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Product Formulas for a Two-Parameter Family of Heckman-Opdam Hypergeometric Functions of Type BC

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Abstract. In this paper we present explicit product formulas for a continuous two-parameter family of Heckman-Opdam hypergeometric functions of type BC on Weyl chambers $C_q \subset \mathbb{R}^q$ of type B. These formulas are related to continuous one-parameter families of probability-preserving convolution structures on $C_q \times \mathbb{R}$. These convolutions on $C_q \times \mathbb{R}$ are constructed via product formulas for the spherical functions of the symmetric spaces $U(p,q)/(U(p) \times SU(q))$ and associated double coset convolutions on $C_q \times \mathbb{T}$ with the torus $\mathbb{T}$. We shall obtain positive product formulas for a restricted parameter set only, while the associated convolutions are always norm-decreasing.

Our paper is related to recent positive product formulas of Rösler for three series of Heckman-Opdam hypergeometric functions of type BC as well as to classical product formulas for Jacobi functions of Koornwinder and Trimeche for rank $q = 1$.
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1. Introduction

It is well-known by the work of Heckman and Opdam ([H], [HS], [O1], [O2]) that the spherical functions on the Grassmann manifolds of rank $q \geq 1$ over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ may be regarded as Heckman-Opdam hypergeometric functions F_{BC_q} of type BC on the closed Weyl chambers

$$C_q := \{t = (t_1, \cdots, t_q) \in \mathbb{R}^q : \; t_1 \geq t_2 \geq \cdots \geq t_q \geq 0\}$$

of type B. The associated product formulas for the spherical functions were stated by Rösler [R3] for the dimensions $p \geq 2q$ in a form such that these formulas can be extended by some principle of analytic continuation to all parameters $p \in \mathbb{R}$ with $p > 2q - 1$. In this way, Rösler [R3] obtained three continuous series of product formulas for F_{BC_q} (for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$) as well as associated

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commutative, probability-preserving convolution algebras of measures on C_q, so-called commutative hypergroups. For the theory of hypergroups we refer to [J] and [BH].

In this paper we start with the symmetric spaces $U(p,q)/(U(p)\times SU(q))$ for $\mathbb{F}=\mathbb{C}$ and $p\ge 2q$. Here the spherical functions can be regarded as functions on $C_q\times\mathbb{T}$ with the torus $\mathbb{T}:=\{z\in\mathbb{C}:\ |z|=1\}$. These functions can be expressed in terms of K-spherical functions of the Hermitian symmetric spaces $U(p,q)/(U(p)\times U(q))$ and thus also in terms of the functions F_{BC_q} depending on the integer multiplicity parameter $p\ge 2q$ and some spectral parameter $l\in\mathbb{Z}$; see [Sh], [HS]. Following [R3], we shall write down product formulas for the spherical functions of $U(p,q)/(U(p)\times SU(q))$ in Section 2 of this paper as product formulas on $C_q\times\mathbb{T}$ for integers $p\ge 2q$. We then use these formulas in Section 3 to construct associated product formulas on the universal covering $C_q\times\mathbb{R}$ of $C_q\times\mathbb{T}$ for a class of functions which are defined in terms of the functions F_{BC_q} where the F_{BC_q} depend now on two continuous multiplicity parameters $p\ge 2q-1$ and $l\in\mathbb{R}$. Here, the extension from integers $p\ge 2q$ and $l\in\mathbb{Z}$ to real numbers $p>2q-1$ and $l\in\mathbb{R}$ is carried out by Carleson's theorem, a principle of analytic continuation. The degenerated limit case $p=2q-1$ then follows by continuity. We shall also see in Section 4 that these product formulas on $C_q\times\mathbb{R}$ for $p\ge 2q-1$ lead to commutative hypergroup structures $(C_q\times\mathbb{R},*_p)$. We shall also derive the Haar measures of these hypergroups.

The product formulas on $C_q\times\mathbb{R}$ and the associated hypergroup structures in Sections 3 and 4 form the basis to derive a lot of further product formulas and convolution algebras by taking suitable quotients. In particular, we immediately obtain extensions of the product formulas on $C_q\times\mathbb{T}$ for the spherical functions of $U(p,q)/(U(p)\times SU(q))$ with integers $p\ge 2q$ to real parameters $p\ge 2q-1$. Moreover, we recover Rösler's formulas in [R3] on C_q for $\mathbb{F}=\mathbb{C}$. More generally, we obtain explicit product formulas and convolution structures on C_q for all hypergeometric functions F_{BC_q} with multiplicities

$$k=(k_1,k_2,k_3)=(p-q-l,1/2+l,1)$$

with real parameters $p\ge 2q-1$ and $l\in\mathbb{R}$. It will turn out in Section 5 that these formulas lead to probability-preserving convolutions and thus classical commutative hypergroups for $|l|\le 1/q$ while for arbitrary $l\in\mathbb{R}$ the positivity of the product formulas remain open. On the other hand we shall see in Section 6 that the product formulas for F_{BC_q} lead for all $l\in\mathbb{R}$ at least to norm-decreasing convolution algebras which are associated with certain so-called signed hypergroup structures on C_q. For this notion we refer to [R1], [RV1], and references cited there.

We also notice that for rank $q=1$, our results are closely related with the classical work of Flensted-Jensen and Koornwinder ([F], [FK], [K]) on Jacobi functions and to the convolutions of Trimeche [Tr] on the space $\{z\in\mathbb{C}:\ |z|\ge 1\}$ which is homeomorphic with $[0,\infty[\times\mathbb{T}$.

Before starting with the analysis of $(U(p,q)/(U(p)\times SU(q))$ in Section 2, we recapitulate some notions and facts. For integers $p>q\ge 1$ consider the Grassmann manifolds G/K over $\mathbb{F}=\mathbb{R},\mathbb{C},\mathbb{H}$ with

$$G=SO_0(p,q),\ SU(p,q),\ Sp(p,q)$$

and the maximal compact subgroups

$$K = SO(p) \times SO(q),\ S(U(p) \times U(q)),\ Sp(p) \times Sp(q),$$

respectively. By the well-known KAK decomposition of G, a system of representatives of the K-double cosets on G is given by the matrices

$$a_t = \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & \cosh \underline{t} & \sinh \underline{t} \\ 0 & \sinh \underline{t} & \cosh \underline{t} \end{pmatrix} \tag{1.1}$$

with t in the closed Weyl chamber C_q where $\cosh \underline{t}$, $\sinh \underline{t}$ are the $q \times q$ diagonal matrices

$$\cosh \underline{t} := diag(\cosh t_1, \ldots, \cosh t_q),\ \ \sinh \underline{t} := diag(\sinh t_1, \ldots, \sinh t_q).$$

Therefore, continuous K-biinvariant functions on G are in a natural one-to-one correspondence with continuous functions on C_q. Moreover, by the theory of Heckman and Opdam [H], [HS], [O1], [O2], in this way the spherical functions of (G, K), i.e., the continuous, K-biinvariant functions $\varphi \in C(G)$ satisfying the product formula

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\, dk \quad (g, h \in G,\ dk \text{ the normalized Haar measure of } K), \tag{1.2}$$

are precisely the Heckman-Opdam hypergeometric functions

$$t \mapsto F(\lambda, k; t) := F_{BC_q}(\lambda, k; t) \qquad (t \in C_q)$$

of type BC with $\lambda \in \mathbb{C}^q$ and the multiplicity parameter

$$k = (k_1, k_2, k_3) = (d(p-q)/2, (d-1)/2, d/2)$$

associated with the roots $\pm 2e_i$, $\pm 4e_i$, and $2(\pm e_i \pm e_j)$ of the root system $2 \cdot BC_q$ in the notation of Heckman and Opdam. Here, $d \in \{1, 2, 4\}$ is the dimension of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ over $\mathbb{R}$; see Remark 2.3 of [H] and also [R3].

For given $q \geq 1$ and d, Rösler [R3] derived the product formula (1.2) explicitly as a product formula for the corresponding F_{BC_q} on C_q depending on $p \geq 2q$ such that this formula can be extended to a product formula on C_q for arbitrary real parameters $p > 2q - 1$ by analytic continuation. Moreover, for each real parameter $p > 2q - 1$, each of these product formulas gives rise to a commutative hypergroup structure on C_q, i.e., a probability preserving commutative Banach-*-algebra structure on the Banach space of all bounded signed Borel measures on C_q with total variation norm; see [BH] and [J] for the theory of hypergroups.

We now modify the approach of [R3] for $\mathbb{F} = \mathbb{C}$, i.e. $d = 2$, by considering K-spherical functions according to Ch. I.5 of [HS] and [Sh] as follows: Take the Gelfand pair $(G, \tilde{K}) := (U(p, q), U(p) \times SU(q))$ for $p \geq q$ as well as the maximal compact subgroup $K := U(p) \times U(q) \subset K$. Then

$$G/K := U(p, q)/(U(p) \times U(q)) \equiv SU(p, q)/S(U(p) \times U(q))$$

is a Hermitian symmetric space. It is well-known that the usual spherical functions of $(G,\tilde{K})$ (i.e., satisfying (1.2)) are in a natural way in a one-to-one correspondence with the so-called K-spherical functions on G of type l with $l\in\mathbb{Z}$. Recapitulate that these K-spherical functions of type l are defined as continuous functions $\varphi\in C(G)$ with $\varphi(e)=1$ satisfying twisted invariance conditions as well as twisted product formulas associated with the characters

$$\chi_l\left(\begin{pmatrix} u & 0\\ 0 & v\end{pmatrix}\right) := (\Delta v)^l \qquad (l\in\mathbb{Z}) \tag{1.3}$$

of K, where Δv stands for the determinant of v. The twisted invariance condition reads as

$$\varphi\left(k_1 g k_2\right) = \chi_l(k_1k_2)^{-1}\cdot\varphi(g) \quad \text{for} \quad g\in G, k_1,k_2\in K, \tag{1.4}$$

and the twisted product formula as

$$\varphi(g)\varphi(h) = \int_{S(U(p)\times U(q))} \chi_l(k)\varphi(gkh)\, dk \qquad (g,h\in G). \tag{1.5}$$

These spherical functions φ of type l can be also characterized as eigenfunctions of some algebra $\mathbb{D}(\chi_l)$ of $SU(p,q)$-invariant differential operators in Section 5.1 of [HS] and [Sh] and can be written down therefore explicitly in terms of F_{BC_q}. In particular, if $G//\tilde{K}$ is identified with $C_q\times\mathbb{T}$, we conclude from Theorem 5.2.2 of [HS] that the spherical functions may be regarded as the functions

$$(t,z)\longmapsto z^l\prod_{j=1}^{q} cosh^l t_j\cdot F_{C_q}(\lambda, k(p,l);t) \qquad (\lambda\in\mathbb{C}^q,\ l\in\mathbb{Z}) \tag{1.6}$$

with the multiplicities

$$k(p,l) = (k_1(p,l),k_2(p,l),k_3(p,l)) = (p-q-l, 1/2+l, 1). \tag{1.7}$$

We shall use this characterization in the next section to derive an explicit product formula for these functions.

We finally point out that the approach of this paper is restricted to $\mathbb{F}=\mathbb{C}$ and thus to the multiplicity $k_3=1$. In fact, for $\mathbb{F}=\mathbb{R}$ with $k_3=1/2$, our approach does not seem to lead to product formulas for F_{BC_q} beyond the cases considered in [R3]. The case $\mathbb{F}=\mathbb{H}$ with $k_3=1/2$ is interesting, but more complicated than $\mathbb{F}=\mathbb{C}$. For the rank one case over $\mathbb{H}$ we refer to [Ta].

It is a pleasure to thank Tom Koornwinder for some essential hints to details in the monograph [HS] as well Margit Rösler for many fruitful discussions.

2. Spherical functions of $(U(p,q), U(p)\times SU(q))$ and their product formula

In this section we derive an explicit product formula for the spherical functions for the Gelfand pair $(G,\tilde{K}) := (U(p,q), U(p)\times SU(q))$. In fact, this is a Gelfand pair by standard criteria, see e.g. Corollary 1.5.4 of [GV]. Moreover, this is also a direct consequence of the explicit convolution (2.9) below. We first identify the double coset space $G//\tilde{K}$ with the direct product $C_q\times\mathbb{T}$ of the Weyl chamber C_q and the torus $\mathbb{T}$. This can be done similar to well known case of the Hermitian symmetric space $G/K := U(p,q)/(U(p)\times U(q))$ where, by the KAK-decomposition, a system of representatives of the K-double cosets in G is given by the matrices a_t of Eq. (1.1) with $t\in C_q$.

Lemma 2.1. *A system of representatives of the $\tilde{K}$-double cosets in G is given by the matrices*

$$a_{t,z} = \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & r_q(z)^{-1}\cdot\cosh \underline{t} & \sinh \underline{t} \\ 0 & \sinh \underline{t} & r_q(z)\cdot\cosh \underline{t} \end{pmatrix} \tag{2.1}$$

for $t\in C_q$ and $z\in\mathbb{T}$ where the mapping $r_q:\mathbb{T}\to\mathbb{T}$ is the q-th root on $\mathbb{T}$ with $r_q(e^{it}) := e^{it/q}$ with $t\in[0,2\pi[$.

Proof. We first check that each double coset has a representative of the form $a_{t,z}$. In fact, by the well known KAK-decomposition of G, each $g\in G$ has the form

$$g = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix} \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & \cosh \underline{t} & \sinh \underline{t} \\ 0 & \sinh \underline{t} & \cosh \underline{t} \end{pmatrix} \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}$$

with $t\in C_q$, $u_1,u_2\in U(p)$, $v_1,v_2\in U(q)$. The matrices $v_k\in U(q)$ $(k=1,2)$ can be written as $v_k = z_k\cdot\tilde{v}_k$ with $\tilde{v}_k\in SU(q)$ and $z_k = r_q(\Delta(v_k))\in\mathbb{T}$. Therefore, defining

$$\tilde{u}_1 := u_1\begin{pmatrix} I_{p-q} & 0 \\ 0 & z_2 I_q \end{pmatrix} \qquad\text{and}\qquad \tilde{u}_2 := \begin{pmatrix} I_{p-q} & 0 \\ 0 & z_1 I_q \end{pmatrix} u_2,$$

we obtain

$$g = \begin{pmatrix} \tilde{u}_1 & 0 \\ 0 & \tilde{v}_1 \end{pmatrix} \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & (z_1z_2)^{-1}\cosh \underline{t} & \sinh \underline{t} \\ 0 & \sinh \underline{t} & z_1z_2\cosh \underline{t} \end{pmatrix} \begin{pmatrix} \tilde{u}_2 & 0 \\ 0 & \tilde{v}_2 \end{pmatrix}, \tag{2.2}$$

which shows that each double coset has a representative of the form

$$\tilde{a}_{t,z} := \begin{pmatrix} I_{p-q} & 0 & 0 \\ 0 & z^{-1}\cosh \underline{t} & \sinh \underline{t} \\ 0 & \sinh \underline{t} & z\cosh \underline{t} \end{pmatrix} \tag{2.3}$$

with $z\in\mathbb{T}$. Moreover, this computation also shows that for all q-th roots of unity $z_0\in\mathbb{T}$, the matrices $\tilde{a}_{t,z}$ and $\tilde{a}_{t,zz_0}$ are contained in the same $\tilde{K}$-double coset, i.e., each double coset has a representative of the form $\tilde{a}_{t,z}$ with $z=e^{i\theta}$, $\theta\in[0,2\pi/q[$, as claimed.

In order to show that the $a_{t,z}$ are contained in different double cosets for different (t,z), we briefly discuss how the parameters t,z of a double coset of an arbitrary group element $g\in G$ can be constructed explicitly. This will be important also later on for the convolution. We write any $g \in G$ in $(p \times q)$-block notation as

$$g = \begin{pmatrix} A(g) & B(g) \\ C(g) & D(g) \end{pmatrix}.$$

Moreover, in $(p \times q)$-block notation, we write

$$a_{t,z} = \begin{pmatrix} A_{t,z} & B_{t,z} \\ C_{t,z} & D_{t,z} \end{pmatrix}.$$

Assume now that $g \in G$ has the form (2.2) with $z_1 z_2 = r_q(z)$ with $z = e^{i\theta}$, $\theta \in [0, 2\pi/q[$. Then $D(g)$ has the form

$$D(g) = r_q(z) \cdot \tilde{v}_1 \, \cosh \underline{t} \, \tilde{v}_2 \tag{2.4}$$

with $\tilde{v}_1, \tilde{v}_2 \in SU(q)$. We now consider the singular spectrum

$$\sigma_{sing} : M^{q,q}(\mathbb{C}) \to C_q, \quad \sigma_{sing}(a) := \sqrt{spec(a^* a)} \in \mathbb{R}^q$$

where the singular values are ordered by size. We also consider the map arg, $\mathbb{C}^\times \to \mathbb{T}$, $arg(z) := z/|z|$. Then, by (2.4),

$$t = \operatorname{arcosh}\left(\sigma_{sing}(D(g))\right) \quad \text{in all components} \qquad \text{and} \qquad z = arg(\Delta(D(g))). \tag{2.5}$$

This completes the proof of the lemma. ■

We proceed with the notations of the second part of the proof of the lemma and write the general product formula (1.2) for spherical functions as a product formula on the parameter space $C_q \times \mathbb{T}$. For this, we take $t, s \in C_q$ and $z_1, z_2 \in \mathbb{T}$ and evaluate the integral

$$\int_{\tilde{K}} f(a_{t,z_1} k a_{s,z_2}) \, dk \qquad \text{with} \qquad k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}.$$

We have

$$a_{t,z_1} k a_{s,z_2} = \begin{pmatrix} * & * \\ * & C_{t,z_1} \, u \, B_{s,z_2} + D_{t,z_1} \, v \, D_{s,z_2} \end{pmatrix}$$

and thus $D(a_{t,z_1} \, k \, a_{s,z_2}) =$

$$C_{t,z_1} \, u \, B_{s,z_2} + D_{t,z_1} \, v \, D_{s,z_2} = (0, \sinh \underline{t}) \, u \begin{pmatrix} 0 \\ \sinh \underline{s} \end{pmatrix} + r_q(z_1 z_2) \, \cosh \underline{t} \, v \, \cosh \underline{s}.$$

With the block matrix

$$\sigma_0 := \begin{pmatrix} 0 \\ I_q \end{pmatrix} \in M_{p,q}(\mathbb{C}) \tag{2.6}$$

this can be written as

$$D(a_{t,z} \, k \, a_{s,w}) = \sinh \underline{t} \, \sigma_0^* u \sigma_0 \sinh \underline{s} \, + \, r_q(z_1 z_2) \, \cosh \underline{t} \, v \, \cosh \underline{s}.$$

Therefore, if we regard a $\tilde{K}$-biinvariant function $f \in C(G)$ as a continuous function on $C_q \times \mathbb{T}$,

$$\int_{\tilde{K}} f(a_{t,z_1} k a_{s,z_2})\, dk =$$
$$= \int_{\tilde{K}} f\left(\operatorname{arcosh}\left(\sigma_{sing}(D(a_{t,z_1}\, k\, a_{s,z_2}))\right), arg(\Delta(D(a_{t,z_1}\, k\, a_{s,z_2})))\right)$$
$$= \int_{U(p)} \int_{SU(q)} f\Big(\operatorname{arcosh}\left(\sigma_{sing}(\sinh \underline{t}\, \sigma_0^* u \sigma_0 \sinh \underline{s} + r_q(z_1 z_2) \cosh \underline{t}\, v \cosh \underline{s})\right),$$
$$arg(\Delta(\sinh \underline{t}\, \sigma_0^* u \sigma_0 \sinh \underline{s} + r_q(z_1 z_2) \cosh \underline{t}\, v \cosh \underline{s}))\Big)\, dv\, dw.$$

Notice that $\sigma_0^* u \sigma_0 \in M_q(\mathbb{F})$ is just the lower right $q \times q$-block of σ and is contained in the ball

$$B_q := \{w \in M^{q,q}(\mathbb{C}) : w^* w \leq I_q\},$$

where $w^* w \leq I_q$ means that $I_q - w^* w$ is positive semidefinite. In order to reduce the $U(p)$-integration, we use Lemma 2.1 of [R3] and obtain that for $p \geq 2q$ the integral above is equal to

$$\frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} f\Big(\operatorname{arcosh}\left(\sigma_{sing}(\sinh \underline{t}\, w \sinh \underline{s} + r_q(z_1 z_2) \cosh \underline{t}\, v \cosh \underline{s})\right), \tag{2.7}$$
$$arg(\Delta(\sinh \underline{t}\, w \sinh \underline{s} + r_q(z_1 z_2) \cosh \underline{t}\, v \cosh \underline{s}))\Big) \cdot \Delta(I_q - w^* w)^{p-2q}\, dv\, dw.$$

with

$$\kappa_p := \int_{B_q} \Delta(I_q - w^* w)^{p-2q}\, dw, \tag{2.8}$$

and where dw means integration w.r.t. the Lebesgue measure. After substitution $w \mapsto r_q(z_1 z_2) w$, we arrive at the following explicit product formula:

Proposition 2.2. *Let $p \geq 2q$. If a $\tilde{K}$-spherical function $\varphi \in C(G)$ is regarded as a continuous function on $C_q \times \mathbb{T}$ as described above, then the associated product formula for spherical functions φ has the following form on $C_q \times \mathbb{T}$:*

$$\varphi(t, z_1) \varphi(s, z_2) = \tag{2.9}$$
$$= \frac{1}{\kappa_p} \int_{B_q} \int_{SU(q)} \varphi\Big(\operatorname{arcosh}\left(\sigma_{sing}(\sinh \underline{t}\, w \sinh \underline{s} + \cosh \underline{t}\, v \cosh \underline{s})\right),$$
$$z_1 z_2 \cdot arg(\Delta(\sinh \underline{t}\, w \sinh \underline{s} + \cosh \underline{t}\, v \cosh \underline{s}))\Big) \cdot \Delta(I_q - w^* w)^{p-2q}\, dv\, dw.$$

Notice that for $p \leq 2q - 1$, the integral over the ball B_q in (2.9) does not exist and that here $\kappa_p = \infty$. However, for $p = 2q - 1$, a degenerated version of formula may be written down explicitly similar to other series of symmetric spaces of higher rank associated with motion groups or Heisenberg groups; see Section 3 of [R2] and Remark 2.14 of [V2]. We shall present a degenerated version of (2.9) in the end of Section 3.

In the case of rank $q = 1$, the space $C_q \times \mathbb{T} = [0, \infty[\times \mathbb{T}$ may be identified with the exterior $Z := \{z \in \mathbb{C} : |z| \geq 1\}$ of the unit disk via polar coordinates.

The associated convolution (2.9) in the general case $p \geq 2q$ and in the degenerated case $p = 2q$ was computed in this case by Trimeche [Tr].

We next turn to the classification of all $\tilde{K}$-spherical functions $\varphi \in C(G)$. In order to translate it into a standard version in the Heckman-Opdam theory, we recapitulate the following result:

Lemma 2.3. *For $\varphi \in C(G)$ the following properties are equivalent:*

(1) *φ is $\tilde{K}$-spherical, i.e., $\tilde{K}$-biinvariant with $\varphi(g)\varphi(h) = \int_{\tilde{K}} \varphi(gkh)\,dk$ for $g, h \in G$.*

(2) *$\varphi(e) = 1$, and there exists a unique $l \in \mathbb{Z}$ such that*

$$\varphi(k_1 g k_2) = \chi_l(k_1 k_2)^{-1} \cdot \varphi(g) \qquad \text{for} \qquad g \in G,\ k_1, k_2 \in K \tag{2.10}$$

and

$$\varphi(g)\varphi(h) = \int_K \varphi(gkh)\chi_l(k)\,dk \qquad \text{for} \qquad g, h \in G. \tag{2.11}$$

(3) *φ is a spherical function of type χ_l for some $l \in \mathbb{Z}$ in the sense of Definition 5.2.1 of Heckman [HS], i.e., φ is an eigenfunction with respect to all members of a certain algebra of G-invariant differential operators.*

Proof. For abbreviation define the diagonal matrix $d_z := \begin{pmatrix} I_p & 0 \\ 0 & zI_q \end{pmatrix} \in K$ for $z \in \mathbb{T}$.

For (1) $\Longrightarrow$ (2) consider a $\tilde{K}$-spherical function φ. Then $\varphi|_K$ is $\tilde{K}$-spherical, and as $K/\tilde{K} \equiv \mathbb{T}$, we find a unique $l \in \mathbb{Z}$ with $\varphi|_K = \chi_l^{-1}$. Moreover, as $\varphi(kg) = \varphi(k)\varphi(g) = \varphi(gk)$ for $g \in G$ and $k \in K$, (2.10) is clear. For (2.11), we use the normalized Haar measure dz on $\mathbb{T}$ and observe that for $g, h \in G$,

$$\begin{aligned}\int_K \varphi(gkh) \cdot \chi_l(k)\,dk &= \int_{\mathbb{T}} \int_{\tilde{K}} \varphi\left(g d_z k h\right) \chi_l\left(d_z k\right)\,dk\,dz = \int_{\mathbb{T}} \varphi\left(g d_z\right) \varphi(h) \chi_l\left(d_z\right)\,dz \\ &= \varphi(g)\varphi(h) \int_{\mathbb{T}} \varphi\left(d_z\right) \chi_l\left(d_z\right)\,dz \;=\; \varphi(g)\varphi(h)\end{aligned}$$

as claimed.

For (2) $\Longrightarrow$ (1) consider φ as in (2). Then by (2.10), $\varphi|_K = \chi_l^{-1}$, and φ is $\tilde{K}$-biinvariant. Moreover, for $g, h \in G$ and $z \in \mathbb{T}$, $\tilde{K} g d_z \tilde{K} = \tilde{K} d_z g \tilde{K}$, and thus

$$\begin{aligned}\varphi(g)\varphi(h) &= \int_K \varphi(gkh) \cdot \chi_l(k)\,dk = \int_{\mathbb{T}} \int_{\tilde{K}} \varphi\left(g d_z k h\right) \chi_l\left(d_z k\right)\,dk\,dz \\ &= \int_{\mathbb{T}} \int_{\tilde{K}} \varphi\left(d_z g k h\right) \chi_l\left(d_z\right)\,dk\,dz = \int_{\tilde{K}} \varphi\left(gkh\right)\,dk\end{aligned}$$

as claimed.

The equivalence of (2) and (3) is already mentioned in Section 5.2 of Heckman [HS] and can be checked in the same way as for classical spherical functions as it is carried out e.g. in Section IV.2 of Helgason [Hel]. ■

The elementary spherical functions φ on G of type χ_l for $l \in \mathbb{Z}$ are classified by Heckman in Section 5 of [HS]. For a description, we consider the root system $R := 2 \cdot BC_q$ with the positive roots

$$R_+ := \{2e_i, 4e_i : \ i = 1, \ldots, q\} \cup \{2(e_i - e_j) : \ 1 \le i < j \le q\}$$

as well as the associated Heckman-Opdam hypergeometric functions according to [H], [HS], [O1], [O2] which we denote by $F_{BC_q}(\lambda, k; t)$ with $\lambda \in \mathbb{C}^q$, $t \in C_q$, and with multiplicity $k = (k_1, k_2, k_3)$ where the k_i belong to the roots as in the ordering of R_+ above. Note that the Heckman-Opdam hypergeometric functions $F_{BC_q}(\lambda, k; t)$ exist for all $\lambda \in \mathbb{C}^q$, $t \in C_q$ whenever the multiplicity k is contained in some open regular subset $K^{reg} \subset \mathbb{C}^3$. It is well-known (see Remark 4.4.3 of [HS]) that, in our notation,

$$\{k = (k_1, k_2, k_3) : \ Re\, k_3 \ge 0, \ Re\, k_1 + k_2 \ge 0\} \subset K^{reg}. \tag{2.12}$$

Taking Lemma 2.3 into account, we obtain the following known classification from Theorem 5.2.2 of [HS]:

Theorem 2.4. *If $\tilde{K}$-spherical functions on G are regarded as functions on $C_q \times \mathbb{T}$ as above, then the $\tilde{K}$-spherical functions on G are given precisely by*

$$\varphi^p_{\lambda,l}(t,z) = z^l \cdot \prod_{j=1}^q \cosh^l t_j \cdot F_{BC_q}(i\lambda, k(p,q,l); t)$$

with $\lambda \in \mathbb{C}^q$, $l \in \mathbb{Z}$, and the multiplicity

$$k(p,q,l) = (p - q - l, \ \frac{1}{2} + l, \ 1) \in K^{reg}.$$

Example 2.5. For $q = 1$, the parameter k_3 is irrelevant and usually suppressed. If one compares the one-dimensional example of Heckman-Opdam functions on p. 89f of [O1] with the classical definition of the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ with $\alpha = k_1 + k_2 - 1/2$, $\beta = k_2 - 1/2$ e.g. in [K], one obtains

$$F_{BC_1}(i\lambda, k; t) = \varphi_\lambda^{(\alpha,\beta)}(t);$$

see also Example 3.4 in [R3]. Therefore, by Theorem 2.4, the $U(p)$-spherical functions on $U(p,1)$ are given by

$$\varphi^p_{\lambda,l}(t,z) = z^l \cdot \cosh^l t \cdot \varphi_\lambda^{(p-1,l)}(t) \qquad (t \ge 0, z \in \mathbb{T})$$

with $l \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$. This is a classical result of Flensted-Jensen [F].

Let us summarize the results above. For integers $q \ge 2q$, we obtain from Proposition 2.2 that the double coset convolution of point measures on the double coset space

$$U(p,q)//(U(p) \times SU(q)) \simeq C_q \times \mathbb{T}$$

is given by

$$(\delta_{(s,z_1)} *_p \delta_{(t,z_2)})(f) := \tag{2.13}$$
$$= \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f\Big(d(t,s;v,w), z_1z_2\cdot h(t,s;v,w)\Big)\cdot\Delta(I_q - w^*w)^{p-2q}\, dv\, dw$$

for $s,t \in C_q$, $z_1, z_2 \in \mathbb{T}$ and all bounded continuous functions $f \in C_b(C_q \times \mathbb{T})$ with the abbreviations

$$d(t,s;v,w) := (d_j(t,s;v,w))_{j=1,\ldots,q}$$
$$:= \text{arcosh}\,(\sigma_{sing}(\sinh \underline{t}\, w\, \sinh \underline{s} + \cosh \underline{t}\, v\, \cosh \underline{s})) \in C_q \tag{2.14}$$

and

$$h(t,s;v,w) := \Delta(\sinh \underline{t}\, w\, \sinh \underline{s} + \cosh \underline{t}\, v\, \cosh \underline{s}). \tag{2.15}$$

It is well known (see e.g. [J]) that this double coset convolution uniquely extends to a bilinear, weakly continuous convolution on the Banach space $M_b(C_q \times \mathbb{T})$ of all bounded regular Borel measures on $C_q \times \mathbb{T}$, and that $(C_q \times \mathbb{T}, *_p)$ forms a commutative hypergroup. Moreover, the functions $\varphi^p_{\lambda,l}$ $(\lambda \in \mathbb{C}^q, l \in \mathbb{Z})$, by Theorem 2.4 form the set of all multiplicative functions on these hypergroups.

Remark 2.6. For integers $p \geq 2q$, the image $\omega_p \in M^+(C_q \times \mathbb{T})$ of the Haar measure on $U(p,q)$ under the canonical projection

$$U(p,q) \to U(p,q)//(U(p) \times SU(q)) \simeq C_q \times \mathbb{T}$$

is given, as a measure on $C_q \times \mathbb{T}$, by

$$d\omega_p(t,z) = const \cdot \prod_{j=1}^{q} \sinh^{2p-2q+1} t_j \cosh t_j \cdot \prod_{1\leq i<j\leq q} \big|\cosh(2t_i) - \cosh(2t_j)\big|^2\, dt\, dz \tag{2.16}$$

with the Lebesgue measure dt on C_q and the uniform distribution dz on $\mathbb{T}$. This measure is, by its construction (see [J]), the Haar measure of the hypergroup $(C_q \times \mathbb{T}, *_p)$.
Eq. (2.16) can be derived analogously to the known case $U(p,q)//(U(p)\times U(q)) \simeq C_q$ and is likely to be known; see Section 5 of Heckman [HS]. We shall derive a more general formula for Haar measures on associated hypergroups on $C_q \times \mathbb{R}$ in Section 4 by using the known Haar measures on the associated hypergroups on C_q due to Rösler [R3]; this formula contains (2.16). We thus skip a proof here.

Remark 2.7. For fixed rank q, the groups $G_p := U(p,q)$ with the compact subgroups $K_p := U(p) \times SU(q)$ satisfy $G_p \subset G_{p+1}$ with $K_p = K_{p+1} \cap G_p$ for $p \geq q$. The associated inductive limits (G_∞, K_∞) thus form a so-called Olshanski-spherical pair. Like for finite parameters p, K_∞-biinvariant continuous functions $\varphi : G_\infty \to \mathbb{C}$ are in a one-to-one correspondence with continuous functions on $C_q \times \mathbb{T}$. A K_∞-biinvariant continuous function φ on G_∞ is called Olshanski-spherical if it satisfies the product formula

$$\varphi(g)\cdot\varphi(h) = \lim_{n\to\infty}\int_{K_n} \varphi(gkh)\, dk \qquad \text{for} \quad g,h \in G_\infty.$$

If we regard φ as a function on $C_q \times \mathbb{T}$, we conclude from Proposition 2.2 and the fact that the probability measures $\frac{1}{\kappa_p}\Delta(I_q - w^*w)^{p-2q}\,dw$ on B_q tend for $p \to \infty$ to the point measure δ_0 that this functional equation reads as

$$\varphi(t, z_1)\varphi(s, z_2) = \int_{SU(q)} \varphi\Big(\text{arcosh}\,(\sigma_{sing}(\cosh \underline{t}\; v\; \cosh \underline{s})), z_1 z_2\Big)\, dv. \tag{2.17}$$

Therefore, φ has the form

$$\varphi(t, z) = z^k \cdot \psi_\lambda(\cosh \underline{t}) \tag{2.18}$$

with $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}^q$ where the functions ψ_λ are the spherical functions of the Gelfand pair $(GL(q, \mathbb{C}), U(q, \mathbb{C}))$. This observation is completely analog to the classification of the Olshanski-spherical function for other Olshanski spherical pairs of fixed finite rank as in Section 7 of [RKV] and in [RV2].

3. Product formulas on $C_q \times \mathbb{R}$

In this section, we extend the product formula (2.9) on $C_q \times \mathbb{T}$ for the spherical functions $\varphi^p_{\lambda,l}(t, z)$ of Theorem 2.4 in several ways.

For a fixed dimension parameter $p \geq 2q$, we first write it as a product formula for functions on the universal covering $C_q \times \mathbb{R}$ of $C_q \times \mathbb{T}$. For this we define the functions

$$\psi_{\lambda,l}(t, \theta) := \psi^p_{\lambda,l}(t, \theta) := e^{i\theta l} \cdot \prod_{j=1}^{q} \cosh^l t_j \cdot F_{BC_q}(i\lambda, k(p, q, l); t) \qquad (t \in C_q,\ \theta \in \mathbb{R}) \tag{3.1}$$

with $\lambda \in \mathbb{C}^q$, $l \in \mathbb{Z}$, and the multiplicity $k(p, q, l) = (p - q - l,\ \frac{1}{2} + l,\ 1)$ as above. These functions are related to the spherical functions $\varphi^p_{\lambda,l}$ of the preceding section by

$$\varphi^p_{\lambda,l}(t, e^{i\theta}) = \psi_{\lambda,l}(t, \theta) \qquad (t \in C_q,\ \theta \in \mathbb{R},\ l \in \mathbb{Z}). \tag{3.2}$$

In a second step we notice that both sides of this product formula depend analytically on the parameters l and p and extend the formula to a positive product formula for all $l \in \mathbb{C}$ and all $p \in \mathbb{R}$ with $p > 2q - 1$ by some principle of analytic continuation. For this step we shall employ Carlson's theorem on analytic continuation which we recapitulate from [Ti], p.186, for the convenience of the reader:

Theorem 3.1. *Let $f(z)$ be holomorphic in a neighbourhood of the closed half-plane $\{z \in \mathbb{C} : \text{Re}\, z \geq 0\}$ satisfying $f(z) = O\big(e^{c|z|}\big)$ on $\text{Re}\, z \geq 0$ for some $c < \pi$. If $f(z) = 0$ for all nonnegative integers z, then f is identically zero for $\text{Re}\, z > 0$.*

To state our product formula on $C_q \times \mathbb{R}$, we need the fact from complex analysis that an analytic function $f : G \to \mathbb{C}$ on a connected, simply connected set $G \subset \mathbb{C}^n$ with $0 \notin f(G)$ admits an analytic logarithm $g : G \to \mathbb{C}$ with $f = e^g$. In fact, this known result can be shown like in the well-known one-dimensional case by using the fact that a closed 1-form is exact on a simply connected domain. We also recapitulate that $SU(q)$ is simply connected. These results and the results of Section 2 lead to the following extended product formula:

Theorem 3.2. *Let $q \geq 1$ be an integer. For all $l \in \mathbb{C}$ and $p \in]2q-1, \infty[$, the functions $\psi^p_{\lambda,l}$ of Eq. (3.1) satisfy the product formula*

$$\psi^p_{\lambda,l}(t,\theta_1)\psi^p_{\lambda,l}(s,\theta_2) = \tag{3.3}$$
$$= \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} \psi^p_{\lambda,l}\Big(d(s,t;v,w), \theta_1+\theta_2+\mathrm{Im}\ \ln \mathrm{h(s,t;v,w)}\Big)\cdot\Delta(\mathrm{I_q}-\mathrm{w^*w})^{\mathrm{p}-2\mathrm{q}}\,\mathrm{dv\,dw}.$$

for all $s,t \in C_q$, $\theta_1,\theta_2 \in \mathbb{R}$ where κ_p, dw, the functions d,h, and other data are defined as in Section 2, and where ln *denotes the unique analytic branch of the logarithm of the function*

$$(s,t,w,v) \mapsto h(s,t;v,w) = \Delta(\sinh \underline{t}\, w \,\sinh \underline{s} + \cosh \underline{t}\, v\, \cosh \underline{s})$$

on the simply connected domain $C_q \times C_q \times B_q \times SU(q)$ with $\ln \Delta(I_q) = 0$.

Proof. In a first step take a parameter $l \in \mathbb{Z}$ and an integer $p \geq 2q$. In this case, (3.3) follows immediately from (3.2) and the product formula (2.9).

We next observe that both sides of (3.3) are analytic in the variables p, l, λ. We now want to employ Carleson's theorem to extend (3.3) to $p \in]2q-1, \infty[$ and $l \in \mathbb{C}$. However, for this we need some exponential growth estimates for the hypergeometric functions $F_{BC_q}(i\lambda, k(p,q,l); t)$ with respect to the parameters p and l in some suitable right half planes, and such suitable exponential estimates are available only for real, nonnegative multiplicities; see Proposition 6.1 of [O1], [Sch], and Section 3 of [RKV]. We thus proceed in several steps, follow the proof of Theorem 4.1 of [R3], and restrict our attention first to a discrete set of spectral parameters λ for which F_{BC_q} is a product of the c-function and Jacobi polynomials such that in this case the growth condition can be checked. Carleson's theorem then leads to (3.3) for this discrete set of spectral parameters λ and all $p \in]2q-1, \infty[$ and $l \in \mathbb{C}$. In a further step we fix $p \in]2q-1, \infty[$ and $l \in [-1/2, p-q]$ and extend (3.3) by Carleson's theorem to all spectral parameters $\lambda \in \mathbb{C}^q$. Finally, usual analytic continuation leads to the general result in the theorem for $l \in \mathbb{C}$.

Let us go into details. We need some notations and facts from [O1], [O2], and [HS]. For our root system $R := 2 \cdot BC_q$ with the set R_+ of positive roots as in Section 2, we define the half sum of roots

$$\rho(k) := \frac{1}{2}\sum_{\alpha \in R_+} k(\alpha)\alpha = (k_1 + 2k_2)\sum_{j=1}^q e_j + 2k_3 \sum_{j=1}^q (q-j)e_j \tag{3.4}$$

as well as the c-function

$$c(\lambda,k) := \prod_{\alpha \in R_+} \frac{\Gamma(\langle \lambda, \alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle \lambda, \alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha))} \cdot \prod_{\alpha \in R_+} \frac{\Gamma(\langle \rho(k), \alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha))}{\Gamma(\langle \rho(k), \alpha^\vee\rangle + \frac{1}{2}k(\frac{\alpha}{2}))} \tag{3.5}$$

with the usual inner product on $\mathbb{C}^q$ and the conventions $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and $k(\frac{\alpha}{2}) = 0$ for $\frac{\alpha}{2} \notin R$. Notice that c is meromorphic on $\mathbb{C}^q \times \mathbb{C}^3$. We now consider the dual root system $R^\vee = \{\alpha^\vee : \alpha \in R\}$, the coroot lattice $Q^\vee = \mathbb{Z}.R^\vee$, and the weight lattice $P = \{\lambda \in \mathbb{R}^q : \langle \lambda, \alpha^\vee\rangle \in \mathbb{Z}\ \forall \alpha \in R\}$ of R. Further, denote by $P_+ = \{\lambda \in P : \langle \lambda, \alpha^\vee \rangle \geq 0 \; \forall \, \alpha \in R_+\}$ the set of dominant weights associated with R_+. Then, by Eq. (4.4.10) of [HS] and by [O1], we obtain for all $k \in K^{reg}$ and all $\lambda \in P_+$,

$$F_{BC_q}(\lambda + \rho(k), k; t) = c(\lambda + \rho(k), k) P_\lambda(k; t) \tag{3.6}$$

where $c(\lambda, k)$ is the c-function (3.5) which is meromorphic on $\mathbb{C}^q \times K$, and where the P_λ are the Heckman-Opdam Jacobi polynomials of type BC_q. We now consider the parameters

$$k_{p,l} := (p - q - l, 1/2 + l, 1) \in K^{reg}$$

(see (2.12)) as well as the associated half sum of roots

$$\rho(k_{p,l}) = (p - q + l + 1) \sum_{j=1}^{q} e_j \; + 2 \sum_{j=1}^{q} (q - j) e_j. \tag{3.7}$$

Using the asymptotics of the gamma function, we now check the growth of $c(\lambda + \rho(k_{p,l}), k_{p,l})$ for fixed $\lambda \in P_+$ and parameters $p, l \to \infty$ in suitable half planes. Indeed, by Stirling's formula,

$$\Gamma(z + a)/\Gamma(z) \sim z^a \qquad \text{for} \qquad z \to \infty, \quad \text{Re}\, z \geq 0.$$

Moreover, for $\rho = \rho(k_{p,l})$,

$$\begin{aligned} c(\lambda + \rho, k) \; &= \\ = \prod_{i=1}^{q} \frac{\Gamma(\lambda_i + \rho_i)\,\Gamma(\rho_i + k_1)}{\Gamma(\lambda_i + \rho_i + k_1)\,\Gamma(\rho_i)} &\cdot \prod_{i=1}^{q} \frac{\Gamma\big(\frac{\lambda_i + \rho_i}{2} + \frac{1}{2}k_1\big)\,\Gamma\big(\frac{\rho_i}{2} + \frac{1}{2}k_1 + k_2\big)}{\Gamma\big(\frac{\lambda_i + \rho_i}{2} + \frac{1}{2}k_1 + k_2\big)\,\Gamma\big(\frac{\rho_i}{2} + \frac{1}{2}k_1\big)} \\ \cdot \prod_{i<j} \frac{\Gamma\big(\frac{\lambda_i + \rho_i - \lambda_j - \rho_j}{2}\big)\,\Gamma\big(\frac{\rho_i - \rho_j}{2} + 1\big)}{\Gamma\big(\frac{\lambda_i + \rho_i - \lambda_j - \rho_j}{2} + 1\big)\,\Gamma\big(\frac{\rho_i - \rho_j}{2}\big)} &\cdot \prod_{i<j} \frac{\Gamma\big(\frac{\lambda_i + \rho_i + \lambda_j + \rho_j}{2}\big)\,\Gamma\big(\frac{\rho_i + \rho_j}{2} + 1\big)}{\Gamma\big(\frac{\lambda_i + \rho_i + \lambda_j + \rho_j}{2} + 1\big)\,\Gamma\big(\frac{\rho_i + \rho_j}{2}\big)}. \end{aligned}$$

For $p, l \to \infty$ we obtain that the first product is asymptotically equal to $\prod_{i=1}^{q} \big(\frac{p+l}{2p}\big)^{\lambda_i}$, while the second is asymptotically equal to $\prod_{i=1}^{q} \big(\frac{p}{p+l}\big)^{\lambda_i/2}$. Further, the third product is independent of p, l, and the last product is asymptotically equal to 1. In summary, for fixed λ, the function $c(\lambda + \rho, k)$ behaves like $\big(\frac{p+l}{p}\big)^a$ for some a, i.e., $c(\lambda + \rho, k)^{-1}$ has polynomial growth.

We now observe for $s, t \in C_q$, $w \in B_q$, $v \in SU(q)$, and $d(t, s; v, w) \in C_q$ that

$$\|d(t, s; v, w)\|_\infty \leq \|s\|_\infty + \|t\|_\infty = s_1 + t_1. \tag{3.8}$$

In fact, this follows easily from the submultiplicativity of the spectral norm; c.f. the proof of Theorem 5.2(1) of [R3]. We now use Eq. (3.3) for integers $p \geq 2q$ and $l \in \mathbb{Z}$, and observe that for integers $p \geq 2q$ and $l \in \mathbb{Z}$ and all $s, t \in C_q$, $\lambda \in P_+$,

$$\begin{aligned} &\frac{\Big(\prod_{j=1}^{q} \cosh t_j \cdot \cosh s_j\Big)^l}{e^{q(t_1+s_1)l}} P_\lambda(k_{p,l}; t) P_\lambda(k_{p,l}; s) = \\ &= \frac{1}{\kappa_p \cdot c(\lambda + \rho(k_{p,l}), k_{p,l})} \int_{B_q} \int_{SU(q)} \frac{\Big(\prod_{j=1}^{q} \cosh d(t, s; v, w)\Big)^l}{e^{q(t_1+s_1)l}} \cdot P_\lambda(k_{p,l}; d(t, s; v, w)) \cdot \\ &\qquad \cdot arg(h(t, s; v, w)) \cdot \Delta(I - w^* w)^{p-2q} dv dw. \end{aligned} \tag{3.9}$$

The Jacobi polynomials $P_\lambda(k;.)$ have rational coefficients in k with respect to the monomial basis e^ν, $\nu \in P$; see Section 11 of [M] or the explicit determinantal construction in Theorem 5.4 of [DLM]. Carlson's theorem now yields that formula (3.9) holds for all $l \in \mathbb{C}$. Moreover, as derived in the proof of Theorem 3.6 of [R2], the normalized integral

$$\frac{1}{|\kappa_p|}\int_{B_q} |\Delta(I-w^*w)^{p-2q}|dw$$

converges exactly if $\mathrm{Re}\, p > 2q-1$ and is of polynomial growth for $p \to \infty$ in the right halfplane $\{p \in \mathbb{C}: \ \mathrm{Re}\, p \geq 2q\}$. Thus for fixed t, s, both sides of (3.9) are holomorphic and of polynomial growth for $p \to \infty$ in this halfplane. Moreover, they coincide for all integers $p \geq 2q$. Another application of Carlson's theorem yields that (3.9) also holds for all p in this halfplane. This proves the stated result for all spectral parameters $\lambda + \rho(k)$ with $\lambda \in P_+$.

In the final step we extend the product formula with respect to the spectral parameter. For this we fix $s, t \in C_q$, $p \geq 2q$ a real number, and $l \in [-1/2, q-p]$. Then $k = k_{p,l}$ is nonnegative, and we have the estimate

$$|F_{BC_q}(\lambda, k; t)| \;\leq\; |W|^{1/2} e^{max_{w\in W}\mathrm{Re}\langle w\lambda, t\rangle}$$

by Proposition 6.1 of [O1]. We now can proceed precisely as in the last step of the proof of Theorem 4.1 on pp. 2791f of [R3]. As in Eq. (3.9), we can rewrite (3.3) with some suitable exponential growth correction which ensures that by a q-fold application of Carlson's theorem, (3.3) can be extended to all $\lambda \in \mathbb{C}^q$. We skip the details.

In a final step, usual analytic continuation yields the theorem for all $l \in \mathbb{C}^q$, which completes the proof. ∎

It should be noticed that the precise choice of the complex logarithm in (3.3) is not essential for this product formula, as it has no influence on the analyticity of the formulas above with respect to p, l, λ. However, our choice of the complex logarithm in (3.3) will be essential in the next section when we prove that (3.3) induces an associative convolution algebra on the Banach space of all bounded signed probability measures on $C_q \times \mathbb{R}$.

Remark 3.3. For $p = 2q-1$, a degenerate version of the product formula (3.3) is available. For this we need some notations and facts. We here follow Section 3 of [R2] and Remark 2.14 of [V2] where also such limit cases of product formulas for Bessel and Laguerre functions on matrix cones were considered.

We fix the dimension q and consider the matrix ball $B_q := \{w \in M^{q,q}(\mathbb{C}) : w^*w \leq I_q\}$ as above as well as the ball $B := \{y \in \mathbb{C}^q : \ \|y\|_2 < 1\}$ and the sphere $S := \{y \in \mathbb{C}^q : \ \|y\|_2 = 1\}$. By Lemma 3.6 and Corollary 3.7 of [R2], the mapping $P : B^q \to B_q$ from the direct product B^q to B_q with

$$P(y_1,\ldots,y_q) := \begin{pmatrix} y_1 \\ y_2(I_q - y_1^*y_1)^{1/2} \\ \vdots \\ y_q(I_q - y_{q-1}^*y_{q-1})^{1/2}\cdots(I_q - y_1^*y_1)^{1/2} \end{pmatrix} \tag{3.10}$$

establishes a diffeomorphism such that the image of the measure

$$\Delta(I_q - w^* w)^{p-2q} dw$$

under P^{-1} is given by $\prod_{j=1}^{q}(1-\|y_j\|_2^2)^{p-q-j} dy_1 \dots dy_q$. Therefore, for $p > 2q-1$, the product formula (3.3) may be rewritten as

$$\begin{aligned}
&\psi^p_{\lambda,l}(t,\theta_1)\psi^p_{\lambda,l}(s,\theta_2) = \\
&= \frac{1}{\kappa_p}\int_{B^q}\int_{SU(q)} \psi^p_{\lambda,l}\Big(d(t,s;v,P(y)), \theta_1+\theta_2 + \mathrm{Im}\,\ln h(t,s;v,P(y))\Big)\cdot \\
&\qquad \cdot \prod_{j=1}^{q}(1-\|y_j\|_2^2)^{p-q-j} dy_1 \dots dy_q\, dw \qquad (3.11)
\end{aligned}$$

with $y = (y_1, \dots, y_q) \in B^q$, where $dy_1, \dots, dy_q$ means integration with respect to the Lebesgue measure on $\mathbb{C}^q$. Moreover, for $p \downarrow 2q-1$, we obtain from (3.11) by continuity the following degenerated product formula:

$$\begin{aligned}
&\psi^{2q-1}_{\lambda,l}(t,\theta_1)\psi^{2q-1}_{\lambda,l}(s,\theta_2) = \\
&= \frac{1}{\kappa_{2q-1}}\int_{B^{q-1}}\int_{S}\int_{SU(q)} \psi^{2q-1}_{\lambda,l}\Big(d(t,s;v,P(y)), \theta_1+\theta_2 + \mathrm{Im}\,\ln h(t,s;v,P(y))\Big)\cdot \\
&\qquad \cdot \prod_{j=1}^{q-1}(1-\|y_j\|_2^2)^{q-1-j} dy_1 \dots dy_{q-1}\, d\sigma(y_q)\, dw \qquad (3.12)
\end{aligned}$$

where $\sigma \in M^1(S)$ is the uniform distribution on S and

$$\kappa_{2q-1} := \int_{B^q} \prod_{j=1}^{q-1}(1-\|y_j\|_2^2)^{q-1-j} dy_1 \dots dy_{q-1}\, d\sigma(y_q).$$

4. Commutative hypergroups on $C_q \times \mathbb{R}$

The positive product formulas (3.3) for real parameters $p > 2q-1$ and (3.12) for $p = 2q-1$ lead to a continuous series of probability-preserving convolution algebras on $C_q \times \mathbb{R}$ parametrized by $p \geq 2q-1$. In fact, these convolutions form commutative hypergroups, which have the functions $\psi^p_{\lambda,l}$ $(\lambda \in \mathbb{C}^q, l \in \mathbb{C})$ as multiplicative functions. For $q = 1$, our convolution structures are closely related to those of Trimeche [Tr].

Before going into details, we briefly recapitulate some notions from hypergroup theory. For more details we refer to [J] and the monograph [BH]. Hypergroups generalize the convolution of bounded measures on locally compact groups such that the convolution $\delta_x * \delta_y$ of two point measures δ_x, δ_y is a probability measure with compact support, but not necessarily a point measure.

Definition 4.1. A hypergroup is a locally compact Hausdorff space X with a weakly continuous, associative, bilinear convolution $*$ on the Banach space $M_b(X)$ of all bounded regular Borel measures on X such that the following properties hold:

(1) For all $x,y\in X$, $\delta_x*\delta_y$ is a compactly supported probability measure on X such that the support $\text{supp}\,(\delta_x*\delta_y)$ depends continuously on x,y with respect to the Michael topology on the space of all compacta in X (see [J] for details).

(2) There exists a neutral element $e\in X$ with $\delta_x*\delta_e=\delta_e*\delta_x=\delta_x$ for all $x\in X$.

(3) There exits a continuous involution $x\mapsto\bar{x}$ on X such that for all $x,y\in X$, $e\in\text{supp}\,(\delta_x*\delta_y)$ holds if and only if $y=\bar{x}$.

(4) If for $\mu\in M_b(X)$, μ^- is the image of μ under the involution, we require that $(\delta_x*\delta_y)^-=\delta_{\bar{y}}*\delta_{\bar{x}}$ for all $x,y\in X$.

Due to weak continuity and bilinearity, the convolution of arbitrary bounded measures on a hypergroup is determined uniquely by the convolution of point measures.

A hypergroup is called commutative if so is the convolution. We recapitulate from [J] that for a Gelfand pair (G,K), the double coset convolution on the double coset space $G//K$ forms a commutative hypergroup.

For a commutative hypergroup we define the space

$$\chi(X)=\{\varphi\in C(X):\ \varphi\not\equiv 0,\ \varphi(x*y):=(\delta_x*\delta_y)(\varphi)=\varphi(x)\varphi(y)\ \forall\, x,y\in X\}$$

of all nontrivial continuous multiplicative functions on X, as well as the dual space

$$\widehat{X}:=\{\varphi\in\chi(X):\ \varphi \text{ is bounded and } \varphi(\overline{x})=\overline{\varphi(x)}\ \forall\, x\in X\}.$$

The elements of $\widehat{X}$ are called characters.

Using the positive product formulas (3.3) and (3.12) for $p>2q-1$ and $p=2q-1$ respectively, we now introduce the convolutions of point measures on $X:=C_q\times\mathbb{R}$ depending on p: For $s,t\in C_q$ and $\theta_1,\theta_2\in\mathbb{R}$, we define the probability measures $(\delta_{(s,\theta_1)}*_p\delta_{(t,\theta_2)})$ with compact supports by

$$(\delta_{(s,\theta_1)}*_p\delta_{(t,\theta_2)})(f):= \tag{4.1}$$
$$=\frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f\Big(d(t,s;v,w),\theta_1+\theta_2+\text{Im}\,\ln h(t,s;v,w)\Big)\cdot\Delta(I_q-w^*w)^{p-2q}\,dv\,dw$$

for $p>2q-1$, and by

$$(\delta_{(s,\theta_1)}*_{2q-1}\delta_{(t,\theta_2)})(f):= \tag{4.2}$$
$$=\frac{1}{\kappa_{2q-1}}\int_{B^q}\int_{SU(q)} f\Big(d(t,s;v,P(y)),\theta_1+\theta_2+\text{Im}\,\ln h(t,s;v,P(y))\Big)\cdot$$
$$\cdot\prod_{j=1}^{q-1}(1-\|y_j\|_2^2)^{q-1-j}dy_1\ldots dy_{q-1}\,d\sigma(y_q)\,dw$$

for $p=2q-1$ for all $f\in C(C_q\times\mathbb{R})$, where the functions d,h and the other data are given as in Sections 2 and 3.

Theorem 4.2. *Let* $q \geq 1$ *be an integer and* $p \in [2q-1, \infty[$. *Then* $*_p$ *can be extended uniquely to a bilinear, weakly continuous convolution on the Banach space* $M_b(C_q \times \mathbb{R})$. *This convolution is associative, and* $(C_q \times \mathbb{R}, *_p)$ *is a commutative hypergroup with* $(0,0)$ *as identity and with the involution* $\overline{(r,a)} := (r, -a)$.

Proof. It is clear by the definition of the convolution that the mapping

$$(C_q \times \mathbb{R}) \times (C_q \times \mathbb{R}) \to M_b(C_q \times \mathbb{R}), \quad ((s,\theta_1),(t,\theta_2)) \mapsto \delta_{(s,\theta_1)} *_p \delta_{(t,\theta_2)}$$

is probability preserving and weakly continuous. It is now standard (see [J]) to extend this convolution uniquely in a bilinear, weakly continuous way to a probability preserving convolution on $M_b(C_q \times \mathbb{R})$.

To prove commutativity, it suffices to consider point measures. But in this case, for $p > 2q-1$, commutativity follows easily by using transposition of matrices and the fact that the integration in (3.3) over $w \in B_q$ and $v \in SU(q)$ remains invariant under transposition. The commutativity for $p > 2q-1$ yields in the limit also commutativity for $p = 2q-1$.

We now turn to associativity. Again it is sufficient to consider point measures. We first consider integers $p \geq 2q$. In this case, we first identify $C_q \times \mathbb{T}$ with the double coset space $U(p,q)//(U(p) \times SU(q))$ as in Section 2 where the associated double coset convolution on $C_q \times \mathbb{T}$ is given by the product formula (2.9). After bilinear, weakly continuous extension, this double coset convolution is associative by its very construction. We now compare convolution products w.r.t. this convolution on $C_q \times \mathbb{T}$ with the corresponding one defined by (3.3) on $C_q \times \mathbb{R}$ for $(r,\theta_1),(s,\theta_2),(t,\theta_3) \in C_q \times \mathbb{R}$ for small $r,s,t \in C_q$. Taking (3.8) into account, we see readily that the complex logarithm in (3.3) for the triple products

$$(\delta_{(r,\theta_1)} *_p \delta_{(s,\theta_2)}) *_p \delta_{(t,\theta_3)} \quad \text{and} \quad \delta_{(r,\theta_1)} *_p (\delta_{(s,\theta_2)} *_p \delta_{(t,\theta_3)})$$

is just the usual main branch of the logarithm on the open right halfplane $\{z : \text{Re} z > 0\}$ for all integration variables in B_q and $SU(q)$ respectively. Using this elementary logarithm, we see immediately that the associativity of the convolution on $C_q \times \mathbb{T}$ implies the associativity on $C_q \times \mathbb{R}$ for small $r,s,t \in C_q$. We now extend the associativity to arbitrary $(r,\theta_1),(s,\theta_2),(t,\theta_3) \in C_q \times \mathbb{R}$. For this we use (3.8) and find an open, relatively compact set $K \subset C_q \times \mathbb{R}$ with

$$\text{supp}\,((\delta_{(\tilde r,\tilde\theta_1)} *_p \delta_{(\tilde s,\tilde\theta_2)}) *_p \delta_{(\tilde t,\tilde\theta_3)}) \cup \text{supp}\,(\delta_{(\tilde r,\tilde\theta_1)} *_p (\delta_{(\tilde s,\tilde\theta_2)} *_p \delta_{(\tilde t,\tilde\theta_3)})) \subset K$$

for all $\tilde r, \tilde s, \tilde t \in C_q$ and $\tilde\theta_1, \tilde\theta_2, \tilde\theta_3 \in \mathbb{R}$ with

$$\|\tilde r\|_\infty, \|\tilde s\|_\infty, \|\tilde t\|_\infty \leq 2\max(\|r\|_\infty, \|s\|_\infty, \|t\|_\infty) \text{ and } |\tilde\theta_1|, |\tilde\theta_2|, |\tilde\theta_3| \leq 2\max(|\theta_1|, |\theta_2|, |\theta_3|).$$

Let $f \in C_c(C_q \times \mathbb{R})$ be a continuous function with compact support which is analytic on K. Then by analyticity of the product formulas (3.3) and (3.12) w.r.t. the variables in $C_q \times \mathbb{R}$,

$$((\delta_{(\tilde r,\tilde\theta_1)} *_p \delta_{(\tilde s,\tilde\theta_2)}) *_p \delta_{(\tilde t,\tilde\theta_3)})(f) \quad \text{and} \quad (\delta_{(\tilde r,\tilde\theta_1)} *_p (\delta_{(\tilde s,\tilde\theta_2)} *_p \delta_{(\tilde t,\tilde\theta_3)}))(f) \tag{4.3}$$

are analytic in the variables $\tilde r, \tilde s, \tilde t, \tilde\theta_1, \tilde\theta_2, \tilde\theta_3$ where both expressions are equal for small $\tilde r, \tilde s, \tilde t$. Therefore, they are equal in general for all such functions f. As both measures have compact support, a Stone-Weierstrass argument leads to the general associativity for integers $p \geq 2q-1$. We now extend the associativity to arbitrary $p > 2q-1$ by Carleson's theorem. For this we compare both sides of (4.3) for functions f as above which is analytic also in the variable $p \in \mathbb{C}$ with $\operatorname{Re} p > 2q-1$ where the boundedness condition in Carleson's theorem can be obtained from the fact that

$$\frac{1}{|\kappa_p|}\int_{B_q} |\Delta(I-w^*w)^{p-2q}| dw$$

is of polynomial growth for $p \to \infty$ in the right halfplane $\{p \in \mathbb{C} : \operatorname{Re} p \geq 2q\}$; see Theorem 3.6 of [R2]. This completes the proof of associativity.

For the remaining hypergroup axioms we first notice that $(0,0)$ is obviously the neutral element. Moreover, for $(t,\theta) \in C_q \times \mathbb{R}$ we have

$$(0,0) \in \operatorname{supp}(\delta_{t,\theta} *_p \delta_{t,-\theta})$$

by taking the integration variables $v \in SU(q)$ as the identity matrix I_q and $w := -I_q \in B_q$ in the convolution (4.1) and $y_1 := -e_1, \ldots, y_q := -e_q$ for the usual unit vectors in $\mathbb{C}^q$ in (4.2).

We next check the converse part of axiom (3) of a hypergroup. For this take $(s,\theta_1), (t,\theta_2) \in C_q \times \mathbb{R}$ with $(0,0) \in \operatorname{supp}(\delta_{s,\theta_1} *_p \delta_{t,\theta_2})$. As the support is independent of $p \in]2q-1, \infty[$ with

$$\operatorname{supp}(\delta_{s,\theta_1} *_{2q-1} \delta_{t,\theta_2}) \subset \operatorname{supp}(\delta_{s,\theta_1} *_{2q} \delta_{t,\theta_2})$$

by (4.1) and (4.2), we may restrict our attention to integers $p \geq 2q$. In this case we now compare the convolution (4.1) on $C_q \times \mathbb{R}$ with the convolution (2.13) on $C_q \times \mathbb{T}$ which is the double coset convolution for $U(p,q)//(U(p) \times SU(q))$. As here axiom (3) is available automatically, we conclude from our assumption that $s = t$ and $\theta_1 - \theta_2 \in 2\pi\mathbb{Z}$ holds. For the proof of $\theta_1 = -\theta_2$, we analyze (2.13) more closely: Recapitulate from Section 2 that the identification $C_q \times \mathbb{T} \simeq U(p,q)//(U(p) \times SU(q))$ is done via the representatives $a_{t,z} \in U(p,q)$ ($t \in C_q$, $z \in \mathbb{T}$) of double cosets. It is clear that for all $t \in C_q$ and $z \in \mathbb{T}$, the matrix

$$J := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \in U(p) \times SU(q)$$

is the only element of $U(p) \times SU(q)$ with $a_{t,z} \cdot J \cdot a_{t,z^{-1}} = I_{p+q} = a_{0,1}$. By the proof of Proposition 2.2 this means that for $v \in SU(q)$ and $w = 0\sigma_0^* u \sigma_0 \in B_q$ with σ_0 as in (2.6), we have

$$d(t,t;v,w) = 0 \quad \text{and} \quad arg\, h(t,t;v,w) = arg\,\Delta(\sinh \underline{t}\, w \sinh \underline{t} + \cosh \underline{t} v \cosh \underline{t}) = 1$$

only for $v = I_q$ and $w = -I_q$. In this case, $d(t,t;I_q,-I_q) = 0$ and $h(t,t;I_q,-I_q) = 1$, where for all $t \in C_q$ obviously the branch of the complex logarithm in the product formula (4.1) satisfies $\ln h(t,t;I_q,-I_q) = 1$. This proves $\theta_1 = -\theta_2$ above and completes the proof of axiom (3).

Finally, axiom (4) is clear, and the continuity of the supports of convolution products is straightforward, but technical. We skip the details. ■

We next turn to subgroups of the commutative hypergroups $(C_q\times\mathbb{R}, *_p)$ for $p\geq 2q-1$. For this we recapitulate that a closed non-empty subset $H\subset C_q\times\mathbb{R}$ is a subhypergroup if $\text{supp}\,(\delta_x *_p \delta_{\bar y})\subset H$ holds for all $x,y\in H$. Moreover, H is called a subgroup, if the convolution restricted to H is the convolution of a group structure on H. It is clear from (3.3) and (3.12), that $\{0\}\times\mathbb{R}$ is a subgroup of $(C_q\times\mathbb{R}, *_p)$ which is isomorphic to the group $(\mathbb{R},+)$.

Now let H be a subgroup of a commutative hypergroup $(X,*)$. Then the cosets

$$x*H := \bigcup_{y\in H}\text{supp}\,(\delta_x *_p \delta_y) \qquad (x\in X)$$

form a disjoint decomposition of X, and the quotient $X/H := \{x*H:\ x\in X\}$ is again a locally compact Hausdorff space with respect to the quotient topology. Moreover,

$$(\delta_{x*H}*\delta_{y*H})(f) := \int_X f(z*H)\, d(\delta_x*\delta_y)(z) \qquad (x,y\in X,\ f\in C_b(X/H)), \tag{4.4}$$

establishes a well-defined quotient convolution and an associated quotient hypergroup $(X/H,*)$. For these quotient convolutions we refer to [J], [R] and [V1]. We now apply this concept to the subgroups $\{0\}\times\mathbb{R}$ and $\{0\}\times\mathbb{Z}$ of our hypergroups $(C_q\times\mathbb{R}, *_p)$ for $p\geq 2q-1$. By (4.4) and (3.3) we can identify the quotient spaces in the obvious way with C_q and $C_q\times\mathbb{T}$ respectively, and we obtain immediately:

Lemma 4.3. *Let $p>2q-1$.*

(1) $(C_q\times\mathbb{R})/(\{0\}\times\mathbb{R})\simeq C_p$ *is a commutative hypergroup with the convolution*

$$(\delta_s *_p \delta_t)(f) := \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f\Big(d(t,s;v,w)\Big)\cdot\Delta(I_q-w^*w)^{p-2q}\, dv\, dw \tag{4.5}$$

for $s,t\in C_q$ and $f\in C_b(C_q)$. This is precisely the hypergroup studied in Section 5 of [R3]. For integers $p\geq 2q$, this is just the double coset hypergroup $U(p,q)//(U(p)\times U(q))$.

(2) $(C_q\times\mathbb{R})/(\{0\}\times\mathbb{Z})\simeq C_p\times\mathbb{Z}$ *is a commutative hypergroup with the convolution $*_p$ of Eq. (2.13), but here for arbitrary real numbers $p>2q-1$. In particular, for integers $p\geq 2q$, this is just the double coset hypergroup $U(p,q)//(U(p)\times SU(q))$.*

For $p=2q-1$, a corresponding result holds on the basis of (3.12).

By using Weil's integral formula for Haar measures on hypergroups (see [Her] and [V1]), we now can determine the Haar measures on the hypergroups $(C_q\times\mathbb{R}, *_p)$ from the known Haar measures on the hypergroups of Lemma 4.3(1) in Theorem 5.2 of [R3]. For this recapitulate that each commutative hypergroup $(X,*)$ admits a (up to a multiplicative constant unique) Haar measure ω_X, which is characterized by the condition $\omega_X(f)=\omega_X(f_x)$ for all continuous functions $f\in C_c(X)$ with compact support and $x\in X$, where the translate $f_x\in C_c(X)$ is given by $f_x(y) := (\delta_y*\delta_x)(f)$.

Proposition 4.4. *For $p \geq 2q-1$, the Haar measure of the commutative hypergroup $(C_q \times \mathbb{R}, *_p)$, is given by*

$$d\omega_p(t,\theta) = const \cdot \prod_{j=1}^{q} \sinh^{2p-2q+1} t_j \cosh t_j \cdot \prod_{1\le i<j\le q} \left|\cosh(2t_i) - \cosh(2t_j)\right|^2 dt d\theta \quad (4.6)$$

with the Lebesgue measure dt on C_q and the Lebesgue measure $d\theta$ on $\mathbb{R}$.

Moreover, for $p \geq 2q-1$, the Haar measure of the commutative quotient hypergroup $(C_q \times \mathbb{R})/(\{0\} \times \mathbb{Z}) \simeq C_p \times \mathbb{T}$ of the preceding lemma is given precisely by the measure of Eq. (2.16) on $C_q \times \mathbb{T}$.

Proof. Let H be a subgroup of a commutative hypergroup $(X,*)$, and let ω_H and $\omega_{X/H}$ be Haar measures of the group H and the quotient hypergroup X/H respectively. Then, by [Her] and [V1], for $f \in C_c(X)$ the function $T_H f(x*H) := \int_H (\delta_x * \delta_h)\, d\omega_H(h)$ establishes a well-defined function $T_H f \in C_c(X/H)$, and Weil's formula

$$\omega_X(f) := \int_{X/H} T_H f(x*H)\, d\omega_{X/H}(x*H)$$

defines (up to a multiplicative constant) the Haar measure ω_X of $(X,*)$. If we apply this construction to our commutative hypergroup $(C_q \times \mathbb{R}, *_p)$ with the subgroup $\{0\} \times \mathbb{R}$, and take the Haar measure in Theorem 5.2(2) of [R3] for the quotient $(C_q \times \mathbb{R})/(\{0\} \times \mathbb{R}) \simeq C_p$, the first part of the proposition is clear.

The second statement follows immediately from the first statement and the fact that a Haar measure on the quotient $(C_q \times \mathbb{R})/(\{0\} \times \mathbb{Z}) \simeq C_p \times \mathbb{T}$ is given as the image of a Haar measure on $C_q \times \mathbb{R}$ under the canonical projection. ■

Remark 4.5. (1) By the preceding results, the functions $\psi^p_{\lambda,l}$ $(\lambda \in \mathbb{C}^q, l \in \mathbb{C})$ are continuous multiplicative functions on the hypergroups $(C_q \times \mathbb{R}, *_p)$ for $p \geq 2q-1$. We conjecture that in fact each continuous multiplicative function on $(C_q \times \mathbb{R}, *_p)$ has this form. In fact, one has to prove similar to Lemma 5.3 of [R3] that continuous multiplicative functions are eigenfunctions of a corresponding family of differential operators discussed in Section I.5 of [HS].

(2) The multiplicative functions $\psi^p_{\lambda,l}$ satisfy several symmetry conditions in the spectral variables which are immediate consequences of corresponding symmetries for F_{BC_q} in [HS], [O1], [O2]. In particular, similar to [R3], we have: For $\lambda, \tilde\lambda \in \mathbb{C}^q$, $l, \tilde l \in \mathbb{C}$, and the Weyl group W_q of type B_q acting on $\mathbb{C}^q$,

$$\psi^p_{\lambda,l} = \psi^p_{\tilde\lambda,\tilde l} \quad \text{on} \quad C_q \qquad \Longleftrightarrow \qquad \tilde\lambda \in W\lambda, \quad \tilde l = l.$$

Moreover,

$$\overline{\psi^p_{\lambda,l}} = \psi^p_{\bar\lambda,-l}.$$

In particular, $\psi^p_{\lambda,l}$ satisfies

$$\psi^p_{\lambda,l}((t,\theta)^-) = \overline{\psi^p_{\lambda,l}(t,\theta)} \qquad \text{for all} \quad (t,\theta) \in C_q \times \mathbb{R} \qquad (4.7)$$

if and only if $l \in \mathbb{R}$ and $\bar\lambda \in W\lambda$ holds.

(3) It is an interesting task to determine the dual space $(C_q\times\mathbb{R})^\wedge$ which consists of all bounded, continuous multiplicative functions satisfying (4.7). For the corresponding hypergroups on C_q, we refer to [R3] and [NPP] for this problem.

5. Product formulas for Heckman-Opdam functions

We fix the dimension $q\ge 1$, a real parameter $p>2q-1$ and some index $l\in\mathbb{R}$. We know from Section 2 that the functions $\psi_{\lambda,l}$ with

$$\psi^p_{\lambda,l}(t,0)=\prod_{j=1}^q \cosh^l t_j\cdot F_{BC_q}(i\lambda,k(p,q,l);t)$$

satisfy the product formula (3.3). We now apply

$$\prod_{j=1}^q \cosh^l d_j(s,t;v,w)\cdot e^{il\cdot \mathrm{Im}\ \ln \mathrm{h(s,t;v,w)}}=h(s,t;v,w)^l=\Delta(\sinh \underline{t}\, w\ \sinh \underline{s}+\cosh \underline{t} v\cosh \underline{s})^l$$

for $s,t\in C_q$, $w\in B_q$, $v\in SU(q)$ and the analytical branch of the l-th power function associated with the analytical branch of the logarithm in the setting of (3.3). This branch will be taken always from now on. This leads immediately to the following product formula for the hypergeometric functions $\varphi^{p,l}_\lambda(t):=F_{BC_q}(i\lambda,k(p,q,l);t)$:

Theorem 5.1. *Fix an integer $q\ge 1$, $p\in[2q-1,\infty[$, and all $l\in\mathbb{R}$. Then the functions $\varphi^{p,l}_\lambda$ satisfy the product formula*

$$\varphi^{p,l}_\lambda(s)\cdot\varphi^{p,l}_\lambda(t)=\frac{1}{\kappa_p\prod_{j=1}^q\Big(\cosh t_j\cdot\cosh s_j\Big)^l}\cdot \tag{5.1}$$
$$\cdot\int_{B_q}\int_{SU(q)}\varphi^{p,l}_\lambda(d(t,s;v,w))\cdot \mathrm{Re}(h(t,s;v,w)^l)\cdot\Delta(I_q-w^*w)^{p-2q}\,dv\,dw$$

for $s,t\in C_q$ and all $\lambda\in\mathbb{C}$.

Proof. Our considerations above lead to

$$\varphi^{p,l}_\lambda(s)\cdot\varphi^{p,l}_\lambda(t)=\frac{1}{\kappa_p\prod_{j=1}^q\Big(\cosh t_j\cdot\cosh s_j\Big)^l}\cdot$$
$$\cdot\int_{B_q}\int_{SU(q)}\varphi^{p,l}_\lambda(d(t,s;v,w))\cdot h(t,s;v,w)^l\cdot\Delta(I_q-w^*w)^{p-2q}\,dv\,dw.$$

for $s,t\in C_q$, $\lambda\in\mathbb{C}$. Now take $\lambda\in\mathbb{R}^q$ in which case $\varphi^{p,l}_\lambda$ is real on C_q. Therefore, taking real parts above, we obtain the product formula of the theorem for $\lambda\in\mathbb{R}^q$. The general case follows by analytic continuation. ■

We next present a condition on l which ensures positivity of the product formula (5.1) for all $s,t\in C_q$. It is based on the following:

Lemma 5.2. *For all $l \in \mathbb{R}$ with $|l| \le 1/q$ and all $s,t \in C_q$, $w \in B_q$, $v \in SU(q)$,*

$$\mathrm{Re}\left((\Delta(\sinh \underline{t}\, w\, \sinh \underline{s} + \cosh \underline{t}\, v\, \cosh \underline{s}))^l\right) \ge 0.$$

Proof. We have

$$\Delta(\sinh \underline{t}\, w\, \sinh \underline{s} + \cosh \underline{t}\, v\, \cosh \underline{s}) = \Delta(\cosh \underline{t} \cdot \cosh \underline{s}) \cdot \Delta(\tilde{w} + I_q) \tag{5.2}$$

for the matrix $\tilde{w} := v^{-1} \cdot \tanh \underline{s}\, w\, \tanh \underline{t}$. We now check $\tilde{w} \in B_q$, i.e., $\tilde{w}^* \tilde{w} \le I_q$. In fact, this is equivalent to $\tanh \underline{t}\, w^* \tanh^2 \underline{s}\, w \tanh \underline{t} \le I_q$, which is clearly a consequence of $w^* \tanh^2 \underline{s}\, w \le I_q$ which is obviously correct.

As all eigenvalues $\tau \in \mathbb{C}$ of a matrix $\tilde{w} \in B_q$ satisfy $|\tau| \le 1$, we obtain that all eigenvalues of $\tilde{w} + I_q$ are contained in $\{z \in \mathbb{C} : \mathrm{Re}\, z \ge 0\}$. The lemma now follows from (5.2). ■

We notice that it can be easily seen that Lemma 5.2 is not correct for a larger range of parameters $l \in \mathbb{R}$. It is however unclear for which precise range of parameters $l \in \mathbb{R}$ there is a positive product formula for the functions $\varphi_\lambda^{p,l}$. We expect that this range depends on q and p; see also the example for $q = 1$ below.

We also remark that the results above for $p > 2q - 1$ are also available in a corresponding way for $p = 2q - 1$, and that for $l \in \mathbb{R}$ with $|l| \le 1/q$ and $p \ge 2q - 1$, our positive product formulas for the $\varphi_\lambda^{p,l}$ lead to commutative hypergroup structures on C_q. This can be shown in the same way as in Section 4. We here skip the details and remark only that $\varphi_{i\rho}^{p,l} \equiv 1$ for $\rho = \rho(k_{p,l})$ as in (3.7) ensures that the corresponding positive measures on the right hand sides are in fact probability measures.

In summary:

Theorem 5.3. *For all integers $q \ge 1$, all $p \in [2q-1, \infty[$ and all $l \in \mathbb{R}$ with $|l| \le 1/q$, the Heckman-Opdam hypergeometric functions $\varphi_\lambda^{p,l}$ ($\lambda \in \mathbb{C}$) satisfy some positive product formula (namely (5.1) for $p > 2q-1$ and a corresponding one for $p = 2q-1$). Moreover, in this case the $\varphi_\lambda^{p,l}$ ($\lambda \in \mathbb{C}$) are multiplicative functions of some associated unique commutative hypergroup structures $(C_q, *_{p.l})$.*

It is not difficult to determine the Haar measures of these hypergroups:

Proposition 5.4. *For integers $q \ge 1$, $p \in [2q-1, \infty[$ and $l \in \mathbb{R}$ with $|l| \le 1/q$, the Haar measure on the hypergroup $(C_q, *_{p.l})$ is given by*

$$d\omega_{p,l}(t) = const \cdot \prod_{j=1}^{q} \sinh^{2p-2q+1} t_j \cosh^{2l+1} t_j \cdot \prod_{1 \le i < j \le q} \left|\cosh(2t_i) - \cosh(2t_j)\right|^2 dt. \tag{5.3}$$

We shall postpone the proof of this result to Section 6.

Example 5.5. Consider $q = 1$ and $p \ge 2q - 1 = 1$ as in Section 2.5. We here have

$$\alpha := k_1 + k_2 - 1/2 = p - 1 \ge 0, \qquad \beta := k_2 - 1/2 = l,$$

and $\varphi_\lambda^{p,l}(t) = \varphi_\lambda^{(\alpha,\beta)}(t)$ $(t \in [0,\infty[)$ for the Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}$ in [K]. We now use the parameters $\alpha \geq 0$ and $\beta \in \mathbb{R}$ instead of p, l. Moreover, for $q = 1$, we have $SU(1) = \{1\}$, $B_1 = \{e^{i\theta} r : \theta \in [-\pi, \pi], r \in [0,1]\}$ and $dw = r\,dr\,d\theta$ in the product formula (5.1). Therefore, (5.1) for $\alpha > 0$, the symmetry of the integral w.r.t. $\theta \in [-\pi, \pi]$ and the correct integration constant lead to the product formula

$$\varphi_\lambda^{(\alpha,\beta)}(s) \cdot \varphi_\lambda^{(\alpha,\beta)}(t) =$$
$$= \frac{2\alpha}{\pi \cdot (\cosh s \cdot \cosh t)^\beta} \int_0^1 \int_0^\pi \varphi_\lambda^{(\alpha,\beta)}(\mathrm{arcosh}|re^{i\theta} \sinh t \sinh s \,+\, \cosh t \cosh s|)$$
$$\cdot \mathrm{Re}\Big((re^{i\theta} \sinh t \sinh s \,+\, \cosh t \cosh s)^\beta\Big) \cdot (1 - r^2)^{\alpha-1} r\,dr\,d\theta \tag{5.4}$$

for $s, t \geq 0$, $\lambda \in \mathbb{C}$. For $\alpha = 0$ we obtain the degenerate formula

$$\varphi_\lambda^{(0,\beta)}(s) \cdot \varphi_\lambda^{(0,\beta)}(t) =$$
$$= \frac{1}{\pi(\cosh s \cdot \cosh t)^\beta} \int_0^\pi \varphi_\lambda^{(0,\beta)}(\mathrm{arcosh}|re^{i\theta} \sinh t \sinh s \,+\, \cosh t \cosh s|)$$
$$\cdot \mathrm{Re}\Big((re^{i\theta} \sinh t \sinh s \,+\, \cosh t \cosh s)^\beta\Big)\,d\theta \tag{5.5}$$

For $\beta = 0$, these formulas coincide with the well known product formulas for Jacobi functions; see Section 7 of [K]. However, for $\beta \neq 0$, (5.4) and (5.5) do not seem to be much used in literature. In fact, to our knowledge, they are only considered in Section 6 of [RV1]. It should be noticed that some details in Section 6 are not correct.

Let us compare our product formulas with those of Koornwinder [K]. Our Eqs. (5.4) and (5.5) are available for all $\alpha \geq 0$ and $\beta \in \mathbb{R}$, and they are positive for $\alpha \geq 0$ and $|\beta| \leq 1$. On the other hand, Koornwinder's formulas in Section 7 of [K] are available with positivity for $\alpha \geq \beta \geq -1/2$. It is well known (see Section 7 of [K], and [J] and [BH] for the hypergroup background) that for all $\alpha \geq \beta \geq -1/2$, Koornwinder's product formulas for the $\varphi_\lambda^{(\alpha,\beta)}$ are associated with unique commutative hypergroup structures on $[0,\infty[$, the so-called Jacobi hypergroups $([0,\infty[, *_{\alpha,\beta})$. As here injectivity of the Jacobi transform (as a special case of the injectivity of the Fourier transform on commutative hypergroups in [J]) ensures that for α, β, s, t there is at most one bounded signed measure $\mu_{s,t}^{\alpha,\beta} \in M_b([0,\infty[)$ with

$$\varphi_\lambda^{(0,\beta)}(s)\varphi_\lambda^{(0,\beta)}(t) = \int \varphi_\lambda^{(0,\beta)}(u)\,d\mu_{s,t}^{\alpha,\beta}(u)$$

for all $\lambda \in \mathbb{C}$, we conclude that for $\alpha \geq \max(\beta, 0)$ and $\beta \geq -1/2$, the product formulas (5.4) and (5.5) are equivalent to those of [K] and thus positive.

This in particular shows that for $q = 1$ the range of parameters p, l in Theorem 5.3 with a positive product formula is larger than described there. We expect that this holds also for $q \geq 2$.

Taking our results and the results of Koornwinder into account, we obtain in summary that the Jacobi functions $\varphi_\lambda^{(0,\beta)}$ $(\lambda \in \mathbb{C})$ admit associated commutative hypergroup structures for the set of parameters

$$\{(\alpha, \beta) \in \mathbb{R}^2 :\ \alpha \geq \beta \geq -1/2 \quad \text{or} \quad \alpha \geq 0,\ \beta \in [-1, 1]\}.$$

By classical results in the case of Koornwinder (see [BH] for details on the Jacobi hypergroups) and by (5.3) in our case, the Haar measures on these hypergroups are given in both cases by

$$const \cdot \sinh^{2\alpha+1} t \cdot \cosh^{2\beta+1} t \qquad (t \geq 0).$$

6. Signed hypergroups on C_q

We show in this section that for all $p \geq 2q-1$ and $l \in \mathbb{R}$, the (not necessarily positive) product formulas of Section 5 for the Heckman-Opdam functions $\varphi_\lambda^{p,l}$ on C_q are related to some so-called signed hypergroup structure $(C_q, \bullet_{p,l})$. For this we return to the commutative hypergroups $(C_q \times \mathbb{R}, *_p)$ of Section 4 for $p \geq 2q-1$ which have the functions $\psi_{\lambda,l}^p$ as multiplicative functions. As in Section 4, we consider the closed subgroup $G := \{0\} \times \mathbb{R}$ of $(C_q \times \mathbb{R}, *_p)$, and identify the quotient $(C_q \times \mathbb{R})/G$ with C_q.

For $l \in \mathbb{R}$ consider the functions $\sigma_l \in C_b(C_q \times \mathbb{R})$ with $\sigma_l(t,\theta) := e^{il\theta}$. They satisfy

$$|\sigma_l(t,\theta)| = 1, \quad \sigma_l(t,-\theta) = \overline{\sigma_l(t,\theta)}, \quad \text{and} \quad (\delta_{(0,\tau)} *_p \delta_{(t,\theta)})(\sigma_l) = \sigma_l(t,\theta) \cdot \sigma_l(0,\tau)$$

for $t \in C_q$ and $\theta, \tau \in \mathbb{R}$, i.e., the σ_l are partial characters on $(C_q \times \mathbb{R}, *_p)$ w.r.rt. G in the sense of Definition 4.1 of [RV1]. For $l \in \mathbb{R}$ we now consider the mapping

$$\begin{aligned}(C_q \times \mathbb{R})/G \times (C_q \times \mathbb{R})/G &\longrightarrow M_b(C_q \times \mathbb{R})/G) \\ ((s,\theta_1) * G, (t,\theta_2) * G) &\longmapsto \delta_{(s,\theta_1)*G} \bullet_{p,l} \delta_{(t,\theta_2)*G} := \\ &:= \overline{\sigma_l(s,\theta_1)} \cdot \overline{\sigma_l(t,\theta_2)} \cdot pr(\sigma_l \cdot (\delta_{(s,\theta_1} *_p \delta_{(t,\theta_2)}))\end{aligned}$$

with pr as the canonical projection $pr : C_q \times \mathbb{R} \to (C_q \times \mathbb{R})/G$ as well as its extension to images of measures. It can be easily checked (see also Section 4 of [RV1] for a general theory) that this mapping is well-defined, i.e., independent of representatives of the cosets, and weakly continuous. Moreover, by Section 4 of [RV1], it can be uniquely extended in a bilinear, weakly continuous way to an associative convolution $*_l$ on $M_b((C_q \times \mathbb{R})/G)$. Moreover, $(M_b((C_q \times \mathbb{R})/G), *_l)$ is a commutative Banach-$*$-algebra with respect to the total variation norm with the identity $\delta_{(0,0)*G}$ as neutral element where the involution on $M_b((C_q \times \mathbb{R})/G)$ is inherited from that on the hypergroup $C_q \times \mathbb{R}$. We also consider the image $\tilde{\omega}_p := pr(\omega_p) \in M^+((C_q \times \mathbb{R})/G)$ of the Haar measure ω_p on $C_q \times \mathbb{R}$ in (4.6). By Theorem 4.6 of [RV1], the quotient $(C_q \times \mathbb{R})/G$ with the convolution $\bullet_{p,l}$ and the measure $\tilde{\omega}_p$ forms a so-called commutative signed hypergroup $((C_q \times \mathbb{R})/G, \bullet_{p,l}, \tilde{\omega}_p)$ with the identity mapping as involution, i.e., a so-called hermitian commutative signed hypergroup. For details on this notion we refer to [R1], [RV1] and references cited there. We only remark that for a Haar measure ω_p in this setting we require the conjugation relation

$$\int_{(C_q \times \mathbb{R})/G} T_x f \cdot g \, d\omega_p = \int_{(C_q \times \mathbb{R})/G} T_x g \cdot f \, d\omega_p \tag{6.1}$$

for all $f, g \in C_c((C_q \times \mathbb{R})/G)$ and $x \in (C_q \times \mathbb{R})/G$ where the translates T_x are defined by $T_x f(y) := (\delta_x \bullet_{p,l} \delta_y)(f)$. It is well-known (see [J]) that for usual hermitian commutative hypergroups, the usual Haar measures satisfy (6.1), and that by [RV1], Haar measures on commutative signed hypergroups in the above sense are unique up to a constant.

We now identify $(C_q \times \mathbb{R})/G$ with C_q as usual. In this case, we obtain from (4.1) that for $p > 2q-1$ the convolution $\bullet_{p,l}$ on C_q is given by

$$(\delta_s \bullet_{p,l} \delta_t)(f) = \int_{C_q\times\mathbb{R}} e^{il\theta}\, d(\delta_{s,0} *_p \delta_{s,0})(t,\theta) \tag{6.2}$$
$$= \frac{1}{\kappa_p}\int_{B_q}\int_{SU(q)} f(d(s,t;v,w))\cdot (arg\, h(s,t;v,w))^l \cdot \Delta(I_q - w^*w)^{p-2q}\, dv\, dw.$$

Moreover, for $p = 2q-1$ we obtain a corresponding formula from (4.2), and the Haar measure $\omega_p \in M^+(C_q)$ of our signed hypergroups is independent of l and is given by

$$d\tilde{\omega}_p(t) = const \cdot \prod_{j=1}^{q} \sinh^{2p-2q+1} t_j \cosh t_j \cdot \prod_{1\le i<j\le q} \left|\cosh(2t_i) - \cosh(2t_j)\right|^2 dt. \tag{6.3}$$

Furthermore, using the multiplicative functions $\psi^p_{\lambda,l}$ on the hypergroups $(C_q \times \mathbb{R}, *_p)$, we obtain immediately that for $l \in \mathbb{R}$, the functions

$$\tilde{\varphi}^{p,l}_\lambda(t) := \prod_{j=1}^{q} \cosh^l t_j \cdot F_{BC_q}(i\lambda, k(p,q,l); t) \qquad (\lambda \in \mathbb{C})$$

are multiplicative functions of our signed quotient hypergroups.

It should be noticed that except for $l = 0$, i.e., the case of a classical quotient hypergroup, our signed quotient hypergroups together with their multiplicative functions $\tilde{\varphi}^{p,l}_\lambda$ are different, but closely related to the convolution of measures on C_q which is induced by the product formulas for the functions $\varphi^{p,l}_\lambda$ in Section 5.

In fact, we have

$$\tilde{\varphi}^{p,l}_\lambda(t) = \prod_{j=1}^{q} \cosh^l t_j \cdot \varphi^{p,l}_\lambda(t)$$

and, for $s, t \in C_q$,

$$\delta_s \bullet_{p,l} \delta_t = \frac{f_0(s)\cdot f_0(t)}{f_0}(\delta_s *_{p,l} \delta_t)$$

with the positive function

$$f_0(t) := \prod_{j=1}^{q} \cosh^l t_j = \tilde{\varphi}^{p,l}_{i\rho}(t).$$

By the following lemma, the Haar measures $\tilde{\omega}_p$ associated with $\bullet_{p,l}$ are related with the Haar measures $\omega_{p,l}$ of the hypergroups $(C_q, *_{p,l})$ of Section 5 for $|l| \le 1/q$. This observation together with (6.3) then lead immediately to the Haar measures in Proposition 5.4.

Lemma 6.1. *Let X be a locally compact Hausdorff space and $h \in C(X)$ a positive continuous function on X. Assume we have two signed hermitian commutative hypergroup structures $(X, *, \omega_*)$ and $(X, \bullet, \omega_\bullet)$ on X with the convolutions $*, \bullet$ and associated Haar measures $\omega_*, \omega_\bullet$. Assume that the convolutions are related by*

$$\delta_x \bullet \delta_y = \frac{h}{h(x)h(y)} \cdot (\delta_x * \delta_y) \qquad \text{for} \quad x, y \in X,$$

then, up to a positive multiplicative constant, the Haar measures are related by $\omega_\bullet = h^2 \omega_*$. Moreover, a function $f \in C(X)$ is multiplicative w.r.t. the convolution $*$ if and only on f/h is multiplicative w.r.t. $\bullet$.

Proof. Let ω_* be a Haar measure associated with $*$, and denote the translates of $f \in C_c(X)$ by $x \in X$ w.r.t. $*$ and $\bullet$ by $T_x^* f$ and $T_x^\bullet f$ respectively. Then, by the conjugation relation for ω_*, we have for $f, g \in C_c(X)$, $x \in X$ and $\omega_\bullet := h^2 \omega_*$ that

$$\int_X T_x^\bullet f \cdot g \, d\omega_\bullet = \int_X \frac{T_x^*(hf)(y)}{h(x)h(y)} \cdot g(y) h^2(y) \, d\omega_* = \frac{1}{h(x)} \int_X T_x^*(hf)(y) \cdot g(y) h(y) \, d\omega_*$$
$$= \frac{1}{h(x)} \int_X h(y) f(y) \cdot T_x^*(hg)(y) \, d\omega_* = \ldots = \int_X T_x^\bullet g \cdot f \, d\omega_\bullet.$$

This yields the conjugation relation for $\omega_\bullet$. As Haar measures on signed hypergroups are unique up to a constant by [RV1], the first part of the lemma follows. The second part of the lemma is also clear. ∎

Let us summarize the preceding observations in Sections 5 and 6. In principle, for $l \in \mathbb{R}$ and $p \geq 2q - 1$, we have two choices to define convolution structures for measures on C_q associated with the hypergeometric functions F_{BC_q}, namely the convolutions of Section 5 for the functions F_{BC_q} directly as well as the convolutions of Section 6 for the functions $\tilde{\varphi}_\lambda^{p,l}$. These both convolutions are equal for $l = 0$ in which case C_q carries the usual classical quotient convolution which was studied in [R3].

For $l \neq 0$ with $l \in [-1/q, 1/q]$, we have a positive convolution. In this case, the convolution of Section 5 for the functions F_{BC_q} is probability preserving and generates classical hypergroup structures, which is not the case for the convolutions of Section 6, which are only positive and norm-decreasing. We thus prefer the point of view of Section 5 in this case.

On the other hand, for $l \in \mathbb{R} \setminus [-1/q, 1/q]$, it is unclear (at least for $q \geq 2$) in which cases our convolutions are positive. It is also unclear in this case whether the convolutions of Section 5 are norm-decreasing (w.r.t. total variation norm), while this is the case always for the convolutions of Section 6 by their very construction above. We thus prefer the convolution of Section 6 in those cases, where no positivity is known.

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