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Complete Integrability, Orbital Linearizability and Independent Normalizers for Local Vector Fields in $\mathbb{R}^n$

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Abstract. In this paper we study how are related three of the basic concepts in the rather non-generic phenomenon of integrability of analytic local vector fields $\mathcal{X}$ around an equilibrium in $\mathbb{R}^n$, namely: complete integrability, orbital linearizability and number of independent normalizers (Lie symmetries). The work relates and extends several results existing in the literature of the subject.
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1. Introduction, background and main results

The main purpose of this work is to show some relationships, consequences and the interplay between the three linked concepts of complete integrability, orbital linearizability and existence of sufficiently many independent normalizers in the analytical setting for local analytic vector fields $\mathcal{X}$ defined in a neighborhood $U \subset \mathbb{R}^n$ of an equilibrium point. We will locate always that equilibrium at the origin using a translation without loss of generality.

We shall use the following notation. $\mathcal{X}_A$ will denote the linear vector field with associated $n \times n$ matrix $A \neq 0$. Moreover an analytic vector field $\mathcal{X}$ with linear part $\mathcal{X}_A$ is expressed as $\mathcal{X} = \mathcal{X}_A + \cdots$, where the dots denote an analytic vector field without linear terms.

A classical problem in normal form theory asks whether the foliation defined by the orbits of $\mathcal{X} = \mathcal{X}_A + \cdots$ can be analytically transformed into the foliation of an orbital linear vector field. Then $\mathcal{X}$ is called analytically *orbitally linearizable* in U if there exists a near-identity analytic change of coordinates $\Phi(x) = x + \cdots$ in U such that $\mathcal{X}$ is conjugated to a resonant vector field in the normal form $\Phi_*\mathcal{X} = f(x)\mathcal{X}_A$ with $f : U \subset \mathbb{R}^n \to \mathbb{R}$ an analytic function which is a unit at the origin, i.e., such that $f(0) = 1$. Here Φ_* denotes the push-forward associated

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to the diffeomorphism Φ, that is $\Phi_* \mathcal{X}|_{\Phi(x)} = D\Phi(x)\, \mathcal{X}|_x$ where $D\Phi(x)$ is the derivative of Φ at the point $x \in U$.

On the other hand, $\mathcal{X}$ is called (locally) *completely integrable* in the neighborhood U of the origin if there are $n-1$ functionally independent analytical first integrals almost everywhere in U, that is in a full Lebesgue measure (dense) open subset of U. Therefore this implies the existence of $n-1$ analytic functions $H_i : U \to \mathbb{R}$ such that $\mathcal{X}(H_i) \equiv 0$ for $i = 1, \ldots, n-1$, and $\nabla H_1 \wedge \ldots \wedge \nabla H_{n-1} \neq 0$ almost everywhere in U. Here ∇H_i denotes the gradient vector of the function H_i and we have used the wedge (exterior) product of vector fields to express the functionally independence of the first integrals. Obviously $\mathcal{X}$ can have at most $n-1$ nontrivial functionally independent first integrals. In [3] it is proved that the number of functionally independent analytic first integrals for $\mathcal{X} = \mathcal{X}_A + \cdots$ does not exceed the maximal number of linearly independent elements of the $\mathbb{Z}^+$*–resonant set* of A which is defined as the set $\Sigma_\lambda = \{k \in (\mathbb{Z}^+)^n :< k, \lambda >= 0, k \neq 0\}$. Here $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$; $\lambda \in \mathbb{C}^n$ is the n-tuple of eigenvalues of A and $<,>$ denotes the standard scalar product. If the nondegenerate condition $\lambda \neq 0$ holds (thus A possesses at least one non-zero eigenvalue) it is clear that the rank of the set Σ_λ is at most $n-1$.

The rather non-generic phenomenon of the existence of a non-constant first integral H for $\mathcal{X}$ is important because the orbits of $\mathcal{X}$ leave invariant the level sets of the function H implying thus a strong constraint on the dynamical behavior of $\mathcal{X}$. But unfortunately, it is generally very difficult to compute H in closed form. Therefore it is necessary to introduce other tools in order to analyze the integrability of $\mathcal{X}$. One of these auxiliary objects is the set $\mathfrak{N}(\mathcal{X})$ of all analytic *normalizers* of the vector analytic field $\mathcal{X}$ which is defined as $\mathfrak{N}(\mathcal{X}) = \{\mathcal{Z} \in C^w(U) : [\mathcal{X}, \mathcal{Z}] = \Lambda \mathcal{X}\}$ for certain function $\Lambda : U \to \mathbb{R}$. The notion of normalizer was introduced by Lie and Engel, see the modern translation [13]. Here $[\mathcal{X}, \mathcal{Z}] = \mathcal{X}\mathcal{Z} - \mathcal{Z}\mathcal{X}$ stands for the Lie bracket. Normalizers are also called orbital symmetries because the flow associated to $\mathcal{Z}$ interchanges (or leaves invariant) the orbits of $\mathcal{X}$. Here we want to emphasize that there is no algorithmic procedure to know if $\mathfrak{N}(\mathcal{X})$ is nontrivial, that is to know if $\mathfrak{N}(\mathcal{X}) \neq \mathbb{R}\,\mathcal{X}$. The set $\mathfrak{N}(\mathcal{X})$ is a Lie algebra which is, in general, infinite-dimensional.

There is an abundant literature linking both notions: first integrals and normalizers. The most direct relationship is that if H is a first integral of $\mathcal{X}$ then $\mathcal{Z}(H)$ yields another (possibly trivial) first integral of $\mathcal{X}$ provided that $\mathcal{Z} \in \mathfrak{N}(\mathcal{X})$. The pioneering Lie's works showed how to compute first integrals in terms of quadratures if we know normalizers, see for instance the excellent textbook on the subject [14].

One of the key results in the problem of closed-form integrability of nonlinear vector fields is the following classical result of Lie (see also [1]): if the n-dimensional Lie algebra $\mathfrak{g}$ of vector fields in $\mathbb{R}^n$ is solvable, that is there is a basis $\{\mathcal{X}_1, \ldots, \mathcal{X}_n\} \subset \mathbb{R}^n$ of $\mathfrak{g}$ such that $[\mathcal{X}_1, \mathcal{X}_j] = c^1_{1,j}\mathcal{X}_1$, $[\mathcal{X}_2, \mathcal{X}_j] = c^1_{1,j}\mathcal{X}_1 + c^2_{2,j}\mathcal{X}_2$, $\ldots$, $[\mathcal{X}_n, \mathcal{X}_j] = \sum_{i=1}^n c^i_{i,j}\mathcal{X}_i$ for $j = 1, \ldots, n$ where $c^k_{i,j} \in \mathbb{R}$ are the structure constants of $\mathfrak{g}$, then the vector field $\mathcal{X}_1$ is integrable by quadratures. In fact [8] proves that under these hypothesis $\mathcal{X}_1$ is completely integrable admitting thus $n-1$ functionally independent first integrals. Recently [9] proves that actually all the vector fields $\mathcal{X}_j$ are integrable by quadratures. In particular, this means that if $\mathcal{Z}_i$ ($i = 1, \dots, n-1$) is a set of commuting vector fields which are independent almost everywhere in $\mathbb{R}^n$ and moreover $[\mathcal{Z}_j, \mathcal{X}] = c_j \mathcal{Z}_j$ with $c_j \in \mathbb{R}$ then $\mathcal{X}$ is integrable by quadratures.

It is interesting to remark the recent approach [10] showing an explicit formula for the first integral in terms of one normalizer and one Jacobi last multiplier (integrating factor in arbitrary dimension n), where there is no need of quadratures.

The work [6] focuses on the interplay between the structure of $\mathfrak{N}(\mathcal{X})$ and the property that $\mathcal{X}$ be analytically orbitally linearizable. That work generalizes one result of [7] in the planar case $n = 2$ to arbitrary dimension and to a more general linear part of the normalizer involved. We notice that in [4] it is collected a lot of interesting results connecting commuting vector fields (with $\Lambda \equiv 0$) and linearizability (hence with $f \equiv 1$).

Very interesting results which are close to the results presented here are given in [11] and [16] and [17]. We emphasize that every linear vector field is completely Darboux integrable, i.e., it has $n-1$ functionally independent Darboux first integrals, see [5]. Hence, clearly any analytically orbitally linearizable vector field is completely (of course not necessarily analytically) integrable. The next theorem proved in [17] characterizes the complete analytic integrability of vector fields with linear part having at least one nonzero eigenvalue.

Theorem 1.1 ([17]). *Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be an analytic vector field in a neighborhood of the origin in $\mathbb{R}^n$ with A having at least one nonzero eigenvalue. Then $\mathcal{X}$ is analytically completely integrable if and only if the $\mathbb{Z}^+$–resonant set Σ_λ of A has $n-1$ linearly independent elements and the distinguished normal form $\hat{\mathcal{X}}$ of $\mathcal{X}$ is analytically orbitally linear. More precisely, $\hat{\mathcal{X}} = \hat{H}\, \mathcal{X}_D$ where $\hat{H}(0) = 1$ and $\hat{H}(x)$ is an analytic function of x^k (in multiindex notation) with $k \in \Sigma_\lambda$. Here $D = \mathrm{diag}[\lambda_1, \dots, \lambda_n]$ where λ_i are all the eigenvalues of A for $i = 1, \dots, n$.*

Actually, [16] proves exactly the same former theorem but in the nondegenerate case, that is, with A having no eigenvalues equal to zero. Moreover, it implies that the linear part A of a completely analytically integrable vector field must be semisimple provided that it has at least one nonzero eigenvalue.

Remark 1.2. As mentioned before, following [3] we know that if $\mathcal{X} = \mathcal{X}_A + \cdots$ is completely integrable then the $\mathbb{Z}^+$–resonant set Σ_λ of A has $n-1$ linearly independent elements. This maximal $\mathbb{Z}^+$–resonant condition is only fulfilled when the spectrum $\sigma(A)$ of A is of the form $\sigma(A) \subset \nu\,\mathbb{Z}$ for some $\nu \in \mathbb{C}$. But taking into account moreover that $\mathcal{X}$ is a real vector field (thus the entries of A are real and so the possible complex eigenvalues of A appears in pairs together with its complex conjugate), it follows that actually $\sigma(A)$ is entirely composed by either real or pure imaginary eigenvalues, i.e., either $\sigma(A) \subset \mu\,\mathbb{Z}$ or $\sigma(A) \subset i\,\mu\,\mathbb{Z}$ with $i^2 = -1$ for some $\mu \in \mathbb{R}$.

Our first result is the following one.

Theorem 1.3. *Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be an analytic vector field in a neighborhood of the origin in $\mathbb{R}^n$ with A having at least one nonzero eigenvalue and such that $\mathcal{X}$ is analytically completely integrable. Then there exist $n-1$ independent normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i = 1, \ldots, n-1$ of the form $\mathcal{Z}_i = \mathcal{Z}_{B_i} + \cdots$ with linearly independent matrices B_i and satisfying $[\mathcal{X}, \mathcal{Z}_i] = \Lambda_i \mathcal{X}$ with Λ_i analytic first integrals of $\mathcal{X}$.*

In [6] it is proved that the analytical orbital linearizability of $\mathcal{X}$ forces the existence of normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i = 1, \ldots, n$ such that $\mathcal{Z}_i = \mathcal{Z}_{B_i} + \cdots$ where the matrices B_i are linearly independent. The following theorem improves this result in several ways and additionally no matter the linear parts of the normalizers involved.

Theorem 1.4. *Consider the analytic vector field $\mathcal{X} = \mathcal{X}_A + \cdots$ defined in a neighborhood of the origin in $\mathbb{R}^n$ with A having at least one nonzero eigenvalue. Then the following holds:*

(i) If $\mathcal{X}$ is analytically orbitally linearizable then there exist $n-1$ normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i = 1, \ldots, n-1$, such that the independent condition

$$\mathcal{X} \wedge \mathcal{Z}_1 \wedge \ldots \wedge \mathcal{Z}_{n-1} \neq 0$$

is satisfied almost everywhere in a neighborhood of the origin.

(ii) If there exist $n-1$ normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i = 1, \ldots, n-1$, such that the independent condition $\mathcal{X} \wedge \mathcal{Z}_1 \wedge \cdots \wedge \mathcal{Z}_{n-1} \neq 0$ is satisfied everywhere in a neighborhood of the origin then $\mathcal{X}$ is analytically orbitally linearizable.

The rest of the paper is dedicated to present the proofs of Theorems 1.3 and 1.4.

2. The proofs

Proof of Theorem 1.3. First we recall that the set $\mathcal{L}_c(\mathcal{X}_A) \subseteq \mathfrak{N}(\mathcal{X}_A)$ of linear centralizer (commutators) of the linear vector field $\mathcal{X}_A$ is defined as $\mathcal{L}_c(\mathcal{X}_A) = \{\mathcal{X}_B : [\mathcal{X}_A, \mathcal{X}_B] = 0\}$. Obviously $\mathcal{L}_c(\mathcal{X}_A)$ is a finite-dimensional real vector space. Actually $n \leq \dim \mathcal{L}_c(\mathcal{X}_A) \leq n^2$ where the upper bound is obvious from linear algebra and a proof of the lower bound can be found for instance in Lemma 2.1 in pag. 32 of [12]. Hence always exist n linearly independent matrices A and B_i for $i = 1, \ldots, n-1$ such that $\mathcal{X}_{B_i} \in \mathcal{L}_c(\mathcal{X}_A)$.

Now since $\mathcal{X} = \mathcal{X}_A + \cdots$ is analytic, A possesses at least one nonzero eigenvalue and moreover $\mathcal{X}$ is analytically completely integrable, from Theorem 1.1 we know that there is a distinguished normalization $\hat{\Phi}$ such that $\hat{\Phi}_* \mathcal{X} = \hat{\mathcal{X}} = \hat{H}\, \mathcal{X}_D$ with $D = \text{diag}[\lambda_1, \ldots, \lambda_n]$ where λ_i are the eigenvalues of A for all $i = 1, \ldots, n$. We emphasize that since $\hat{H}$ is an analytic function of a resonant monomial x^k (in multiindex notation) with $k \in \Sigma_\lambda$ then it follows that $\hat{H}$ is an analytic first integral of $\mathcal{X}_A$ in diagonal form . This is because straightforward computations show that any resonant monomial x^k with $k \in \Sigma_\lambda$ is a first integral of $\mathcal{X}_A$ because $\mathcal{X}_A(x^k) =< \lambda, k > x^k = 0$ for any semisimple matrix A in diagonal form $A = \text{diag}[\lambda_1, \ldots, \lambda_n]$.

As particular case of the former arguments we conclude with the existence of an analytic diffeomorphism Φ such that $\Phi_*\mathcal{X} = H\,\mathcal{X}_A$ where $H(0)=1$ and H is an analytic first integral of $\mathcal{X}_A$.

Using $\mathcal{X}_A(H)=0$ and $[\mathcal{X}_A, \mathcal{X}_{B_i}]=0$ we have

$$[\Phi_*\mathcal{X}, H\mathcal{X}_{B_i}] = [H\,\mathcal{X}_A, H\mathcal{X}_{B_i}] = -\mathcal{X}_{B_i}(H)\,H\,\mathcal{X}_A = -\mathcal{X}_{B_i}(H)\,\Phi_*\mathcal{X}.$$

Hence $H\mathcal{X}_{B_i} \in \mathfrak{N}(\Phi_*\mathcal{X})$. Recalling that $\mathcal{X}_{B_i}(H)$ are also analytic first integrals of $\mathcal{X}_A$ and pulling-back we obtain that $[\mathcal{X}, \tilde{\mathcal{Z}}_i] = \tilde{\Lambda}_i\,\mathcal{X}$ where $\tilde{\mathcal{Z}}_i = \Phi^*(H\,\mathcal{X}_{B_i}) = \tilde{H}\,\mathcal{Z}_i$, $\mathcal{Z}_i = \mathcal{X}_{B_i} + \cdots$ and $\tilde{H}$ is an analytic first integrals of $\mathcal{X}$ being additionally a unit at the origin. Moreover $\tilde{\Lambda}_i = \Phi^*(-\mathcal{X}_{B_i}(H))$ are also analytic first integrals of $\mathcal{X}$ whose underlying structure is $\tilde{\Lambda}_i = -\mathcal{Z}_i(\tilde{H})$.

In short we have $[\mathcal{X}, \mathcal{Z}_i] = \Lambda_i\,\mathcal{X}$ where $\Lambda_i = \tilde{\Lambda}_i/\tilde{H}_i$. Hence Λ_i is analytic near the origin an satisfies $\mathcal{X}(\Lambda_i)\equiv 0$. ■

Proof of Theorem 1.4. Assume that $\mathcal{X}$ is analytically orbitally linearizable in U. Then following [6] there are normalizers $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i=1,\ldots,n-1$ such that $\mathcal{Z}_i = \mathcal{Z}_{B_i} + \cdots$ where the matrices B_i are linearly independent. In particular, using properties of determinants we have that

$$\det\{\mathcal{X}, \mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}\} = \det\{\mathcal{X}_A, \mathcal{Z}_{B_1}, \ldots, \mathcal{Z}_{B_{n-1}}\} + \cdots \not\equiv 0$$

because $\det\{\mathcal{X}_A, \mathcal{Z}_{B_1}, \ldots, \mathcal{Z}_{B_{n-1}}\} \not\equiv 0$ since the matrices B_i and A are linearly independent. Therefore $\mathcal{X}\wedge\mathcal{Z}_1\wedge\ldots\wedge\mathcal{Z}_{n-1} \neq 0$ almost everywhere in U. This proves statement (i)

Conversely, assume the existence of the analytic vector fields $\mathcal{Z}_i \in \mathfrak{N}(\mathcal{X})$ for $i=1,\ldots,n-1$, with $\mathcal{X}\wedge\mathcal{Z}_1\wedge\ldots\wedge\mathcal{Z}_{n-1}\neq 0$ everywhere in the neighborhood $U\subseteq\mathbb{R}^n$ of the origin. Since U is a simply connected domain, it is a well known fact in differential geometry (see [15] for a proof) that the existence of the former $n-1$ normalizers $\mathcal{Z}_i$ of $\mathcal{X}$ such that $\{\mathcal{X}, \mathcal{Z}_1, \ldots, \mathcal{Z}_{n-1}\}$ are independent vector fields in U implies the existence of $n-1$ functionally independent analytic first integrals $\{H_1, \ldots, H_{n-1}\}$ of $\mathcal{X}$ in U, thus satisfying $\nabla H_1\wedge\ldots\wedge\nabla H_{n-1}\neq 0$ almost everywhere in U. In other words, $\mathcal{X}$ is analytically completely integrable in U. Now we are under the hypothesis of Theorem 1.1 so that $\mathcal{X}$ is analytically orbitally linearizable proving thus statement (ii). ■

Remark 2.1. May be there is a stronger version of Theorem 1.4 where (ii) is just the converse of (i), that is, where we can change the word "everywhere" by "almost everywhere" in statement (ii). But unfortunately, at the moment we do not have the right proof of this fact. Notice that if you only assume the existence of the normalizers $\mathcal{Z}_i\in\mathfrak{N}(\mathcal{X})$ for $i=1,\ldots,n-1$ satisfying $\mathcal{X}\wedge\mathcal{Z}_1\wedge\ldots\wedge\mathcal{Z}_{n-1}\neq 0$ almost everywhere in the simply connected domain U then you cannot assure the existence of $n-1$ functionally independent analytic first integrals of $\mathcal{X}$ in U as the following trivial example shows. Let $\mathcal{X} = x\partial_x + y\partial_y$ and $\mathcal{Z} = -y\partial_x + x\partial_y$ which clearly commute $[\mathcal{X}, \mathcal{Y}] = 0$ but $\mathcal{X}$ does not have any analytic (nor continuous) first integral in a neighborhood $U\subset\mathbb{R}^2$ of the origin. The reason is that $\mathcal{X}\wedge\mathcal{Z} = x^2+y^2$ and therefore it vanishes at the origin.

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