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The Adjoint Homology of Heisenberg Lie Algebras

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Abstract. By considering the Heisenberg Lie algebra $\mathfrak{h}_{2n-1}$ as the nilradical of a parabolic subalgebra $\mathfrak{p}$ of A_n, we give a full description of its adjoint homology as a module over a Levi factor of $\mathfrak{p}$.
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Introduction

If $\mathfrak{p}$ is a parabolic subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{n}$ is its nilradical, then $\mathfrak{p}$ is the semidirect product $\mathfrak{g}_1 \ltimes \mathfrak{n}$, where $\mathfrak{g}_1$ is a Levi factor of $\mathfrak{p}$. In 1961, Kostant [6] computed the $\mathfrak{g}_1$-structure of the (co)homology of these nilradicals with coefficients in a representation of $\mathfrak{n}$ which is the restriction of a representation of $\mathfrak{g}$. The adjoint (co)homology of these nilradicals is not covered by Kostant's result since the adjoint representation of $\mathfrak{n}$ is not the restriction of any representation of $\mathfrak{g}$, unless $\mathfrak{n}$ is abelian. The description of the $\mathfrak{g}_1$-module structure of $H(\mathfrak{n},\mathfrak{n})$ remains open. About this problem, we can mention the works of Cagliero - Tirao [2], and Alvarez - Tirao [1], where they computed the adjoint homology of some nilradicals of parabolic subalgebras of simple Lie algebras of type B and A respectively.

The Heisenberg Lie algebras appear as nilradicals of parabolic subalgebras of every classical family of simple Lie algebras. The trivial cohomology of these algebras was computed in the work of Santharoubane [7] and it is also contained in the classical work of Kostant [6]. The adjoint cohomology can be found in the works of Magnin [4] and Cagliero - Tirao [3]. In this work we consider the Heisenberg Lie algebras $\mathfrak{h}_{2n-1}$ as nilradicals of parabolic subalgebras $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{h}_{2n-1}$ of simple Lie algebras of type A, to give a full description of their adjoint homology $H(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1})$ as a module over the Levi factor $\mathfrak{g}_1$ of $\mathfrak{p}$.

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1. Preliminaries

Parabolic subalgebras and nilradicals

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of finite dimension, $\mathfrak{h}$ a Cartan subalgebra and $\Delta = \Delta(\mathfrak{g},\mathfrak{h})$ the set of roots. A *Borel subalgebra* of $\mathfrak{g}$ is a subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{t}$, where $\mathfrak{t} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ for some positive system $\Delta^+ \subset \Delta$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing a Borel subalgebra is called a *parabolic subalgebra* of $\mathfrak{g}$. Let Π be the simple system determining Δ^+ and $\mathfrak{t}$. Since $\mathfrak{p} \supset \mathfrak{h}$ and roots spaces are 1-dimensional, $\mathfrak{p}$ is of the form

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha, \tag{1}$$

for a subset Γ of Δ containing Δ^+.

If for $\Pi_0 \subset \Pi$ one defines

$$\Gamma = \Delta^+ \cup \{\alpha \in \Delta : \alpha \in \operatorname{span}(\Pi\backslash\Pi_0)\}, \tag{2}$$

then the parabolic subalgebras $\mathfrak{p}$ containing $\mathfrak{b}$ are parameterized by the set of subsets of simple roots; the one corresponding to Π_0 is of the form (1) with Γ as in (2).

The parabolic subalgebra $\mathfrak{p}$ decomposes as the semidirect product $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$, of its Levi factor which is reductive and its nilradical, where

$$\mathfrak{g}_1 = \sum_{\alpha\in\Delta_1^+} \mathfrak{g}_{-\alpha} \oplus \mathfrak{h} \oplus \sum_{\alpha\in\Delta_1^+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n} = \sum_{\alpha\in\Delta(\mathfrak{n})} \mathfrak{g}_\alpha,$$

with $\Delta(\mathfrak{n}) = \{\alpha \in \Delta^+ : \alpha \notin \operatorname{span}(\Pi\backslash\Pi_0)\}$ and $\Delta_1^+ = \Delta^+\backslash\Delta(\mathfrak{n})$.

Homology of nilradicals

Given a semisimple Lie algebra $\mathfrak{g}$, a parabolic subalgebra $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ and an irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ, the $\mathfrak{g}_1$-module structure of the cohomology of $\mathfrak{n}$ with coefficients in V^λ, $H^\bullet(\mathfrak{n}, V^\lambda)$, where V^λ is an $\mathfrak{n}$-module by restriction, is a well known result from Kostant [6]. However, if $\mathfrak{n}$ is not abelian, the adjoint representation of $\mathfrak{n}$ is not the restriction of a representation of $\mathfrak{g}$ (see Proposition 3.3 in [2]). Since adjoint homology and coadjoint cohomology, and vice versa, are related via $H_\bullet(\mathfrak{n},\mathfrak{n}) \simeq H^\bullet(\mathfrak{n},\mathfrak{n}^*)^*$, the description of the adjoint (co)homology of nilradicals remains an open problem.

Let $\mathfrak{n}$ be a Lie algebra of dimension d. The trivial homology of $\mathfrak{n}$, $H_\bullet(\mathfrak{n}) = \bigoplus_{p=0}^{d} H_p(\mathfrak{n})$, is the homology of the complex $(\Lambda^\bullet\mathfrak{n}, \partial_0)$, where $\partial_0^p : \Lambda^p\mathfrak{n} \to \Lambda^{p-1}\mathfrak{n}$ is defined by

$$\partial_0^p(v_1 \wedge \ldots \wedge v_p) = \sum_{1\le i<j\le p} (-1)^{i+j+1}[v_i, v_j] \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge \widehat{v_j} \wedge \ldots \wedge v_p.$$

The adjoint homology of $\mathfrak{n}$, $H_\bullet(\mathfrak{n},\mathfrak{n}) = \bigoplus_{p=0}^{d} H_p(\mathfrak{n},\mathfrak{n})$ is the homology of the complex $(\Lambda^\bullet\mathfrak{n}\otimes\mathfrak{n},\partial)$, where $\partial^p : \Lambda^p\mathfrak{n}\otimes\mathfrak{n}\to\Lambda^{p-1}\mathfrak{n}\otimes\mathfrak{n}$ is defined by $\partial^p = \partial_0^p\otimes 1 + \partial_1^p$ and

$$\partial_1^p(v_1\wedge\ldots\wedge v_p\otimes m) = \sum_{i=1}^{p}(-1)^{i+1} v_1\wedge\ldots\wedge\widehat{v_i}\wedge\ldots\wedge v_p\otimes[v_i,m].$$

If $\mathfrak{n} = V\oplus\mathfrak{z}$ is a 2-step nilpotent Lie algebra and $\mathfrak{z}$ is the center of $\mathfrak{n}$, then the short exact sequence of trivial $\mathfrak{n}$-modules ($V\simeq\mathfrak{n}/\mathfrak{z}$)

$$0\to\mathfrak{z}\to\mathfrak{n}\to V\to 0$$

induces the long exact sequence

$$\cdots\to H_{p+1}(\mathfrak{n})\otimes V\xrightarrow{\delta_{p+1}} H_p(\mathfrak{n})\otimes\mathfrak{z}\to H_p(\mathfrak{n},\mathfrak{n})\to H_p(\mathfrak{n})\otimes V\xrightarrow{\delta_p} H_{p-1}(\mathfrak{n})\otimes\mathfrak{z}\to\cdots \quad (3)$$

Also we have:

Lemma 1.1. *Let $\delta_p : H_p(\mathfrak{n})\otimes V\to H_{p-1}(\mathfrak{n})\otimes\mathfrak{z}$ be the connecting morphism in the above long exact sequence and let $v\in\Lambda^p(\mathfrak{n})\otimes V$ such that $[v]\in H_p(\mathfrak{n})\otimes V$. Then*

$$\delta_p([v]) = [\partial_1^p(v)].$$

Proof. Standard. ∎

Therefore one can see that, as $\mathfrak{n}$-modules:

$$H_p(\mathfrak{n},\mathfrak{n})\simeq\ker\delta_p\oplus\operatorname{coker}\delta_{p+1}. \quad (4)$$

This paper is organized as follows: in Section 2 we obtain the highest weight vectors for the trivial homology of $\mathfrak{n}$, and in Section 3 we construct highest weight vectors of particular tensor products. By using the previous results and equation (4), we obtain the adjoint homology of $\mathfrak{n}$ by computing $\ker\delta$ and $\operatorname{coker}\delta$ in Section 4.

2. The family $\mathfrak{h}_{2n-1}$ and its trivial homology

Let $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{C})$ and let $\mathfrak{h}$ be the subalgebra of all diagonal matrices in $\mathfrak{g}$.

Define the matrix $E_{i,j}$ to be 1 in the (i,j)-th place and 0 elsewhere, and define e_j in the dual space $\mathfrak{h}^*$ by

$$e_j\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_{n+1}\end{pmatrix} = h_j.$$

For each $H\in\mathfrak{h}$ and $E_{i,j}$ with $i\neq j$ we have

$$(\operatorname{ad}H)E_{i,j} = [H,E_{i,j}] = (e_i(H)-e_j(H))\,E_{i,j}.$$

This means that $E_{i,j}$ is a simultaneous eigenvector for all $\operatorname{ad} H$, with eigenvalue $e_i(H) - e_j(H)$. It turns out that the eigenvalue, namely $e_i - e_j$, is a linear functional on $\mathfrak{h}$. The $(e_i - e_j)$'s, for $i \neq j$, are the *roots* for $\mathfrak{g}$.

Let Δ^+ and Π be as usual, that is

$$\Delta^+ = \{e_i - e_j : 1 \leq i < j \leq n+1\} \quad \text{and} \quad \Pi = \{\alpha_i : 1 \leq i \leq n\},$$

where $\alpha_i = e_i - e_{i+1}$.

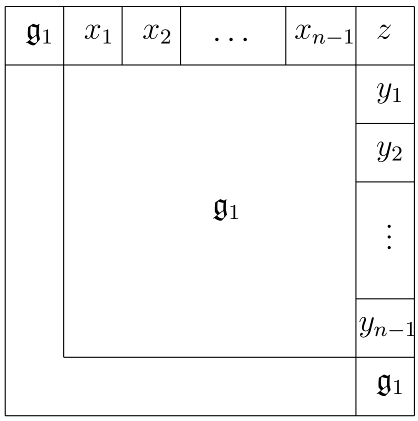
Let us consider the nilradical $\mathfrak{n}$ corresponding to the choice of $\Pi_0 = \{\alpha_1, \alpha_n\}$. Then we have,

$$\begin{aligned} \Delta(\mathfrak{n}) =& \{e_1 - e_{n+1}\} \cup \{e_1 - e_j : 2 \leq j \leq n\} \cup \{e_j - e_{n+1} : 2 \leq j \leq n\}, \\ \Delta_1^+ =& \{e_i - e_j : 2 \leq i < j \leq n\}. \end{aligned}$$

If we denote $z = E_{1,n+1}$, $x_j = E_{1,j+1}$ and $y_j = E_{j+1,n+1}$ for $j = 1, \ldots, n-1$, then $\{z, x_1, ..., x_{n-1}, y_1, ..., y_{n-1}\}$ is a basis of $\mathfrak{n}$, and the only nonzero brackets of basis elements are $[x_i, y_i] = z$, for $i = 1, ..., n-1$. Therefore $\mathfrak{n}$ is the Heisenberg Lie algebra of dimension $2n-1$, denoted by $\mathfrak{h}_{2n-1}$.

Let us call $X = \langle\{x_1, ..., x_{n-1}\}\rangle$, $Y = \langle\{y_1, ..., y_{n-1}\}\rangle$ and $\mathfrak{z} = \langle z\rangle$. Notice that $\mathfrak{z}$ is the center of $\mathfrak{n}$.

The parabolic $\mathfrak{p} = \mathfrak{g}_1 \ltimes \mathfrak{n}$ corresponding to Π_0 inside $\mathfrak{g}$ is:



One can see that the semisimple part of $\mathfrak{g}_1$ is $\mathfrak{g}_1^{ss} = [\mathfrak{g}_1, \mathfrak{g}_1] \simeq \mathfrak{sl}(n-1)$. Let us fix the set $\{H_1, H_2\}$ as a basis for the center of $\mathfrak{g}_1$, where

$$\begin{aligned} H_1 =& E_{1,1} - E_{n+1,n+1}, \\ H_2 =& \sum_{i=2}^{n} E_{i,i} - (n-1)E_{n+1,n+1}. \end{aligned}$$

Consider $\{e_2, \ldots, e_{n-1}\}$ as a basis of the dual of the Cartan subalgebra of $\mathfrak{sl}(n-1)$, i.e., $\langle\{e_2, \ldots, e_n\}\rangle / \langle\{e_2 + \cdots + e_n = 0\}\rangle$. Given a vector v of weight λ, we associate to it the 3-tuple:

$$(\lambda(H_1), \lambda(H_2), (\lambda_1, \lambda_2, \ldots, \lambda_{n-2})). \tag{5}$$

where the pair $(\lambda(H_1), \lambda(H_2))$ is the action of the center of $\mathfrak{g}_1$ on v, and $\lambda_1 e_2 + \lambda_2 e_3 + \cdots + \lambda_{n-2} e_{n-1} = \lambda\,|_{\mathfrak{sl}(n-1)}$.

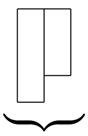
The subspaces $\mathfrak{z}$, X and Y of $\mathfrak{n}$ are $\mathfrak{g}_1$-irreducible, with z, x_{n-1} and y_1 respectively as highest weight vectors. Then we have:

$$z \simeq (2, n-1, \underbrace{(0, \dots, 0)}_{n-2});$$
$$x_{n-1} \simeq (1, -1, \underbrace{(1, \dots, 1)}_{n-2});$$
$$y_1 \simeq (1, n, \underbrace{(1, 0, \dots, 0)}_{n-2}).$$

The trivial homology of $\mathfrak{h}_{2n-1}$

Let $\{\omega_1, \dots, \omega_{n-2}\}$ be the fundamental weights of $\mathfrak{sl}(n-1)$. It is convenient to represent the irreducible representations of $\mathfrak{sl}(n-1)$ using Young diagrams. Such a diagram is an arrangement of boxes representing a highest weight. For instance, for $0 \le k \le n-2$, we will denote by D_k the one column diagram with k boxes. For $k = 0$, D_0 is the trivial representation, and for $1 \le k \le n-2$ D_k is the representation of highest weight ω_k. Diagrams $D_{k,l}$ and $D_{k,l,m}$ are defined analogously, i.e., $D_{k,l}$ is the representation of highest weight $\omega_k + \omega_l$ with $1 \le l \le k \le n-2$ and $D_{k,l,m}$ is the representation of highest weight $\omega_k + \omega_l + \omega_m$ with $1 \le m \le l \le k \le n-2$.

For example,

$$D_{k,l} =$$

k,l
$$\sim (\underbrace{2, \dots, 2}_{l}, \underbrace{1, \dots, 1}_{k-l}, \underbrace{0, \dots, 0}_{n-2-k}).$$

An irreducible $\mathfrak{g}_1$-module W_λ will be identified by a triple $(\lambda(H_1), \lambda(H_2), D)$ (as in (5)), where D is the Young diagram determining the action of the simple part of $\mathfrak{g}_1$.

We have for instance,

$$\mathfrak{z} \simeq (2, n-1, D_0), \quad X \simeq (1, -1, D_{n-2}), \quad Y \simeq (1, n, D_1).$$

We define the following vectors in $\Lambda^\bullet \mathfrak{h}_{2n-1}$, that will help us to characterize the trivial homology of $\mathfrak{h}_{2n-1}$:

1. $u_0 = 1$,
2. $u_a = x_{n-1} \wedge x_{n-2} \wedge \dots \wedge x_{n-a}$, for $1 \le a \le n-1$,
3. $v_0 = 1$,
4. $v_b = y_1 \wedge y_2 \wedge \dots \wedge y_b$, for $1 \le b \le n-1$,

and set $u_a \wedge 1 = u_a$ and $v_a \wedge 1 = v_a$.

Next we state two known results. The first one can be obtained from Kostant's Theorem [6], and the second one is due to Santharoubane [7].

Theorem 2.1. *The highest weight vectors for the trivial homology $H_p(\mathfrak{h}_{2n-1})$ are:*

- *For $0 \le p \le n-1$ and $0 \le a \le p$:*
$$r_a = u_a \wedge v_{p-a} \quad \simeq \quad (p, n(p-a) - a, D_{n-a-1,p-a}).$$
- *For $n \le p \le 2n-1$ and $0 \le a \le p-1$:*
$$t_a = z \wedge u_a \wedge v_{p-a-1} \quad \simeq \quad (p+1, n(p-a) - a - 1, D_{p-a-1,n-a-1}).$$

Theorem 2.2 (Santharoubane). *For every $0 \le p \le n-1$ we have,*
$$\dim H_p(\mathfrak{h}_{2n-1}) = \binom{2n-2}{p} - \binom{2n-2}{p-2}$$
and the rest of the terms are obtained according to Poincaré's duality.

We will denote the irreducible submodules of the p-th homology group of $\mathfrak{h}_{2n-1}$ with highest weight vectors r_a and t_a by $H_p^{r_a}$ and $H_p^{t_a}$ respectively. Then:

- For $0 \le p \le n-1$:
$$H_p(\mathfrak{h}_{2n-1}) = \sum_{a=0}^{p} H_p^{r_a}.$$
- For $n \le p \le 2n-1$:
$$H_p(\mathfrak{h}_{2n-1}) = \sum_{a=0}^{n-1} H_p^{t_a}.$$

3. Highest weight vectors for tensor products

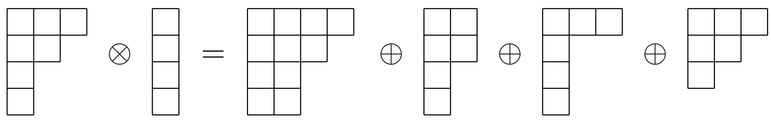
In this section we show how to construct highest weight vectors for all the irreducible summands of the tensor product of an arbitrary irreducible representation of $\mathfrak{sl}(n-1)$ and the last fundamental representation of $\mathfrak{sl}(n-1)$.

Let us begin by recalling the Littlewood-Richardson rule in terms of Young diagrams (see for instance [8], Appendix A).

Littlewood-Richardson rule. Let V_D be an irreducible representation of $\mathfrak{sl}(n-1)$ represented by the Young diagram D and let V_{n-2} be the last fundamental representation of $\mathfrak{sl}(n-1)$ represented by the Young diagram D_{n-2} (one column with $n-2$ boxes).

The tensor product $V_D \otimes V_{n-2}$ decomposes as the direct sum of the irreducible representations represented by the Young diagram obtained from D by adding one box in each row, plus all Young diagrams that can be constructed from D by removing one box.

Example 3.1. For $n=6$, the Littlewood-Richardson rule gives the following for this particular tensor product of representations of $\mathfrak{sl}(6-1)$.



The representation V_{n-2}. Let ν_{n-1} be a highest weight vector of V_{n-2} and let, for $i=n-1,\ldots,2$, ν_{i-1} be defined by $\nu_{i-1}=-E_{i,i-1}\cdot\nu_i$. Then $\{\nu_{n-1},\nu_{n-2},\ldots,\nu_1\}$ is a basis of weight vectors of V_{n-2}; ν_{n-1} is of weight $(1,\ldots,1)$ and, for $1\leq i\leq n-2$, ν_i is of weight $(0,\ldots,\underbrace{-1}_{i},\ldots,0)$.

Recall that $E_{j,j+1}\cdot\nu_i=\begin{cases}-\nu_{j+1} & \text{if } i=j,\\ 0 & \text{if } i\neq j.\end{cases}$.

In what follows we construct a highest weight vector for each of the irreducible components of $V_D\otimes V_{n-2}$. We state a Definition and a Theorem that can be found in [1].

Definition 3.2. Let $[[n-1]]$ be the set of integers from 1 to $n-1$. Given a nonempty set $J=\{j_1,\ldots,j_k\}\subset[[n-1]]$, such that $j_1<\ldots<j_k$, let $U_J\in\mathcal{U}(\mathfrak{sl}(n-1)^-)$ be defined by

$$U_J=\begin{cases}1, & k=1;\\ E_{j_k,j_{k-1}}E_{j_{k-1},j_{k-2}}\ldots E_{j_3,j_2}E_{j_2,j_1}, & k>1.\end{cases}$$

Theorem 3.3 ([1])**.** *We denote by v a highest weight vector of V_D of weight $(\lambda_1,\lambda_2,\ldots,\lambda_{n-2})$. Let $S=\{n-1\}\cup\{s:\lambda_s>\lambda_{s+1},1\leq s\leq n-2\}$ and set $\lambda_{n-1}=0$. For $s\in S$ define $A_{(s)}=\{J\subset[[n-1]]:\min(J)=s\}$. For $s\leq i\leq n-1$ let $\sigma_i=\lambda_s-\lambda_i-(s-i)-1$ and $\sigma_J=\prod_{i\in J}\sigma_i$. Then*

$$w=\sum_{J\in A_{(s)}}\frac{1}{\sigma_J}U_J(v)\otimes\nu_{\max(J)}$$

is a highest weight vector for an irreducible submodule $(V_D\otimes V_{n-2})(s)$ of weight

$$\begin{cases}(\lambda_1,\lambda_2,\ldots,\lambda_s-1,\ldots,\lambda_{n-2}), & \text{if } s<n-1,\\ (\lambda_1+1,\lambda_2+1,\ldots,\lambda_{n-2}+1), & \text{if } s=n-1.\end{cases}$$

Moreover, $V_D\otimes V_{n-2}=\bigoplus_{s\in S}(V_D\otimes V_{n-2})(s)$.

4. The adjoint homology of $\mathfrak{h}_{2n-1}$

A necessary condition for $\delta_{p+1}(W_0) = W_1$ is that W_0 and W_1 have the same action of the center of $\mathfrak{g}_1$ and the same Young diagrams.

The following table describes the action of the center of $\mathfrak{g}_1$ on the tensor products that are needed to compute the adjoint homology of $\mathfrak{n}$.

w_a	$H_p^{w_a} \otimes X$	$H_p^{w_a} \otimes Y$	$H_{p-1}^{w_a} \otimes \mathfrak{z}$
r_a	$(p+1, n(p-a)-a-1)$	$(p+1, n(p+1-a)-a)$	$(p+1, n(p-a)-a-1)$
t_a	$(p+2, n(p-a)-a-2)$	$(p+2, n(p+1-a)-a-1)$	$(p+2, n(p-a)-a-2)$

Table 1: The action of the center of $\mathfrak{g}_1$

Lemma 4.1. *The decomposition of $H_p(\mathfrak{h}_{2n-1}) \otimes X$ is as follows:*

1. *For $0 \le p \le n-1$ and $0 \le a \le p$:*

   $$H_p^{r_a} \otimes X = (H_p^{r_a} \otimes X)(n-1) \oplus (H_p^{r_a} \otimes X)(p-a) \oplus (H_p^{r_a} \otimes X)(n-a-1),$$

   where the second summand on the right only exists for $a \le p-1$, and the third summand only exists for $p \le n-2$ and $a \ge 1$.

2. *For $n \le p \le 2n-1$ and $0 \le a \le n-1$:*

   $$H_p^{t_a} \otimes X = (H_p^{t_a} \otimes X)(n-1) \oplus (H_p^{t_a} \otimes X)(n-a-1) \oplus (H_p^{t_a} \otimes X)(p-a-1),$$

   where the second summand on the right only exists for $1 \le a \le n-2$, and the third summand only exists for $p \ge n+1$.

Proof. 1. By Theorem 2.1, r_a has weight $\lambda = (\underbrace{2,\dots,2}_{p-a},\underbrace{1,\dots,1}_{n-p-1},\underbrace{0,\dots,0}_{a-1})$. Then by Theorem 3.3 the possible s for r_a are

- $n-1$;
- $p-a$, if $p-a \ge 1$; and
- $n-a-1$, if $p-a < n-a-1 \le n-2$.

Therefore we obtain

$$H_p^{r_a} \otimes X = (H_p^{r_a} \otimes X)(n-1) \oplus (H_p^{r_a} \otimes X)(p-a) \oplus (H_p^{r_a} \otimes X)(n-a-1),$$

with the above restrictions.

2. Follows analogously to item 1. ∎

Let us introduce a Lemma that will be used in what follows:

Lemma 4.2. *Let s be according to Theorem 3.3. Then*

$$\sum_{J \in A_{(s)}} \frac{1}{\sigma_J} < 0.$$

Proof. From Theorem 3.3, $s = n-1$ or s is such that $\lambda_s > \lambda_{s+1}$.

- If $s = n-1$, then $J = \{n-1\}$ and $\sum_{J \in A_{(s)}} \frac{1}{\sigma_J} = -1 < 0$.
- If $s \neq n-1$, for every $J \in A_{(s)}$ and each $i \in J$, $i > s$ we have

$$\sigma_i = \lambda_s - \lambda_i - (s-i) - 1 > 0$$

and if $i = s$,

$$\sigma_s = \lambda_s - \lambda_s - (s-s) - 1 = -1.$$

Therefore

$$\sigma_J = \prod_{i \in J} \sigma_i = -\prod_{\substack{i \in J \\ i > s}} \sigma_i < 0,$$

and then $\sum_{J \in A_{(s)}} \frac{1}{\sigma_J} < 0$. ■

Proposition 4.3. *Let $\delta_{p+1} : H_{p+1}(\mathfrak{h}_{2n-1}) \otimes V \to H_p(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}$ be the connecting morphism in the long exact sequence (3). Then* $\operatorname{coker} \delta_{p+1}$ *is as follows:*

1. *For $0 \leq p \leq n-2$,*
$$\operatorname{coker} \delta_{p+1} = 0$$
2. *For $p = n-1$,.*
$$\operatorname{coker} \delta_{p+1} = H_{n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}.$$
3. *For $n \leq p \leq 2n-2$,*
$$\operatorname{coker} \delta_{p+1} = 0.$$
4. *For $p = 2n-1$,*
$$\operatorname{coker} \delta_{p+1} = H_{2n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}.$$

Proof. We will prove the following:

1. For $0 \leq p \leq n-2$ and $0 \leq a \leq p$,
$$\delta_{p+1}((H_{p+1}^{r_a} \otimes X)(p+1-a)) = H_p^{r_a} \otimes \mathfrak{z}.$$
2. For every $0 \leq a \leq n-1$, $H_{n-1}^{r_a} \otimes \mathfrak{z}$ has no preimage by δ_n.
3. For $n \leq p \leq 2n-2$ and $0 \leq a \leq n-1$,
$$\delta_{p+1}((H_{p+1}^{t_a} \otimes X)(p-a)) = H_p^{t_a} \otimes \mathfrak{z}.$$
4. For every $0 \leq a \leq 2n-2$, $H_{2n-1}^{t_a} \otimes \mathfrak{z}$ has no preimage by δ_{2n}.

Since $\mathfrak{z}$ is the trivial representation, we can obtain the decomposition of $H_p(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}$ from Theorem 2.1:

- For $0 \le p \le n-2$:

$$H_p(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z} = \sum_{a=0}^{p} H_p^{r_a} \otimes \mathfrak{z}.$$

- For $p = n-1$:

$$H_{n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z} = \sum_{a=0}^{n-1} H_{n-1}^{r_a} \otimes \mathfrak{z}.$$

- For $n \le p \le 2n-2$:

$$H_p(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z} = \sum_{a=0}^{n-1} H_p^{t_a} \otimes \mathfrak{z}.$$

- For $p = 2n-1$:

$$H_{2n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z} = \sum_{a=0}^{n-1} H_{2n-1}^{t_a} \otimes \mathfrak{z}.$$

1. By Theorems 2.1 and 3.3, a highest weight vector of $(H_{p+1}^{r_a} \otimes X)(p+1-a)$ is

$$w_1 = \sum_{J \in A_{(p+1-a)}} \frac{1}{\sigma_J} U_J(x_{n-1} \wedge \ldots \wedge x_{n-a} \wedge y_1 \wedge \ldots \wedge y_{p+1-a}) \otimes x_{\max(J)}$$

Let $k = |J|$ and $U_J = E_{j_k, j_{k-1}} E_{j_{k-1}, j_{k-2}} \ldots E_{j_3, j_2} E_{j_2, p+1-a}$. Recall that $r_a = u_a \wedge v_{p+1-a}$ and define:

$$U_i = E_{j_{i+1}, j_i} E_{j_i, j_{i-1}} \ldots E_{j_3, j_2} E_{j_2, p+1-a}, \qquad U_i^c = E_{j_k, j_{k-1}} E_{j_{k-1}, j_{k-2}} \ldots E_{j_{i+2}, j_{i+1}}.$$

Then we have

$$U_J(r_a) = U_J(u_a) \wedge v_{p+1-a} + u_a \wedge U_J(v_{p+1-a}) + \sum_{i=1}^{k-2} U_i^c(u_a) \wedge U_i(v_{p+1-a})$$
$$+ \sum_{i=1}^{k-2} U_i(u_a) \wedge U_i^c(v_{p+1-a}).$$

Since $j_{i+1} > p+1-a$ for $i \ge 1$ then $U_i^c(v_{p+1-a}) = 0$ and

$$U_J(r_a) = U_J(u_a) \wedge v_{p+1-a} + u_a \wedge U_J(v_{p+1-a}) + \sum_{i=1}^{k-2} U_i^c(u_a) \wedge U_i(v_{p+1-a}).$$

Therefore we have,

$$\begin{aligned} U_J(r_a) =& (-1)^{\alpha_1}\ x_{n-1} \wedge \ldots \wedge \widehat{x_{\max(J)}} \wedge \ldots \wedge x_{n-a} \wedge x_{p+1-a} \wedge v_{p+1-a} \\ &+ u_a \wedge y_1 \wedge \ldots \wedge y_{p-a} \wedge y_{\max(J)} \\ &+ \sum_{\substack{1 \le i \le k-2 \\ j_{i+2} \ge n-a}} (-1)^{\alpha_2}\ x_{n-1} \wedge \ldots \wedge \widehat{x_{\max(J)}} \wedge \ldots \wedge x_{n-a} \wedge x_{j_{i+1}} \wedge \\ &\qquad \wedge y_1 \wedge \ldots \wedge y_{p-a} \wedge y_{j_{i+1}}, \end{aligned}$$

where $(-1)^{\alpha_1}$ and $(-1)^{\alpha_2}$ are the changes of signs due to the action of U_J and the reordering of the vector.

Since $\partial_1^{p+1}(U_J(r_a)\otimes x_{\max(J)})$ is nonzero if and only if $y_{\max(J)}$ is a factor of a term of $U_J(r_a)$, we obtain

$$\begin{aligned}\partial_1^{p+1}(w_1) &= \sum_{J\in A_{(p+1-a)}} \frac{1}{\sigma_J}(-1)^p\, x_{n-1}\wedge\ldots\wedge x_{n-a}\wedge y_1\wedge\ldots\wedge y_{p-a}\otimes \\ &\qquad\qquad\otimes[y_{\max(J)},x_{\max(J)}] \\ &= \left(\sum_{J\in A_{(p+1-a)}}\frac{1}{\sigma_J}\right)(-1)^{p+1}\left(u_a\wedge v_{p-a}\otimes z\right).\end{aligned}$$

By Lemma 4.2 we have that $\displaystyle\sum_{J\in A_{(p+1-a)}}\frac{1}{\sigma_J}<0$, then by Theorem 3.3, $\partial_1^{p+1}(w_1)$ is a highest weight vector of $H_p^{r_a}\otimes\mathfrak{z}$ for $0\le a\le p$. Therefore $\delta_{p+1}((H_{p+1}^{r_a}\otimes X)(p+1-a)) = H_p^{r_a}\otimes\mathfrak{z}$.

2. Let $p=n$. Then, by Table 1, the action of the center of $\mathfrak{g}_1$ on $H_{n-1}^{r_a}\otimes\mathfrak{z}$ is $(n+1, n(n-a)-a-1)$ while the actions on $H_n^{t_a}\otimes X$ and $H_n^{t_a}\otimes Y$ are $(n+2, n(n+1-a)-a-1)$ and $(n+2, n(n-a)-a-2)$ respectively. Therefore $H_{n-1}^{r_a}\otimes\mathfrak{z}\in\operatorname{coker}\delta_n$ for every $0\le a\le n-1$.

3. By Theorems 2.1 and 3.3, a highest weight vector of $(H_{p+1}^{t_a}\otimes X)(p-a)$ is

$$w_2=\sum_{J\in A_{(p-a)}}\frac{1}{\sigma_J}U_J(z\wedge u_a\wedge v_{p-a})\otimes x_{\max(J)}$$

Therefore we have,

$$\begin{aligned}\partial_1^{p+1}(w_2) &= \sum_{J\in A_{(p-a)}}\frac{1}{\sigma_J}(-1)^p z\wedge u_a\wedge y_1\wedge\ldots\wedge y_{p-a-1}\otimes[y_{\max(J)},x_{\max(J)}] \\ &= \left(\sum_{J\in A_{(p-a)}}\frac{1}{\sigma_J}\right)(-1)^{p+1}\left(z\wedge u_a\wedge v_{p-a-1}\otimes z\right)\end{aligned}$$

which, by Theorem 3.3 and Lemma 4.2, is a highest weight vector of $H_p^{t_a}\otimes\mathfrak{z}$ for $0\le a\le n-1$. Therefore $\delta_{p+1}((H_{p+1}^{t_a}\otimes X)(p-a)) = H_p^{t_a}\otimes\mathfrak{z}$ for $0\le a\le n-1$.

4. Since $\delta_{2n}=0$ then $H_{2n-1}^{t_a}\otimes\mathfrak{z}\in\operatorname{coker}\delta_{2n}$ for $0\le a\le 2n-2$. ∎

Every irreducible submodule in $H_p(\mathfrak{h}_{2n-1})\otimes(X\oplus Y)$ that has not been considered in Proposition 4.3 (or someone isomorphic to it) must be in $\ker\delta_p$. Therefore, using Lemma 4.1, we have:

1. For $0\le p\le n-2$,

$$\ker\delta_p\simeq\sum_{a=0}^{p}(H_p^{r_a}\otimes X)(n-1)\oplus\sum_{a=1}^{p}(H_p^{r_a}\otimes X)(n-a-1)\oplus H_p(\mathfrak{h}_{2n-1})\otimes Y.$$

2. For $p = n-1$,

$$\ker \delta_{n-1} \simeq \sum_{a=0}^{n-1} (H_{n-1}^{r_a} \otimes X)(n-1) \oplus H_{n-1}(\mathfrak{h}_{2n-1}) \otimes Y.$$

3. For $p = n$,

$$\ker \delta_n = H_n(\mathfrak{h}_{2n-1}) \otimes X \oplus H_n(\mathfrak{h}_{2n-1}) \otimes Y.$$

4. For $n+1 \le p \le 2n-1$,

$$\ker \delta_p \simeq \sum_{a=0}^{n-1} (H_p^{t_a} \otimes X)(n-1) \oplus \sum_{a=1}^{n-2} (H_p^{t_a} \otimes X)(n-a-1) \oplus H_p(\mathfrak{h}_{2n-1}) \otimes Y.$$

Finally, we have the main result:

Theorem 4.4. *The adjoint homology of* $\mathfrak{h}_{2n-1}$ *is as follows:*

1. *For* $0 \le p \le n-2$,

$$H_p(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) \simeq \sum_{a=0}^{p} (H_p^{r_a} \otimes X)(n-1) \oplus \sum_{a=1}^{p} (H_p^{r_a} \otimes X)(n-a-1) \oplus H_p(\mathfrak{h}_{2n-1}) \otimes Y.$$

2. *For* $p = n-1$,

$$H_{n-1}(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) \simeq \sum_{a=0}^{n-1} (H_{n-1}^{r_a} \otimes X)(n-1) \oplus H_{n-1}(\mathfrak{h}_{2n-1}) \otimes Y \oplus H_{n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}.$$

3. *For* $p = n$,

$$H_n(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) = H_n(\mathfrak{h}_{2n-1}) \otimes X \oplus H_n(\mathfrak{h}_{2n-1}) \otimes Y,$$

4. *For* $n+1 \le p \le 2n-2$,

$$H_p(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) \simeq \sum_{a=0}^{n-1} (H_p^{t_a} \otimes X)(n-1) \oplus \sum_{a=1}^{n-2} H_p^{t_a} \otimes X(n-a-1) \oplus H_p(\mathfrak{h}_{2n-1}) \otimes Y.$$

5. *For* $p = 2n-1$,

$$H_{2n-1}(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) \simeq \sum_{a=0}^{n-1} (H_{2n-1}^{t_a} \otimes X)(n-1) \oplus \sum_{a=1}^{n-2} (H_{2n-1}^{t_a} \otimes X)(n-a-1)$$
$$\oplus \; H_{2n-1}(\mathfrak{h}_{2n-1}) \otimes Y \oplus H_{2n-1}(\mathfrak{h}_{2n-1}) \otimes \mathfrak{z}.$$

Proof. Follows from Propositions 4.3 and Lemma 4.1. ■

One can obtain the dimensions of the adjoint cohomology groups of $\mathfrak{h}_{2n-1}$ by direct computation using Theorems 4.4 and 2.2. However, since $\mathfrak{h}_{2n-1}$ is a nilpotent Lie algebra, there is a vector space isomorphism (see equation (2.4) in [5]):

$$H_p(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1}) \simeq H^{2n-1-p}(\mathfrak{h}_{2n-1}, \mathfrak{h}_{2n-1})$$

and the dimensions for the adjoint homology can be obtained from the ones of the adjoint cohomology in [4].

Corollary 4.5 (Magnin). *Let b_p be the dimension of the p-th adjoint homology group for the Heisenberg Lie algebra $\mathfrak{h}_{2n-1}$. Then*

1. *For $0 \leq p \leq n-2$,*

$$b_p = (2n-2)\left[\binom{2n-2}{p} - \binom{2n-2}{p-2}\right] - \binom{2n-2}{p-1} + \binom{2n-2}{p-3}.$$

2. *For $p = n-1$,*

$$b_{n-1} = (2n-1)\left[\binom{2n-2}{n-1} - \binom{2n-2}{n-3}\right] - \binom{2n-2}{n-2} + \binom{2n-2}{n-4}.$$

3. *For $n \leq p \leq 2n-2$,*

$$b_p = (2n-1)\binom{2n-1}{p} - \binom{2n-1}{p-1} - (2n-2)\binom{2n-1}{p+1}.$$

4. *For $p = 2n-1$,*

$$b_{2n-1} = 1.$$

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