# Relationship between Nichols Braided Lie Algebras and Nichols algebras

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**Abstract.**     We establish the relationship among Nichols algebras, Nichols braided Lie algebras and Nichols Lie algebras. We prove two results: (i) The Nichols algebra $\mathfrak{B}(V)$ is finite-dimensional if and only if the Nichols braided Lie algebra $\mathfrak{L}(V)$ is finite-dimensional if there does not exist any $m$-infinity element in $\mathfrak{B}(V)$; (ii) the Nichols Lie algebra $\mathfrak{L}^-(V)$ is infinite dimensional if $D^-$ is infinite. We give sufficient conditions for the Nichols braided Lie algebra $\mathfrak{L}(V)$ to be a homomorphic image of a braided Lie algebra generated by $V$ with defining relations.
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## 1.    Introduction

The theory of Lie superalgebras has been developed systematically, which includes the representation theory and classifications of simple Lie superalgebras and their varieties [Ka77] . In many physical applications or in pure mathematical interest, one has to consider not only $\mathbb{Z}_2$– or $\mathbb{Z}$– grading but also G-grading of Lie algebras, where G is an abelian group equipped with a skew symmetric bilinear form given by a 2-cocycle. Lie algebras in symmetric and more general categories were discussed in [GRR95, Gu86, ZZ04]. A sophisticated multilinear version of the Lie bracket was considered in [Kh99a, Pa98]. Various generalized Lie algebras have already appeared under different names, e.g. Lie color algebras, $\epsilon$ Lie algebras [Sc79], quantum and braided Lie algebras [Ma94, KS97], generalized Lie algebras [BFM96] and H-Lie algebras [BFM01]. In [Ar11], a Milnor–Moore type theorem for primitively generated braided bialgebras was obtained by means of braided Lie algebras. The question of finite-dimensionality of Nichols algebras dominates an important part of the recent developments in the theory of (pointed) Hopf algebras. The interest in this problem comes from the Lifting Method by Andruskiewitsch

and Schneider to classify finite (Gelfand-Kirillov) dimensional pointed Hopf algebras, which are generalizations of quantized enveloping algebras of semi-simple Lie algebras. The classification of finite dimensional pointed Hopf algebras was studied in [AS02, AHS08, AS10, He05, He06a, He06b, WZZ].

This paper provides a new method to determine whether a Nichols algebra is finite dimensional or not.

Let $\mathfrak{B}(V)$ be the Nichols algebra of vector space $V$. Let $\mathfrak{L}(V)$, $\mathfrak{L}^-(V)$ and $\mathfrak{L}_c(V)$ denote the braided Lie algebras generated by $V$ in $\mathfrak{B}(V)$ under Lie operations $[x, y] = yx - p_{yx}xy$, $[x, y]^- = xy - yx$ and $[x, y]_c = xy - p_{xy}yx$, respectively, for any homogeneous elements $x, y \in \mathfrak{B}(V)$. $(\mathfrak{L}(V), [\,])$, $(\mathfrak{L}^-(V), [\,]^-)$ and $(\mathfrak{L}_c(V), [\,]_c)$ are called Nichols braided Lie algebra, Nichols Lie algebra and Nichols braided m-Lie algebra of $V$, respectively. It is clear that $(\mathfrak{L}(V), [\,])$ and $(\mathfrak{L}_c(V), [\,]_c)$ are equivalent as vector spaces. If $\mathfrak{B}(V)$ is finite dimensional then $\mathfrak{B}(V)$ is nilpotent, so $(\mathfrak{L}(V), [\,])$ and $(\mathfrak{L}^-(V), [\,]^-)$ also are nilpotent.

In this paper we prove the following two results: (i) $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)$ is finite-dimensional when there does not exist any $m$-infinity element; (ii) $\mathfrak{L}^-(V)$ is infinite dimensional if $D^-$ is infinite. We give the sufficient conditions for Nichols braided Lie algebra $\mathfrak{L}(V)$ to be a homomorphic image of a braided Lie algebra generated by $V$ with defining relations.

This paper is organized as follows. In section 2 we recall some results on Nichols algebras and fix the notation. In section 3 we show that $\mathfrak{L}^-(V)$ is infinite dimensional if $D^-$ is infinite. In section 4 we prove that $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)$ is finite-dimensional when there does not exist any $m$-infinity element in $\mathfrak{B}(V)$. In section 5 we present the condition for $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$. In section 6 we give the sufficient conditions for Nichols braided Lie algebra $\mathfrak{L}(V)$ to be a homomorphic image of a braided Lie algebra generated by $V$ with defining relations.

Throughout, $\mathbb{Z} =: \{x | x \text{ is an integer }\}$. $\mathbb{N}_0 =: \{x | x \in \mathbb{Z}, x \geq 0\}$. $\mathbb{N} =: \{x | x \in \mathbb{Z}, x > 0\}$. $F$ denotes the base field of characteristic zero.

## 2.   Preliminaries

In this section we recall some results on Nichols algebras ( see [AHS08]).

**Lemma 2.1.**     *(see [AHS08])If $(V, \alpha, \delta)$ is a $FG$-YD module, then tensor algebra $T(V)$ over $V$ is a $FG$-YD module.*

If $\{x_1, \cdots, x_n\}$ is a basis of vector space $V$ and $C(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ with $q_{ij} \in F$, then $V$ is called a braided vector space of diagonal type, $\{x_1, \cdots, x_n\}$ is called canonical basis and $(q_{ij})_{n \times n}$ is called braided matrix. Throughout this paper all of braided vector spaces are connected and of diagonal type without special announcement. Let $G = \mathbb{Z}^n$ and

$$E = \{e_1, e_2, \cdots, e_n\}, e_i =: (\overbrace{0, \cdots 0, 1}^{i}, \cdots 0) \in G \ , \ 1 \leq i \leq n.$$

Let $\chi$ be a bicharacter of $G$ such that $\chi(e_i, e_j) = p_{ij}$ and $C(x_i \otimes x_j) = \chi(e_i, e_j)x_j \otimes x_j$. Let $S_m \in End_k(T(V)^m)$ and $S_{1,j} \in End_k(T(V)^{j+1})$ denote the maps

$$S_m = \prod_{j=1}^{m-1} (id^{\otimes m-j-1} \bigotimes S_{1,j}),$$

$$S_{1,j} = id + C_{12}^{-1} + C_{12}^{-1}C_{23}^{-1} + \cdots + C_{12}^{-1}C_{23}^{-1} \cdots C_{j,j+1}^{-1}$$

(in leg notation) for $m \geq 2$ and $j \in \mathbb{N}$. Then the subspace $S = \bigoplus_{m=2}^{\infty} ker S_m$ of the tensor $T(V) = \bigoplus_{m=0}^{\infty} T(V)^{\otimes m}$ is a two-sided ideal, and algebra $\mathfrak{B}(V) = T(V)/S$ is termed the Nichols algebra associated to $(V, C)$. Define linear map $p$ from $\mathfrak{B}(V) \otimes \mathfrak{B}(V)$ to $F$ such that $p(u \otimes v) = \chi(deg(u), deg(v))$, for any homogeneous element $u, v \in \mathfrak{B}(V)$. For convenience, $p(u \otimes v)$ is denoted by $p_{uv}$. Let $A =: \{x_1, x_2, \cdots, x_n\}$ be an alphabet, $A^*$ the set of all of words in $A$ and $A^+ =: A^* \setminus 1$. Define $x_1 < x_2 < \cdots < x_n$ and the order on $A^*$ is the lexicographic ordering. For the concept of words refer [Lo83]. Let $|u|$ denote the length of word $u$.

**Definition 2.2.** ([Kh99b, Def. 1]) A word $u$ is called a Lyndon word if $|u| = 1$ or $|u| \geq 2$, and for each representation $u = u_1 u_2$, where $u_1$ and $u_2$ are nonempty words, the inequality $u < u_2 u_1$ holds.

Any word $u \in A^*$ has a unique decomposition into the product of non-increasing sequence of Lyndon words by [Lo83, Th.5.1.5]. If $u$ is a Lyndon word with $|u| > 1$, then there uniquely exist two Lyndon words $v$ and $w$ such that $u = vw$ and $v$ is shortest (see [Lo83, Prop. 5.1.3] and [He07]) (the composition is called the Shirshov decomposition of $u$).

**Definition 2.3.** We inductively define a linear map $[\ ]$ from $A^+$ to $\mathfrak{B}(V)$ as follows: (1) $[u] =: u$ when $u$ is a letter; (2) $[u] =: [w][v] - p_{wv}[v][w]$ when $u$ is a Lyndon word with $|u| > 1$ and $u = vw$ is a Shirshov decomposition; (3) $[u] =: [[[l_1, l_2], l_3] \cdots l_m]$, when $u = l_1 l_2 \cdots l_m$ is a non-increasing product of Lyndon words, i.e. $l_1 \geq l_2 \geq l_3 \geq \cdots \geq l_m$, and $l_i$ is a Lyndon word for any $1 \leq i \leq m$.

Similarly, we inductively define a linear map $[\ ]^-$ from $A^+$ to $\mathfrak{B}(V)$ as follows: (1) $[u]^- =: u$ when $u$ is a letter; (2) $[u]^- =: [w]^-[v]^- - [v]^-[w]^-$ when $u$ is a Lyndon word with $|u| > 1$ and $u = vw$ is a Shirshov decomposition; (3) $[u]^- =: [[[l_1, l_2]^-, l_3]^- \cdots l_m]^-$, when $u = l_1 l_2 \cdots l_m$ is a non-increasing product of Lyndon words.

$[u]$ is called a nonassociative word for any $u \in A^+$, $[u]$ is called a standard nonassociative word if $u$ is a Lyndon word. Every standard nonassociative word is also called a super-letter.

**Definition 2.4.** ([Kh99b, Def. 6]) A super-letter $[u]$ is said to be hard if it is not a linear combination of products $[u_1][u_2] \cdots [u_i], i \in \mathbb{N}$, where $[u_j]$ are super-letter with $[u] < [u_j], 1 \leq j \leq i$.

**Definition 2.5.** ([Kh99b, Def. 7] or [He07, before Th. 10]) We say that the height of a super-letter $[u]$ with degree $d$ equals a natural number $h$ if $h$ is least with the following properties:

(1) $p_{uu}$ is a primitive root of unity of degree $t \geq 1$, and $h = t$ ;

(2) super-word $[u]^h$ is a linear combination of super-words of degree $hd$ in greater super-letters than $[u]$.

If the number $h$ with above properties does not exist then we say that the height of $[u]$ is infinite.

Let $h_u$ denote the height of $u$. Let $\mathrm{ord}(p_{uu})$ denote the order of $p_{uu}$ with respect to multiplication. We define $D =: \{[u] | [u]$ is a hard super-letter $\}$. If $[u] \in D$ and $\mathrm{ord}(p_{u,u}) = m > 1$ with $h_u = \infty$, then $[u]$ is called an $m$-infinity element. We now let $P$ denote the set of all $[u_1]^{k_1}[u_2]^{k_2} \cdots [u_s]^{k_s}$ such that $[u_i] \in D$, $k_i, s \in \mathbb{N}_0$, $0 \leq k_i < h_{u_i}$ where $1 \leq i \leq s$, $u_s < u_{s-1} < \cdots < u_1$. Next $\Delta^+(\mathfrak{B}(V)) := \{\deg(u) \mid [u] \in D\}$; now $\Delta(\mathfrak{B}(V)) := \Delta^+(\mathfrak{B}(V)) \cup \Delta^-(\mathfrak{B}(V))$, which is called the root system of $V$. If $\Delta(\mathfrak{B}(V))$ is finite, then it is called an arithmetic root system. Let
$$E'_e =: \tfrac{1}{2}|\{([u],[v]) \in D \times D \mid [u],[v] \in D, p_{u,v}p_{v,u} \neq 1\}|.$$
Let $D^- := \{[u]^- \mid [u] \in D\}$ and $\deg(D^-) := \{\deg([u]^-) \mid [u] \in D\}$. Let $E_e$ denote the number of edges of generalized Dynkin diagram. If $u = vw$ is the shirshov decomposition of $[u] \in D$, then $[v], [w] \in D$, which are called sons of $u$. If $[u_1], [u_2], \cdots, [u_m] \in D$ and $u_{i+1}$ is a son of $u_i$ for $1 \leq i \leq m-1$, then $u_2, u_3, \cdots, u_m$ are called descendants of $u_1$.

**Remark 2.6.**    There does not exist any $m$-infinity element in $\mathfrak{B}(V)$ if and only if Property (P) in [He05, Section 2.2] holds.

**Theorem 2.7.**    *([Kh99b, Th. 2] or [He07, Th. 10]) $P$ is a basis of $\mathfrak{B}(V)$.*

## 3.    Relationship between Nichols algebras and Nichols Lie algebras

In this section it is proved that the Nichols Lie algebra $\mathfrak{L}^-(V)$ is infinite dimensional if $D^-$ is infinite.

**Lemma 3.1.**    *Assume that $u = vw$ is a Shirshov decomposition of $u$. If $[u] \in D$, then $[[v],[w]]^- \neq 0$. Furthermore, if $[v],[w] \in \mathfrak{L}^-(V)$ (e.g. $\mid u \mid = 2$), then $[u]^- \neq 0$.*

**Proof.**    If $[[v],[w]]^- = 0$, then $[v][w] = [w][v]$, we know $[[v],[w]] = [w][v] - p_{wv}[v][w] = (1 - p_{wv})[w][v]$, it contradicts to $[u] \in P$ and $[w][v] \in P$. ∎

**Theorem 3.2.**    *(i) $\dim \mathfrak{L}^-(V) \geq \mid \deg(D^-) \mid -1 \geq n + E_e - 1$, where $E_e$ is the number of edges in generalized Dynkin diagram of $V$. (ii) If $D^-$ is infinite, then $\dim \mathfrak{L}^-(V) = \infty$.*

**Proof.**    If $u_1, u_2, \cdots, u_m$ in $D^- \setminus 0$ with different degrees, then $u_1, u_2, \cdots, u_m$ is linearly independent. It is clear that $D^- \subseteq L^-(V)$. Consequently,
$$\dim L^-(V) \geq \mid \deg(D^-)\mid.$$
Obviously, there exists a line between $x_i$ and $x_j$ if and only if $[x_i x_j] \in D$ with

$i < j$, which implies $| \deg(D^-) | - 1 \geq n + E_e - 1$. ∎

## 4.  Relationship between Nichols algebras and Nichols braided Lie algebras

In this section it is proved that $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)$ is finite-dimensional when there does not exist any $m$-infinity element in $\mathfrak{B}(V)$. Let $l_u(v) := [u, v]$ and $r_u(v) := [v, u]$ for any $u, v \in \mathfrak{B}(V)$.

**Lemma 4.1.**    *If $[u]$ is a nonassociative word, then $[u] \in \mathfrak{L}(V)$.*

**Proof.**    By the definition of nonassociative words, we have $[u] \in \mathfrak{L}(V)$. ∎

**Remark 4.2.**    If $|D| = \infty$, then $\dim \mathfrak{L}(V) = \infty$.

**Lemma 4.3.**    *If $[u]$ is a nonassociative word with $t \in \mathbb{N}$, $t \leq ord(p_{uu})$, then $[u]^t \in \mathfrak{L}(V)$.*

**Proof.**    Let $l^0_{[u]}[u] =: [u], l^i_{[u]}[u] =: [[u], l^{i-1}_{[u]}[u]], i \geq 1$. Obviously $l^i_{[u]}[u] \in \mathfrak{L}(V)$. It is clear $l^1_{[u]}[u] = [[u], [u]] = [u]^2 - p_{uu}[u]^2 = (1 - p_{uu})[u]^2$. By means of induction, we obtain $l^k_{[u]}[u] = (1 - p_{uu})(1 - p^2_{uu}) \cdots (1 - p^k_{uu})[u]^{k+1}, \forall\, 1 < k \in \mathbb{N}$. We have $(1 - p_{uu})(1 - p^2_{uu}) \cdots (1 - p^{t-1}_{uu}) \neq 0$ since $t \leq ord(p_{uu})$, which implies $[u]^t \in \mathfrak{L}(V)$. ∎

**Theorem 4.4.**    *If there does not exist any $m$-infinity element in $\mathfrak{B}(V)$ and $1 < ord(p_{uu}) < \infty$ for any $u \in D$, then the following conditions are equivalent: (i) $\mathfrak{B}(V)$ is finite-dimensional; (ii) $\mathfrak{L}(V)$ is finite-dimensional; (iii) $\Delta(\mathfrak{B}(V))$ is an arithmetic root system.*

**Proof.**    It follows from [He05, Section 2.2] that (i) and (iii) are equivalent. $(i) \implies (ii)$. Assume that $\mathfrak{B}(V)$ is finite-dimensional. Since $\mathfrak{L}(V) \subseteq \mathfrak{B}(V)$, we have that $\mathfrak{L}(V)$ are finite-dimensional. $(ii) \implies (i)$. Assume that $\mathfrak{L}(V)$ is finite-dimensional. By Lemma 4.1, $D \subseteq \mathfrak{L}(V)$. Obviously, $D \subseteq P$. Therefore $D$ is linearly independent and $|D| \leq \dim \mathfrak{L}(V) < \infty$, $h_u < \infty$ since $1 < ord(p_{uu}) < \infty$ for $[u] \in D$. It follows from Theorem 2.7 that $\dim \mathfrak{B}(V) < \infty$. ∎

**Proposition 4.5.**    *Assume that $V$ is a Cartan type with generalized Cartan matrix $(a_{ij})_{n \times n}$ and $1 < ord(p_{uu}) < \infty$ for any $[u] \in D$. If there does not exist any $m$-infinity element in $\mathfrak{B}(V)$, then the following conditions are equivalent. (i) $\mathfrak{L}(V)$ is finite dimensional; (ii) $(a_{ij})_{n \times n}$ is a Cartan matrix; (iii) $\dim \mathfrak{B}(V) < \infty$.*

**Proof.**    It follows from [He05, Th. 2.10.2], Theorem 4.4 and Lemma 4.1. ∎

**Proposition 4.6.**    *If there exists $[u] \in D$ such that $ord(p_{uu}) = \infty$, then $\dim \mathfrak{B}(V) = \infty$ and $\dim \mathfrak{L}(V) = \infty$.*

**Proof.**    By Theorem 2.7, $\dim \mathfrak{B}(V) = \infty$. By Lemma 4.3, $\dim \mathfrak{L}(V) = \infty$. ∎

**Proposition 4.7.**　*If there exist* $[u], [v]$ *such that* $p_{uu}^i p_{uv} p_{vu} \neq 1$ *for* $\forall\, 0 \leq i \leq 2k-2$, $\forall\, k \in \mathbb{N}$, *then* $[v][u]^k, [u][v][u]^{k-1}, \dots, [u]^k[v] \in \mathfrak{L}(V)$.

**Proof.**　We first show

$$p_{uu}^{k-t+i} p_{uv} r_{[u]}^{t-i-1}([u]^i l_{[u]}^{k-t+1}[v]) + r_{[u]}^{t-i}([u]^i l_{[u]}^{k-t}[v])$$
$$= (1 - p_{uu}^{2(k-t)+i} p_{uv} p_{vu}) r_{[u]}^{t-i-1}([u]^{i+1} l_{[u]}^{k-t}[v]) \qquad (4.1)$$

for $\forall\, 1 \leq t \leq k$, $\forall\, 0 \leq i \leq t-1$. In fact,

$$\text{left hand side of (4.1)} = p_{uu}^{k-t+i} p_{uv} r_{[u]}^{t-i-1}([u]^i[[u], l_{[u]}^{k-t}[v]]) + r_{[u]}^{t-i-1}([[u]^i l_{[u]}^{k-t}[v], [u]])$$

$$= r_{[u]}^{t-i-1}\left( p_{uu}^{k-t+i} p_{uv}[u]^i l_{[u]}^{k-t}[v][u] - p_{uu}^{k-t+i} p_{uv}[u]^i l_{[u]}^{k-t}[v][u] \right.$$

$$\left. - p_{uu}^{k-t+i} p_{uv} p_{uu}^{k-t} p_{vu}[u]^{i+1} l_{[u]}^{k-t}[v] + [u]^{i+1} l_{[u]}^{k-t}[v] \right)$$

$$= (1 - p_{uu}^{2(k-t)+i} p_{uv} p_{vu}) r_{[u]}^{t-i-1}([u]^{i+1} l_{[u]}^{k-t}[v])$$

$$= \text{right hand side of (4.1).}$$

Let $B^{(i)} = (b_{rs}^{(i)})_{(k-i+1) \times (k-i+1)}$ be real matrices such that

$$\begin{pmatrix} [u]^i l_{[u]}^{k-i}[v] \\ r_{[u]}^1([u]^i l_{[u]}^{k-i-1}[v]) \\ r_{[u]}^2([u]^i l_{[u]}^{k-i-2}[v]) \\ \cdots \\ r_{[u]}^{k-1}([u]^i[v]) \end{pmatrix} = B^{(i)} \begin{pmatrix} [u]^i[v][u]^{k-i} \\ [u]^{i+1}[v][u]^{k-i-1} \\ [u]^{i+2}[v][u]^{k-i-2} \\ \cdots \\ [u]^k[v] \end{pmatrix} \quad \text{and } b_{11}^{(i)} = 1 \text{ for }$$

$0 \leq i \leq t-1$.

We know $|B^{(i)}| = \prod_{t=i+1}^{k} (1 - p_{uu}^{2(k-t)+i} p_{uv} p_{vu}) |B^{(i+1)}|$ by (4.1). Consequently,

$|B^{(0)}| = \prod_{i=0}^{k-1} \prod_{t=i+1}^{k} (1 - p_{uu}^{2(k-t)+i} p_{uv} p_{vu})$, and $|B^{(0)}| = 0$ is equivalent to

$\prod_{i=0}^{2k-2} (1 - p_{uu}^i p_{uv} p_{vu}) = 0$. This completes the proof. ∎

According to the above Proposition, we obtain immediately,

**Corollary 4.8.**　*If there exist* $[u], [v] \in D$ *such that* $p_{u,u} = 1$ *and* $p_{uv} p_{vu} \neq 1$, *then* $\dim \mathfrak{B}(V) = \infty$, $\dim \mathfrak{L}(V) = \infty$.

**Corollary 4.9.**　*If* $[u], [v] \in D$, *such that* $p_{u,u} = 1$ *and* $p_{uv} p_{vu} = 1$, *then* $d_1 d_2 \cdots d_k[v] \in Fl_{[u]}^k[v]$ *for* $\forall\, d_i = l_{[u]}$ *or* $r_{[u]}, 1 \leq i \leq k$.

**Proof.**　We know

$$p_{uv} r_{[u]}^s l_{[u]}^{k-s}[v] + r_{[u]}^{s+1} l_{[u]}^{k-s-1}[v] = (1 - p_{uv} p_{vu})[u] r_{[u]}^s l_{[u]}^{k-s-1}[v] \qquad (4.2)$$

for $\forall\, 0 \leq s < k$ by Definition 2.4. Then $l_{[u]}^k[v] = -p_{vu} r_{[u]}^1 l_{[u]}^{k-1}[v] = \dots = (-p_{vu})^k r_{[u]}^k[v]$. On the other hand, $r_{[u]} l_{[u]}[v] = l_{[u]} r_{[u]}[v]$. This proves the corollary. ∎

**Proposition 4.10.**     *If there exists $[u] \in D$ such that $p_{uv}p_{vu} = 1$ for $\forall v \in D$ with $[u] \neq [v]$, then $\Delta(\mathfrak{B}(V))$ is not an arithmetic root system while $\mathfrak{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$.*
     *Moreover, $\dim \mathfrak{B}(V) = \infty$, $\dim \mathfrak{L}(V) = \infty$.*

**Proof.**     There exists a basis $\pi$ of $\Delta(\mathfrak{B}(V))$ such that $\deg u \in \pi$. Since $\mathfrak{B}(V)$ is connected Nichols algebra and $n = \mid \pi \mid > 1$, there exists $\beta \in \pi \setminus \{\deg u\}$ such that $\chi(\deg u, \beta)\chi(\beta, \deg u) \neq 1$. By the definition of $\Delta(\mathfrak{B}(V))$ there exists $[v] \in D \setminus \{u\}$ with $\deg v \in \{\beta, -\beta\}$. This yields a contradiction to $p_{uv}p_{vu} = 1$. ∎

**Theorem 4.11.**     *If $\mathfrak{B}(V)$ is connected Nichols algebra of diagonal type with $\dim V > 1$ and there does not exist any $m$-infinity elements, then $\mathfrak{B}(V)$ is finite-dimensional if and only if $\mathfrak{L}(V)$ is finite-dimensional.*

**Proof.**     It follows from Proposition 4.4, Proposition 4.6 , Corollary 4.8 and Proposition 4.10. ∎

By [ZZ04], $(\mathfrak{B}(V), [\ ]_c)$ is a braided m-Lie algebra and we have the braided Jacobi identity as follows:

$$[[u, v], w] = [u, [v, w]] + p_{vw}^{-1}[[u, w], v] + (p_{wv} - p_{vw}^{-1})v \cdot [u, w]. \tag{4.3}$$

**Lemma 4.12.**     *If $u$ and $v$ are homogeneous elements in $\mathfrak{L}(V)$ with $p_{uv}p_{vu} \neq 1$, then $uv, vu \in \mathfrak{L}(V)$. Furthermore, if $u, v \in \mathfrak{L}(V)$, then $[u, v]^- \in \mathfrak{L}(V)$.*

**Proof.**     $[u, v] = vu - p_{v,u}uv$ and $[v, u] = uv - p_{u,v}vu$, which implies that $uv$ and $vu$ are a linear combination of $[u, v]$ and $[v, u]$. ∎

**Proposition 4.13.**     $\dim \mathfrak{L}(V) \geq \sum_{[u] \in D}(h_u - 1) + E'_e$.

**Proof.**     It follows from Lemma 4.3 and Lemma 4.1. ∎

Recall the dual $\mathfrak{B}(V^*)$ of Nichols algebra $\mathfrak{B}(V)$ of rank $n$ in [He05, Section 1.3] and [He06b]. Let $y_i$ be a dual basis of $x_i$. $\delta(y_i) = g_i^{-1} \otimes y_i$, $g_i \cdot y_j = p_{ij}^{-1}y_j$ and $\Delta(y_i) = g_i^{-1} \otimes y_i + y_i \otimes 1$. There exists a bilinear map $\langle, \rangle$ from $(\mathfrak{B}(V^*)\#kG) \times \mathfrak{B}(V)$ to $\mathfrak{B}(V)$ such that $\langle y_i, uv \rangle = \langle y_i, u \rangle v + g_i^{-1}.u\langle y_i, v \rangle$ and $\langle y_i, \langle y_j, u \rangle \rangle = \langle y_i y_j, u \rangle$ for any $u, v \in \mathfrak{B}(V)$. Furthermore, for any $u \in \oplus_{i=1}^{\infty}\mathfrak{B}(V)_{(i)}$, one has that $u = 0$ if and only if $\langle y_i, u \rangle = 0$ for any $1 \leq i \leq n$. Let $i$ denote $x_i$ in short, sometimes.

**Lemma 4.14.**     *Let $l_i^0[j] = [j]$ , $l_i^k[j] = [i, l_i^{k-1}[j]]$ , $r_i^0[j] = [j]$ , $r_i^k[j] = [r_i^{k-1}[j], i]$ , $k \geq 1$ . Then we have*
*(i) $\langle y_j, l_i^k[j] \rangle = 0, \langle y_i, r_i^k[j] \rangle = 0, \forall k \geq 1$.*
*(ii) The following conditions are equivalent:*

$$\text{(1) } l_i^k[j] = 0. \quad \text{(2) } r_i^k[j] = 0. \quad \text{(3) } (k)_{p_{ii}}^! \prod_{t=0}^{k-1}(p_{ii}^t p_{ji}p_{ij} - 1) = 0.$$

**Proof.**    (i) It is clear that $\langle y_j, l_i^k[j] \rangle = 0$, and

$\langle y_i, r_i^k[j] \rangle = \langle y_i, i r_i^{k-1}[j] - p_{ii}^{k-1} p_{ij} r_i^{k-1}[j]i \rangle = r_i^{k-1}[j] - p_{ii}^{k-1} p_{ij} p_{ii}^{-(k-1)} p_{ij}^{-1} r_i^{k-1}[j] = 0$.

    (ii) By means of induction, we obtain

$$\langle y_i, l_i^k[j] \rangle = p_{ii}^{-(k-1)} p_{ij}^{-1} (1 - p_{ii}^{k-1} p_{ij} p_{ji})(1 + p_{ii} + \cdots p_{ii}^{k-1}) l_i^{k-1}[j],$$

    then

$$\langle y_j y_i^k, l_i^k[j] \rangle = p_{ii}^{-\sum\limits_{i=1}^{k-1} i} p_{ij}^{-k} \prod_{t=0}^{k-1} (1 - p_{ii}^t p_{ij} p_{ji})(1 + p_{ii} + \cdots p_{ii}^t),$$

    and so (1) is equivalent to (3) by (i). On the other hand,

$$\langle y_j, r_i^k[j] \rangle = p_{ji}^{-k} \prod_{t=0}^{k-1} (1 - p_{ii}^t p_{ij} p_{ji})[i]^k,$$

$$\langle y_i^k y_j, r_i^k[j] \rangle = p_{ii}^{-\sum\limits_{i=1}^{k-1} i} p_{ji}^{-k} \prod_{t=0}^{k-1} (1 - p_{ii}^t p_{ij} p_{ji})(1 + p_{ii} + \cdots p_{ii}^t).$$

One knows that (2) is equivalent to (3) by (i). This proves the lemma.    ∎

## 5.    Conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

In this section we give sufficient conditions for $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

**Lemma 5.1.**    ([WZZ, Lemma 3.1]) (i) If $|u| = |v|$, then $u < v$ if and only if $uw < vw$. (ii) If $u = vw$ is the Shirshov decomposition of Lyndon word $u$ and $[u]$ is hard, then both $[v]$ and $[w]$ are hard, too.

**Lemma 5.2.**    If there exist $x_i, x_j, i \neq j$ such that $p_{ij} p_{ji} = 1$, then $\mathfrak{B}(V) \neq F \oplus \mathfrak{L}(V)$.

**Proof.**    It is clear $[x_i, x_j] = [x_j, x_i] = 0$ and $x_i x_j = p_{ij} x_j x_i \neq 0$. Then $x_i x_j$ or $x_j x_i \in P$ and $x_i x_j, x_j x_i \notin \mathfrak{L}(V)$.    ∎

**Corollary 5.3.**    If $\mathfrak{B}(V)$ is connected Nichols algebra of rank $> 3$ of diagonal type and $\Delta(\mathfrak{B}(V))$ is arithmetic root systems, then $\mathfrak{B}(V) \neq F \oplus \mathfrak{L}(V)$.

**Proof.**    It is clear from [He05, Table A.1], [He05, Table A.2], [He06a, Table B] and [He06a, Table C].    ∎

**Example 5.4.**    If $\overset{\zeta}{\bullet}\overset{-\zeta}{\rule{2cm}{0.4pt}}\overset{-1}{\bullet}$ ,   $\zeta \in R_3$, then
$D = \{[x_1], [x_2], [x_1, x_2], [x_1, [x_1, x_2]]\}, \dim \mathfrak{B}(V) = 2^2 3^2$ and $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$.

**Proof.**    Assume that $[u]$ is a hard super-letter or zero and $u = vw$ is the Shirshov decomposition of $u$ when $[u] \neq 0$. We show $[u] \in D$ step by step for the length $|u|$ of $u$.

    (a) $|u| = 2$, then $u = [1, 2]$ by Lemma 4.14.

    (b) $|u| = 3$, then $[u] = [1, [1, 2]]$ and $[[1, 2], 2] = 0$ by Lemma 4.14.

    (c) $|u| = 4$. then $[1, [1, [1, 2]]] = 0$ by Lemma 4.14.

(d) $|u| = 5$, then $<y_1, [[1, [1,2]], [1,2]]>$

$$= \; <y_1, [1,2][1,[1,2]] - p_{11}^2 p_{12} p_{21}^2 p_{22}[1,[1,2]][1,2] >$$
$$= \; p_{12}^{-1}(1 - p_{12}p_{21})2[1,[1,2]]$$
$$+ p_{11}^{-1}p_{12}^{-1}[1,2]p_{11}^{-1}p_{12}^{-1}(1 - p_{11}p_{12}p_{21})(1 + p_{11})[1,2]$$
$$- p_{11}^2 p_{12} p_{21}^2 p_{22} p_{11}^{-1} p_{12}^{-1}(1 - p_{11}p_{12}p_{21})(1 + p_{11})[1,2][1,2]$$
$$- p_{11}^2 p_{12} p_{21}^2 p_{22} p_{11}^{-2} p_{12}^{-1}[1,[1,2]]p_{12}^{-1}(1 - p_{12}p_{21})2$$
$$= \; p_{12}^{-1}(1 - p_{12}p_{21})[[1,[1,2]],2]$$
$$+ (p_{11}^{-1}p_{12}^{-1} - p_{11}^2 p_{12} p_{21}^2 p_{22})p_{11}^{-1}p_{12}^{-1}$$
$$\times (1 - p_{11}p_{12}p_{21})(1 + p_{11})[1,2]^2$$
$$= \; p_{12}^{-1}(1 - p_{12}p_{21})\left\{ p_{12}^{-1}p_{22}^{-1}[[1,2],[1,2]] \right.$$
$$\left. + (p_{21}p_{22} - p_{12}^{-1}p_{22}^{-1})[1,2]^2 \right\}$$
$$+ (1 - p_{11}^3 p_{12}^2 p_{21}^2 p_{22})p_{11}^{-2}p_{12}^{-2}(1 - p_{11}p_{12}p_{21})(1 + p_{11})[1,2]^2$$
$$= \; p_{12}^{-1}(1 - p_{12}p_{21})(p_{21}p_{22} - p_{11}p_{21})[1,2]^2$$
$$+ (1 - p_{11}^3 p_{12}^2 p_{21}^2 p_{22})p_{12}^{-2}$$
$$\times (p_{11}^{-2} + p_{11}^{-1} - p_{11}^{-1}p_{12}p_{21} - p_{12}p_{21})[1,2]^2$$
$$= \; p_{12}^{-2}\{p_{12}p_{21}(1 - p_{12}p_{21})(p_{22} - p_{11})$$
$$+ (1 - p_{11}^3 p_{12}^2 p_{21}^2 p_{22})$$
$$\times (p_{11}^{-2} + p_{11}^{-1} - p_{11}^{-1}p_{12}p_{21} - p_{12}p_{21})\}[1,2]^2$$
$$= \; p_{12}^{-2}\left(p_{11}^{-2}(1 - p_{11}p_{12}p_{21})(1 - p_{11}^2 p_{12}p_{21})(1 + p_{11}^2 p_{12}p_{21}p_{22})\right.$$
$$\left. + p_{11}^{-1}(1 - p_{11}^2 p_{12}p_{21})(1 - p_{11}p_{12}^2 p_{21}^2 p_{22})\right)[1,2]^2$$
$$= \; p_{12}^{-2}(1 - p_{11}^2 p_{12}p_{21})\{p_{11}^{-2}(1 - p_{11}p_{12}p_{21})(1 + p_{11}^2 p_{12}p_{21}p_{22})$$
$$+ p_{11}^{-1}(1 - p_{11}p_{12}^2 p_{21}^2 p_{22})\}[1,2]^2$$
$$= \; p_{12}^{-2}(1 - p_{11}^2 p_{12}p_{21})\{p_{11}^{-2}(1 - p_{11}p_{12}p_{21} + p_{11})$$
$$+ p_{12}p_{21}p_{22}(1 - p_{11}p_{12}p_{21} - p_{12}p_{21})\}[1,2]^2$$
$$= \; p_{12}^{-2}(1 - p_{11}^2 p_{12}p_{21})(1 + p_{11})(p_{11} - p_{11}^2 p_{12}p_{21} + p_{12}^2 p_{21}^2)[1,2]^2$$
$$= \; p_{12}^{-2}(1 - p_{11}^2 p_{12}p_{21})(1 + p_{11})(\zeta + \zeta^2\zeta + \zeta^2)[1,2]^2 = 0.$$

(e) $|u| = 6$. $[u]$ does not exist. Then we show that
$$D = \{[u_4] = [1], [u_1] = [2], [u_2] = [1,2], [u_3] = [1,[1,2]]\},$$
$$p_{u_1 u_2}p_{u_2 u_1} = -\zeta, p_{u_1 u_3}p_{u_3 u_1} = \zeta^2, p_{u_1 u_4}p_{u_4 u_1} = -\zeta, p_{u_2 u_3}p_{u_3 u_2}$$
$$= -\zeta, p_{u_2 u_4}p_{u_4 u_2} = -1, p_{u_3 u_4}p_{u_4 u_3} = -\zeta^2,$$
and
$$p_{u_i u_i} = -1, \zeta^2, -1, \zeta, \; ord(p_{u_i u_i}) = 2, 3, 2, 3, \; i = 1, 2, 3, 4.$$
Considering Lemma 4.12, we have

$$P \setminus \{1\} = \{u_1, u_2, u_2^2 = u_2 u_2, u_3, u_4, u_4^2 = u_4 u_4, u_1 u_2, u_1 u_2^2, u_1 u_3, u_1 u_4, u_1 u_4^2, u_2 u_3,$$
$$u_2^2 u_3, u_2 u_4, u_2^2 u_4 = u_2(u_2 u_4), u_2 u_4^2 = (u_2 u_4)u_4, u_2^2 u_4^2 = (u_2^2 u_4)u_4, u_3 u_4,$$
$$u_3 u_4^2, (u_1 u_2)u_3, u_1 u_2^2 u_3 = (u_1 u_2^2)u_3, (u_1 u_2)u_4, (u_1 u_2^2)u_4, (u_1 u_2)u_4^2,$$
$$u_1(u_2^2 u_4^2), u_1(u_3 u_4), u_1(u_3 u_4^2), (u_2 u_3)u_4, (u_2^2 u_3)u_4, (u_2 u_3)u_4^2, (u_2^2 u_3)u_4^2,$$
$$(u_1 u_2 u_3)u_4, u_1(u_2 u_3 u_4^2), u_1(u_2^2 u_3 u_4), u_1 u_2^2 u_3 u_4^2 = (u_1 u_2)u_2 u_3 u_4^2\}.$$

Thus $\mathfrak{B}(V) = F \oplus \mathfrak{L}(V)$. ∎

**Proposition 5.5.** *Assume that $\mathfrak{B}(V)$ is connected Nichols algebra of diagonal type and $\Delta(\mathfrak{B}(V))$ is arithmetic root systems. If $u, v, w \in D$ with $\deg u = \deg v + \deg w$ ( specially, if $u = vw$ is the Shirshov decomposition of $u \in D$), then $p_{wv}p_{vw} \neq 1$ except for the following cases:*

   (i)  $p_{ww} = p_{vv}, p_{vv} \neq \pm 1$;

   (ii)  $p_{ww} = -p_{vv}^{-1}, p_{vv} \neq \pm 1$;

   (iii)  $p_{vv} = -p_{ww}^2, p_{ww} \in R_{18}$;

   (iv)  $p_{vv} = -p_{ww}^{-4}, p_{ww} \in R_{18}$;

   (v)  $p_{vv} = -p_{ww}^{-4}, p_{ww} \in R_{10}$;

   (vi)  $p_{ww} = -p_{vv}^2, p_{vv} \in R_{18}$;

   (vii)  $p_{ww} = -p_{vv}^{-4}, p_{vv} \in R_{18}$;

   (viii)  $p_{ww} = -p_{vv}^{-4}, p_{vv} \in R_{10}$.

**Proof.**   (i) $\deg(u) \in \Delta(\chi; \deg(v), \deg(w))$ $\overset{p_{vv} \quad\;\; p_{vw}p_{wv} \quad\;\; p_{ww}}{\bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \bullet}$. By [He05, Prop. 2.7.1] and [He05, Lemma.2.7.2], it is clear $p_{wv}p_{vw} \neq 1$.

    (ii) If exist some $k \in \mathbb{N}$ such that $\deg(v) - k \deg(w) \in \mathbb{N} \cdot \Delta^+(\mathfrak{B}(V))$, let $k_1 \in \mathbb{N}$ be the maximum integer such that $\deg(v) - k_1 \deg(w) := k_2 \deg(v_1) \in \mathbb{N} \cdot \Delta^+(\mathfrak{B}(V))$. We know $\deg(v_1) - k \deg(w) \notin \mathbb{N} \cdot \Delta(\mathfrak{B}(V))$ for $\forall k \in \mathbb{N}$ by the maximality of $k_1$. Then we obtain $\deg(u) \in \Delta(\chi; \deg(v_1), \deg(w))$ with $\overset{p_{v_1 v_1} \quad\; p_{v_1 w}p_{wv_1} \quad\; p_{ww}}{\bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \bullet}$ by [He05, Prop. 2.7.1] and [He05, Lemma 2.7.2]. In this case, let $\alpha = \deg(v_1)$. If $\deg(v) - k \deg(w) \notin \mathbb{N} \cdot \Delta^+(\mathfrak{B}(V))$ for $\forall k \in \mathbb{N}$ and there exists some $k \in \mathbb{N}$ such that $\deg(v) - k \deg(w) \in \mathbb{N} \cdot \Delta^-(\mathfrak{B}(V))$. Let $k_1 \in \mathbb{N}$ be the maximum integer such that $\deg(v) - k_1 \deg(w) := -k_2 \deg(v_1) \in \mathbb{N} \cdot \Delta^-(\mathfrak{B}(V))$. We know $-\deg(v_1) - k \deg(w) \notin \mathbb{N} \cdot \Delta(\mathfrak{B}(V))$ for $\forall k \in \mathbb{N}$ by the maximality of $k_1$. Then we obtain $\deg(u) \in \Delta(\chi; -\deg(v_1), \deg(w))$ $\overset{p_{v_1 v_1} \quad\; p_{v_1 w}p_{wv_1} \quad\; p_{ww}}{\bullet \!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\! \bullet}$ by [He05, Prop. 2.7.1] and [He05, Lemma 2.7.2]. In these cases, $\deg(v) = -k_2 \deg(v_1) + k_1 \deg(w)$, $\deg(u) = -k_2 \deg(v_1) + (k_1 + 1) \deg(w)$, and $2 \nmid k_2$ by [He05, Cor. 2.5.4]. In this case, let $\alpha = -\deg(v_1)$.

    (iii) Set $\deg(w) = e_2, \alpha = e_1$. Then $p_{vw}p_{wv} = p_{ww}^{2k_1}(p_{vw}p_{wv})^{k_2} = p_{22}^{2k_1}(p_{12}p_{21})^{k_2}$.

    T4(1).   $p_0 = p_{12}p_{21}p_{11} \in R_{12}$, $p_{11} = p_0^4$, $p_{22} = -p_0^2$, $p_{12}p_{21} = p_0 p_{11}^{-1} = p_0^{-3} = -p_0^3$. $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (-p_0^2)^{2k_1}(-p_0^3)^{k_2} = (-1)^{k_2}(p_0)^{4k_1 + 3k_2} \neq 1$ since $2 \nmid k_2$.

    T4(2).   $p_{12}p_{21} \in R_{12}$, $p_{11} = p_{22} = -(p_{12}p_{21})^2$, $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2}$ $(-(p_{12}p_{21})^2)^{2k_1} = (p_{12}p_{21})^{4k_1 + k_2} \neq 1$ since $2 \nmid k_2$.

    T5(1).   $p_{12}p_{21} \in R_{12}$, $p_{11} = -(p_{12}p_{21})^2$, $p_{22} = -1$, $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

    T5(2).   $p_0 = p_{12}p_{21}p_{11} \in R_{12}$, $p_{11} = p_0^4$, $p_{22} = -1$, $p_{12}p_{21} = p_0 p_{11}^{-1} = p_0^{-3} = -p_0^3$. $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (-p_0^3)^{k_2} = (-1)^{k_2}(p_0)^{3k_2} \neq 1$ since $2 \nmid k_2$.

T7(1).     $p_{11} \in R_{12}$, $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-3})^{k_2} = p_{11}^{-3k_2} \neq 1$ since $2 \nmid k_2$.

T7(2).     $p_{12}p_{21} \in R_{12}$, $p_{11} = (p_{12}p_{21})^{-3}$, $p_{22} = -1$. $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$;

T8(2)$_1$.     $(p_{12}p_{21})^4 = -1$, $p_{22} = -1$, $p_{12}p_{21} = -p_{11}$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T8(2)$_2$.     $(p_{12}p_{21})^4 = -1$, $p_{22} = -1$, $p_{11} = (p_{12}p_{21})^{-2}$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T8(3).  $(p_{12}p_{21})^4 = -1$, $p_{11} = (p_{12}p_{21})^2$, $p_{22} = (p_{12}p_{21})^{-1}$;
$p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2}(p_{12}p_{21})^{-2k_1} = (p_{12}p_{21})^{k_2-2k_1} \neq 1$ since $2 \nmid k_2$.

T10.   $p_{12}p_{21} \in R_{24}$, $p_{11} = (p_{12}p_{21})^{-6}$, $p_{22} = (p_{12}p_{21})^{-8}$;
$p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2}((p_{12}p_{21})^{-8})^{2k_1} = (p_{12}p_{21})^{-16k_1+k_2} \neq 1$ since $2 \nmid k_2$.

T11(2).     $p_{11} \in R_{20}$, $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-3})^{k_2} = p_{11}^{-3k_2} \neq 1$ since $2 \nmid k_2$.

T12.  $p_{11} \in R_{30}$, $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = -p_{11}^5$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (-p_{11}^5)^{2k_1}(p_{11})^{-3k_2}$
$= (p_{11})^{10k_1-3k_2} \neq 1$ since $2 \nmid k_2$.

T13.   $p_{12}p_{21} \in R_{24}$, $p_{11} = (p_{12}p_{21})^6$, $p_{22} = (p_{12}p_{21})^{-1}$;
$p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2}(p_{12}p_{21})^{-2k_1} = (p_{12}p_{21})^{-2k_1+k_2} \neq 1$ since $2 \nmid k_2$.

T15.   $p_{12}p_{21} \in R_{30}$, $p_{11} = -(p_{12}p_{21})^{-3}$, $p_{22} = (p_{12}p_{21})^{-1}$;
$p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2}(p_{12}p_{21})^{-2k_1} = (p_{12}p_{21})^{-2k_1+k_2} \neq 1$ since $2 \nmid k_2$.

T16(2).     $p_{12}p_{21} \in R_{20}$, $p_{11} = (p_{12}p_{21})^{-4}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T17.     $p_{12}p_{21} \in R_{24}$, $p_{11} = -(p_{12}p_{21})^4$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T18.     $p_{12}p_{21} \in R_{30}$, $p_{11} = -(p_{12}p_{21})^5$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T19.     $p_{11} \in R_{14}$, $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-3})^{k_2} = p_{11}^{-3k_2} \neq 1$ since $2 \nmid k_2$.

T20.     $p_{12}p_{21} \in R_{30}$, $p_{11} = (p_{12}p_{21})^{-6}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{12}p_{21})^{k_2} \neq 1$ since $2 \nmid k_2$.

T21 and T22.     $p_{11} \in R_{24}$ or $p_{11} \in R_{14}$, $p_{12}p_{21} = p_{11}^{-5}$, $p_{22} = -1$;
$p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-5})^{k_2} = p_{11}^{-5k_2} \neq 1$ since $2 \nmid k_2$.

T2.   $\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2\}$.

T3.   $\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2\}$.

T6.   $p_{11} \in R_{18}$, $p_{12}p_{21} = p_{11}^{-2}$, $p_{22} = -p_{11}^3$, $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-2})^{k_2}(-p_{11}^3)^{2k_1}$
$= p_{11}^{6k_1-2k_2}$.
$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2, e_1 + 2e_2, 3e_1 + 2e_2\}$ by [An, Ex. 2.5]. If
$\deg(v) = e_1 + e_2$, it is clear $\deg(u) = e_1 + 2e_2$, $(p_{11})^{6k_1-2k_2} = (p_{11})^4 \neq 1$,

T8(1).     $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = p_{11}^3$, $p_{11} \in \cup_{m=4}^{\infty} R_m$. $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-3})^{k_2}(p_{11}^3)^{2k_1} = p_{11}^{6k_1-3k_2}$. $\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2, 3e_1 + e_2, 3e_1 + 2e_2\}$.
If $\deg(v) = 3e_1 + e_2$, then it is clear $\deg(u) = 3e_1 + 2e_2$ and $p_{11}^{6k_1-3k_2} = (p_{11})^{-3} \neq 1$.

T9.   $p_{12}p_{21} \in R_9$, $p_{11} = (p_{12}p_{21})^{-3}$, $p_{22} = -1$;
$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2, 4e_1 + 3e_2, 3e_1 + 2e_2\}$ by [An, Ex. 2.5],

T11(1).     $p_{11} \in R_5$, $p_{12}p_{21} = p_{11}^{-3}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11}^{-3})^{k_2} = p_{11}^{-3k_2}$; $\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, 3e_1 + e_2, 2e_1 + e_2, 5e_1 + 3e_2, 4e_1 + 3e_2, 3e_1 + 2e_2, e_1 + e_2\}$

by [An, Ex. 2.7]. If $\deg(v) = 3e_1 + e_2$, it is clear $\deg(u) = 3e_1 + 2e_2$, then $(p_{11})^{-3k_2} = (p_{11})^{-9} \neq 1$.

    T14.     $p_{11} \in R_{18}$, $p_{12}p_{21} = p_{11}^{-4}$, $p_{22} = -1$;
$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, e_1 + e_2, 2e_1 + e_2, 4e_1 + e_2, 3e_1 + e_2\}$ by [An, Ex. 2.5].

    T16(1).     $p_{11} \in R_{10}$, $p_{12}p_{21} = p_{11}^{-4}$, $p_{22} = -1$; $p_{22}^{2k_1}(p_{12}p_{21})^{k_2} = (p_{11})^{-4k_2} = (p_{11})^{-4k_2}$. $\Delta^+(\mathfrak{B}(V)) = \{e_1, e_2, 3e_1 + e_2, 2e_1 + e_2, 5e_1 + 2e_2, 4e_1 + e_2, 3e_1 + 2e_2, e_1 + e_2\}$ by [An, Ex. 2.7]. If $\deg(v) = 3e_1 + e_2$, it is clear $\deg(u) = 3e_1 + 2e_2$, $(p_{11})^{-4k_2} = (p_{11})^{-12} \neq 1$.

    (iv) Set $\deg(w) = e_1, \alpha = e_2$. In these cases, $\deg(v) = k_1 e_1 + k_2 e_2$ and $2 \nmid k_2$ by [He05, Cor. 2.5.4]. It is clear that $|u| \geq 3$ and $|v| \geq 2$. Then $p_{vw}p_{wv} = p_{ww}^{2k_1}(p_{vw}p_{wv})^{k_2} = p_{11}^{2k_1}(p_{12}p_{21})^{k_2}$.

    Arguments for T4 - T16 are similar to those above except for the following additional cases:

    T3(1)$_1$.     $p_{12}p_{21} = p_{11}^{-2}$, $p_{22} = p_{11}^2$, $p_{11} \neq \pm 1$, if $\deg(v) = e_1 + e_2$, it is clear $\deg(u) = 2e_1 + e_2$, then $p_{11}^2(p_{12}p_{21})^1 = 1$.

    T3(1)$_2$.     $p_{12}p_{21} = p_{11}^{-2}$, $p_{22} = -1$, $p_{11} \neq \pm 1$, if $\deg(v) = e_1 + e_2$, it is clear $\deg(u) = 2e_1 + e_2$, then $p_{11}^2(p_{12}p_{21})^1 = 1$.

    T6.     $p_{11} \in R_{18}$, $p_{12}p_{21} = p_{11}^{-2}$, $p_{22} = -p_{11}^3$, If $\deg(v) = e_1 + e_2$, it is clear $\deg(u) = 2e_1 + e_2$, then $(p_{11})^{2k_1-2k_2} = (p_{11})^0 = 1$.

    T14.     $p_{11} \in R_{18}$, $p_{12}p_{21} = p_{11}^{-4}$, $p_{22} = -1$; If $\deg(v) = 2e_1 + e_2$, it is clear $\deg(u) = 3e_1 + e_2$, then $(p_{11})^{2k_1-4k_2} = (p_{11})^0 = 1$.

    T16(1).     $p_{11} \in R_{10}$, $p_{12}p_{21} = p_{11}^{-4}$, $p_{22} = -1$; If $\deg(v) = 2e_1 + e_2$, it is clear $\deg(u) = 3e_1 + e_2$, then $(p_{11})^{2k_1-4k_2} = (p_{11})^0 = 1$.

    Similarly, we have (v)–(viii).               ■

**Remark 5.6.**     (i) If $[u][v] \neq 0$, then the following conditions are equivalent: (1) $[[u], [v]] = 0$, $[[v], [u]] = 0$; (2) $1 - p_{uv}p_{vu} = 0$, $[[v], [u]] = 0$; (3) $1 - p_{uv}p_{vu} = 0$, $[[u], [v]] = 0$.

    (ii) If $[[u], [v]] = 0$, then $[[v], [u]] = (1 - p_{uv}p_{vu})[u][v]$.

**Remark 5.7.**     There is a braided m-Lie algebra which is not a Nichols braided Lie algebra or Nichols braided m-Lie algebra. In fact, let $L = L_{\bar{0}} + L_{\bar{1}}$ be a super-Lie algebra with $L_{\bar{0}} = sl(2)$, $L_{\bar{1}} = 0$. It is clear that $L$ is a finite dimensional m-braided Lie algebra of diagonal type in $_{k\mathbb{Z}_2}^{k\mathbb{Z}_2}\mathcal{Y}D$. Because $L$ is not nilpotent and every finite dimensional $\mathfrak{L}(V)$ is nilpotent, $L$ is not a Nichols braided Lie algebra or Nichols braided m-Lie algebra.

## 6.    Cartan type

In this section we give the sufficient conditions for Nichols braided Lie algebra $\mathfrak{L}(V)$ to be a homomorphic image of a braided Lie algebra generated by $V$ with defining relations. Basic field $F$ is the complex field $\mathbb{C}$.

    Let $\Phi^+$ denote the positive root system of simple Lie algebras. $E_e = n-1, n-1, n-1, n-1, 5, 6, 7, 3, 1$ in $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$, respectively, i.e. $E_e = \text{rank}\Phi - 1$. In fact, $\Phi^+ = \Delta^+(\mathfrak{B}(V))$ by [He06b]. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ be a

normal orthogonal basis of $\mathbb{R}^n$; $I =: \sum_{i=1}^{n} \mathbb{Z}\epsilon_i$ and $I' = I + \mathbb{Z}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_n)/2$. Let $\Delta = \{e_1, \cdots, e_n\}$ be a prime root system.

**Lemma 6.1.**    *Let $\Omega$ be the class of connected components of $V$. then $\mathfrak{B}(V) = \oplus_{J \in \Omega}\mathfrak{B}(V_J)$ and $\mathfrak{L}(V) = \oplus_{J \in \Omega}\mathfrak{L}(V_J)$.*

**Lemma 6.2.**    *Assume that $V$ is a braided vector space of diagonal type with braided matrix $(q_{ij})_{n \times n}$ and basis $x_1, x_2, \cdots, x_n$. Then $\mathrm{ord}(q_{ij})$ is finite for any $1 \leq i, j \leq n$ if and only if there exists a finite abelian group $G$ such that $V$ becomes a $kG$- YD module.*

**Proof.**    The necessity is clear. The sufficiency. Let $N$ be the least common multiple of $\{ord q_{ij} \mid 1 \leq i, j \leq n\}$ and $G = (g_1) \times (g_2) \times \cdots \times (g_n)$ with $\mathrm{ord}(g_i) = N$ for $1 \leq i \leq n$. Let $\chi(g_1^{k_1} \cdots g_n^{k_n}, g_1^{k'_1} \cdots g_n^{k'_n}) := \prod_{1 \leq i,j \leq n} q_{ij}^{k_i k'_j}$. It is clear that $\chi$ is a bicharacter on $G \times G$. Therefore $V$ becomes a $kG$- YD module.  ∎

**Lemma 6.3.**    *Assume that $V$ is a braided vector space of diagonal type with braided matrix $(q_{ij})_{n \times n}$. If $\dim \mathfrak{B}(V) < \infty$, then there exists a braided matrix $(q'_{ij})_{n \times n}$, which is twisted equivalent to $(q_{ij})_{n \times n}$, such that $\mathrm{ord}(q'_{ij}) < \infty$ for any $1 \leq i, j \leq n$.*

**Proof.**    We show this by two steps.

(i) $\mathrm{ord}(q_{ij}q_{ji}) < \infty$ for any $1 \leq i, j \leq n$. In fact, it is clear $\mathrm{ord}(q_{ii}^2) < \infty$ for any $1 \leq i \leq n$. If there exist $i$ and $j$ with $i < j$ such that $\mathrm{ord}(q_{ij}q_{ji}) = \infty$. Obviously, $[u] := [x_i, x_j] \in D$. Consequently, $\mathrm{ord}(p_{u,u}) = \infty$ and $\dim \mathfrak{B}(V) = \infty$, which is a contradiction. (ii) Set $q'_{ij} := \sqrt{q_{ij}q_{ji}}$ for any $1 \leq i, j \leq n$.  ∎

**Lemma 6.4.**    *Assume that $(V, (q_{ij})_{n \times n})$ is of connected Cartan type with Cartan matrix $(a_{ij})_{n \times n}$. Then*

*(i) For $A_n, D_n, E_8, E_7, E_6$: $p_{\alpha,\alpha} = q$, $p_{\alpha,\gamma}p_{\gamma,\alpha} = q^{(\alpha,\gamma)}$, $p_{\alpha,\beta}p_{\beta,\alpha} = q^{\pm 1}$ or $1$ for any $\alpha, \beta, \gamma \in \Delta^+(\mathfrak{B}(V))$ with $\alpha \neq \beta$.*

*(ii) For $B_n$:*

$$\Delta^+(\mathfrak{B}(V)) \;=\; \left\{\sum_{i=n}^{n} e_i, \sum_{i=n-1}^{n} e_i, \sum_{i=n-2}^{n} e_i, \cdots, \sum_{i=1}^{n} e_i\right\}$$

$$\cup \left\{\sum_{i=1}^{1} e_i + \sum_{i=2}^{n} 2e_i, \sum_{i=1}^{2} e_i + \sum_{i=3}^{n} 2e_i, \sum_{i=1}^{3} e_i + \sum_{i=4}^{n} 2e_i,\right.$$

$$\cdots, \sum_{i=1}^{n-1} e_i + \sum_{i=n}^{n} 2e_i, \sum_{i=2}^{2} e_i + \sum_{i=3}^{n} 2e_i, \sum_{i=2}^{3} e_i + \sum_{i=4}^{n} 2e_i,$$

$$\left.\cdots, \sum_{i=2}^{n-1} e_i + \sum_{i=n}^{n} 2e_i, \cdots, \sum_{i=n-1}^{n-1} e_i + \sum_{i=n}^{n} 2e_i\right\}$$

$$\cup \left\{\sum_{i=1}^{1} e_i, \sum_{i=1}^{2} e_i, \sum_{i=1}^{3} e_i, \cdots, \sum_{i=1}^{n-1} e_i, \sum_{i=2}^{2} e_i, \sum_{i=2}^{3} e_i,\right.$$

$$\left.\cdots, \sum_{i=2}^{n-1} e_i, \cdots, \sum_{i=n-2}^{n-2} e_i, \sum_{i=n-2}^{n-1} e_i, \sum_{i=n-1}^{n-1} e_i\right\} := Q \cup S \cup T,$$

$p_{\alpha,\alpha} = q$ for $\forall \alpha \in Q$ and $p_{\alpha,\alpha} = q^2$ for $\forall \alpha \in S$ or $T$.

    (iii) For $C_n$:



$$\Delta^+(\mathfrak{B}(V)) \;=\; \left\{\sum_{i=1}^{n-1} 2e_i + e_n, \sum_{i=2}^{n-1} 2e_i + e_n, \cdots, \sum_{i=n-1}^{n-1} 2e_i + e_n, e_n\right\}$$

$$\cup \left\{\sum_{i=1}^{1} e_i + \sum_{i=2}^{n-1} 2e_i + e_n, \sum_{i=1}^{2} e_i + \sum_{i=3}^{n-1} 2e_i + e_n,\right.$$

$$\sum_{i=1}^{3} e_i + \sum_{i=4}^{n-1} 2e_i + e_n, \cdots, \sum_{i=1}^{n-1} e_i + e_n, \sum_{i=2}^{2} e_i + \sum_{i=3}^{n-1} 2e_i + e_n,$$

$$\left.\sum_{i=2}^{3} e_i + \sum_{i=4}^{n-1} 2e_i + e_n, \cdots, \sum_{i=2}^{n-1} e_i + e_n, \cdots, \sum_{i=n-1}^{n-1} e_i + e_n\right\}$$

$$\cup \left\{\sum_{i=1}^{1} e_i, \sum_{i=1}^{2} e_i, \sum_{i=1}^{3} e_i, \cdots, \sum_{i=1}^{n-1} e_i, \sum_{i=2}^{2} e_i, \sum_{i=2}^{3} e_i,\right.$$

$$\left.\cdots, \sum_{i=2}^{n-1} e_i, \cdots, \sum_{i=n-2}^{n-2} e_i, \sum_{i=n-2}^{n-1} e_i, \sum_{i=n-1}^{n-1} e_i\right\} := Q \cup S \cup T,$$

$p_{\alpha,\alpha} = q^2$ for $\forall \alpha \in Q$ and $p_{\alpha,\alpha} = q$ for $\forall \alpha \in S$ or $T$.

    (iv) For $F_4$:

$$\Delta^+(\mathfrak{B}(V)) \;=\; \{e_2 + 2e_3 + 2e_4, e_1 + e_2 + 2e_3 + 2e_4, e_1 + 2e_2 + 2e_3 + 2e_4,$$
$$e_1, e_1 + e_2, e_2, 2e_1 + 3e_2 + 4e_3 + 2e_4, e_1 + 3e_2 + 4e_3 + 2e_4,$$
$$e_1 + 2e_2 + 4e_3 + 2e_4, e_1 + 2e_2 + 2e_3, e_1 + e_2 + 2e_3, e_2 + 2e_3\}$$
$$\cup \{e_1 + 2e_2 + 3e_3 + e_4, e_2 + 2e_3 + e_4, e_1 + e_2 + 2e_3 + e_4,$$
$$e_1 + 2e_2 + 2e_3 + e_4, e_3 + e_4, e_2 + e_3 + e_4, e_1 + e_2 + e_3 + e_4, e_4,$$
$$e_1 + 2e_2 + 3e_3 + 2e_4, e_1 + e_2 + e_3, e_2 + e_3, e_3\} := Q \cup S,$$

$p_{\alpha,\alpha} = q^2$ for $\forall \alpha \in Q$ and $p_{\alpha,\alpha} = q$ for $\forall \alpha \in S$.

(v) For $G_2$:

$$\overset{q \quad\; q^{-3} q^3}{\underset{1 \qquad\; 2}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet}}$$

$\Delta^+(\mathfrak{B}(V)) = \{e_1, e_1 + e_2, 2e_1 + e_2\} \cup \{3e_1 + e_2, 3e_1 + 2e_2, e_2\} := Q \cup S$. $p_{\alpha\alpha} = q$ for $\forall \alpha \in Q$ and $p_{\alpha\alpha} = q^3$ for $\forall \alpha \in S$.

**Proof.**     (i) By [Hu72, Section 12.1], the root system
$$\Delta^+(\mathfrak{B}(V)) = \{\beta \in I \mid (\beta, \beta) = 2\}.$$
Let $\alpha = k_1 e_1 + \cdots + k_n e_n, \gamma = k'_1 e_1 + \cdots + k'_n e_n \in \Delta^+(\mathfrak{B}(V))$.

$$(\alpha, \gamma) = \sum_{i=1}^{n} 2k_i k'_i + \sum_{i \neq j} a_{ij} k_i k'_j \tag{6.1}$$

and

$$p_{\alpha,\gamma} p_{\gamma,\alpha} = q^{\sum_{i=1}^n 2k_i k'_i + \sum_{i \neq j} a_{ij} k_i k'_j} = q^{(\alpha,\gamma)}. \tag{6.2}$$

Consequently, $p_{\alpha,\alpha} = q$ by (6.2). By [Hu72, Section 9.4, Table 1], $(\alpha, \beta) = 1$ or $-1$ or $0$.

(ii) - (v) are clear.     ∎

Recall that $(T(V), [\,])$ is a braided Lie algebra. The braided Lie algebra $(FL(V), [\,])$ generated by $V$ in $(T(V), [\,])$ is called the free braided Lie algebra of $V$. If $f_1, f_2, \cdots, f_r \in FL(V)$ and $I$ is an ideal $I$ generated by $f_1, f_2, \cdots, f_r$ in $(FL(V), [\,])$, then $(FL(V)/I, [\,])$ is called the braided Lie algebra generated by $x_1, x_2, \cdots, x_n$ with the defining relations $f_1, f_2, \cdots, f_r$.

**Lemma 6.5.**     *Assume that $(V, (q_{ij})_{n \times n})$ is a connected braided vector space of finite Cartan type with Cartan matrix $(a_{ij})_{n \times n}$. If $\mathrm{ord}(q_{ii})$ is prime to 3 when $q_{ij} q_{ji} \in \{q_{ii}^3, q_{jj}^3\}$, then $\mathrm{ord}(p_{\alpha,\alpha}) = N$ for root vector $x_\alpha$ with $\alpha \in \Delta^+(\mathfrak{B}(V))$ and $N = \mathrm{ord}(q_{11})$, where root vectors were defined in [Lu90].*

**Proof.**     By [AS00, Th.1.1(i)], $\mathrm{ord}(q_{ii}) = N$ for $1 \leq i \leq n$. $\mathrm{ord}(p_{\alpha,\alpha}) = N$ for any root $\alpha$ by Lemma 6.4 and $x_\alpha^i \in \mathfrak{L}(V)$ for $1 \leq i \leq N$ by Lemma 4.3.     ∎

Let $ad_c xy := [x, y]_c = xy - p_{x,y} yx$.

**Theorem 6.6.** *If $V$ is a finite Cartan type with Cartan matrix $(a_{ij})_{n \times n}$ and the following conditions satisfied for any $1 \leq i, j \leq n$ : (i) $\mathrm{ord}(q_{ii})$ is odd; (ii) $\mathrm{ord}(q_{ii})$ is prime to 3 when $q_{ij}q_{ji} \in \{q_{ii}^3, q_{jj}^3\}$; (iii) $\mathrm{ord}(q_{ij}) < \infty$. then Nichols braided Lie algebra $\mathfrak{L}(V)$ is a homomorphic image of the braided Lie algebra generated by $x_1, x_2, \cdots, x_n$ with defining relations: (iv) $\mathrm{ad}_c x_i^{1-a_{ij}} x_j$, $i \neq j$. (v) $x_\alpha^N$ for any $\alpha \in \Delta^+(\mathfrak{B}(V))$, where $N$ is order of $q_{11}$.*

**Proof.** By [AS02, Section 4.1], $x_\alpha \in FL(V)$. It follows from Lemma 6.5 and Lemma 4.3 that $x_\alpha^N \in FL(V)$. Let $I$ and $J$ denote ideals generated by elements of (iv) and (v) in $T(V)$ as algebras and in $FL(V)$ as braided Lie algebras with bracket [ ]. Consequently, using [AS10, Theorem 5.1], we have that the map from $FL(V)/J$ to $\mathfrak{L}(V)$ by sending $x + J$ to $x + I$ is a epimorphism. ∎

**Theorem 6.7.** *If $\dim \mathfrak{B}(V) < \infty$ and the following conditions satisfied for any $1 \leq i, j \leq n$ : (i) $\mathrm{ord}(q_{ii})$ is odd; (ii) $\mathrm{ord}(q_{ii})$ is prime to 3 when $q_{ij}q_{ji} \in \{q_{ii}^3, q_{jj}^3\}$; (iii) $\mathrm{ord}(q_{ii}) > 3$; (iv) $\mathrm{ord}(q_{ij}) < \infty$. Then $V$ is a finite Cartan type with Cartan matrix $(a_{ij})_{n \times n}$ and Nichols braided Lie algebra $\mathfrak{L}(V)$ is a homomorphic image of the braided Lie algebra generated by $x_1, x_2, \cdots, x_n$ with the defining relation, (v) $\mathrm{ad}_c x_i^{1-a_{ij}} x_j$, $i \neq j$; (vi) $x_\alpha^N$ for any $\alpha \in \Delta^+(\mathfrak{B}(V))$, where $N$ is order of $q_{11}$.*

**Proof.** By the proof of [AS10, Theorem 5.3], $V$ is a finite Cartan type. Using Theorem 6.6 we complete the proof. ∎

**Lemma 6.8.** *Under the conditions of Theorem 6.7 or Theorem 6.6, for any $\alpha \in \Delta^+(\mathfrak{B}(V))$, there exists a unique hard super-letter $[u]$ such that $\deg(u) = \alpha$.*

**Proof.** Considering the dimensional formulas of $\mathfrak{B}(V)$ in Theorem 2.7 and [AS00, Th. 1.1(i)], we complete the proof. ∎

**Lemma 6.9.** *Assume that $(V, (q_{ij})_{n \times n})$ is of a connected Cartan type. If $\alpha, \beta, \gamma \in \Delta^+(\mathfrak{B}(V))$ with $\alpha + \beta = \gamma$, then $p_{\alpha,\beta}p_{\beta,\alpha} = 1$ if and only if $(\alpha, \beta)$ or $(\beta, \alpha) \in X$ and $X$ is defined in the following cases:*
(i) *For $F_4$, $X := \{(\epsilon_i, \epsilon_j) \mid i \neq j\} \cup$*
$\left\{ \left(\frac{1}{2}(\epsilon_1 + (-1)^{k_2}\epsilon_2 + (-1)^{k_3}\epsilon_3 + (-1)^{k_4}\epsilon_4), \frac{1}{2}(\epsilon_1 + (-1)^{k_2'}\epsilon_2 + (-1)^{k_3'}\epsilon_3 + (-1)^{k_4'}\epsilon_4)\right) \mid$
$(-1)^{k_2} + (-1)^{k_2'}$ *or* $(-1)^{k_3} + (-1)^{k_3'}$ *or* $(-1)^{k_4} + (-1)^{k_4'}$ *is* $1\}$.
(ii) *For $B_n$, $X := \{(\epsilon_i, \epsilon_j) \mid 1 \leq i \neq j \leq n\}$.*
(iii) *For $C_n$, $X := \{(\epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j) \mid 1 \leq i < j \leq n\}$.*

**Proof.** By Proposition 5.5, it is clear that $p_{\alpha,\beta}p_{\beta,\alpha} = 1$ if and only if

$$p_{\gamma,\gamma} = q^2, p_{\alpha,\alpha} = p_{\beta,\beta} = q; \tag{6.3}$$

$$\text{or} \quad p_{\gamma,\gamma} = q, p_{\alpha,\alpha} = p_{\beta,\beta} = q^2, q^3 = 1. \tag{6.4}$$

By Lemma 6.5, $p_{\alpha,\alpha} = q$ and $p_{\alpha,\beta}p_{\beta,\alpha} \neq 1$ for $A_n, D_n, E_8, E_7, E_6$. We can complete the proof by simple computation. ∎

**Proposition 6.10.**    *Assume* $[u] \in D$. *(i) If* $p_{vw}p_{wv} \neq 1$ *for any two descendants* $v$ *and* $w$ *of* $u$, *then* $[u]^- \in \mathfrak{L}(V)$; *(ii) If connected* $(V, (q_{ij})_{n \times n})$ *is of* $A_n, D_n, E_6, E_7, E_8, G_2$, *then* $[u]^- \in \mathfrak{L}(V)$.

**Proof.**    (i) By induction and Lemma 4.12, we obtain (i). (ii) follows from (i) and Lemma 6.9. $\blacksquare$

**Theorem 6.11.**    *Under the conditions of Theorem 6.7 or Theorem 6.6, assume that connected* $(V, (q_{ij})_{n \times n})$ *is of* $A_n, D_n, E_6, E_7, E_8, G_2$, *and* $q = q_{ii}$ *with* $N := \mathrm{ord}(q_{11})$.

(i) *If* $u, v \in D$, *then* $p_{uu}^i p_{u,uv} p_{uv,u} \neq 1$ *for* $1 \leq i \leq 2([\frac{N}{2}] - 1) - 2$.

(ii) $\dim \mathfrak{L}(V) \geq (N-1)^{|\Phi^+|} + ([\frac{N}{2}] - 1)^{\frac{|\Phi^+|(|\Phi^+|-1)}{2}}$.

**Proof.**    (i) By Lemma 6.5 (i), $p_{uu}^i p_{u,uv} p_{uv,u} = p_{uu}^i p_{uu}^2 p_{uv} p_{vu} = q^{i+2+(\deg(u),\deg(v))} \neq 1$ for $1 \leq i \leq 2([\frac{N}{2}] - 1) - 2$.    (ii) Let $D := \{u_1, u_2, \cdots, u_r\}$ with $u_r < u_{r-1} < \cdots < u_1$. By Lemma 6.8, $r = |\Phi^+|$. It follows from Lemma 4.7 that $u^i uv \in \mathfrak{L}(V)$ for any $u, v \in D$ and $1 \leq i \leq [\frac{N}{2}] - 1$. Consequently, $B := \{u_1^j, u_2^j, \cdots, u_r^j; u_1^i u_1 u_2, u_1^i u_1 u_3, \cdots, u_1^i u_1 u_r; u_2^i u_2 u_3, u_2^i u_2 u_4, \cdots, u_2^i u_2 u_r; \cdots; u_{r-1}^i u_{r-1} u_r\}$ $\subset P \cap \mathfrak{L}(V)$, which implies $\dim \mathfrak{L}(V) \geq (N-1)^{|\Phi^+|} + ([\frac{N}{2}] - 1)^{\frac{|\Phi^+|(|\Phi^+|-1)}{2}}$. $\blacksquare$

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