \rho^*_{g^{-1}}(\alpha_{uv})) \in T^*_{(h,v)}S^R \qquad (28) \\
T^*_{(g,u)}R^R_{(h,v)}(\alpha_{gh}, \alpha_{uv}) &= (T^*_g R_h(\alpha_{gh}) - T^*(L_{g^{-1}} \circ R_{g^{-1}}) \cdot T^*_{g^{-1}}\rho_v(\alpha_{uv}), \alpha_{uv}),
\end{aligned}$$

where $(\alpha_{gh}, \alpha_{uv}) \in T^*_{(gh,uv)} S^R$. Adjoint actions of S^R and $\mathfrak{s}^R$ on the dual space $\mathfrak{s}^{R*}$ are

$$
\begin{aligned}
Ad^{R*}_{(g,u)} (\mu, \alpha_0) &= \left(Ad^{R*}_{g^{-1}} \mu - T^*_e \rho_u \circ \rho^*_g (\alpha_0), \rho^*_g (\alpha_0) \right) \\
ad^{R*}_{(\xi,V_0)} (\mu, \alpha_0) &= \left(ad^{R*}_{\xi} \mu - T^*_e \rho_{V_o} (\alpha_0), T^*_e \rho_{\xi} (\alpha_0) \right)
\end{aligned}
$$

where $(\mu, \alpha_0) \in \mathfrak{g}^* \times W^*$ and we used the notation in Eq.(13).

2.3. Tangent Group.

The tangent bundle TG of a Lie group G is also a Lie group with the induced multiplication

$$
\varpi_{TG} (V_g, U_h) = T_g R_h V_g + T_h L_g U_h, \tag{29}
$$

where $V_g \in T_g G$, $U_h \in T_h G$ and $\varpi_{TG} (V_g, U_h) \in T_{gh} G$ [26, 35, 52, 69]. The identity element of TG is the zero vector in $T_e G$ and the inverse of V_g is

$$
(V_g)^{-1} = -T_g \left(R_{g^{-1}} \circ L_{g^{-1}} \right) V_g \in T_{g^{-1}} G, \tag{30}
$$

which is a vector starting at g^{-1} and in the reverse direction of V_g. The mappings

$$
\begin{aligned}
tr^R_{TG} &= (\tau_G, T_g R_{g^{-1}}) \circ \mathfrak{D} : TG \to G \times \mathfrak{g} : V_g \to (g, T_g R_{g^{-1}} V_g) \tag{31} \\
tr^L_{TG} &= (\tau_G, T_g L_{g^{-1}}) \circ \mathfrak{D} : TG \to G \times \mathfrak{g} : V_g \to (g, T_g L_{g^{-1}} V_g), \tag{32}
\end{aligned}
$$

are called right and left global trivializations of the tangent bundle TG, respectively [12, 46, 54]. Here, $\mathfrak{D}$ is the diagonalization map defined in Eq.(5). We can define two Lie group structures $\left(tr^R_{TG} \right)^* \varpi_{TG}$ and $\left(tr^L_{TG} \right)^* \varpi_{TG}$ on $G \times \mathfrak{g}$. The pull back $\left(tr^R_{TG} \right)^* \varpi_{TG}$ by the right trivialization gives the group operation

$$
L^R_{(g,\xi)} (h, \eta) = R^R_{(h,\eta)} (g, \xi) = \left(gh, \xi + Ad^R_{g^{-1}} \eta \right) \tag{33}
$$

on $G \times \mathfrak{g}$. The operation in Eq.(33) is an example of the semidirect product in Eq.(23) with $V = \mathfrak{g}$, $\rho = Ad^R$ and $T_e \rho_\eta (\xi) = [\xi, \eta]^R_{\mathfrak{g}}$. The tangent group with this structure will be denoted by $G \circledS_R \mathfrak{g}$. The identity is $(e, 0)$ and the inverse is

$$
(g, \xi)^{-1} = \left(g^{-1}, -Ad^R_g \xi \right).
$$

The Lie algebra $Lie \left(G \circledS_R \mathfrak{g} \right)$ consists of two pairs $(\xi_1, \xi_2) \in \mathfrak{g} \circledS_R \mathfrak{g}$ with the semidirect product Lie algebra bracket

$$
[(\xi_1, \xi_2), (\eta_1, \eta_2)]_{\mathfrak{g} \circledS_R \mathfrak{g}} = \left(ad^R_{\xi_1} \eta_1, ad^R_{\xi_1} \eta_2 - ad^R_{\eta_1} \xi_2 \right). \tag{34}
$$

The actions of the group $G \circledS_R \mathfrak{g}$ on itself and to tangent spaces are

$$
\begin{aligned}
I^R_{(g,\xi)} (h, \eta) &= \left(I^R_g h, Ad^R_g \left(\eta - \xi + Ad^R_{h^{-1}} \xi \right) \right) \\
T_{(h,\eta)} L^R_{(g,\xi)} (V_h, V_\eta) &= \left(T_h L_g V_h, Ad^R_{g^{-1}} V_\eta \right) \\
T_{(g,\xi)} R^R_{(h,\eta)} (V_g, V_\xi) &= \left(T_g R_h V_g, V_\xi + [Ad^R_{g^{-1}} \eta, TR_{g^{-1}} V_g]^R_{\mathfrak{g}} \right) \tag{35} \\
Ad^R_{(g,\xi)} (\eta, \zeta) &= \left(Ad^R_g \eta, Ad^R_g \left(\zeta - [\eta, \xi]^R_{\mathfrak{g}} \right) \right) \tag{36}
\end{aligned}
$$

where $(V_h, V_\eta) \in T_{(h,\eta)} (G\circledS_R\mathfrak{g})$ and $(V_g, V_\xi) \in T_{(g,\xi)} (G\circledS_R\mathfrak{g})$. The cotangent lifted actions are

$$
\begin{aligned}
T^*_{(h,\eta)} L^R_{(g,\xi)} (\alpha_{gh}, \alpha_{\xi\eta}) &= \left(T^*_h L_g \alpha_{gh}, Ad^{R*}_g \alpha_{\xi\eta} \right) \\
T^*_{(g,\xi)} R^R_{(h,\eta)} (\alpha_{gh}, \alpha_{\xi\eta}) &= \left(T^*_g R_h \alpha_{gh} + T^*_g R_{g^{-1}} \circ ad^{R*}_{\left(Ad^R_{g^{-1}}\eta\right)} \alpha_{\xi\eta}, \alpha_{\xi\eta} \right)
\end{aligned}
$$

where $\alpha_{gh} \in T_{gh}G$ and $\alpha_{\xi\eta} \in T^*_{\xi\eta}\mathfrak{g} \simeq \mathfrak{g}^*$ and, $\xi\eta = \xi + Ad^R_{g^{-1}}\eta$. The dual of $\mathfrak{g}\circledS_R\mathfrak{g}$ is the product $\mathfrak{g}^* \times \mathfrak{g}^*$ of two copies of $\mathfrak{g}^*$. Coadjoint actions of $G\circledS_R\mathfrak{g}$ and $\mathfrak{g}\circledS_R\mathfrak{g}$ on the dual space $\mathfrak{g}^* \times \mathfrak{g}^*$ are

$$
\begin{aligned}
Ad^{R*}_{(g,\xi)} (\nu, \mu) &= \left(Ad^{R*}_g \left(\nu - ad^{R*}_\xi \mu \right), Ad^{R*}_g \mu \right) & (37) \\
ad^{R*}_{(\xi,\eta)} (\nu, \mu) &= \left(ad^{R*}_\xi \nu + ad^{R*}_\eta \mu, ad^{R*}_\xi \mu \right), & (38)
\end{aligned}
$$

where $(\nu, \mu) \in \mathfrak{g}^* \times \mathfrak{g}^*$. These are obtained by dualization of the adjoint actions $Ad^R_{(g,\xi)}$ and $ad^R_{(\xi,\eta)}$ in Eqs.(36) and (34), respectively, and the identity in Eq.(22) is used in calculation of $Ad^{*R}_{(g,\xi)}$.

A vector field $X^{G\circledS_R\mathfrak{g}}$ on $G\circledS_R\mathfrak{g}$ is a section of the tangent bundle $T(G\circledS_R\mathfrak{g})$. Since the tangent bundle $T(G\circledS_R\mathfrak{g})$ is isomorphic to $TG \times T\mathfrak{g}$ as a vector space, we take $X^{G\circledS_R\mathfrak{g}}$ in coordinates as

$$
X^{G\circledS_R\mathfrak{g}} (g, \xi) = \left(X^G (g, \xi), X^\mathfrak{g} (g, \xi) \right),
$$

where $X^G : (G\circledS_R\mathfrak{g}) \to TG$ and $X^\mathfrak{g} : (G\circledS_R\mathfrak{g}) \to T\mathfrak{g}$. A right invariant vector field can be defined by the right translation of a Lie algebra element $(\eta, \zeta) \in \mathfrak{g}\circledS_R\mathfrak{g}$, that is,

$$
X^{G\circledS_R\mathfrak{g}}_{(\eta,\zeta)} (g, \xi) = T_{(e,0)} R^R_{(g,\xi)} (\eta, \zeta) = \left(T_e R_g \eta, \zeta + [\xi, \eta]^R_\mathfrak{g} \right). \qquad (39)
$$

By replacing G with $G\circledS_R\mathfrak{g}$ in Eq.(19) and making necessary modifications, we arrive at the following result.

Proposition 2.2. *The Jacobi-Lie bracket of two right invariant vector fields, in the form of Eq.(39), on $G\circledS_R\mathfrak{g}$ is given by*

$$
\begin{aligned}
[X^{G\circledS_R\mathfrak{g}}_{(\eta_1,\eta_2)}, X^{G\circledS_R\mathfrak{g}}_{(\zeta_1,\zeta_2)}] (g, \xi) &= X^{G\circledS_R\mathfrak{g}}_{[(\eta_1,\eta_2),(\zeta_1,\zeta_2)]_{\mathfrak{g}\circledS_R\mathfrak{g}}} (g, \xi) \\
&= \left(T_e R_g [\eta_1, \zeta_1]^R_\mathfrak{g}, [\eta_1, \zeta_2]^R_\mathfrak{g} - [\zeta_1, \eta_2]^R_\mathfrak{g} + [\xi, [\eta_1, \zeta_1]^R_\mathfrak{g}]^R_\mathfrak{g} \right),
\end{aligned}
$$

where, in the first line, the bracket designated the Lie algebra element associated to the vector on the right hand side is the Lie algebra bracket in Eq.(34).

2.4. Cotangent Group.

The cotangent bundle T^*G of G is a Lie group with the group multiplication

$$
\varpi_{T^*G} (\alpha_g, \beta_h) = T^*_{gh} R_{h^{-1}} \alpha_g + T^*_{gh} L_{g^{-1}} \beta_h \qquad (40)
$$

for $\alpha_g \in T^*_g G$ and $\beta_h \in T^*_h G$ [26, 35, 52]. Note that $\varpi_{T^*G} (\alpha_g, \beta_h) \in T^*_{gh}G$. Here, T^*R_g stands for the cotangent lift of the right translation R_g on G, that is, $T^*_{gh}R_g$

is the pointwise dual of the tangent mapping $T_{gh}R_g$. The identity element is the zero covector at e, and the inverse is

$$(\alpha_g)^{-1} = -T^* (L_g \circ R_g) \, \alpha_g.$$

The trivialization maps

$$\begin{aligned}
tr_{T^*G}^L &= (\pi_G, T_e^* L_g) \circ \mathfrak{D} : T^*G \to G \times \mathfrak{g}^* : \alpha_g \to (g, T_e^* L_g \alpha_g), &\text{(41)} \\
tr_{T^*G}^R &= (\pi_G, T_e^* R_g) \circ \mathfrak{D} : T^*G \to G \times \mathfrak{g}^* : \alpha_g \to (g, T_e^* R_g \alpha_g) &\text{(42)}
\end{aligned}$$

of T^*G into $G \times \mathfrak{g}^*$ are defined by the assumption that the natural pairing between T_g^*G and T_gG is right, respectively, left invariant. Here, $\mathfrak{D}$ is the diagonalization map. If we require that the right trivialization of T^*G be a Lie group isomorphism, then we obtain the group multiplication

$$L_{(g,\mu)}^R (h,\nu) = R_{(h,\nu)}^R (g,\mu) = (g,\mu)(h,\nu) = \left(gh, \mu + Ad_{g^{-1}}^{R*}\nu\right) \tag{43}$$

on $G \circledS_R \mathfrak{g}^*$. This is a particular case of semidirect product structure in Eq.(23) with $V = \mathfrak{g}^*$, $\rho = Ad^{R*}$ and $T_e\rho_\mu(\xi) = -ad_\xi^*\mu$. The identity element is $(e,0)$ and the inverse of an element (g,μ) is

$$(g,\mu)^{-1} = \left(g^{-1}, -Ad_g^{R*}\mu\right). \tag{44}$$

The tangent space to $G \circledS_R \mathfrak{g}^*$ at the identity is the underlying vector space for the Lie algebra $\mathfrak{g} \circledS_R \mathfrak{g}^*$. We calculate the following actions

$$\begin{aligned}
I_{(g,\mu)}^R (h,\nu) &= \left(I_g^R h, Ad_g^{R*}\left(\nu - \mu + Ad_{h^{-1}}^{R*}\mu\right)\right) \\
Ad_{(g,\mu)}^R (\eta,\nu) &= \left(Ad_g^R \eta, Ad_g^{R*}\left(\nu + ad_\eta^{R*}\mu\right)\right) &\text{(45)} \\
ad_{(\xi,\mu)}^R (\eta,\nu) &= [(\xi,\mu),(\eta,\nu)]_{\mathfrak{g}\circledS_R\mathfrak{g}^*} = \left([\xi,\eta]_\mathfrak{g}^R, ad_\eta^{R*}\mu - ad_\xi^{R*}\nu\right) &\text{(46)} \\
T_{(h,\nu)}L_{(g,\mu)}^R (V_h, V_\nu) &= \left(TL_g^R V_h, Ad_{g^{-1}}^{R*} V_\nu\right) \\
T_{(h,\nu)}^* L_{(g,\mu)}^R (\alpha_{gh}, \alpha_{\mu\nu}) &= \left(T_h^* L_g(\alpha_{gh}), Ad_g^R \alpha_{\mu\nu}\right) \\
T_{(h,\nu)}R_{(g,\mu)}^R (V_h, V_\nu) &= \left(TR_g V_h, V_\nu + Ad_{h^{-1}}^{R*} \circ ad_{TL_{h^{-1}}V_h}^{R*}\mu\right) &\text{(47)} \\
T_{(h,\nu)}^* R_{(g,\mu)}^R (\alpha_{hg}, \alpha_{\nu\mu}) &= \left(T^* R_g(\alpha_{hg}) - T^* L_{h^{-1}} \circ ad_{Ad_h\alpha_{\nu\mu}}^{R*}\mu, \alpha_{\nu\mu}\right) &\text{(48)}
\end{aligned}$$

where $V_h \in T_hG$ and $V_\nu \in T_\nu\mathfrak{g}^* \simeq \mathfrak{g}^*$, $\alpha_{\nu\mu} \in T_{\nu\mu}^*\mathfrak{g}^* \simeq \mathfrak{g}$ and, $\nu\mu = \nu + Ad_{h^{-1}}^{R*}\mu$.

We take the dual of the Lie algebra $\mathfrak{g} \circledS_R \mathfrak{g}^*$ to be $\mathfrak{g}^* \times \mathfrak{g}$ by assuming the reflexivity condition $\mathfrak{g}^{**} \simeq \mathfrak{g}$. Coadjoint actions of $G \circledS_R \mathfrak{g}^*$ and $\mathfrak{g} \circledS_R \mathfrak{g}^*$ on the dual space are

$$\begin{aligned}
Ad_{(g,\mu)}^{R*} (\nu,\eta) &= \left(Ad_g^{R*}\left(\nu + ad_\eta^{R*}\mu\right), Ad_g\eta\right) &\text{(49)} \\
ad_{(\xi,\mu)}^{R*} (\nu,\eta) &= \left(ad_\xi^{R*}\nu - ad_\eta^{R*}\mu, [\eta,\xi]_\mathfrak{g}^R\right), &\text{(50)}
\end{aligned}$$

where $(\nu,\eta) \in \mathfrak{g}^* \times \mathfrak{g}$. Define a mapping s that changes the order of direct product

$$s : \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}^* : (\nu,\eta) \to (\eta,\nu), \tag{51}$$

which is an involution, that is, s^2 is the identity. The following proposition establishes the link between the adjoint and coadjoint actions.

Proposition 2.3. *The adjoint and coadjoint actions of $G\circledS_R\mathfrak{g}^*$ on $\mathfrak{g}\circledS_R\mathfrak{g}^*$ and $\mathfrak{g}^* \times \mathfrak{g}$, given in Eqs.(45) and (49), respectively, satisfy*

$$Ad^{R*}_{(g,\mu)} \circ s = s \circ Ad^R_{(g,\mu)},$$

where s is the involution in Eq.(51). The infinitesimal adjoint and coadjoint actions of $\mathfrak{g}\circledS_R\mathfrak{g}^*$ on itself and its dual $\mathfrak{g}^* \times \mathfrak{g}$, given in Eqs.(46) and (50), respectively, satisfy

$$ad^R_{(\xi,\mu)} \circ s = -s \circ ad^{R*}_{(\xi,\mu)}.$$

A right invariant vector field on $G\circledS_R\mathfrak{g}^*$ is defined by

$$X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}(g,\mu) = T_{(e,0)}R^R_{(g,\mu)}(\xi,\nu) = \left(T_eR_g\xi, \nu + ad^{R*}_\xi\mu\right) \tag{52}$$

which is the right translation of the Lie algebra element (ξ,ν). The following proposition defines a Lie algebra structure on right invariant sections of the tangent bundle $T\left(G\circledS_R\mathfrak{g}^*\right) \to G\circledS_R\mathfrak{g}^*$.

Proposition 2.4. *Jacobi-Lie bracket on $G\circledS_R\mathfrak{g}^*$ of two right invariant vector fields $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ and $X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}$ in the form of Eq.(52) is*

$$\begin{aligned}\left[X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}\right](g,\mu) &= X^{G\circledS_R\mathfrak{g}^*}_{[(\xi,\nu),(\eta,\lambda)]_{\mathfrak{g}\circledS_R\mathfrak{g}^*}}(g,\mu) \tag{53}\\ &= \left(T_eR_g\left([\xi,\eta]^R_\mathfrak{g}\right), ad^{R*}_\eta\nu - ad^{R*}_\xi\lambda + ad^{R*}_{[\xi,\eta]^R_\mathfrak{g}}\mu\right).\end{aligned}$$

A right invariant one-from on $G\circledS_R\mathfrak{g}^*$ follows from translation of an element (ν,ξ) of the dual $\mathfrak{g}^* \times \mathfrak{g}$ of the Lie algebra by (g,μ)

$$\theta^{G\circledS_R\mathfrak{g}^*}_{(\nu,\xi)}(g,\mu) = T^*_{(g,\mu)}R_{(g,\mu)^{-1}}(\nu,\xi) = \left(T^*R_{g^{-1}}\nu + T^*L_{g^{-1}} \circ Ad^{R*}_g \circ ad^{R*}_\xi\mu, \xi\right)$$

where we used the identity in Eq.(22). Note that, the pairing of a right invariant vector field $X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}$ and a right invariant one-form $\theta^{G\circledS_R\mathfrak{g}^*}_{(\nu,\xi)}$ is

$$\left\langle\theta^{G\circledS\mathfrak{g}^*}_{(\nu,\xi)}, X^{G\circledS\mathfrak{g}^*}_{(\eta,\lambda)}\right\rangle(g,\mu) = \langle\nu,\eta\rangle + \langle\lambda,\xi\rangle$$

which is independent of (g,μ) and hence a constant function on $G\circledS_R\mathfrak{g}^*$.

Trivialization of the canonical symplectic structure: Being a cotangent bundle, T^*G carries an exact symplectic two-form $\Omega_{T^*G} = d\theta_{T^*G}$. We refer to Appendix A for definitions of these canonical forms. We carry θ_{T^*G} and Ω_{T^*G} to $G\circledS\mathfrak{g}^*$ by push forward with the right trivialization map $tr^R_{T^*G}$. We consider a (not necessarily right invariant) vector field $X^{G\circledS_R\mathfrak{g}^*}(g,\mu) = (V_g, V_\mu)$ on $G\circledS\mathfrak{g}^*$ and pair with $\theta_{G\circledS_R\mathfrak{g}^*} = \left(tr^R_{T^*G}\right)_*\theta_{T^*G}$ to obtain the explicit expression for $\theta_{G\circledS\mathfrak{g}^*}$.

At the point (g, μ), we compute

$$
\begin{aligned}
& \left\langle \theta_{G\circledS_R\mathfrak{g}^*}, X^{G\circledS_R\mathfrak{g}^*} \right\rangle (g, \mu) \\
=\ & \left\langle \left(tr_{T^*G}^R \right)_* \theta_{T^*G}, X^{G\circledS_R\mathfrak{g}^*} \right\rangle (g, \mu) \\
=\ & \left\langle \theta_{T^*G}, \left(tr_{T^*G}^R \right)^* X^{G\circledS_R\mathfrak{g}^*} \right\rangle (T_e R_g \mu) \\
=\ & \left\langle \theta_{T^*G} \circ \left(tr_{T^*G}^R \right)^{-1}, T \left(tr_{T^*G}^R \right)^{-1} \circ X^{G\circledS_R\mathfrak{g}^*} \right\rangle (g, \mu) \\
=\ & \left\langle \tau_{T^*G} \circ T \left(tr_{T^*G}^R \right)^{-1} \circ X^{G\circledS_R\mathfrak{g}^*}, T\pi_G \circ T \left(tr_{T^*G}^R \right)^{-1} \circ X^{G\circledS_R\mathfrak{g}^*} \right\rangle (g, \mu) \\
=\ & \left\langle T^* R_{g^{-1}} \mu, V_g \right\rangle = \left\langle \mu, T R_{g^{-1}} V_g \right\rangle
\end{aligned}
\tag{54}
$$

where $\left(tr_{T^*G}^R \right)^{-1} (g, \mu) = T_g^* R_{g^{-1}} \mu$ is the reconstruction [43]. Note that, if $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ is a right invariant vector field generated by (ξ, ν), that is $V_g = T_e R_g \xi$, then the value of the canonical form $\theta_{G\circledS\mathfrak{g}^*}$ on $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ is the linear function $\langle \mu, \xi \rangle$. Hence, the value of the two-form $\Omega_{G\circledS_R\mathfrak{g}^*} = d\theta_{G\circledS_R\mathfrak{g}^*}$ on two right invariant vector fields $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ and $X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}$ can be computed using invariant definition of the exterior derivative operator

$$
\begin{aligned}
& \left\langle \Omega_{G\circledS_R\mathfrak{g}^*}; \left(X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)} \right) \right\rangle (g, \mu) \\
=\ & X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)} \left(\left\langle \theta_{G\circledS\mathfrak{g}^*}, X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)} \right\rangle \right) (g, \mu) - X^{G\circledS\mathfrak{g}^*}_{(\eta,\lambda)} \left(\left\langle \theta_{G\circledS_R\mathfrak{g}^*}, X^{G\circledS\mathfrak{g}^*}_{(\xi,\nu)} \right\rangle \right) (g, \mu) \\
& - \left\langle \theta_{G\circledS_R\mathfrak{g}^*}, \left[X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)} \right] \right\rangle (g, \mu) \\
=\ & X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)} \left(\langle \mu, \eta \rangle \right) - X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)} \left(\langle \mu, \xi \rangle \right) - \left\langle \theta_{G\circledS_R\mathfrak{g}^*}, X^{G\circledS_R\mathfrak{g}^*}_{[(\xi,\nu),(\eta,\lambda)]} \right\rangle (g, \mu) \\
=\ & \left\langle \nu + ad_\xi^{R*} \mu, \eta \right\rangle - \left\langle \lambda + ad_\eta^{R*} \mu, \xi \right\rangle - \langle \mu, [\xi, \eta] \rangle \\
=\ & \langle \nu, \eta \rangle - \langle \lambda, \xi \rangle + \left\langle \mu, [\xi, \eta]_{\mathfrak{g}}^R \right\rangle,
\end{aligned}
\tag{55}
$$

where in calculating the value of $\theta_{G\circledS_R\mathfrak{g}^*}$ on the Jacobi-Lie bracket of $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ and $X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}$, we used Eq.(53). Since push forward is a natural operation, $\Omega_{G\circledS_R\mathfrak{g}^*}$ is closed and (weakly) non-degenerate, hence defines a symplectic structure on $G\circledS_R\mathfrak{g}^*$ invariant under the left action. In other words, $tr_{T^*G}^R$ is a symplecto-morphism between the symplectic manifolds (T^*G, Ω_{T^*G}) and $(G\circledS_R\mathfrak{g}^*, \Omega_{G\circledS_R\mathfrak{g}^*})$. Musical isomorphism $\Omega^\flat$ maps vector fields to the one-form sections and it is defined by

$$
\left\langle \Omega^\flat (X), Y \right\rangle = \langle \Omega; (X, Y) \rangle.
$$

For $\Omega_{G\circledS_R\mathfrak{g}^*}$ in Eq.(55), the musical isomorphism is computed pointwise to be

$$
\begin{aligned}
\Omega^\flat_{G\circledS_R\mathfrak{g}^*} (g, \mu) \ &:\ T_{(g,\mu)} (G\circledS_R\mathfrak{g}^*) \to T^*_{(g,\mu)} (G\circledS_R\mathfrak{g}^*) \\
&:\ \left(T_e R_g \xi, \nu + ad_\xi^* \mu \right) \to \left(T_g^* R_{g^{-1}} \nu, -\xi \right)
\end{aligned}
\tag{56}
$$

for the right invariant vector fields. We summarize the results of this subsection in the following proposition.

Proposition 2.5. $G\circledS_R\mathfrak{g}^*$ *is an exact symplectic manifold with the potential one-form*

$$
\left\langle \theta_{G\circledS_R\mathfrak{g}^*}, X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)} \right\rangle (g, \mu) = \langle \mu, \xi \rangle
$$

and the symplectic two-form

$$\left\langle \Omega_{G\circledS_R\mathfrak{g}^*}; \left(X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)} \right) \right\rangle (g,\mu) = \langle \nu, \eta \rangle - \langle \lambda, \xi \rangle + \left\langle \mu, [\xi, \eta]^R_{\mathfrak{g}} \right\rangle,$$

where $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,\nu)}$ and $X^{G\circledS_R\mathfrak{g}^*}_{(\eta,\lambda)}$ are right invariant vector fields in the form of Eq.(52). The musical isomorphism induced by $\Omega_{G\circledS_R\mathfrak{g}^*}$ is

$$\Omega^\flat_{G\circledS\mathfrak{g}^*}(g,\mu) \quad : \quad T_{(g,\mu)}(G\circledS\mathfrak{g}^*) \to T^*_{(g,\mu)}(G\circledS\mathfrak{g}^*)$$

$$: \quad (V_g, V_\mu) \to \left(T^*R_{g^{-1}} \left(V_\mu - ad^*_{TR_{g^{-1}}V_g}\mu \right), -TR_{g^{-1}}V_g \right). \quad (57)$$

     Left action and symplectic reduction of $G\circledS^*_R\mathfrak{g}$: The group G is a subgroup of the trivialized cotangent group $G\circledS_R\mathfrak{g}^*$ with multiplication as in Eq.(43). The embedding of G into $G\circledS_R\mathfrak{g}^*$ leads us to define the trivialization of lifted left action of G on T^*G given by

$$G \times (G\circledS_R\mathfrak{g}^*) \to G\circledS_R\mathfrak{g}^* : (h; (g,\mu)) \to (hg, Ad^*_{h^{-1}}\mu). \quad (58)$$

The infinitesimal generators of this action are the right invariant vector fields $X^{G\circledS_R\mathfrak{g}^*}_{(\xi,0)}$ generated by $(\xi, 0) \in \mathfrak{g}\circledS_R\mathfrak{g}^*$. It is immediate to see that the mapping

$$\mathfrak{g} \to \mathfrak{X}^R(G\circledS_R\mathfrak{g}^*) : \xi \to X^{G\circledS_R\mathfrak{g}^*}_{(\xi,0)}$$

from $\mathfrak{g}$ with the Lie algebra bracket $[\ ,\ ]^R_\mathfrak{g}$ to the space $\mathfrak{X}^R(G\circledS_R\mathfrak{g}^*)$ of right invariant vector fields with the Jacobi-Lie bracket in Eq.(53), is a Lie algebra homomorphism.

     The action in Eq.(58) is a symplectic action with the momentum map

$$\mathbf{J}_{G\circledS_R\mathfrak{g}^*} \quad : \quad G\circledS_R\mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$

$$\langle \mathbf{J}_{G\circledS_R\mathfrak{g}^*}(g,\mu), \xi \rangle = \left\langle \theta_{G\circledS_R\mathfrak{g}^*}, X^{G\circledS_R\mathfrak{g}^*}_{(\xi,0)} \right\rangle = \langle \mu, \xi \rangle$$

(c.f. Eq.(136) in Appendix A) where, $\theta_{G\circledS_R\mathfrak{g}^*}$ is the potential one-form in Eq.(54). $\mathbf{J}_{G\circledS_R\mathfrak{g}^*}$ is thus the projection

$$\mathbf{J}_{G\circledS_R\mathfrak{g}^*} : (g,\mu) \to \mu$$

to the second factor. The inverse image $\mathbf{J}^{-1}_{G\circledS_R\mathfrak{g}^*}(\mu) \subset G\circledS_R\mathfrak{g}^*$ of a regular value $\mu \in \mathfrak{g}^*$ consists of two-tuples (g,μ) for $g \in G$ and fixed $\mu \in \mathfrak{g}^*$. Hence, we may identify $\mathbf{J}^{-1}_{G\circledS_R\mathfrak{g}^*}(\mu)$ with the group G.

     The isotropy group G_μ of the coadjoint action is

$$G_\mu = \left\{ g \in G : Ad^{R*}_{g^{-1}}\mu = \mu \right\}. \quad (59)$$

The quotient space $\mathbf{J}^{-1}_{G\circledS_R\mathfrak{g}^*}(\mu)\big/ G_\mu$ is isomorphic to the coadjoint orbit

$$\mathcal{O}_\mu = \left\{ Ad^{R*}_{g^{-1}}\mu \in \mathfrak{g}^* : g \in G \right\} \quad (60)$$

through the point $\mu \in \mathfrak{g}^*$. Note that, the identification $G/G_\mu \longleftrightarrow \mathcal{O}_\mu$ is given by $[g]_\mu \longleftrightarrow Ad_{g^{-1}}^{R*}\mu$. We have the reduction

$$
\begin{array}{ccc}
\mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu) = G & \xrightarrow{\quad\imath\quad} & G\circledS_R\mathfrak{g}^* \\[2mm]
\Big\downarrow{\scriptstyle\chi_{G\circledS_R\mathfrak{g}^*}} & \swarrow{\scriptstyle p_{G\circledS_R\mathfrak{g}^*}} & \\[2mm]
\mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu)/G_\mu = \mathcal{O}_\mu & &
\end{array}
$$

where, $\imath$ is the inclusion of $\mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu)$ into $G\circledS_R\mathfrak{g}^*$, $\chi_{G\circledS_R\mathfrak{g}^*}$ and $p_{G\circledS_R\mathfrak{g}^*}$ are the surjective projections

$$
\begin{aligned}
\chi_{G\circledS_R\mathfrak{g}^*} &\;:\; \mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu) \to \mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu)/G_\mu : (g,\mu) \to \left([g]_\mu \longleftrightarrow Ad_{g^{-1}}^{R*}\mu\right) \\
p_{G\circledS_R\mathfrak{g}^*} &\;:\; G\circledS_R\mathfrak{g}^* \to \mathcal{O}_\mu : (g,\nu) \to Ad_{g^{-1}}^{R*}\mu.
\end{aligned}
$$

According to the Marsden-Weinstein symplectic reduction theorem (c.f. Subsection 2 in Appendix A), the reduced space $\mathcal{O}_\mu$ in Eq.(60) is a symplectic manifold [41, 48]. We denote the reduced symplectic two-form on $\mathcal{O}_\mu$ by $\Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash}$ which is the Kostant-Kirillov-Souriou two-form [31, 32, 33, 34, 36, 37, 58]. The reduced symplectic two-form $\Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash}$ is defined by

$$
(\chi_{G\circledS_R\mathfrak{g}^*})^* \,\Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash} = (\imath_{G\circledS_R\mathfrak{g}^*})^* \,\Omega_{G\circledS_R\mathfrak{g}^*}, \tag{61}
$$

where $\Omega_{G\circledS_R\mathfrak{g}^*}$ is the symplectic two-form on $G\circledS_R\mathfrak{g}^*$ in Eq.(55). To obtain an explicit expression for $\Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash}$, we observe the followings: the second entry of a tangent vector on $\mathbf{J}_{G\circledS_R\mathfrak{g}^*}^{-1}(\mu)$ must vanish in order not to encounter any movement over μ. Hence, it is in the form of $X_{(\xi,-ad_\xi^{R*}\mu)}^{G\circledS_R\mathfrak{g}^*}(g,\mu)$, and the push-forward

$$
(\chi_{G\circledS_R\mathfrak{g}^*})_* X_{(\xi,-ad_\xi^{R*}\mu)}^{G\circledS_R\mathfrak{g}^*}(g,\mu) = ad_\xi^{R*}\mu = \xi_{\mathfrak{g}^*}(\mu)
$$

is obtained by deriving $Ad_{g^{-1}}^{R*}\mu$ at the identity in the direction of ξ. In other words, the infinitesimal generators of the coadjoint action Ad^{R*} of G on $\mathfrak{g}^*$ are tangent vectors on $\mathcal{O}_\mu$. The definition in Eq.(61) gives the value of $\Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash}(\mu)$ on $\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu) \in T_\mu\mathcal{O}_\mu$ as

$$
\left\langle \Omega_{G\circledS_R\mathfrak{g}^*}^{G\backslash}; (\xi_{\mathfrak{g}^*},\eta_{\mathfrak{g}^*}) \right\rangle(\mu) = -\left\langle \mu, [\xi,\eta]_\mathfrak{g}^R \right\rangle \tag{62}
$$

which is the well-known expression of the coadjoint orbit symplectic two-form.

3. Iterated Bundles

Throughout, we shall refer trivialization of the first kind (denoted by superscript 1) to the semidirect product decomposition of an iterated bundle and trivialization of the second kind (denoted by superscript 2) to the functorial decomposition.

In the first subsection, trivialization of $TTG \simeq T(G \circledS \mathfrak{g})$ into the product of G and three copies of its Lie algebra $\mathfrak{g}$ will be presented. A canonical involution $\bar{\kappa}_G$ on $T(G \circledS \mathfrak{g})$ and its *dual* $\bar{\sigma}_G$ will be defined. In the second subsection, trivialization of cotangent bundle $T^*TG \simeq T^*(G \circledS \mathfrak{g})$ and group structure on trivialized space ${}^1T^*TG$ will be presented. Symplectic two-form on ${}^1T^*TG$ will be derived and the Marsden-Weinstein symplectic reduction theorem will be applied with symplectic action of the group G on ${}^1T^*TG$. In the third subsection, $T^*T^*G \simeq T^*(G \circledS \mathfrak{g}^*)$ will be treated similarly. In the last subsection, trivialization of the Tulczyjew's symplectic space $TT^*G \simeq T(G \circledS \mathfrak{g}^*)$ will be presented. Special symplectic structures will be defined and the symplectic reduction theorem will be applied to ${}^1TT^*G$ for the canonical left action of G.

3.1. Tangent bundle of tangent group.

There are two possible ways to decompose $TTG \simeq T(G \circledS \mathfrak{g})$ into a semidirect product of G and three copies of $\mathfrak{g}$. The first trivialization uses the tangent group structure on $T(G \circledS \mathfrak{g})$. We decompose $T(G \circledS \mathfrak{g})$ as a semidirect product of the group $(G \circledS \mathfrak{g})$ and its Lie algebra $\mathfrak{g} \circledS \mathfrak{g}$

$$
\begin{aligned}
tr^1_{T(G \circledS \mathfrak{g})} \quad &: \quad T(G \circledS \mathfrak{g}) \to (G \circledS \mathfrak{g}) \circledS (\mathfrak{g} \circledS \mathfrak{g}) =: {}^1TTG \\
&: \quad (V_g, V_\xi) \to \left(\tau_{G \times \mathfrak{g}}(V_g, V_\xi) ; TR_{(g,\xi)^{-1}}(V_g, V_\xi) \right) \\
&: \quad (V_g, V_\xi) \to (g, \xi, TR_{g^{-1}}V_g, V_\xi - [\xi, TR_{g^{-1}}V_g]), \quad (63)
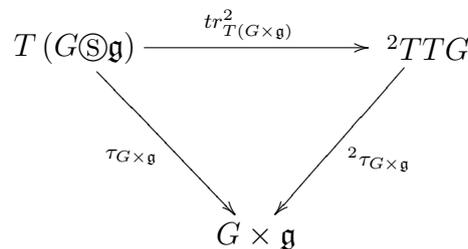\end{aligned}
$$

where we apply the procedure in Eq.(31) by replacing G with $G \circledS \mathfrak{g}$, [6, 65]. To obtain the second trivialization, we distribute the tangent functor T to the tangent Lie group $G \circledS \mathfrak{g}$ and then apply the operator $(tr_{TG} \times id)$ to the product $TG \times T\mathfrak{g}$

$$
\begin{aligned}
tr^2_{T(G \circledS \mathfrak{g})} \quad &: \quad T(G \circledS \mathfrak{g}) \simeq TG \times T\mathfrak{g} \to G \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} =: {}^2TTG \\
&: \quad (V_g, V_\xi) \to (tr_{TG}(V_g), V_\xi) = (g, T_g R_{g^{-1}} V_g, \xi, V_\xi) \quad (64)
\end{aligned}
$$

which results in an interchange of second and third projections. Namely, the tangent bundle projection

$$
{}^2\tau_{G \times \mathfrak{g}} : {}^2TTG \to G \times \mathfrak{g} : (g, \eta, \xi, \zeta) \to (g, \xi)
$$

makes the diagram

$$
\begin{array}{ccc}
T(G \circledS \mathfrak{g}) & \xrightarrow{\quad tr^2_{T(G \times \mathfrak{g})} \quad} & {}^2TTG \\
& {}_{\tau_{G \times \mathfrak{g}}} \searrow & \downarrow {}^2\tau_{G \times \mathfrak{g}} \\
& G \times \mathfrak{g} &
\end{array}
$$

commutative.

Canonical involution: TTG has two different fibrations over TG [1, 64, 56]. The first one is obtained by considering TTG as the tangent bundle of TG with the canonical tangent bundle projection τ_{TG}. The second one, denoted by $T\tau_G$, is obtained by lifting the projection τ_G. This double vector bundle structure

of TTG is known as dual tangent rhombic. This structure implies the existence of a canonical involution κ_G on TTG satisfying $T\tau_G = \tau_{TG} \circ \kappa_G$ [62], or equivalently, makes the diagram

$$
\begin{array}{ccc}
TTG & \xrightarrow{\ \kappa_G\ } & TTG \\
& \searrow{\scriptstyle T\tau_G} \quad {\scriptstyle \tau_{TG}}\swarrow & \\
& TG &
\end{array}
\tag{65}
$$

commutative. We shall use the diagram

$$
\begin{array}{ccc}
TTG & \xrightarrow{\ \kappa_G\ } & TTG \\
{\scriptstyle Ttr_{TG}}\big\downarrow & & \big\downarrow{\scriptstyle Ttr_{TG}} \\
T\left(G\circledS\mathfrak{g}\right) & \xrightarrow{\ \bar{\kappa}_G\ } & T\left(G\circledS\mathfrak{g}\right)
\end{array}
$$

to define the canonical involution $\bar{\kappa}_G$ on $T\left(G\circledS\mathfrak{g}\right)$.

To this end, we first need a characterization of elements of $T\left(G\circledS\mathfrak{g}\right)$. Consider $g = g\left(t, \epsilon\right)$ for real parameters t and ϵ, such that $g\left(0,0\right) = g$ and

$$
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} g = \delta g\left(t\right), \quad \left.\frac{d}{dt}\right|_{t=0} g = \dot{g}\left(\epsilon\right)
$$

are elements of $T_g G$. Using trivialization in Eq.(31), we trivialize these two vectors as

$$
\begin{aligned}
\delta g\left(t\right) &\rightarrow \left(g\left(t\right), \xi\left(t\right)\right), \quad \xi\left(t\right) = T_{g(t)}R_{g^{-1}(t)}\delta g\left(t\right), \\
\dot{g}\left(\epsilon\right) &\rightarrow \left(g\left(\epsilon\right), \eta\left(\epsilon\right)\right), \quad \eta\left(\epsilon\right) = T_{g(\epsilon)}R_{g^{-1}(\epsilon)}\dot{g}\left(\epsilon\right)
\end{aligned}
$$

and will denote $\xi\left(0\right) = \xi$ and $\eta\left(0\right) = \eta$. Note that, the derivatives of the curves $\left(g\left(t\right), \xi\left(t\right)\right)$ and $\left(g\left(\epsilon\right), \eta\left(\epsilon\right)\right)$ on $G\circledS\mathfrak{g}$ give the following vectors

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0}\left(g\left(t\right), \xi\left(t\right)\right) &= \left(\dot{g}, \dot{\xi}\right) \in T_{(g,\xi)}\left(G\circledS\mathfrak{g}\right) \\
\left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\left(g\left(\epsilon\right), \eta\left(\epsilon\right)\right) &= \left(\delta g, \delta\eta\right) \in T_{(g,\eta)}\left(G\circledS\mathfrak{g}\right),
\end{aligned}
\tag{66}
$$

respectively. It follows that, the difference of derivatives

$$
\left.\frac{d}{dt}\right|_{t=0}\xi\left(t\right) - \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\eta\left(\epsilon\right) = -\left[\eta, \xi\right]_{\mathfrak{g}},
$$

is the Lie bracket [9, 10, 29, 43, 46], and we have

$$
\delta\eta = \dot{\xi} + \left[\eta, \xi\right]_{\mathfrak{g}}.
\tag{67}
$$

Proposition 3.1. *[46] The canonical involution $\bar{\kappa}_G$ on $T(G\text{⑤}\mathfrak{g})$ is defined as to satisfy*

$$\bar{\kappa}_G \circ Ttr_{TG} = Ttr_{TG} \circ \kappa_G,$$

where κ_G is the canonical involution defined by the commutative diagram (65). In terms of trivializations in Eqs.(66), $\bar{\kappa}_G$ is given by

$$\bar{\kappa}_G : T_{(g,\xi)}(G\text{⑤}\mathfrak{g}) \to T_{(g,\eta)}(G\text{⑤}\mathfrak{g}) : \left(\dot{g}, \dot{\xi}\right) \to (\delta g, \delta \eta) = \left(\delta g, \dot{\xi} + [\eta, \xi]_{\mathfrak{g}}\right). \quad (68)$$

Eq.(67) is very well known as the reduced variational principle on $\mathfrak{g}$ [43, 10]. The compositions of $\bar{\kappa}_G$ with trivializations $tr^1_{T(G\text{⑤}\mathfrak{g})}$ and $tr^2_{T(G\text{⑤}\mathfrak{g})}$ give the same canonical involutions

$$^{1,2}\bar{\kappa}_G : (g, \xi, \eta, \zeta) \to \left(g, \eta, \xi, \zeta + [\xi, \eta]_{\mathfrak{g}}\right) \quad (69)$$

on ^1TTG and ^2TTG, respectively [46].

Existence of the endomorphism $\bar{\kappa}_G$ in Eq.(68) leads to an isomorphism $\bar{\sigma}_G$ from $T(G\text{⑤}\mathfrak{g}^*)$ to $T^*(G\text{⑤}\mathfrak{g})$. To obtain $\bar{\sigma}_G$ we first define, following reference [60], a pairing $\langle\,,\,\rangle^{\tilde{}}$ between $T(G\text{⑤}\mathfrak{g}^*)$ and $T(G\text{⑤}\mathfrak{g})$ as tangent lift of the pairing

$$\langle (g, \mu), (g, \xi) \rangle = \langle \mu, \xi \rangle,$$

between $G\text{⑤}\mathfrak{g}^*$ and $G\text{⑤}\mathfrak{g}$, where the pairing on the right hand side is the one between $\mathfrak{g}^*$ and $\mathfrak{g}$. In coordinates, for $(V_g, V_\mu) \in T_{(g,\mu)}(G\text{⑤}\mathfrak{g}^*)$ and $(U_g, U_\xi) \in T_{(g,\xi)}(G\text{⑤}\mathfrak{g})$, the pairing $\langle\,,\,\rangle^{\tilde{}}$ is defined by

$$\langle (V_g, V_\mu), (U_g, U_\xi) \rangle^{\tilde{}} = \langle \mu, U_\xi \rangle + \langle V_\mu, \xi \rangle. \quad (70)$$

Proposition 3.2. *The dual $\bar{\sigma}_G$ of the map $\bar{\kappa}_G$ is defined by*

$$\langle V, \bar{\kappa}_G(U) \rangle^{\tilde{}} = \langle \bar{\sigma}_G(V), U \rangle,$$

for $U_{(g,\xi)} \in T_{(g,\xi)}(G\text{⑤}\mathfrak{g})$ and $V_{(g,\mu)} \in T_{(g,\mu)}(G\text{⑤}\mathfrak{g}^*)$. In coordinates,

$$\begin{aligned}
\bar{\sigma}_G \;:\; & T_{(g,\mu)}(G\text{⑤}\mathfrak{g}^*) \to T^*_{(g,\xi)}(G\text{⑤}\mathfrak{g}) \\
\;:\; & (V_g, V_\mu) \to \left(T_g^* R_{g^{-1}}\left(V_\mu - ad_\xi^* \mu\right), \mu\right)
\end{aligned} \quad (71)$$

where $T_g R_{g^{-1}} V_g = \xi \in \mathfrak{g}$, and $\mu \in T_\xi^* \mathfrak{g} \simeq \mathfrak{g}^*$.

3.2. Cotangent bundle of tangent group.

Regarding $T^*(G\text{⑤}\mathfrak{g}^*)$ as a cotangent group, we trivialize $T^*(G\text{⑤}\mathfrak{g}^*)$ as the semidirect product group of $G\text{⑤}\mathfrak{g}^*$ and the dual $T^*_{(e,0)}(G\text{⑤}\mathfrak{g}^*) = \mathfrak{g}^* \times \mathfrak{g}^*$ of its Lie algebra. This trivialization of the first kind gives

$$\begin{aligned}
tr^1_{T^*(G\text{⑤}\mathfrak{g})} \;:\; & T^*(G\text{⑤}\mathfrak{g}) \to (G\text{⑤}\mathfrak{g})\text{⑤}(\mathfrak{g}^* \times \mathfrak{g}^*) =: \,^1T^*TG \\
\;:\; & (\alpha_g, \alpha_\xi) \to \left(\pi_{(G\text{⑤}\mathfrak{g})}(\alpha_g, \alpha_\xi), T^*_{(e,0)} R_{(g,\xi)}(\alpha_g, \alpha_\xi)\right) \\
\;:\; & (\alpha_g, \alpha_\xi) \to \left(g, \xi, T_e^* R_g(\alpha_g) + ad_\xi^* \alpha_\xi, \alpha_\xi\right), \quad (72)
\end{aligned}$$

where $\pi_{(G\text{\textcircled{S}}\mathfrak{g})}$ is the cotangent bundle projection of the fibration $T^*(G\text{\textcircled{S}}\mathfrak{g}) \to (G\text{\textcircled{S}}\mathfrak{g})$ and $T^*_{(e,0)}R_{(g,\xi)}$ is the cotangent lifted right multiplication in Eq.(48) whose image lies in the dual space $\mathfrak{g}^* \times \mathfrak{g}^*$. Here, α_ξ is a cotangent space element on $\mathfrak{g}$, that is, $\alpha_\xi \in T^*_\xi \mathfrak{g} \simeq \mathfrak{g}^*$.

Alternatively, we distribute the cotangent functor T^* to the product $(G\text{\textcircled{S}}\mathfrak{g}^*)$ and obtain the trivialization of the second kind

$$tr^2_{T^*(G\text{\textcircled{S}}\mathfrak{g})} \; : \; T^*(G\text{\textcircled{S}}\mathfrak{g}) \simeq T^*G \times T^*\mathfrak{g} \to (G\text{\textcircled{S}}\mathfrak{g}^*) \times \mathfrak{g} \times \mathfrak{g}^* = \; {}^2T^*TG,$$
$$(\alpha_g, \alpha_\xi) \to (g, T^*_e R_g(\alpha_g), \xi, \alpha_\xi). \tag{73}$$

For trivialization of the first kind in Eq.(72), we have a group structure similar to the cotangent group operation in Eq.(43) where the underlying group is $G\text{\textcircled{S}}\mathfrak{g}$ in place of G and the dual space, in place of $\mathfrak{g}^*$, is $\mathfrak{g}^*\text{\textcircled{S}}\mathfrak{g}^*$. This time the coadjoint action is the one given in Eq.(37). Then, the group multiplication on ${}^1T^*TG$ is

$$(g, \xi, \mu_1, \mu_2)(h, \eta, \nu_1, \nu_2) \tag{74}$$
$$= \left(gh, \xi + Ad_{g^{-1}}\eta, \mu_1 + Ad^*_{g^{-1}}\left(\nu_1 + ad^*_{Ad_g\xi}\nu_2\right), \mu_2 + Ad^*_{g^{-1}}\nu_2\right)$$

from which the following subgroup structures follow.

Proposition 3.3. *The embeddings*

$$\begin{aligned} G &\to {}^1T^*TG : g \to (g,0,0,0) \\ G\text{\textcircled{S}}\mathfrak{g} &\to {}^1T^*TG : (g,\xi) \to (g,\xi,0,0) \\ G\text{\textcircled{S}}\mathfrak{g}^* &\to {}^1T^*TG : (g,\mu) \to (g,0,\mu,0) \\ G\text{\textcircled{S}}(\mathfrak{g} \times \mathfrak{g}^*) &\to {}^1T^*TG : (g,\xi,\mu) \to (g,\xi,\mu,0) \end{aligned}$$

define submanifolds G, $G\text{\textcircled{S}}\mathfrak{g}$ and $G\text{\textcircled{S}}(\mathfrak{g} \times \mathfrak{g}^*)$ of ${}^1T^*TG$, respectively. Here, the group structure on $G\text{\textcircled{S}}\mathfrak{g}$ is the one in Eq.(33), the group structure on $G\text{\textcircled{S}}\mathfrak{g}^*$ is in Eq.(43), and the group structure on $G\text{\textcircled{S}}(\mathfrak{g} \times \mathfrak{g}^*)$ is

$$(g, \xi, \mu)(h, \eta, \nu) = \left(gh, \xi + Ad_{g^{-1}}\eta, \mu + Ad^*_{g^{-1}}\nu\right). \tag{75}$$

Symplectic structure: To calculate canonical symplectic structure on ${}^1T^*TG$, we shall first derive right invariant vector fields on ${}^1T^*TG$. For an explicit expression of a right invariant vector field, we shall proceed as follows. Take $H = G\text{\textcircled{S}}\mathfrak{g}$, $\mathfrak{h} = \mathfrak{g}\text{\textcircled{S}}\mathfrak{g}$ and $\mathfrak{h}^* = \mathfrak{g}^* \times \mathfrak{g}^*$, and recall the definition of right invariant vector field on trivialized cotangent bundle in Eq.(52). This gives

$$\begin{aligned} X^{{}^1T^*TG}_{(\eta,\zeta,\lambda_1,\lambda_2)}(g,\xi,\mu,\nu) &= \left(T_{(e,0)}R_{(g,\xi)}(\eta,\zeta),(\lambda_1,\lambda_2) + ad^*_{(\eta,\zeta)}(\mu,\nu)\right) \\ &= \left(T_e R_g\eta, \zeta + [\xi,\eta], \lambda_1 + ad^*_\eta\mu + ad^*_\zeta\nu, \lambda_2 + ad^*_\eta\nu\right) \end{aligned}$$

as a right invariant vector field over the cotangent group

$$X^{{}^1T^*TG}_{(\eta,\zeta,\lambda_1,\lambda_2)}(g,\xi,\mu,\nu) \in T_{(g,\xi,\mu,\nu)}((G\text{\textcircled{S}}\mathfrak{g})\text{\textcircled{S}}(\mathfrak{g}^* \times \mathfrak{g}^*)).$$

Here, the coadjoint representation $ad^*_{(\xi,\eta)}$ is the one in Eq.(38) and the tangent mapping $T_{(e,0)}R_{(g,\xi)}$ is in Eq.(35). This, of course, can be obtained as infinitesimal generator of the left action on $^1T^*TG$ with multiplication given by Eq.(74).

From the definition of canonical one-form in Eq.(54) over the trivialized cotangent bundle, we compute the value of canonical one-form

$$\langle \theta_{^1T^*TG}, X^{^1T^*TG}_{(\eta,\zeta,\lambda_1,\lambda_2)}\rangle(g,\xi,\mu,\nu) = \langle(\mu,\nu),(\eta,\zeta)\rangle = \langle\mu,\eta\rangle + \langle\nu,\zeta\rangle \qquad (76)$$

which is a constant function on $G\,Ⓢ\,\mathfrak{g}$. Using Eq.(55) for the value of canonical symplectic structure on two right invariant vector fields we obtain the value of symplectic two form $\Omega_{^1T^*TG} = d\theta_{^1T^*TG}$ on two right invariant vector fields

$$\left\langle \Omega_{^1T^*TG}; \left(X^{^1T^*TG}_{(\eta,\zeta,\lambda_1,\lambda_2)}, X^{^1T^*TG}_{(\bar\eta,\bar\zeta,\bar\lambda_1,\bar\lambda_2)}\right)\right\rangle(g,\xi,\mu,\nu)$$
$$= \langle(\lambda_1,\lambda_2),(\bar\eta,\bar\zeta)\rangle - \langle(\bar\lambda_1,\bar\lambda_2),(\eta,\zeta)\rangle + \left\langle(\mu,\nu),[(\eta,\zeta),(\bar\eta,\bar\zeta)]_{\mathfrak{g}Ⓢ\mathfrak{g}}\right\rangle$$
$$= \langle\lambda_1,\bar\eta\rangle + \langle\lambda_2,\bar\zeta\rangle - \langle\bar\lambda_1,\eta\rangle - \langle\bar\lambda_2,\zeta\rangle + \langle\mu,[\eta,\bar\eta]\rangle + \langle\nu,[\eta,\bar\zeta]-[\bar\eta,\zeta]\rangle, (77)$$

where $[\ ,\ ]_{\mathfrak{g}Ⓢ\mathfrak{g}}$ is the Lie algebra bracket in Eq.(34).

Proposition 3.4. *The trivialized bundle $^1T^*TG$ is an exact symplectic manifold with the potential one-form $\theta_{^1T^*TG}$ as given by Eq.(76) and the symplectic two-form $\Omega_{^1T^*TG}$ defined by Eq.(77). The mapping*

$$tr^1_{T^*(GⓈ\mathfrak{g})} \circ T^*tr^{-1}_{TG} : (T^*TG, \Omega_{T^*TG}) \to \left(^1T^*TG, \Omega_{^1T^*TG}\right)$$

is a symplectomorphism, where $T^*tr^{-1}_{TG}$ is the cotangent lift of the inverse

$$tr^{-1}_{TG} : (g,\xi) \to T_eR_g\xi \qquad (78)$$

of tr_{TG} in Eq.(31) and $tr^1_{T^*(GⓈ\mathfrak{g})}$ is the trivialization in Eq.(72).

Left action of G and symplectic reduction: From the multiplication in Eq.(74), we find the lifted left action of G on $^1T^*TG$ to be

$$G \times {}^1T^*TG \to {}^1T^*TG : (g;(h,\eta,\mu,\nu)) \to \left(gh, Ad_{g^{-1}}\eta, Ad^*_{g^{-1}}\mu, Ad^*_{g^{-1}}\nu\right). \quad (79)$$

The infinitesimal generator of this is given by the right invariant vector field $X^{^1T^*TG}_{(\eta,0,0,0)}$ which, in turn, is generated by the Lie algebra element $(\eta,0,0,0) \in Lie\left(^1T^*TG\right)$. Note that, we have the Lie algebra homomorphism $\eta \to (\eta,0,0,0)$ from the Lie algebra $\mathfrak{g}$ into the algebra of right invariant vector fields $\mathfrak{X}^R\left(^1T^*TG\right)$. Each orbit of the action in Eq.(79) is isomorphic to the product

$$G \backslash \left(^1T^*TG\right) \simeq \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^*. \qquad (80)$$

The action in Eq.(79) is symplectic and has an Ad^*-equivariant momentum mapping $\mathbf{J}_{^1T^*TG}$ defined by

$$\langle \mathbf{J}_{^1T^*TG}(g,\xi,\mu,\nu),\eta\rangle = \langle\theta_{^1T^*TG}, X^{^1T^*TG}_{(\eta,0,0,0)}\rangle = \langle\mu,\eta\rangle$$

(c.f. Eq.(136) in Appendix A), is given by the projection

$$\mathbf{J}_{{}^1T^*TG} : {}^1T^*TG \to \mathfrak{g}^* : (g, \xi, \mu, \nu) \to \mu$$

to the second factor in the trivialization ${}^1T^*TG$. Assuming $\mu \in \mathfrak{g}^*$ is a regular value, the inverse image $\mathbf{J}^{-1}_{{}^1T^*TG}(\mu)$ of μ becomes a submanifold of ${}^1T^*TG$ which can be identified with $G \times \mathfrak{g} \times \mathfrak{g}^*$ by the embedding

$$\imath_{{}^1T^*TG} : G \times \mathfrak{g} \times \mathfrak{g}^* \to \mathbf{J}^{-1}_{{}^1T^*TG}(\mu) : (g, \xi, \nu) \to (g, \xi, \mu, \nu). \tag{81}$$

The isotropy group G_μ in Eq.(59) of μ acts on $\mathbf{J}^{-1}_{{}^1T^*TG}(\mu)$. Under the assumption that this action is free and proper, we arrive at the fact that the quotient

$$G_\mu \backslash \mathbf{J}^{-1}_{{}^1T^*TG}(\mu) \simeq G_\mu \backslash (G \times \mathfrak{g} \times \mathfrak{g}^*) = (G_\mu \backslash G) \times \mathfrak{g} \times \mathfrak{g}^* = \mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^* \tag{82}$$

has a manifold structure. The reduction diagram is

$$\mathbf{J}^{-1}_{{}^1T^*TG}(\mu) \simeq G \times \mathfrak{g} \times \mathfrak{g}^* \xrightarrow{\imath_{{}^1T^*TG}} {}^1T^*TG \tag{83}$$

where $\chi_{{}^1T^*TG}$ (down), $p_{{}^1T^*TG}$ (diagonal)

$$\mathbf{J}^{-1}_{{}^1T^*TG}(\mu)/G_\mu = \mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$$

where $\imath_{{}^1T^*TG}$ is the inclusion in Eq.(81), and the projections

$$p_{{}^1T^*TG} \quad : \quad {}^1T^*TG \to \mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^* : (g, \xi, \mu, \nu) \to \left(Ad^*_{g^{-1}}\mu, \xi, \nu\right) \tag{84}$$

$$\chi_{{}^1T^*TG} \quad : \quad G \times \mathfrak{g} \times \mathfrak{g}^* \to \mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^* : (g, \xi, \nu) \to \left(Ad^*_{g^{-1}}\mu, \xi, \nu\right)$$

make the diagram commutative. The Marsden-Weinstein symplectic reduction theorem defines the reduced symplectic two-form $\Omega^{G\backslash}_{{}^1T^*TG}$ on $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$ by

$$\left(\chi_{{}^1T^*TG}\right)^* \Omega^{G\backslash}_{{}^1T^*TG} = \left(\imath_{{}^1T^*TG}\right)^* \Omega_{{}^1T^*TG}, \tag{85}$$

where $\Omega_{{}^1T^*TG}$ is the symplectic two-form on ${}^1T^*TG$ in Eq.(77). A direct calculation proves the following proposition.

Proposition 3.5. *The reduced space $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$ in Eq.(82) is a symplectic manifold with the reduced symplectic two-form $\Omega^{G\backslash}_{{}^1T^*TG}$ whose value on two vectors $\left(\eta_{\mathfrak{g}^*}(\mu), \zeta, \lambda\right)$ and $\left(\bar\eta_{\mathfrak{g}^*}(\mu), \bar\zeta, \bar\lambda\right)$ over $\mathcal{O}_\mu \times \mathfrak{g} \times \mathfrak{g}^*$ is*

$$\Omega^{G\backslash}_{{}^1T^*TG}\left(\left(\eta_{\mathfrak{g}^*}(\mu), \zeta, \lambda\right), \left(\bar\eta_{\mathfrak{g}^*}(\mu), \bar\zeta, \bar\lambda\right)\right) = \langle\lambda, \bar\zeta\rangle - \langle\bar\lambda, \zeta\rangle - \langle\mu, [\eta, \bar\eta]\rangle. \tag{86}$$

3.3. Cotangent bundle of cotangent group.

For trivialization of the first kind, we first decompose $T^*(G\circledS\mathfrak{g}^*)$ into a semidirect product consisting of the group $(G\circledS\mathfrak{g}^*)$ in Eq.(43) and the dual $\mathfrak{g}^* \times \mathfrak{g}$ of its Lie algebra $\mathfrak{g}\circledS\mathfrak{g}^*$

$$\begin{aligned} tr^1_{T^*(G\circledS\mathfrak{g}^*)} \quad &: \quad T^*(G\circledS\mathfrak{g}^*) \to (G\circledS\mathfrak{g}^*)\circledS(\mathfrak{g}^* \times \mathfrak{g}) := {}^1T^*T^*G \\ &: \quad (\alpha_g, \alpha_\mu) \to \left(\pi_{G\circledS\mathfrak{g}^*}(\alpha_g, \alpha_\mu), T^*_{(e,0)}R_{(g,\mu)}(\alpha_g, \alpha_\mu)\right) \\ &: \quad (\alpha_g, \alpha_\mu) \to \left(g, \mu, T^*_e R_g(\alpha_g) - ad^*_{\alpha_\mu}\mu, \alpha_\mu\right), \tag{87} \end{aligned}$$

where $\pi_{G\circledS\mathfrak{g}^*} : T^*(G\circledS\mathfrak{g}^*) \to (G\circledS\mathfrak{g}^*)$ is the cotangent bundle projection, and $T^*_{(e,0)}R_{(g,\mu)}$ is the cotangent lift of the right multiplication given in Eq.(48). α_μ is in the cotangent space of $\mathfrak{g}^*$, that is, $\alpha_\mu \in T^*_\mu\mathfrak{g}^* \simeq \mathfrak{g}$. Alternatively, distribution of the functor T^* leads to trivialization of the second kind

$$
\begin{aligned}
tr^2_{T^*(G\circledS\mathfrak{g}^*)} \quad &: \quad T^*(G\circledS\mathfrak{g}^*) \simeq T^*G \times T^*\mathfrak{g}^* \to G \times \mathfrak{g}^* \times \mathfrak{g}^* \times \mathfrak{g} \\
&: \quad (\alpha_g, \alpha_\mu) \to (g, T^*_e R_g(\alpha_g), \mu, \alpha_\mu).
\end{aligned}
\tag{88}
$$

On the trivialization ${}^1T^*T^*G$ in Eq.(87), we have the group structure with multiplication

$$
\begin{aligned}
&(g, \mu, \lambda_1, \xi_1)(h, \nu, \lambda_2, \xi_2) \\
&= \left(gh, \mu + Ad^*_{g^{-1}}\nu, \lambda_1 + Ad^*_{g^{-1}}\lambda_2 - ad^*_{Ad_{g^{-1}}\xi_2}\mu, \xi_1 + Ad_{g^{-1}}\xi_2\right).
\end{aligned}
\tag{89}
$$

Direct calculations prove the following proposition.

Proposition 3.6. *The following embeddings*

$$
\begin{aligned}
G &\to {}^1T^*T^*G : g \to (g, 0, 0, 0) \\
G\circledS\mathfrak{g}^* &\to {}^1T^*T^*G : (g, \mu) \to (g, \mu, 0, 0) \\
G\circledS\mathfrak{g} &\to {}^1T^*T^*G : (g, \xi) \to (g, 0, 0, \xi) \\
G\circledS(\mathfrak{g} \times \mathfrak{g}^*) &\to {}^1T^*T^*G : (g, \xi, \mu) \to (g, 0, \mu, \xi)
\end{aligned}
$$

define submanifolds of ${}^1T^*T^*G$ with multiplication in Eq.(89). The group structure on $G\circledS(\mathfrak{g} \times \mathfrak{g}^*)$ is the one in Eq.(75).

Symplectic Structure: We shall compute the canonical one-form $\theta_{{}^1T^*T^*G}$ and the symplectic two-form $\Omega_{{}^1T^*T^*G}$ on ${}^1T^*T^*G$. To this end, we take $H = G\circledS\mathfrak{g}^*$ and ${}^1T^*T^*G$ becomes $H\circledS\mathfrak{h}^*$ with $\mathfrak{h}^* = \mathfrak{g}^* \times \mathfrak{g}$ being the dual of the Lie algebra $\mathfrak{h} = \mathfrak{g}\circledS\mathfrak{g}^*$ endowed with the bracket in Eq.(46). At the point (g, μ, ν, ξ), using Eq.(52), we obtain the right invariant vector field

$$
\begin{aligned}
X^{{}^1T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)} &= \left(TR_{(g,\mu)}(\eta, \lambda_1), (\lambda_2, \zeta) + ad^*_{(\eta, \lambda_1)}(\nu, \xi)\right) \\
&= \left(TR_g\eta, \lambda_1 + ad^*_\eta\mu, \lambda_2 + ad^*_\eta\nu - ad^*_\xi\lambda_1, \zeta + [\xi, \eta]\right)
\end{aligned}
\tag{90}
$$

on ${}^1T^*T^*G$, where $TR_{(g,\mu)}(\eta, \lambda_1)$ is the right translation and, is the coadjoint representation. The definitions in Eqs.(54) and (55) of canonical forms over trivialized cotangent bundle lead to the value

$$
\langle \theta_{{}^1T^*T^*G}, X^{{}^1T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)}\rangle = \langle(\eta, \lambda_1), (\nu, \xi)\rangle = \langle\eta, \nu\rangle + \langle\lambda_1, \xi\rangle
\tag{91}
$$

of $\theta_{{}^1T^*T^*G}$ on $X^{{}^1T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)}$ and, the value of symplectic two form $\Omega_{{}^1T^*TG} = d\theta_{{}^1T^*TG}$ on two right invariant vector fields is given by

$$
\begin{aligned}
&\langle\Omega_{{}^1T^*T^*G}; \left(X^{{}^1T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)}, X^{{}^1T^*T^*G}_{(\bar\eta, \bar\lambda_1, \bar\lambda_2, \bar\zeta)}\right)\rangle(g, \mu, \nu, \xi) \\
&= \langle(\lambda_2, \zeta), (\bar\eta, \bar\lambda_1)\rangle - \langle(\bar\lambda_2, \bar\zeta), (\eta, \lambda_1)\rangle + \left\langle(\nu, \xi), [(\eta, \lambda_1), (\bar\eta, \bar\lambda_1)]_{\mathfrak{g}\circledS\mathfrak{g}^*}\right\rangle \\
&= \langle\lambda_2, \bar\eta\rangle + \langle\zeta - [\eta, \xi], \bar\lambda_1\rangle - \langle\bar\lambda_2, \eta\rangle + \langle[\bar\eta, \xi] - \bar\zeta, \lambda_1\rangle + \langle\nu, [\eta, \bar\eta]\rangle.
\end{aligned}
\tag{92}
$$

Proposition 3.7.   *The trivialized bundle* $^1T^*T^*G$ *is an exact symplectic manifold with the potential one-form* $\theta_{^1T^*T^*G}$ *in Eq.(91) and the symplectic two-form* $\Omega_{^1T^*T^*G}$ *in Eq.(92). The mapping*

$$tr^1_{T^*(G\circledS\mathfrak{g}^*)} \circ T^*tr^{-1}_{T^*G} : (T^*T^*G, \Omega_{T^*T^*G}) \rightarrow \left(\, ^1T^*T^*G, \Omega_{^1T^*T^*G}\right)$$

is a symplectomorphism. Here, $T^*tr^{-1}_{T^*G}$ *is obtained from the cotangent lift of the inverse of* tr_{T^*G}, $tr^1_{T^*(G\circledS\mathfrak{g}^*)}$ *is the trivialization in Eq.(87).*

Left action of G **and symplectic reduction:** It follows from Eq.(89) that the left action of G on $^1T^*T^*G$ is

$$(g, (h, \nu, \lambda_2, \xi_2)) \rightarrow \left(gh, Ad^*_{g^{-1}}\nu, Ad^*_{g^{-1}}\lambda_2, Ad_{g^{-1}}\xi_2\right) \tag{93}$$

with the infinitesimal generator $X^{^1T^*T^*G}_{(\eta,0,0,0)}$ being a right invariant vector field as in Eq.(90) generated by $(\eta, 0, 0, 0)$ for $\eta \in \mathfrak{g}$. This action is symplectic with Ad^*-equivariant momentum map $\mathbf{J}_{^1T^*T^*G}$ defined by

$$\langle \mathbf{J}_{^1T^*T^*G}\left(g, \mu, \nu, \xi\right), \eta\rangle = \theta_{^1T^*T^*G}\left(X^{^1T^*T^*G}_{(\eta,0,0,0)}\right) = \langle \eta, \nu\rangle ,$$

(c.f. Eq.(136) in Appendix A) which is the projection

$$\mathbf{J}_{^1T^*T^*G} : \ ^1T^*T^*G \rightarrow \mathfrak{g}^* : (g, \mu, \nu, \xi) \rightarrow \nu$$

to the third factor in $^1T^*T^*G$. For a regular value of $\nu \in \mathfrak{g}^*$, the inverse image $\mathbf{J}^{-1}_{^1T^*T^*G}(\nu)$ is an embedded submanifold of $^1T^*T^*G$ defined by the inclusion

$$\imath_{^1T^*T^*G} : G \times \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbf{J}^{-1}_{^1T^*T^*G}(\nu) : (g, \mu, \xi) \rightarrow (g, \mu, \nu, \xi) .$$

The isotropy group G_ν acts on $\mathbf{J}^{-1}_{^1T^*T^*G}(\nu)$ and the quotient is

$$\mathbf{J}^{-1}_{^1T^*T^*G}(\nu) \big/ G_\nu \simeq G \times \mathfrak{g}^* \times \mathfrak{g} / G_\nu = G/G_\nu \times \mathfrak{g}^* \times \mathfrak{g} = \mathcal{O}_\nu \times \mathfrak{g}^* \times \mathfrak{g} \tag{94}$$

where $\mathfrak{g}_\nu$ is the coadjoint orbit of ν in Eq.(60). We have the following commutative diagram which summarizes the reduction

$$\mathbf{J}^{-1}_{^1T^*T^*G}(\nu) = G \times \mathfrak{g}^* \times \mathfrak{g} \xrightarrow{\imath_{^1T^*T^*G}} {}^1T^*T^*G \tag{95}$$

$$\chi_{^1T^*TG}\Big\downarrow \qquad\qquad p_{^1T^*T^*G}$$

$$\mathbf{J}^{-1}_{^1T^*T^*G}(\nu)/G_\nu = \mathcal{O}_\nu \times \mathfrak{g} \times \mathfrak{g}$$

where the projections are

$$p_{^1T^*T^*G} \ : \ (g, \mu, \nu, \xi) \rightarrow \left(Ad^*_{g^{-1}}\nu, \mu, \xi\right) \tag{96}$$

$$\chi_{^1T^*T^*G} \ : \ (g, \mu, \xi) \rightarrow \left(Ad^*_{g^{-1}}\nu, \mu, \xi\right) .$$

According to the symplectic reduction theorem, $\mathfrak{g}^*_\nu \times \mathfrak{g} \times \mathfrak{g}^*$ has a symplectic structure $\Omega^{G\backslash}_{^1T^*T^*G}$ defined by

$$(\pi_{^1T^*T^*G})^* \Omega^{G\backslash}_{^1T^*T^*G} = (\imath_{^1T^*T^*G})^* \Omega_{^1T^*T^*G}, \tag{97}$$

with $\Omega_{^1T^*T^*G}$ being the symplectic two-form in Eq.(92). Direct calculation gives

Proposition 3.8. *The reduced space $\mathcal{O}_\nu \times \mathfrak{g} \times \mathfrak{g}^*$ in Eq.(94) carries the symplectic two-form $\Omega^{G\backslash}_{{}^1T^*T^*G}$ with values*

$$\Omega^{G\backslash}_{{}^1T^*T^*G}\left(\left(\eta_{\mathfrak{g}^*}\left(\nu\right),\lambda,\zeta\right),\left(\bar{\eta}_{\mathfrak{g}^*}\left(\nu\right),\bar{\lambda},\bar{\zeta}\right)\right) = \langle\zeta,\bar{\lambda}\rangle - \langle\bar{\zeta},\lambda\rangle - \langle\nu,[\eta,\bar{\eta}]\rangle, \qquad (98)$$

on two elements $\left(\eta_{\mathfrak{g}^*}\left(\nu\right),\lambda,\zeta\right)$ and $\left(\bar{\eta}_{\mathfrak{g}^*}\left(\nu\right),\bar{\lambda},\bar{\zeta}\right)$ of $T\mathcal{O}_\nu \times \mathfrak{g} \times \mathfrak{g}^*$.

3.4. Tangent bundle of cotangent group.

For $TT^*G \simeq T\left(G\circledS\mathfrak{g}^*\right)$, trivialization of the first kind as tangent group results in a semidirect product of $G\circledS\mathfrak{g}^*$ and its Lie algebra $\mathfrak{g}\circledS\mathfrak{g}^*$

$$tr^1_{T(G\circledS\mathfrak{g}^*)} \;:\; T\left(G\circledS\mathfrak{g}^*\right) \to \left(G\circledS\mathfrak{g}^*\right)\circledS\left(\mathfrak{g}\circledS\mathfrak{g}^*\right) =: {}^1TT^*G$$
$$:\; (V_g, V_\mu) \to \left(g, \mu, T_{(g,\mu)}R_{(g,\mu)^{-1}}\left(V_g, V_\mu\right)\right)$$
$$:\; (V_g, V_\mu) \to \left(g, \mu, TR_{g^{-1}}V_g, V_\mu - ad^*_{TR_{g^{-1}}V_g}\mu\right), \qquad (99)$$

where $T_{(g,\mu)}R_{(g,\mu)^{-1}}$ is the tangent mapping in Eq.(47). Trivialization of the second kind with tangent functor results in

$$tr^2_{T(G\circledS\mathfrak{g}^*)} \;:\; T\left(G\circledS\mathfrak{g}^*\right) \simeq TG \times T\mathfrak{g}^* \to G \times \mathfrak{g} \times \mathfrak{g}^* \times \mathfrak{g}^* =: {}^2TT^*G$$
$$:\; (V_g, V_\mu) \to \left(g, T_gR_{g^{-1}}V_g, \mu, V_\mu\right). \qquad (100)$$

The semidirect product group multiplication on ${}^1TT^*G$ is

$$(g, \mu, \xi_1, \nu_1)(h, \lambda, \xi_2, \nu_2)$$
$$= \left(gh, \mu + Ad^*_{g^{-1}}\lambda, \xi_1 + Ad_{g^{-1}}\xi_2, \nu_1 + Ad^*_{g^{-1}}\nu_2 - ad^*_{Ad_{g^{-1}}\xi_2}\mu\right) \qquad (101)$$

and some embedded subgroups of ${}^1TT^*G$ follows.

Proposition 3.9. *The embeddings*

$$G \;\to\; {}^1TT^*G : g \to (g, 0, 0, 0)$$
$$G\circledS\mathfrak{g} \;\to\; {}^1TT^*G : (g, \xi) \to (g, 0, \xi, 0)$$
$$G\circledS\mathfrak{g}^* \;\to\; {}^1TT^*G : (g, \mu) \to (g, \mu, 0, 0)$$
$$G\circledS\left(\mathfrak{g}\times\mathfrak{g}^*\right) \;\to\; {}^1TT^*G : (g, \xi, \nu) \to (g, 0, \xi, \nu)$$

define subgroups G, $G\circledS\mathfrak{g}$, $G\circledS\mathfrak{g}^*$ and $G\circledS\left(\mathfrak{g}\times\mathfrak{g}^*\right)$ of ${}^1TT^*G$ with group structures on $G\circledS\mathfrak{g}$, $G\circledS\mathfrak{g}^*$ and $G\circledS\left(\mathfrak{g}\times\mathfrak{g}^*\right)$ defined by Eqs.(33), (43) and (75), respectively.

A right invariant vector field $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ on ${}^1TT^*G$ can be obtained by tangent lift of the group multiplication in Eq.(101), that is, by right translation of a Lie algebra element $(\xi_2, \nu_2, \xi_3, \nu_3)$ as follows. We replace G, $\mathfrak{g}$ and $\mathfrak{g}^*$ in Eq.(39) by $H = G\circledS\mathfrak{g}^*$, $\mathfrak{h} = \mathfrak{g}\circledS\mathfrak{g}^*$ and $\mathfrak{h}^* = \mathfrak{g}^* \times \mathfrak{g}$, respectively, and obtain

$$X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}(g, \mu, \xi, \nu) \qquad (102)$$
$$= \left(TR_g\xi_2, \nu_2 + ad^*_{\xi_2}\mu, \xi_3 + [\xi, \xi_2]_{\mathfrak{g}}, \nu_3 + ad^*_{\xi_2}\nu - ad^*_{\xi}\nu_2\right).$$

A right invariant vector field $X^{T(G\circledS\mathfrak{g}^*)}$ on the tangent group $T(G\circledS\mathfrak{g}^*)$ can be pushed forward to a vector field $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ on its trivialization by the mapping $tr^1_{T(G\circledS\mathfrak{g}^*)}$ in Eq.(99). Conversely, a vector field $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ can be pulled back by $tr^1_{T(G\circledS\mathfrak{g}^*)}$. Using the latter, we compute the following relations

$$
\begin{aligned}
\tau_{T(G\circledS\mathfrak{g}^*)} \circ \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}(g,\mu,\xi,\nu) &= X^{G\circledS\mathfrak{g}^*}_{(\xi,\nu)}(g,\mu) \\
T\tau_{G\circledS\mathfrak{g}^*} \circ \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}(g,\mu,\xi,\nu) &= X^{G\circledS\mathfrak{g}^*}_{(\xi_2,\nu_2)}(g,\mu).
\end{aligned}
\tag{103}
$$

relating vector fields on trivializations ${}^1TT^*G$ and $G\circledS\mathfrak{g}^*$.

Tulczyjew's symplectic structure: Applying the derivation i_T to $\Omega_{G\circledS\mathfrak{g}^*}$ (c.f. Eq.(137) in Appendix A) we obtain the one form $i_T\Omega_{G\circledS\mathfrak{g}^*} = \theta_1$ on $T(G\circledS\mathfrak{g}^*)$. The pull-back of the one-from $i_T\Omega_{G\circledS\mathfrak{g}^*}$ to trivialization ${}^1TT^*G$ of the first kind will be denoted by ${}^1\theta_1$.

Proposition 3.10. *At the point (g,μ,ξ,ν), the one-form*

$$
{}^1\theta_1 = \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)_* i_T\Omega_{G\circledS\mathfrak{g}^*}
\tag{104}
$$

takes the value

$$
\langle {}^1\theta_1, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\rangle(g,\mu,\xi,\nu) = \langle \nu,\xi_2\rangle - \langle \nu_2,\xi\rangle + \left\langle \mu, [\xi,\xi_2]_\mathfrak{g}\right\rangle
\tag{105}
$$

on a vector field $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ in form of Eq.(102).

To prove this, at the point (g,μ,ξ,ν), we need compute

$$
\begin{aligned}
\langle {}^1\theta_1, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\rangle &= \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)_* \left\langle \theta_1, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle \\
&= \left\langle \theta_1, \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle,
\end{aligned}
\tag{106}
$$

where the one-form $\theta_1 = i_T\Omega_{G\circledS\mathfrak{g}^*}$ is obtained by the derivation of the canonical symplectic two-form $\Omega_{G\circledS\mathfrak{g}^*}$ on $G\circledS\mathfrak{g}^*$. The value (c.f. Eq.(138) Appendix A) of the one-form $i_T\Omega_{G\circledS\mathfrak{g}^*}$ over the vector field $(tr^1_{T(G\circledS\mathfrak{g}^*)})^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ equals to the value of $\Omega_{G\circledS\mathfrak{g}^*}$ over the projections $\tau_{T(G\circledS\mathfrak{g}^*)}$ and $T\tau_{G\circledS\mathfrak{g}^*}$ of $(tr^1_{T(G\circledS\mathfrak{g}^*)})^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ given in Eqs.(103) at the point (g,μ). That is,

$$
\begin{aligned}
\left\langle i_T\Omega_{G\circledS\mathfrak{g}^*}, \left(tr^1_{T(G\circledS\mathfrak{g}^*)}\right)^* X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle &= \left\langle \Omega_{G\circledS\mathfrak{g}^*}; \left(X^{G\circledS\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS\mathfrak{g}^*}_{(\xi_2,\nu_2)}\right)\right\rangle \\
&= \langle \nu,\xi_2\rangle - \langle \nu_2,\xi\rangle + \left\langle \mu, [\xi,\xi_2]_\mathfrak{g}\right\rangle.
\end{aligned}
$$

Application of the derivation i_T to the canonical one-form $\theta_{G\circledS\mathfrak{g}^*}$ in Eq.(54) gives a function on $T(G\circledS\mathfrak{g}^*)$ and the push-forward of the latter by the trivialization $tr^1_{T(G\circledS\mathfrak{g}^*)}$ gives the function

$$
f(g,\mu,\xi,\nu) = \langle \mu,\xi\rangle,
$$

on ${}^1TT^*G$. Exterior derivative of f is a one-form on ${}^1TT^*G$ and its value on the right invariant vector field $X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ in Eq.(102) is

$$
\langle df, X^{{}^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\rangle(g,\mu,\xi,\nu) = \langle \nu_2,\xi\rangle + \langle \mu,\xi_3\rangle.
\tag{107}
$$

We now define the one-form $^1\theta_2$ as $^1\theta_1 + df$ by its value on the right invariant vector field $X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$

$$\left\langle\, ^1\theta_2, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle = \langle\mu,\xi_3\rangle + \langle\nu,\xi_2\rangle + \left\langle\mu,[\xi,\xi_2]_{\mathfrak{g}}\right\rangle. \tag{108}$$

Above two step construction of $^1\theta_2$ is the same as the push-forward of $d_T\theta_{G\circledS\mathfrak{g}^*}$ with the trivialization map $tr^1_{T(G\circledS\mathfrak{g}^*)}$. The following proposition indicates that the exterior derivatives of $^1\theta_1$ and $^1\theta_2$ are the same and it is the Tulczyjew's symplectic two-form on $^1TT^*G$.

Proposition 3.11. *The space $^1TT^*G$ with coordinates given as in Eq.(72) is the trivialized Tulczyjew's symplectic manifold with the Tulczyjew's symplectic two-form $\Omega_{^1TT^*G}$ and with two potential one-forms $^1\theta_1 = i_T\Omega_{G\circledS\mathfrak{g}^*}$ and $^1\theta_2 = d_T\theta_{G\circledS\mathfrak{g}^*}$ obtained by deriving $\Omega_{G\circledS\mathfrak{g}^*}$ and $\theta_{G\circledS\mathfrak{g}^*}$, respectively. At a point $(g,\mu,\xi,\nu) \in {}^1TT^*G$, the values of $^1\theta_1$ and $^1\theta_2$, on a right invariant vector field on $^1TT^*G$, are*

$$\left\langle\, ^1\theta_1, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle = \langle\nu,\xi_2\rangle - \langle\nu_2,\xi\rangle + \left\langle\mu,[\xi,\xi_2]_{\mathfrak{g}}\right\rangle \tag{109}$$

$$\left\langle\, ^1\theta_2, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle = \langle\mu,\xi_3\rangle + \langle\nu,\xi_2\rangle + \left\langle\mu,[\xi,\xi_2]_{\mathfrak{g}}\right\rangle, \tag{110}$$

and the value of the symplectic two-form $\Omega_{^1TT^*G}$ at (g,μ,ξ,ν) is

$$\left\langle\Omega_{^1TT^*G}; \left(X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right)\right\rangle \tag{111}$$

$$= \langle\nu_3,\bar\xi_2\rangle + \langle\nu_2,\bar\xi_3\rangle - \langle\bar\nu_2,\xi_3\rangle - \langle\bar\nu_3,\xi_2\rangle + \left\langle\nu,[\xi_2,\bar\xi_2]_{\mathfrak{g}}\right\rangle$$

$$+ \left\langle\mu,[\xi_3,\bar\xi_2]_{\mathfrak{g}} + [\xi_2,\bar\xi_3]_{\mathfrak{g}} + [\xi,[\xi_2,\bar\xi_2]_{\mathfrak{g}}]\right\rangle.$$

Commutation property of the derivation d_T and the exterior derivative d shows that $d\,^1\theta_1 = d\,^1\theta_2$. Hence, to obtain the symplectic two-form $\Omega_{^1TT^*G}$ on $^1TT^*G$, it is enough to take the exterior derivative of one of the one-forms $^1\theta_1$ or $^1\theta_2$. The calculation

$$\left\langle\Omega_{^1TT^*G}; \left(X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right)\right\rangle$$

$$= \left\langle d\left(\,^1\theta_1\right); \left(X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right)\right\rangle$$

$$= X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\left\langle\,^1\theta_1, X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right\rangle - X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\left\langle\,^1\theta_1, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle$$

$$- \left\langle\,^1\theta_1, \left[X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{^1TT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}\right]\right\rangle$$

at (g,μ,ξ,ν) results in the required explicit expression in Eq.(111).

The musical isomorphism $\Omega^\flat_{^1TT^*G}$ maps the tangent bundle of $^1TT^*G$ to cotangent bundle of $^1TT^*G$. At a point $(g,\mu,\xi,\nu) \in {}^1TT^*G$, $\Omega^\flat_{^1TT^*G}$ maps a right invariant vector field $X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ in Eq.(102) to

$$\Omega^\flat_{^1TT^*G}(X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}) = \left(T^*R_{g^{-1}}\left(\nu_3 - ad^*_\xi\nu_2\right), -\left(\xi_3 + [\xi,\xi_2]_{\mathfrak{g}}\right), \nu_2 + ad^*_{\xi_2}\mu, -\xi_2\right)$$

which is an element of $T^*_{(g,\mu,\xi,\nu)}\left(\,^1TT^*G\right)$.

Left action and reduction of Tulczyjew's symplectic space: In the explicit expression of the symplectic two-form $\Omega\,_{^1TT^*G}$, given in Eq.(111), the base component $g \in G$ does not exist. This gives that the left action

$$(g;(h,\lambda,\xi,\nu)) \to \left(gh, Ad^*_{g^{-1}}\lambda, Ad_{g^{-1}}\xi, Ad^*_{g^{-1}}\nu\right) \qquad (112)$$

of G on $^1TT^*G$ is a symplectic action. The infinitesimal generators of this action are right invariant vector fields $X^{^1TT^*G}_{(\eta,0,0,0)}$ in the form of Eq.(102). The momentum map $\mathbf{J}\,_{^1TT^*G}$ is

$$\langle \mathbf{J}\,_{^1TT^*G}\left(g,\mu,\xi,\nu\right),\eta\rangle = \left\langle\,^1\theta_1, X^{^1TT^*G}_{(\eta,0,0,0)}\right\rangle = \left\langle\,^1\theta_2, X^{^1TT^*G}_{(\eta,0,0,0)}\right\rangle \qquad (113)$$

where $^1\theta_1$ and $^1\theta_2$ are the potential one-forms on $^1TT^*G$ given in Eqs.(109) and (110), respectively (c.f. Eq.(136) in Appendix A). In coordinates the momentum mapping $\mathbf{J}\,_{^1TT^*G}$ reads

$$\mathbf{J}\,_{^1TT^*G}:\ ^1TT^*G \to \mathfrak{g}^*:(g,\mu,\xi,\nu) \to \nu + ad^*_\xi\mu.$$

The inverse image $\mathbf{J}^{-1}_{^1TT^*G}$ for a regular value $\lambda = \nu + ad^*_\xi\mu \in \mathfrak{g}^*$ is

$$\begin{aligned}
\mathbf{J}^{-1}_{^1TT^*G}(\lambda) &= \ ^1TT^*G\big/ \sim_\lambda = ((G\,\text{Ⓢ}\,\mathfrak{g}^*)\,\text{Ⓢ}\,(\mathfrak{g}\,\text{Ⓢ}\,\mathfrak{g}^*))/ \sim_\lambda \\
&= \ G\,\text{Ⓢ}\,(\mathfrak{g}^* \times \mathfrak{g} \times \mathfrak{g}^*)/ \sim_\lambda.
\end{aligned}$$

We may consider $\mathbf{J}^{-1}_{^1TT^*G}(\lambda)$ as an embedded submanifold of $^1TT^*G$ by the inclusion

$$\imath\,_{^1TT^*G}:G\times(\mathfrak{g}^*\times\mathfrak{g}) \to \mathbf{J}^{-1}_{^1TT^*G}(\lambda) = (g,\mu,\xi) \to \left(g,\mu,\xi,\lambda - ad^*_\xi\mu\right). \qquad (114)$$

The isotropy group G_λ of $\lambda \in \mathfrak{g}^*$ acts on $\mathbf{J}^{-1}_{^1TT^*G}(\lambda)$ and the quotient is

$$\begin{aligned}
\left(\,^1TT^*G\right)^{/G} &:\ = \mathbf{J}^{-1}_{^1TT^*G}(\lambda)\big/G_\lambda \simeq G\times(\mathfrak{g}^*\times\mathfrak{g})/G_\lambda \\
&\simeq \ G/G_\lambda\times(\mathfrak{g}^*\times\mathfrak{g}) \simeq \mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g}. \qquad (115)
\end{aligned}$$

The reduction diagram becomes

$$\mathbf{J}^{-1}_{^1TT^*G}(\lambda) = G\,\text{Ⓢ}\,(\mathfrak{g}^*\times\mathfrak{g}) \xrightarrow{\ \imath_{^1TT^*G}\ } {}^1TT^*G \qquad (116)$$

$$\chi_{^1TT^*G}\Big\downarrow \qquad\qquad {}^{p_{^1TT^*G}}\!\!\swarrow$$

$$\mathbf{J}^{-1}_{^1TT^*G}(\lambda)/G_\lambda = \mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g}$$

where the projections are

$$p\,_{^1TT^*G}\ :\ \ ^1T^*TG \to \mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g}:(g,\mu,\xi,\nu) \to \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right) \qquad (117)$$

$$\chi\,_{^1TT^*G}\ :\ \ G\,\text{Ⓢ}\,(\mathfrak{g}^*\times\mathfrak{g}) \to \mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g}:(g,\mu,\xi) \to \left(Ad^*_{g^{-1}}\lambda,\mu,\xi\right). \qquad (118)$$

According to the Marsden-Weinstein symplectic reduction theorem, the quotient $\mathcal{O}_\lambda\times\mathfrak{g}^*\times\mathfrak{g}$ is a symplectic manifold with the symplectic two form $\Omega^{G\backslash}_{^1TT^*G}$ satisfying

$$(\chi\,_{^1TT^*G})^*\,\Omega^{G\backslash}_{^1TT^*G} = (\imath\,_{^1TT^*G})^*\,\Omega\,_{^1TT^*G}. \qquad (119)$$

This definition and a direct calculation proves the following proposition.

Proposition 3.12. *The reduced symplectic two-form* $\Omega^{G\backslash}_{^1TT^*G}$ *on*

$$\mathbf{J}^{-1}_{^1TT^*G}(\lambda)\big/ G_\lambda = \mathfrak{g}^*_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \tag{120}$$

is defined by its value

$$\Omega^{G\backslash}_{^1TT^*G}\left((\eta_{\mathfrak{g}^*}(\lambda),\nu,\zeta),(\bar{\eta}_{\mathfrak{g}^*}(\lambda),\bar{\nu},\bar{\zeta})\right) = \langle\nu,\bar{\zeta}\rangle - \langle\bar{\nu},\zeta\rangle + \langle\lambda,[\bar{\eta},\eta]\rangle, \tag{121}$$

on two vectors $(\eta_{\mathfrak{g}^*}(\lambda),\nu,\zeta)$ *and* $(\bar{\eta}_{\mathfrak{g}^*}(\lambda),\bar{\nu},\bar{\zeta})$ *in* $T_\lambda\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^*$.

In obtaining Eq.(121) we take the derivative of the constraint $\lambda = \nu + ad^*_\xi\mu$ and obtained

$$\nu_3 + ad^*_{\xi_2}\nu + ad^*_{\xi_2}\circ ad^*_\xi\mu + ad^*_{\xi_3}\mu = 0,$$

and used this in the definition of $X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ to define elements of the tangent space of $\mathbf{J}^{-1}_{^1TT^*G}(\lambda)$.

4. Tulczyjew's Triplet

We have so far established the trivializations of first order bundles TG and T^*G as well as iterated bundles TTG, T^*T^*G, T^*TG and TT^*G. The symplectic reduction theorem has been applied to the trivializations of T^*G, T^*T^*G, T^*TG and TT^*G under the symplectic left actions of the group G. In this section, we shall conclude all the discussions made so far by drawing two Tulczyjew's triplets, one for the trivializations of iterated bundles and the other for their reductions by G. This, we shall do by trivializing symplectomorphisms in the construction of the triplet and constructing trivialized special symplectic structures.

4.1. Trivialization of the Tulczyjew's triplet.

We start by recalling the musical isomorphism $\Omega^\flat_{G\circledS\mathfrak{g}^*}$ given in Eq.(56). Define the trivialization $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ of $\Omega^\flat_{G\circledS\mathfrak{g}^*}$ by the relation

$$^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\circ tr^1_{T(G\circledS\mathfrak{g}^*)} = tr^1_{T^*(G\circledS\mathfrak{g}^*)}\circ \Omega^\flat_{G\circledS\mathfrak{g}^*},$$

where $tr^1_{T(G\circledS\mathfrak{g}^*)}$ and $tr^1_{T^*(G\circledS\mathfrak{g}^*)}$ are in Eqs.(99) and (87), respectively. More explicitly, we have

$$^1\Omega^\flat_{G\circledS\mathfrak{g}^*}: {}^1TT^*G \to {}^1T^*T^*G : (g,\mu,\xi,\nu) \to (g,\mu,\nu + ad^*_\xi\mu,-\xi). \tag{122}$$

Similarly, the trivialization

$$^1\bar{\sigma}_G: {}^1TT^*G \to {}^1T^*TG : (g,\mu,\xi,\nu) \to (g,\xi,\nu + ad^*_\xi\mu,\mu) \tag{123}$$

of the mapping $\bar{\sigma}_G$ in Eq.(71) is achieved by the relation

$$^1\bar{\sigma}_G \circ tr^1_{T(G\circledS\mathfrak{g}^*)} = tr^1_{T^*(G\circledS\mathfrak{g})}\circ \bar{\sigma}_G,$$

where $tr^1_{T^*(G\circledS\mathfrak{g})}$ is in Eq.(72).

Proposition 4.1. *The diffeomorphisms* $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ *and* $^1\bar\sigma_G$ *satisfy*

$$\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^*\theta_{\,^1T^*T^*G} = \,^1\theta_1, \quad \left(\,^1\bar\sigma_G\right)^*\theta_{\,^1T^*TG} = \,^1\theta_2,$$

where $^1\theta_{\,^1T^*T^*G}$ *and* $^1\theta_{\,^1T^*TG}$ *are canonical one-forms in Eqs.(91) and (76), whereas* $^1\theta_1$ *and* $^1\theta_2$ *are in Eqs.(109) and (110), respectively. Hence,* $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ *and* $^1\bar\sigma_G$ *are symplectic diffeomorphisms satisfying*

$$\begin{aligned}
\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^*\Omega_{\,^1T^*T^*G} &= \Omega_{\,^1TT^*G}\\
\left(\,^1\bar\sigma_G\right)^*\Omega_{\,^1T^*TG} &= \Omega_{\,^1TT^*G},
\end{aligned}$$

where $\Omega_{\,^1T^*TG}$, $\Omega_{\,^1T^*T^*G}$ *and* $\Omega_{\,^1TT^*G}$ *are symplectic two-forms given in Eqs.(77), (92), and (111), respectively.*

To prove the identities involving $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$, we observe

$$\begin{aligned}
\left\langle\,^1\theta_1, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle &= \left\langle\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^*\theta_{\,^1T^*T^*G}, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle\\
&= \left\langle\theta_{\,^1T^*T^*G}, \left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)_* X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle
\end{aligned}$$

and for $\lambda_3 = \nu_3 + ad^*_{\xi_3}\mu - ad^*_\xi\nu_2$ the relation

$$T_{(g,\mu,\xi,\nu)}\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right) \circ X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = X^{^1T^*T^*G}_{(\xi_2,\nu_2,\lambda_3,-\xi_3)}\left(g,\mu,\nu+ad^*_\xi\mu,-\xi\right).$$

Then, the definitions of $^1\theta_1$ and $\theta_{\,^1T^*T^*G}$ in Eqs.(109) and (91) prove the first assertion. The commutation of the exterior derivative and pull-back operation proves that $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ is a symplectic diffeomorphism. For the identities involving $^1\bar\sigma_G$, we note the relations

$$\begin{aligned}
\left\langle\,^1\theta_2, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle &= \left\langle\left(\,^1\bar\sigma_G\right)^*\theta_{\,^1T^*TG}, X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle\\
&= \left\langle\theta_{\,^1T^*TG}, \left(\,^1\bar\sigma_G\right)_* X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}\right\rangle\\
T_{(g,\mu,\xi,\nu)}\left(\,^1\bar\sigma_G\right) \circ X^{^1TT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} &= X^{^1T^*TG}_{(\xi_2,\xi_3,\nu_3,\nu_2)}\left(g,\xi,\nu+ad^*_\xi\mu,\mu\right).
\end{aligned}$$

Similar to the first case, the definitions of $^1\theta_2$ and $\theta_{\,^1T^*TG}$ in Eqs.(110) and (76) prove the first assertion and the commutation of the exterior derivative and pull-back operation proves that $^1\bar\sigma_G$ is a symplectic diffeomorphism. We shall now construct trivialized special symplectic structures associated with trivialized iterated bundles.

Proposition 4.2. $^1TT^*G$ *admits two trivialized special symplectic structures*

$$\left(\,^1TT^*G,\,^1\tau_{G\circledS\mathfrak{g}^*},\,^1T^*T^*G,\,^1\theta_1,\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right) \tag{124}$$

$$\left(\,^1TT^*G,\,^1T\pi_G,\,^1T^*TG,\,^1\theta_2,\,^1\bar\sigma_G\right), \tag{125}$$

where potential one-forms $^1\theta_1$ *and* $^1\theta_2$ *are as in Eqs.(109) and (110), respectively, and the projections are*

$$\begin{aligned}
^1\tau_{G\circledS\mathfrak{g}^*} &: \quad ^1TT^*G \to G\circledS\mathfrak{g}^* : (g,\mu,\xi,\nu) \to (g,\mu)\\
^1T\pi_G &: \quad ^1TT^*G \to G\circledS\mathfrak{g} : (g,\mu,\xi,\nu) \to (g,\xi).
\end{aligned}$$

The following diagram summarizes the construction of Tulczyjew's triplet for a Lie group G via trivialization of the first kind

$$
\begin{array}{ccccc}
{}^1T^*TG & \xleftarrow{\ {}^1\bar{\sigma}_G\ } & {}^1TT^*G & \xrightarrow{\ {}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\ } & {}^1T^*T^*G \\
\end{array}
\qquad (126)
$$

with maps ${}^1\pi_{G\circledS\mathfrak{g}}$, ${}^1T\pi_G$, ${}^1\tau_{G\circledS\mathfrak{g}^*}$, ${}^1\pi_{G\circledS\mathfrak{g}}$ down to $G\circledS\mathfrak{g} \longrightarrow G \longleftarrow G\circledS\mathfrak{g}^*$.

4.2. Reduction of Tulczyjew's triplet.

Recall the reductions of symplectic manifolds

$$
\begin{aligned}
(G\circledS\mathfrak{g}^*, \Omega_{G\circledS\mathfrak{g}^*}) &\;\to\; \left(\mathcal{O}_\lambda, \Omega^{G\backslash}_{G\circledS\mathfrak{g}^*}\right) \\
\left({}^1T^*TG, \Omega_{\,{}^1T^*TG}\right) &\;\to\; \left(\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^*, \Omega^{G\backslash}_{\,{}^1T^*TG}\right) \\
\left({}^1T^*T^*G, \Omega_{\,{}^1T^*T^*G}\right) &\;\to\; \left(\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}, \Omega^{G\backslash}_{\,{}^1T^*T^*G}\right) \\
\left({}^1TT^*G, \Omega_{\,{}^1TT^*G}\right) &\;\to\; \left(\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}, \Omega^{G\backslash}_{\,{}^1TT^*G}\right),
\end{aligned}
$$

with the left actions of G, where the reduced manifolds are described by Eqs.(60), (82), (115) and (94), and the symplectic two-forms are given in Eqs.(62), (86), (98) and (121), respectively.

We will obtain the reduced symplectic isomorphism ${}^1\bar{\sigma}_G^{G\backslash}$ making use of the following commutative diagram

$$
\begin{array}{ccc}
{}^1T^*TG & \xleftarrow{\ {}^1\bar{\sigma}_G\ } & {}^1TT^*G \\
\Big\downarrow{\scriptstyle p_{1T^*TG}} & & \Big\downarrow{\scriptstyle p_{\,{}^1TT^*G}} \\
\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* & \xleftarrow{\ {}^1\bar{\sigma}_G^{G\backslash}\ } & \mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^*
\end{array}
\qquad (127)
$$

where the fibrations $p_{\,{}^1T^*TG}$ and $p_{\,{}^1TT^*G}$ are given in Eqs.(84) and (117), respectively, and ${}^1\bar{\sigma}_G$ is the symplectic isomorphism in Eq.(123). Then, $\left({}^1\bar{\sigma}_G\right)^{G\backslash}$ is

$$
\begin{aligned}
{}^1\bar{\sigma}_G^{\,G\backslash} \;:\;& \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \to \mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* \\
\;:\;& \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^*_{g^{-1}}\lambda, \xi, \mu\right).
\end{aligned}
\qquad (128)
$$

Similarly, the reduced musical isomorphism $\left({}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash}$ is defined by requiring commutativity of the diagram

$$
\begin{array}{ccc}
{}^1TT^*G & \xrightarrow{\ {}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\ } & {}^1T^*T^*G \\
\Big\downarrow{\scriptstyle p_{\,{}^1TT^*G}} & & \Big\downarrow{\scriptstyle p_{\,{}^1T^*T^*G}} \\
\mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} & \xrightarrow{\ \left({}^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash}\ } & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g}
\end{array}
\qquad (129)
$$

where, $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ is the trivialized symplectic isomorphism in Eq.(122). The explicit expression for $\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash}$ is then

$$
\begin{aligned}
\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash} \quad &: \quad \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \to \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \\
&: \quad \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \left(Ad^*_{g^{-1}}\lambda, \mu, -\xi\right)
\end{aligned} \tag{130}
$$

and a direct calculation proves the following proposition.

Proposition 4.3. *The diffeomorphisms* $^1\bar{\sigma}_G^{G\backslash}$ *in Eq.(128) and* $\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash}$ *in Eq.(130) are symplectic and they satisfy*

$$
\begin{aligned}
\left(^1\bar{\sigma}_G^{G\backslash}\right)^* \Omega^{G\backslash}_{^1T^*TG} &= \Omega^{G\backslash}_{^1TT^*G}, \\
\left(\left(\,^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{G\backslash}\right)^* \Omega^{G\backslash}_{^1T^*T^*G} &= \Omega^{G\backslash}_{^1TT^*G}.
\end{aligned}
$$

Thus, we arrive at the Tulczyjew's triplet for a Lie group G reduced by its left action

$$
\begin{array}{ccccc}
\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* & \xleftarrow{\ ^1\bar{\sigma}_G^{G\backslash}\ } & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} & \xrightarrow{\ ^1\Omega^{G\backslash}_{G\circledS\mathfrak{g}}\ } & \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \\
\end{array} \tag{131}
$$

with projections $^1\pi^{G\backslash}_{G\circledS\mathfrak{g}}$, $^1T\pi^{G\backslash}_G$, $^1\tau^{G\backslash}_{G\circledS\mathfrak{g}^*}$, $^1\pi^{G\backslash}_{G\circledS\mathfrak{g}^*}$ onto $\mathfrak{g}$ and $\mathfrak{g}^*$

with the projections

$$
\begin{aligned}
^1\pi^{G\backslash}_{G\circledS\mathfrak{g}} \quad &: \quad \mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g} : \left(Ad^*_{g^{-1}}\lambda, \xi, \mu\right) \to \xi \\
^1T\pi^{G\backslash}_G \quad &: \quad \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g} : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \xi \\
^1\tau^{G\backslash}_{G\circledS\mathfrak{g}^*} \quad &: \quad \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}^* : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \mu \\
^1\pi^{G\backslash}_{G\circledS\mathfrak{g}^*} \quad &: \quad \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \to \mathfrak{g}^* : \left(Ad^*_{g^{-1}}\lambda, \mu, \xi\right) \to \mu.
\end{aligned}
$$

5. Summary and Conclusions

The structure of Tulczyjew's triplet inherits symplectic diffeomorphisms between iterated bundles

$$
T^*\left(G\circledS\mathfrak{g}\right) \xleftarrow{\ \bar{\sigma}_G\ } T\left(G\circledS\mathfrak{g}^*\right) \xrightarrow{\ \Omega^\flat_{G\circledS\mathfrak{g}^*}\ } T^*\left(G\circledS\mathfrak{g}^*\right) \tag{132}
$$

where the diffeomorphism $\bar{\sigma}_G$ in Eq.(71) results from double vector bundle structure of $T\left(G\circledS\mathfrak{g}\right)$, and $\Omega^\flat_{G\circledS\mathfrak{g}^*}$ in Eq.(56) is defined due the canonical symplectic structure on $G\circledS\mathfrak{g}^*$. The triplet also contains two special symplectic structures due to the existence of two potential one-forms for the symplectic manifold $\left(T\left(G\circledS\mathfrak{g}^*\right), \Omega_{T(G\circledS\mathfrak{g}^*)}\right)$. We have trivialized the triplet (132) using the trivializations of the iterated bundles $T^*\left(G\circledS\mathfrak{g}\right)$, $T\left(G\circledS\mathfrak{g}^*\right)$ and $T^*\left(G\circledS\mathfrak{g}^*\right)$ in Eqs.(72),

(99) and (87), respectively. By properly trivializing the diffeomorphisms $\bar{\sigma}_G$ and $\Omega^\flat_{G\circledS\mathfrak{g}^*}$, we have been able to obtain

$$^1T^*TG \xleftarrow{\quad ^1\bar{\sigma}_G \quad} {}^1TT^*G \xrightarrow{\quad ^1\Omega^\flat_{G\circledS\mathfrak{g}^*} \quad} {}^1T^*T^*G \qquad (133)$$

where $^1\bar{\sigma}_G$ and $^1\Omega^\flat_{G\circledS\mathfrak{g}^*}$ are trivialized symplectic diffeomorphisms in Eqs.(123) and (122), respectively. Two special symplectic structures (in Eqs.(124) and (125)) for the exact Tulczyjew symplectic manifold $\left(^1TT^*G, \Omega_{{}_1TT^*G}\right)$ have been derived by trivializing the ones for $\left(T\left(G\circledS\mathfrak{g}^*\right), \Omega_{T(G\circledS\mathfrak{g}^*)}\right)$.

We have applied the Marsden-Weinstein symplectic reduction theorem to the trivialized symplectic manifolds $G\circledS\mathfrak{g}^*$, $^1T^*TG$, $^1T^*T^*G$ and $^1TT^*G$ for the left symplectic action of the underlying group G, and obtained trivialized reduced symplectic spaces. These reductions have been substituted in the triplet (133) and the reduced and trivialized Tulczyjew's triplet

$$\mathcal{O}_\lambda \times \mathfrak{g} \times \mathfrak{g}^* \xleftarrow{\quad \left(^1\bar{\sigma}_G\right)^{/G} \quad} \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \xrightarrow{\quad \left(^1\Omega^\flat_{G\circledS\mathfrak{g}^*}\right)^{/G} \quad} \mathcal{O}_\lambda \times \mathfrak{g}^* \times \mathfrak{g} \qquad (134)$$

has been obtained.

We are now ready to investigate the dynamics on the triplets (133) and (134), [13]. Reduced and trivialized Tulczyjew's triplet is well-suited for a generalized Legendre transformation between Euler-Poincaré and Lie-Poisson equations. Relations between unreduced equations may result from other levels of trivializations. With the present geometric framework, it is also possible to study implicit Euler-Poincaré and Lie-Poisson equations [30, 70, 71, 72, 73] by considering trivializations of the Pontryagin bundle $TG \times T^*G$.

A.   Tulczyjew's Triplet

In this appendix, we shall review the construction of the Tulczyjew's triplet over an arbitrary manifold $\mathcal{M}$. Special symplectic structures and derivations on differential forms will be defined in a general setting. The Marsden-Weinstein symplectic reduction theorem will be presented.

A.1. Special symplectic structures.

A symplectic structure $\Omega_\mathcal{M}$ on a manifold $\mathcal{M}$ is a closed, non-degenerate two-form [1, 2, 3, 39, 43, 66]. The cotangent bundle $\mathcal{M} = T^*Q$ is canonically symplectic. The Liouville one-form θ_{T^*Q} on T^*Q is defined by

$$\langle \theta_{T^*Q}, X_{T^*Q} \rangle := \langle \tau_{T^*Q} \circ X_{T^*Q}, T\pi_Q \circ X_{T^*Q} \rangle, \qquad (135)$$

where X_{T^*Q} is a vector field on T^*Q, $\tau_{T^*Q} : TT^*Q \to T^*Q$ is the tangent bundle projection and $T\pi_Q$ is the tangent map of the cotangent bundle projection $\pi_Q : T^*Q \to Q$. The canonical symplectic two-from is $\Omega_{T^*Q} = d\theta_{T^*Q}$.

A special symplectic structure is a quintuple $(\mathcal{P}, \pi^\mathcal{P}_\mathcal{M}, \mathcal{M}, \theta_\mathcal{P}, \varphi)$ where $\pi^\mathcal{P}_\mathcal{M} : \mathcal{P} \to \mathcal{M}$ is a fibre bundle, $\theta_\mathcal{P}$ is a one-form on $\mathcal{P}$, and $\varphi : \mathcal{P} \to T^*\mathcal{M}$ is a fiber preserving diffeomorphism such that $\varphi^*\theta_{T^*\mathcal{M}} = \theta_\mathcal{P}$. φ can be characterized uniquely by the condition

$$\langle \varphi(z), X_\mathcal{M}(x) \rangle = \langle \theta_\mathcal{P}(z), X_\mathcal{P}(z) \rangle$$

for each $z \in \mathcal{P}$, $\pi^{\mathcal{P}}_{\mathcal{M}}(z) = x$ and for vector fields $X_{\mathcal{M}}$ and $X_{\mathcal{P}}$ satisfying $\left(\pi^{\mathcal{P}}_{\mathcal{M}} \right)_* X_{\mathcal{P}} = X_{\mathcal{M}}$. Here, $(\mathcal{P}, d\theta_{\mathcal{P}})$ is the underlying symplectic manifold of the special symplectic structure.

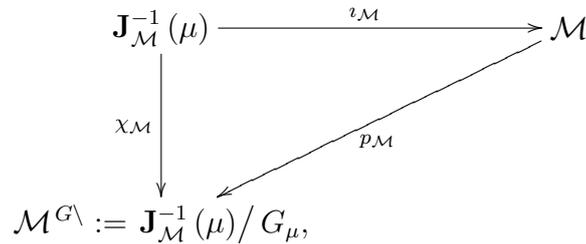## A.2. Symplectic reduction.

Let $(\mathcal{M}, \Omega_{\mathcal{M}} = d\theta_{\mathcal{M}})$ be an exact symplectic manifold and assume that there is a symplectic action of a Lie group G on $(\mathcal{M}, \Omega)$. Then, there is an Ad^*-equivariant momentum map $\mathbf{J}_{\mathcal{M}} : \mathcal{M} \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}_{\mathcal{M}}(q), \xi \rangle = \langle \theta_{\mathcal{M}}, \xi_{\mathcal{M}} \rangle (q), \tag{136}$$

where the pairing on left hand side is the one between $\mathfrak{g}^*$ and $\mathfrak{g}$, and the pairing on right hand side is the one between $T_q^*\mathcal{M}$ and $T_q\mathcal{M}$, [43]. Assume, in addition, that $\mu \in \mathfrak{g}^*$ is a regular value of $\mathbf{J}$ and isotropy group G_μ of μ with Ad^* action acts freely and properly on $\mathbf{J}^{-1}_{\mathcal{M}}(\mu)$. Under these circumstances, the Marsden-Weinstein theorem [41, 48] establishes the reduction of the symplectic structure $(\mathcal{M}, \Omega_{\mathcal{M}} = d\theta_{\mathcal{M}})$.

Theorem A.1. *The reduced manifold $\mathcal{M}^{G\backslash} := \mathbf{J}^{-1}_{\mathcal{M}}(\mu) / G_\mu$ has a unique symplectic structure Ω_μ satisfying $\chi^*_\mu \Omega_\mu = \imath_\mu \Omega_{\mathcal{M}}$, where χ_μ is the surjection $\mathbf{J}^{-1}_{\mathcal{M}}(\mu) \to \mathbf{J}^{-1}_{\mathcal{M}}(\mu) / G_\mu$ and $\imath_\mu$ is the injection $\mathbf{J}^{-1}_{\mathcal{M}}(\mu) \hookrightarrow \mathcal{M}$.*

In the main text, we refer to this theorem through the following reduction diagram

$$
\begin{array}{ccc}
\mathbf{J}^{-1}_{\mathcal{M}}(\mu) & \xrightarrow{\quad \imath_{\mathcal{M}} \quad} & \mathcal{M} \\
{\scriptstyle \chi_{\mathcal{M}}} \downarrow & {\scriptstyle p_{\mathcal{M}}} \nearrow & \\
\mathcal{M}^{G\backslash} := \mathbf{J}^{-1}_{\mathcal{M}}(\mu) / G_\mu, & &
\end{array}
$$

where $p_{\mathcal{M}}$ is the projection from $\mathcal{M}$ to the reduced space $\mathcal{M}^{G\backslash}$ and defined by means of the commutation of the diagram.

A.3. Derivations.

Introduce two derivations i_T and d_T which maps exterior algebra $\Lambda(\mathcal{M})$ on a manifold $\mathcal{M}$ to the exterior algebra $\Lambda(T\mathcal{M})$ on its tangent bundle $T\mathcal{M}$ [60, 61, 62, 63]. The action of i_T on a p-form $\omega^p \in \Lambda^p(\mathcal{M})$ is a $(p-1)$-form $i_T \omega^p \in \Lambda^{p-1}(T\mathcal{M})$ defined by

$$i_T \omega^p \left(X_1^{T\mathcal{M}}, ..., X_{p-1}^{T\mathcal{M}} \right) = \omega^p \left(\tau_{T\mathcal{M}} \circ X_1^{T\mathcal{M}}, T\tau_{\mathcal{M}} \circ X_1^{T\mathcal{M}}, ..., T\tau_{\mathcal{M}} \cdot X_{p-1}^{T\mathcal{M}} \right) \tag{137}$$

where $X_1^{T\mathcal{M}}, ..., X_p^{T\mathcal{M}}$ are vector fields on $T\mathcal{M}$. Here, $\tau_{T\mathcal{M}} : TT\mathcal{M} \to T\mathcal{M}$ is the natural projection and $T\tau_{\mathcal{M}} : TT\mathcal{M} \to T\mathcal{M}$ is the tangent mapping of the natural projection $\tau_{\mathcal{M}} : T\mathcal{M} \to \mathcal{M}$. Using the derivation i_T and exterior derivative d, define degree 0 derivation $d_T : \Lambda(\mathcal{M}) \to \Lambda(T\mathcal{M})$ by $d_T = i_T d + d i_T$. It follows that, d and d_T commutes. In particular, if f, $\theta_{\mathcal{M}}$ and $d\theta_{\mathcal{M}}$ are a function, a

one-form and an exact two-form on $\mathcal{M}$, then

$$
\begin{aligned}
d_T f &= i_T df, \quad d_T \theta_{\mathcal{M}} = i_T d\theta_{\mathcal{M}} + d i_T \theta_{\mathcal{M}}, \\
d_T d\theta_{\mathcal{M}} &= d i_T d\theta_{\mathcal{M}} = d d_T \theta_{\mathcal{M}}
\end{aligned}
$$

are a function, a one-form and a two-form on $T\mathcal{M}$, respectively.

A.4. Tulczyjew's triplet.

Tangent bundle of a symplectic manifold $(\mathcal{M}, \Omega_{\mathcal{M}})$ is a symplectic manifold $(T\mathcal{M}, \Omega_{T\mathcal{M}})$ [60]. Here, the symplectic two-form $\Omega_{T\mathcal{M}}$ on $T\mathcal{M}$ is obtained by the application of the derivation d_T on $\Omega_{\mathcal{M}}$, that is $\Omega_{T\mathcal{M}} = d_T \Omega_{\mathcal{M}}$. In particular, for $\mathcal{M} = T^*Q$, Ω_{T^*Q} leads to an exact symplectic structure Ω_{TT^*Q} with two potential one-forms

$$
\vartheta_1 = i_T \Omega_{T^*\mathcal{Q}}, \quad \vartheta_2 = d_T \theta_{T^*Q} = i_T \Omega_{T^*\mathcal{Q}} + d i_T \theta_{T^*Q} \tag{138}
$$

forming two different special symplectic structures

$$
(TT^*\mathcal{Q}, \tau_{T^*\mathcal{Q}}, T^*\mathcal{Q}, \vartheta_1, \Omega^\flat_{T^*\mathcal{Q}}), \tag{139}
$$

$$
(TT^*\mathcal{Q}, T\pi_{\mathcal{Q}}, T\mathcal{Q}, \vartheta_2, \alpha_{\mathcal{Q}}) \tag{140}
$$

on the Tulczyjew's symplectic manifold $(TT^*\mathcal{Q}, \Omega_{TT^*Q})$. Here, $\Omega^\flat_{T^*\mathcal{Q}}$ is induced from $\Omega_{T^*\mathcal{Q}}$ and $\alpha_{\mathcal{Q}}$ is a diffeomorphism constructed as a *dual* of canonical involution $\kappa_{\mathcal{Q}}$ of $TT\mathcal{Q}$. They satisfy

$$
\left(\Omega^\flat_{T^*\mathcal{Q}}\right)^* \theta_{T^*T^*\mathcal{Q}} = \vartheta_1, \quad \alpha_{\mathcal{Q}}^* \theta_{T^*T\mathcal{Q}} = \vartheta_2, \tag{141}
$$

where $\theta_{T^*T^*\mathcal{Q}}$ and $\theta_{T^*T\mathcal{Q}}$ canonical one-forms on $T^*T^*\mathcal{Q}$ and $T^*T\mathcal{Q}$, respectively. The Tulczyjew's triplet is the diagram
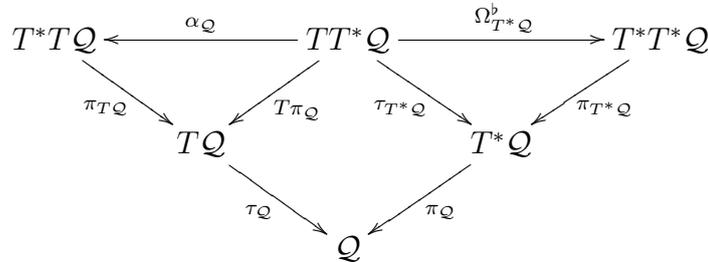
which contains two special symplectic structures in Eqs.(139) and (140) [59, 60, 61, 63]. The triangular diagrams on left and right define special symplectic structures on $TT^*\mathcal{Q}$.

B. Left Trivializations

In this appendix, we shall summarize the results for the case of left actions and left representations. We shall trivialize first order and iterated bundles and derive the canonical mappings between them. The symplectic two-forms on the trivialized bundles $G\circledS_L \mathfrak{g}^*$, ${}^L T^*TG$, ${}^L T^*T^*G$ and ${}^L T^*T^*G$ will be written and under Hamiltonian right action of G, the symplectic reduction will be applied to these symplectic manifolds. Hence, both of the left trivialization and the left reduction

of Tulczyjew's triplet will be obtained. We will continue to obey the notations in Eqs.(2), (3) and (4).

B.1. Tangent and cotangent groups.

Left representation ϱ of a Lie group G on a vector space V satisfies $\varrho_g \circ \varrho_h = \varrho_{gh}$, for all $g, h \in G$. Inner automorphism, given by $I_g^L h = g h g^{-1}$, and both of the adjoint actions

$$Ad_g^L \xi = \frac{d}{dt} I_g^L h^t \big|_{t=0}, \quad h^0 = e, \quad \frac{d}{dt} h^t \big|_{t=0} = \xi$$

$$[\eta, \xi]_{\mathfrak{g}}^L = ad_\eta^L \xi = \frac{d}{dt} Ad_{g^t}^L \xi \big|_{t=0}, \quad g^0 = e, \quad \frac{d}{dt} g^t \big|_{t=0} = \eta$$

of G and its Lie algebra $\mathfrak{g}$ on $\mathfrak{g}$, are left actions. Here, $g, h \in G$ and $\eta, \xi \in \mathfrak{g}$. We will use the superscript L to remark that the action is left. A left invariant vector field is defined by $X_\xi^L = T_e L_g \xi$ and the identity

$$\left[X_{\xi_1}^L, X_{\xi_2}^L \right]_{JL} (e) = [\xi_1, \xi_2]_{\mathfrak{g}}^L$$

implies that $\xi \to X_\xi^L$ is a Lie algebra homomorphism. The action Ad_g^{L*} of G on dual space $\mathfrak{g}^*$ is the linear algebraic dual $Ad_{g^{-1}}^L$, whereas infinitesimal action ad_ξ^{L*} of $\mathfrak{g}$ on $\mathfrak{g}^*$ is linear algebraic dual of ad_ξ^L. We note an important commutation rule

$$ad_\xi^{L*} \circ Ad_{g^{-1}}^{L*} = Ad_{g^{-1}}^{L*} \circ ad_{Ad_g \xi}^{L*},$$

which is valid for g and ξ.

We recall the left trivialization map

$$tr_{TG}^L : TG \to G \times \mathfrak{g} : V_g \to (g, T_g L_{g^{-1}} V_g)$$

in Eq.(32). The pull-back of group structure ϖ_{TG}, in Eq.(29), on TG by the left trivialization map tr_{TG}^L defines a semi-direct product group structure

$$L_{(g,\xi)}^L (h, \eta) = R_{(h,\eta)}^L (g, \xi) = \left(gh, \eta + Ad_{h^{-1}}^L \xi \right)$$

on the product $G \times \mathfrak{g}$ and we will use the notation $G \circledS_L \mathfrak{g}$. The identity element is $(e, 0)$ and inverse of an element (g, ξ) is $\left(g^{-1}, -Ad_g^L \xi \right)$. the Lie algebra of the group $G \circledS_L \mathfrak{g}$ is $\mathfrak{g} \circledS_L \mathfrak{g}$. The dual of the Lie algebra $\mathfrak{g} \circledS_L \mathfrak{g}$ is $\mathfrak{g}^* \times \mathfrak{g}^*$. We have the following list of actions:

$$\begin{aligned}
I_{(g,\xi)}^L (h, \eta) &= \left(I_g^L h, Ad_g^L \left(\eta - \xi + Ad_{h^{-1}}^L \xi \right) \right), \\
T_{(h,\eta)} L_{(g,\xi)}^L (V_h, V_\eta) &= \left(T_h L_g V_h, V_\eta + [Ad_{h^{-1}}^L \xi, T_h L_{h^{-1}} V_h]_{\mathfrak{g}}^L \right), \\
T_{(g,\xi)} R_{(h,\eta)}^L (V_g, V_\xi) &= \left(T_g R_h V_g, Ad_{h^{-1}}^L V_\xi \right), \\
T_{(h,\eta)}^* L_{(g,\xi)}^L (\alpha_{gh}, \alpha_{\xi\eta}) &= \left(T_h^* L_g (\alpha_{gh}) + T_e^* L_{h^{-1}} \circ ad_{Ad_{h^{-1}}^L \xi}^{L*} \alpha_{\xi\eta}, \alpha_{\xi\eta} \right), \\
T_{(g,u)}^* R_{(h,v)}^L (\alpha_{gh}, \alpha_{\xi\eta}) &= \left(T_g^* R_h (\alpha_{gh}), Ad_h^{L*} \alpha_{\xi\eta} \right), \\
Ad_{(g,\xi)}^L (\eta, \zeta) &= \left(Ad_g^L \eta, Ad_g^L \left(\zeta + [\xi, \eta]_{\mathfrak{g}}^L \right) \right), \\
Ad_{(g,\xi)}^{L*} (\mu, \nu) &= \left(Ad_g^{L*} \left(\mu - ad_\xi^* \nu \right), Ad_{g^{-1}}^{L*} \nu \right), \\
[(\chi, \xi), (\eta, \zeta)]_{\mathfrak{g} \circledS_L \mathfrak{g}}^L &= \left([\chi, \eta]_{\mathfrak{g}}^L, [\chi, \zeta]_{\mathfrak{g}}^L - [\eta, \xi]_{\mathfrak{g}}^L \right), \\
ad_{(\chi,\xi)}^{L*} (\mu, \nu) &= \left(ad_\chi^{L*} \mu + ad_\xi^{L*} \nu, ad_\chi^{L*} \nu \right),
\end{aligned}$$

where $(V_h, V_\eta) \in T_{(h,\eta)}(G \textcircled{S}_L \mathfrak{g})$, $(\alpha_{gh}, \alpha_{\xi\eta}) \in T^*_{(gh,\xi\eta)}(G \textcircled{S}_L \mathfrak{g})$, and $\xi\eta = \eta + Ad^L_{h^{-1}}\xi$. The bracket $[\ ,\ ]^L_{\mathfrak{g} \textcircled{S}_L \mathfrak{g}}$ is the Lie algebra structure on $\mathfrak{g} \textcircled{S}_L \mathfrak{g}$. A left invariant vector field on $G \textcircled{S}_L \mathfrak{g}$ is

$$X^{G\textcircled{S}_L \mathfrak{g}}_{(\eta,\zeta)}(g,\xi) = T_{(e,0)}L^L_{(g,\xi)}(\eta,\zeta) = (T_e L_g \eta, \zeta + [\xi, \eta]).$$

We pull the group structure ϖ_{T^*G}, in Eq.(40), back to $G \times \mathfrak{g}^*$ by left trivialization map

$$tr^L_{T^*G} : T^*G \to G \times \mathfrak{g}^* : \alpha_g \to (g, T^*_e L_g \alpha_g)$$

and obtain a semi-direct group structure

$$(g,\mu)(h,\nu) = (gh, \nu + Ad^{L*}_{h^{-1}}\mu) = L_{(g,\mu)}(h,\nu) = R_{(h,\nu)}(g,\mu), \quad (142)$$

on $G \times \mathfrak{g}^*$. With this group multiplication, we denote the product by $G \textcircled{S}_L \mathfrak{g}^*$. The identity is $(e,0)$ and the inverse of (g,μ) is $(g^{-1}, -Ad^{L*}_g \mu)$. Lie algebra of $G \textcircled{S}_L \mathfrak{g}^*$ is $\mathfrak{g} \textcircled{S}_L \mathfrak{g}^*$ and the dual of $\mathfrak{g} \textcircled{S}_L \mathfrak{g}^*$ is $\mathfrak{g}^* \times \mathfrak{g}$, where we assume the reflexivity condition $\mathfrak{g}^{**} \simeq \mathfrak{g}$. We have the following list of actions

$$
\begin{aligned}
I^L_{(g,\mu)}(h,\nu) &= \left(I^L_g h, Ad^{L*}_g\left(\nu - \mu + Ad^{L*}_{h^{-1}}\mu\right)\right), \\
T_{(h,\nu)}L^L_{(g,\mu)}(V_h, V_\nu) &= \left(T_h L_g V_h, V_\nu + Ad^{L*}_{h^{-1}} \circ ad^{L*}_{TR_{h^{-1}}V_h}\mu\right), \\
T_{(g,\mu)}R^L_{(h,\nu)}(V_g, V_\mu) &= \left(T_g R_h V_g, Ad^{L*}_{h^{-1}}V_\mu\right), \\
T^*_{(h,\nu)}L^L_{(g,\mu)}(\alpha_{gh}, \alpha_{\mu\nu}) &= \left(T^*_h L_g \alpha_{gh} - T^*_h R_{h^{-1}} \circ ad^{L*}_{Ad^L_h \alpha_{\mu\nu}}\mu, \alpha_{\mu\nu}\right), \\
T^*_{(g,\mu)}R^L_{(h,\nu)}(\alpha_{gh}, \alpha_{\mu\nu}) &= \left(T^*_g R_h \alpha_{gh}, Ad^L_h \alpha_{\mu\nu}\right), \\
Ad^L_{(g,\mu)}(\eta,\nu) &= \left(Ad^L_g \eta, Ad^{L*}_g\left(\nu + ad^{L*}_\eta \mu\right)\right), \\
Ad^{L*}_{(g,\mu)}(\lambda,\xi) &= \left(Ad^{L*}_g\left(\lambda + ad^{L*}_\xi \mu\right), Ad^L_g \xi\right), \\
[(\xi,\mu),(\eta,\nu)]^L_{\mathfrak{g}\textcircled{S}_L\mathfrak{g}^*} &= \left([\xi,\eta]^L_\mathfrak{g}, ad^{L*}_\eta \mu - ad^{L*}_\xi \nu\right), \\
ad^{L*}_{(\xi,\mu)}(\lambda,\zeta) &= \left(ad^{L*}_\xi \lambda - ad^{L*}_\zeta \mu, [\zeta,\xi]^L_\mathfrak{g}\right).
\end{aligned}
$$

where $(V_h, V_\nu) \in T_{(h,\nu)}(G \textcircled{S}_L \mathfrak{g}^*)$, $(\alpha_{gh}, \alpha_{\mu\nu}) \in T^*_{(gh,\mu\nu)}(G \textcircled{S}_L \mathfrak{g}^*)$, $\alpha_{\mu\nu} \in T^*_{\mu\nu}\mathfrak{g}^* \simeq \mathfrak{g}$, $\mu\nu = \nu + Ad^{L*}_{h^{-1}}\mu$. By recalling the involution s, in Eq.(51), we arrive

$$
\begin{aligned}
Ad^{L*}_{(g,\mu)} \circ s &= s \circ Ad^L_{(g,\mu)}, \\
ad^L_{(\xi,\mu)} \circ s &= -s \circ ad^{L*}_{(\xi,\mu)}.
\end{aligned}
$$

A left invariant vector field on $G \textcircled{S}_L \mathfrak{g}^*$ is in form

$$X^{G\textcircled{S}_L\mathfrak{g}^*}_{(\xi,\nu)}(g,\mu) = T_{(e,0)}L^L_{(g,\mu)}(\xi,\nu) = \left(T_e L_g \xi, \nu + ad^{L*}_\xi \mu\right),$$

and the Jacobi-Lie bracket of two such vector fields are

$$\left[X^{G\textcircled{S}_L\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\textcircled{S}_L\mathfrak{g}^*}_{(\eta,\lambda)}\right] = X^{G\textcircled{S}_L\mathfrak{g}^*}_{[(\xi,\mu),(\eta,\nu)]^L_{\mathfrak{g}\textcircled{S}_L\mathfrak{g}^*}}.$$

If we let the left trivialization map $tr^L_{T^*G}$ be a symplectomorphism, we arrive a canonical one-form $\theta_{G\circledS_L\mathfrak{g}^*}$ and a symplectic two-form $\Omega_{G\circledS_L\mathfrak{g}^*}$ on $G\circledS_L\mathfrak{g}^*$. The values of these forms on right invariant vector fields are

$$\left\langle \theta_{G\circledS_L\mathfrak{g}^*}, X^{G\circledS_L\mathfrak{g}^*}_{(\xi,\nu)} \right\rangle (g,\mu) \;=\; \langle \mu, \xi \rangle$$

$$\left\langle \Omega_{G\circledS_L\mathfrak{g}^*}; \left(X^{G\circledS_L\mathfrak{g}^*}_{(\xi,\nu)}, X^{G\circledS_L\mathfrak{g}^*}_{(\eta,\lambda)} \right) \right\rangle (g,\mu) \;=\; \langle \nu, \eta \rangle - \langle \lambda, \xi \rangle + \left\langle \mu, [\xi,\eta]^L_\mathfrak{g} \right\rangle. \quad (143)$$

Right action

$$(G\circledS_L\mathfrak{g}^*) \times G \to G\circledS_L\mathfrak{g}^* : (g,\mu;h) \to \left(gh, Ad^{L*}_{h^{-1}}\mu \right)$$

of G on $G\circledS_L\mathfrak{g}^*$ generated by the vector field $X^{G\circledS_L\mathfrak{g}^*}_{(\xi,0)}$. This is a Hamiltonian action with Ad^{L*}-invariant momentum mapping $\mathbf{J}_{G\circledS_L\mathfrak{g}^*}(g,\mu) = \mu$. Quotient of the preimage $\mathbf{J}^{-1}_{G\circledS_L\mathfrak{g}^*}(\mu) = G$ to the the isotropy group $G^L_\mu = \left\{ g : Ad^{L*}_{g^{-1}}\mu = \mu \right\}$ is the coadjoint orbit $\mathcal{O}^L_\mu = \left\{ Ad^{L*}_{g^{-1}}\mu : g \in G \right\}$. The value

$$\left\langle \Omega^{/G}_{G\circledS_L\mathfrak{g}^*}; (\xi_{\mathfrak{g}^*}, \eta_{\mathfrak{g}^*}) \right\rangle (\mu) = - \left\langle \mu, [\xi,\eta]^L_\mathfrak{g} \right\rangle.$$

of the reduced symplectic structure $\Omega^{/G}_{G\circledS_L\mathfrak{g}^*}$ on $\xi_{\mathfrak{g}^*}(\mu), \eta_{\mathfrak{g}^*}(\mu) \in T_\mu\mathcal{O}^L_\mu$.

B.2. Lifts of group actions.

We give the list of actions of a Lie group G to its tangent and cotangent bundles and to their trivializations. Left and right actions of G on its tangent bundle are

$$G \times TG \;\to\; TG : (g,U_h) \to T_h L_g U_h$$
$$TG \times G \;\to\; TG : (V_g,h) \to T_g R_h V_g,$$

respectively. Using trivializations, we arrive actions of G on the right $G\circledS_R\mathfrak{g}$ and left $G\circledS_L\mathfrak{g}$ trivializations

$$G \times (G\circledS_R\mathfrak{g}) \;\to\; (G\circledS_R\mathfrak{g}) : (g,(h,\eta)) \to \left(gh, Ad^R_{g^{-1}}\eta \right)$$
$$G \times (G\circledS_L\mathfrak{g}) \;\to\; (G\circledS_L\mathfrak{g}) : (g,(h,\eta)) \to (gh, \eta)$$
$$(G\circledS_R\mathfrak{g}) \times G \;\to\; (G\circledS_R\mathfrak{g}) : ((g,\xi),h) \to (gh, \xi)$$
$$(G\circledS_L\mathfrak{g}) \times G \;\to\; (G\circledS_L\mathfrak{g}) : ((g,\xi),h) \to \left(gh, Ad^L_{h^{-1}}\xi \right),$$

which are called right lift of left action, left lift of left action, right lift of right action and left lift of right action, respectively. Left and right actions of G on its cotangent bundle are

$$G \times T^*G \;\to\; T^*G : (g,\beta_h) \to T^*_{gh}L_{g^{-1}}\beta_h$$
$$T^*G \times G \;\to\; T^*G : (\alpha_g,h) \to T_{gh}R_{h^{-1}}\alpha_g,$$

respectively. Similar to the tangent bundle case, we calculate actions of G on the trivializations as follows

$$G \times (G\circledS_R\mathfrak{g}^*) \;\to\; (G\circledS_R\mathfrak{g}^*) : (g,(h,\nu)) \to \left(gh, Ad^{R*}_{g^{-1}}\nu \right)$$
$$G \times (G\circledS_L\mathfrak{g}^*) \;\to\; (G\circledS_L\mathfrak{g}^*) : (g,(h,\nu)) \to (gh, \nu)$$
$$(G\circledS_R\mathfrak{g}^*) \times G \;\to\; (G\circledS_R\mathfrak{g}^*) : ((g,\mu),h) \to (gh, \mu)$$
$$(G\circledS_L\mathfrak{g}^*) \times G \;\to\; (G\circledS_L\mathfrak{g}^*) : ((g,\xi),h) \to \left(gh, Ad^{L*}_{h^{-1}}\mu \right).$$

B.3. Trivializations of iterated bundles.

We list the left trivializations of iterated bundles $TTG \simeq T(G\circledS_L\mathfrak{g})$, $TT^*G \simeq T(G\circledS_L\mathfrak{g}^*)$, $T^*TG \simeq T^*(G\circledS_L\mathfrak{g})$, $T^*T^*G \simeq T^*(G\circledS_L\mathfrak{g}^*)$ (and use the superscript L to denote the trivialized spaces):

$$
\begin{aligned}
tr^L_{T(G\circledS_L\mathfrak{g})} \;\; &: \;\; T(G\circledS_L\mathfrak{g}) \to (G\circledS_L\mathfrak{g})\circledS_L(\mathfrak{g}\circledS_L\mathfrak{g}) =: {}^LTTG \\
&: \;\; (V_g, V_\xi) \to \left(g, \xi, T_gL_{g^{-1}}V_g, V_\xi - [\xi, TL_{g^{-1}}V_g]^L_{\mathfrak{g}}\right), \quad (144)
\end{aligned}
$$

$$
\begin{aligned}
tr^L_{T(G\circledS_L\mathfrak{g}^*)} \;\; &: \;\; T(G\circledS_L\mathfrak{g}^*) \to (G\circledS_L\mathfrak{g}^*)\circledS_L(\mathfrak{g}\circledS_L\mathfrak{g}^*) =: {}^LTT^*G \\
&: \;\; (V_g, V_\mu) \to \left(g, \mu, T_gL_{g^{-1}}V_g, V_\mu - ad^{L*}_{TL_{g^{-1}}V_g}\mu\right), \quad (145)
\end{aligned}
$$

$$
\begin{aligned}
tr^L_{T^*(G\circledS_L\mathfrak{g})} \;\; &: \;\; T^*(G\circledS_L\mathfrak{g}) \to (G\circledS_L\mathfrak{g})\circledS_L(\mathfrak{g}^* \times \mathfrak{g}^*) =: {}^LT^*TG \\
&: \;\; (\alpha_g, \alpha_\xi) \to \left(g, \xi, T_e^*L_g\alpha_g + ad^{L*}_\xi\alpha_\xi, \alpha_\xi\right), \quad (146)
\end{aligned}
$$

$$
\begin{aligned}
tr^L_{T^*(G\circledS_L\mathfrak{g}^*)} \;\; &: \;\; T^*(G\circledS_L\mathfrak{g}^*) \to (G\circledS_L\mathfrak{g}^*)\circledS_L(\mathfrak{g}^* \times \mathfrak{g}) =: {}^LT^*T^*G \\
&: \;\; (\alpha_g, \alpha_\mu) \to \left(g, \mu, T_e^*L_g\alpha_g - ad^{L*}_{\alpha_\mu}\mu, \alpha_\mu\right), \quad (147)
\end{aligned}
$$

where $\alpha_\mu \in T_\mu^*\mathfrak{g}^* \simeq \mathfrak{g}$. The canonical involution on the iterated tangent bundle $TTG \simeq T(G\circledS\mathfrak{g})$ is given by

$$
\begin{aligned}
\bar{\kappa}^L_G \;\; &: \;\; T_{(g,\xi)}(G\circledS\mathfrak{g}) \to T_{(g,\eta)}(G\circledS\mathfrak{g}) \\
&: \;\; (V_g, V_\xi) \to (U_g, V_\eta) = \left(U_g, V_\xi + [\eta, \xi]^L_{\mathfrak{g}}\right), \quad (148)
\end{aligned}
$$

where $T_gL_{g^{-1}}V_g = \eta$ and $T_gL_{g^{-1}}U_g = \xi$. Note that, the structure of this mapping is the same with $\bar{\kappa}_G$ in Eq.(68) but the definitions of the brackets $[\;,\;]_{\mathfrak{g}}$ are different. Using the paring $\langle\;,\;\rangle^{\tilde{}}$ between $T(G\circledS\mathfrak{g}^*)$ and $T(G\circledS\mathfrak{g})$, in Eq.(??), we define the *dual* of $\bar{\kappa}^L_G$ as

$$
\begin{aligned}
\bar{\sigma}^L_G \;\; &: \;\; T_{(g,\mu)}(G\circledS_L\mathfrak{g}^*) \to T_{(g,\xi)}^*(G\circledS_L\mathfrak{g}) \\
&: \;\; (V_g, V_\mu) \to \left(T_g^*L_{g^{-1}}\left(V_\mu - ad^{L*}_\xi\mu\right), \mu\right). \quad (149)
\end{aligned}
$$

After the trivializations of the iterated bundles, we arrive the trivializations of $\bar{\kappa}^L_G$ and $\bar{\sigma}^L_G$ as

$$
\begin{aligned}
{}^L\bar{\kappa}^L_G \;\; &: \;\; {}^LTTG \to {}^LTTG : (g, \xi, \eta, \zeta) \to \left(g, \eta, \xi, \zeta + [\xi, \eta]^L_{\mathfrak{g}}\right) \\
{}^L\bar{\sigma}^L_G \;\; &: \;\; {}^LTT^*G \to {}^LT^*TG : (g, \mu, \xi, \nu) \to \left(g, \xi, \nu + ad^{L*}_\xi\mu, \mu\right). \quad (150)
\end{aligned}
$$

The musical isomorphism $\Omega^\flat_{G\circledS_L\mathfrak{g}^*}$ associated with the symplectic two $\Omega_{G\circledS_L\mathfrak{g}^*}$, in Eq.(143), is

$$
\begin{aligned}
\Omega^\flat_{G\circledS_L\mathfrak{g}^*} \;\; &: \;\; T(G\circledS_L\mathfrak{g}^*) \to T^*(G\circledS_L\mathfrak{g}^*) \\
&: \;\; (V_g, V_\mu) \to \left(T^*L_{g^{-1}}\left(V_\mu - ad^{L*}_{TL_{g^{-1}}V_g}\mu\right), -TL_{g^{-1}}V_g\right). \quad (151)
\end{aligned}
$$

After the trivializations of the domain and image spaces of $\Omega^\flat_{G\circledS_L\mathfrak{g}^*}$, we arrive the trivialization

$$
{}^L\Omega^\flat_{G\circledS_L\mathfrak{g}^*} : {}^LTT^*G \to {}^LT^*T^*G : (g, \mu, \eta, \nu) \to \left(g, \mu, \nu + ad^{L*}_\eta\mu, -\eta\right). \quad (152)
$$

B.4. Symplectic two-forms and their reductions.

The trivialized bundle $^{L}T^*TG$ is a symplectic manifold with the canonical one-form $\theta_{{}_{L}T^*TG}$ and symplectic two-form $\Omega_{{}_{L}T^*TG}$. At a point $(g, \xi, \mu, \nu) \in {}^{L}T^*TG$, a left invariant vector field $X^{{}^{L}T^*TG}_{(\eta, \zeta, \lambda_1, \lambda_2)}$ on $^{L}T^*TG$ is

$$X^{{}^{L}T^*TG}_{(\eta, \zeta, \lambda_1, \lambda_2)} = \left(TL_g \eta, \zeta + [\xi, \eta]^L_{\mathfrak{g}}, \lambda_1 + ad^{L*}_\eta \mu + ad^{L*}_\zeta \nu, \lambda_2 + ad^{L*}_\eta \nu \right)$$

and the value of the canonical one-form over a left invariant vector field is

$$\left\langle \theta_{{}_{L}T^*TG}, X^{{}^{L}T^*TG}_{(\eta, \zeta, \lambda_1, \lambda_2)} \right\rangle = \langle \mu, \eta \rangle + \langle \nu, \zeta \rangle, \tag{153}$$

whereas the value of the symplectic two-form on two left invariant vector fields is

$$\left\langle \Omega_{{}_{L}T^*TG}; \left(X^{{}^{L}T^*TG}_{(\eta, \zeta, \lambda_1, \lambda_2)}, X^{{}^{L}T^*TG}_{(\bar{\eta}, \bar{\zeta}, \bar{\lambda}_1, \bar{\lambda}_2)} \right) \right\rangle = \langle \lambda_1, \bar{\eta} \rangle + \langle \lambda_2, \bar{\zeta} \rangle$$
$$- \langle \bar{\lambda}_1, \eta \rangle - \langle \bar{\lambda}_2, \zeta \rangle + \left\langle \mu, [\eta, \bar{\eta}]^L_{\mathfrak{g}} \right\rangle + \left\langle \nu, [\zeta, \bar{\eta}]^L_{\mathfrak{g}} - [\bar{\zeta}, \eta]^L_{\mathfrak{g}} \right\rangle.$$

The right action of G on $^{L}T^*TG$ is

$$^{L}T^*TG \times G \to {}^{L}T^*TG : ((g, \xi, \mu, \nu); h) \to \left(gh, Ad^L_{h^{-1}} \xi, Ad^{L*}_{h^{-1}} \mu, Ad^{L*}_{h^{-1}} \nu \right),$$

which is generated by $X^{{}^{L}T^*TG}_{(\eta, 0, 0, 0)}$. This action is a Hamiltonian action with a momentum mapping

$$\mathbf{J}_{{}_{L}T^*TG} : {}^{L}T^*TG \to \mathfrak{g}^* : (g, \xi, \mu, \nu) \to \mu.$$

The symplectic reduction is

$$\left({}^{L}T^*TG, \Omega_{{}_{L}T^*TG} \right) \to \left(\mathbf{J}^{-1}_{{}_{L}T^*TG} (\mu) / G_\mu, \Omega^{/G}_{{}_{L}T^*TG} \right) \tag{154}$$

where the reduced space is

$$\mathbf{J}^{-1}_{{}_{L}T^*TG} (\mu) / G_\mu = \mathcal{O}^L_\mu \times \mathfrak{g} \times \mathfrak{g}^*$$

and reduced symplectic two-form $\Omega^{/G}_{{}_{L}T^*TG}$ takes value

$$\left\langle \Omega^{/G}_{{}_{L}T^*TG}; \left((\eta^{\mathfrak{g}^*}, \zeta, \lambda_2), (\bar{\eta}^{\mathfrak{g}^*}, \bar{\zeta}, \bar{\lambda}_2) \right) \right\rangle = \langle \lambda_2, \bar{\zeta} \rangle - \langle \bar{\lambda}_2, \zeta \rangle - \left\langle \mu, [\eta, \bar{\eta}]^L_{\mathfrak{g}} \right\rangle$$

over $(\eta_{\mathfrak{g}^*}, \zeta, \lambda_2), (\bar{\eta}_{\mathfrak{g}^*}, \bar{\zeta}, \bar{\lambda}_2) \in T_\mu \mathcal{O}^L_\mu \times \mathfrak{g} \times \mathfrak{g}^*$.

The trivialized cotangent bundle $^{L}T^*T^*G$ is a symplectic manifold with a canonical one-form $\theta_{{}_{L}T^*T^*G}$ and a symplectic two-from $\Omega_{{}_{L}T^*T^*G}$. At the point (g, μ, ν, ξ), the value of the one-form is

$$\left\langle \theta_{{}_{L}T^*T^*G}, X^{{}^{L}T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)} \right\rangle = \langle \nu, \eta \rangle + \langle \lambda_1, \xi \rangle \tag{155}$$

where $X^{{}^{L}T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)}$ is a left invariant vector field

$$X^{{}^{L}T^*T^*G}_{(\eta, \lambda_1, \lambda_2, \zeta)} = \left(TL_g \eta, \lambda_1 + ad^{L*}_\eta \mu, \lambda_2 + ad^{L*}_\eta \nu - ad^{L*}_\xi \lambda_1, \zeta + [\xi, \eta]^L_{\mathfrak{g}} \right).$$

The value of the symplectic two-form on two left invariant vector fields is

$$\langle \Omega \,_{^LT^*T^*G}; \left(X^{^LT^*T^*G}_{(\eta,\lambda_1,\lambda_2,\zeta)}, X^{^LT^*T^*G}_{(\bar\eta,\bar\lambda_1,\bar\lambda_2,\bar\zeta)} \right) \rangle = \langle \lambda_2, \bar\eta \rangle + \langle \bar\lambda_1, \zeta \rangle$$
$$- \langle \bar\lambda_2, \eta \rangle - \langle \lambda_1, \bar\zeta \rangle + \langle \nu, [\eta, \bar\eta]^L_{\mathfrak{g}} \rangle + \langle ad^{L*}_{\bar\eta}\lambda_1 - ad^{L*}_{\eta}\bar\lambda_1, \xi \rangle \,.$$

The right action of G on $^LT^*T^*G$ is

$$^LT^*T^*G \times G \to \,^LT^*T^*G : ((g,\mu,\nu,\xi)\,;h) \to \left(gh, Ad^{L*}_{h^{-1}}\mu, Ad^{L*}_{h^{-1}}\nu, Ad_{h^{-1}}\xi \right),$$

which is generated by the vector field $X^{^LT^*T^*G}_{(\eta,0,0,0)}$. The action has a momentum mapping

$$\mathbf{J} \,_{^LT^*T^*G} : (g,\mu,\nu,\xi) \to \nu.$$

The symplectic reduction theorem

$$\left(\,^LT^*T^*G, \Omega \,_{^LT^*T^*G} \right) \to \left(\mathbf{J}^{-1}_{^LT^*T^*G}(\nu)/G_\nu, \Omega^{/G}_{^LT^*T^*G} \right) \tag{156}$$

gives the reduced symplectic manifold

$$\mathbf{J}^{-1}_{^LT^*T^*G}(\nu)/G_\nu = \mathcal{O}^L_\nu \times \mathfrak{g}^* \times \mathfrak{g}$$

and the reduced symplectic two-form

$$\langle \Omega^{/G}_{^LT^*T^*G}; \left((\eta^{\mathfrak{g}^*}, \lambda, \zeta), (\bar\eta^{\mathfrak{g}^*}, \bar\lambda, \bar\zeta) \right) \rangle = \langle \bar\lambda, \zeta \rangle - \langle \lambda, \bar\zeta \rangle - \langle \nu, [\eta, \bar\eta]^L_{\mathfrak{g}} \rangle$$

for $(\eta_{\mathfrak{g}^*}, \lambda, \zeta), (\bar\eta_{\mathfrak{g}^*}, \bar\lambda, \bar\zeta) \in T_\nu \mathcal{O}^L_\nu \times \mathfrak{g}^* \times \mathfrak{g}$.

On $^LTT^*G$, at the point (g,μ,ξ,ν), a left invariant $X^{^LTT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}$ vector field is

$$X^{^LTT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} = \left(TL_g\xi_2, \nu_2 + ad^{L*}_{\xi_2}\mu, \xi_3 + [\xi,\xi_2]^L_{\mathfrak{g}}, \nu_3 + ad^{L*}_{\xi_2}\nu - ad^{L*}_{\xi}\nu_2 \right).$$

Tulczyjew's symplectic two-from $\Omega \,_{^LTT^*G}$ takes the value

$$\langle \Omega \,_{^LTT^*G}; (X^{^LTT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)}, X^{^LTT^*G}_{(\bar\xi_2,\bar\nu_2,\bar\xi_3,\bar\nu_3)}) \rangle = \langle \nu_3, \bar\xi_2 \rangle + \langle \nu_2, \bar\xi_3 \rangle - \langle \bar\nu_2, \xi_3 \rangle - \langle \bar\nu_3, \xi_2 \rangle$$
$$+ \langle \mu, [\xi_3, \bar\xi_2]^L_{\mathfrak{g}} - [\bar\xi_3, \xi_2]^L_{\mathfrak{g}} \rangle + \langle \mu, \left[[\bar\xi_2, \xi_2]^L_{\mathfrak{g}}, \xi \right]^L_{\mathfrak{g}} \rangle - \langle \nu, [\bar\xi_2, \xi_2] \rangle \,.$$

The symplectic from $\Omega \,_{^LTT^*G}$ has two potential one-forms $^L\theta_1$ and $^L\theta_2$ whose values over a left invariant vector field, at the point (g,μ,ξ,ν), are

$$\left\langle \,^L\theta_1, X^{^LTT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} \right\rangle = \langle \nu, \xi_2 \rangle - \langle \nu_2, \xi \rangle + \langle \mu, [\xi, \xi_2] \rangle \tag{157}$$

$$\left\langle \,^L\theta_2, X^{^LTT^*G}_{(\xi_2,\nu_2,\xi_3,\nu_3)} \right\rangle = \langle \nu, \xi_2 \rangle + \langle \mu, \xi_3 \rangle + \langle \mu, [\xi, \xi_2] \rangle \,. \tag{158}$$

The right action of G on $^LTT^*G$

$$^LTT^*G \times G \to \,^LTT^*G : ((g,\mu,\xi,\nu)\,;h) \to \left(gh, \lambda + Ad^{L*}_{h^{-1}}\mu, Ad^L_{h^{-1}}\xi, Ad^{L*}_{h^{-1}}\nu \right)$$

is Hamiltonian action with infinitesimal generators in form $X_{(\xi_2,0,0,0)}^{{}^{L}TT^*G}$. The momentum mapping is

$$\mathbf{J}_{{}^{L}TT^*G} : (g,\mu,\xi,\nu) \rightarrow \nu + ad_{\xi}^{L*}\mu = \lambda.$$

The symplectic reduction theorem gives that

$$\left({}^{L}TT^*G, \Omega_{{}^{L}TT^*G} \right) \rightarrow \left(\mathbf{J}_{{}^{L}TT^*G}^{-1}(\lambda)/G_\lambda, \Omega_{{}^{L}TT^*G}^{/G} \right) \tag{159}$$

where

$$\mathbf{J}_{{}^{L}TT^*G}^{-1}(\lambda)/G_\lambda \ \simeq \ \mathcal{O}_\lambda^L \times \mathfrak{g}^* \times \mathfrak{g}$$
$$\left\langle \Omega_{{}^{L}TT^*G}^{/G}; \left(\left(\eta^{\mathfrak{g}^*},\nu,\zeta\right), \left(\bar{\eta}^{\mathfrak{g}^*},\bar{\nu},\bar{\zeta}\right) \right) \right\rangle \ = \ \left\langle \nu,\bar{\zeta} \right\rangle - \left\langle \bar{\nu},\zeta \right\rangle - \left\langle \lambda, \left[\zeta,\bar{\zeta}\right] \right\rangle$$

for $(\eta^{\mathfrak{g}^*},\nu,\zeta), \left(\bar{\eta}^{\mathfrak{g}^*},\bar{\nu},\bar{\zeta}\right) \in T_\lambda \mathcal{O}_\lambda^L \times \mathfrak{g}^* \times \mathfrak{g}$.

The following proposition summarizes the discussion done so far and determines the left trivialization of Tulczyjew's triplet.

Proposition B.1. ${}^{L}TT^*G$ *admits two trivialized special symplectic structures*

$$\left({}^{L}TT^*G, \ {}^{L}\tau_{G\circledS_L\mathfrak{g}^*}, \ {}^{L}T^*T^*G, \ {}^{L}\theta_1, \ {}^{L}\Omega_{G\circledS_L\mathfrak{g}^*}^\flat \right)$$
$$\left({}^{L}TT^*G, \ {}^{L}T\pi_G, \ {}^{L}T^*TG, \ {}^{L}\theta_2, \ {}^{L}\bar{\sigma}_G^L \right).$$

Here ${}^{L}\theta_1$ *and* ${}^{L}\theta_2$ *are the potential one-froms in Eqs.(157) and (158),* ${}^{L}\Omega_{G\circledS_L\mathfrak{g}^*}^\flat$ *and* ${}^{L}\bar{\sigma}_G^L$ *are symplectomorphisms in Eqs.(152) and (150), and the projections are*

$$\begin{aligned} {}^{L}\tau_{G\circledS_L\mathfrak{g}^*} \ &: \ {}^{L}TT^*G \rightarrow G\circledS_L\mathfrak{g}^* : (g,\mu,\xi,\nu) \rightarrow (g,\mu) \\ {}^{L}T\pi_G \ &: \ {}^{L}TT^*G \rightarrow G\circledS_L\mathfrak{g} : (g,\mu,\xi,\nu) \rightarrow (g,\xi). \end{aligned}$$

After applying symplectic reduction to the triplet due to the right Hamiltonian actions of G, we arrive the reduced Tulczyjew's triplet.

Proposition B.2.   *By reduction, we obtain trivializations*

$$\begin{aligned} \left({}^{L}\bar{\sigma}_G^L \right)^{/G} \ &: \ \left(\mathcal{O}_\lambda^L \times \mathfrak{g}^* \times \mathfrak{g}, \Omega_{{}^{L}TT^*G}^{/G} \right) \rightarrow \left(\mathcal{O}_\lambda^L \times \mathfrak{g} \times \mathfrak{g}^*, \Omega_{{}^{L}T^*TG}^{/G} \right) \\ &: \ \left(Ad_{g^{-1}}^{L*}\lambda, \mu, \xi \right) \rightarrow \left(Ad_{g^{-1}}^{L*}\lambda, \xi, \mu \right) \\ \left({}^{L}\Omega_{G\circledS_L\mathfrak{g}^*}^\flat \right)^{/G} \ &: \ \left(\mathcal{O}_\lambda^L \times \mathfrak{g}^* \times \mathfrak{g}, \Omega_{{}^{L}TT^*G}^{/G} \right) \rightarrow \left(\mathcal{O}_\lambda^L \times \mathfrak{g}^* \times \mathfrak{g}, \Omega_{{}^{L}T^*T^*G}^{/G} \right) \\ &: \ \left(Ad_{g^{-1}}^{L*}\lambda, \mu, \xi \right) \rightarrow \left(Ad_{g^{-1}}^{L*}\lambda, \mu, -\xi \right) \end{aligned}$$

of ${}^{L}\bar{\sigma}_G^L$ *and* ${}^{L}\Omega_{G\circledS_L\mathfrak{g}^*}^\flat$ *both of which are symplectic diffeomorphisms.*

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