

On the Reducibility Points beyond the Ends of Complementary Series of p -adic General Linear Groups

Marko Tadić

Communicated by G. Olshanski

Abstract. In this paper we consider the reducibility points beyond the ends of complementary series of general linear groups over a p -adic field, which start with Speh representations. We describe explicitly the composition series of the representations at these reducibility points. They are multiplicity one representations, and they can be of arbitrary length. We give Langlands parameters of all the irreducible subquotients and determine the lattice of subrepresentations. *Mathematics Subject Classification 2010:* 22E50.

Key Words and Phrases: Non-archimedean local fields, general linear groups, Speh representations, parabolically induced representations, reducibility, composition series, unitarizability.

1. Introduction

Problems of reducibility of parabolically induced representations are very important in the harmonic analysis on reductive groups over local fields (they are of particular importance for the problem of unitarizability). They are also very important for the theory of automorphic forms for number of questions. A closely related (usually very non-trivial) problem is the determination of the composition series at the reducibility points. The knowledge of composition series is equivalent to the corresponding character identity. In this paper we study such type of problems for general linear groups over a local non-archimedean field F .

Speh representations are key representations in the classification of unitary duals of general linear groups (see [14]; for the archimedean case see [13]). One directly gets all the complementary series for general linear groups from the complementary series starting with Speh representations. Composition series at the ends of these complementary series are crucial in determining the topology of the unitary duals. The composition series at the ends of these complementary series played also crucial role in obtaining explicit formula for characters of irreducible unitary representations in terms of standard characters ([17]). These complementary series terminate at the first reducibility point and the representations there have length two (when one starts with a single Speh representation). It is a natural

question to ask what are the composition series at the further reducibility points.

There can exist a significant number of reducibility points beyond the end of complementary series. In all these reducibility points we completely determine the composition series, give the Langlands parameters of the irreducible subquotients and determine the lattice of subrepresentations. These representations are always multiplicity one representations (when one starts from a single Speh representation), and can be of arbitrary length. For example, if we want to get a representation of length 1000 supported by the minimal parabolic subgroup, we shall need to start with Speh representations of $GL(999\,000, F)$ ¹, and consider the complementary series of $GL(1\,998\,000, F)$. It is interesting that it is very easy to write down the Langlands parameters of all the irreducible subquotients (they are given by the simple formula (2); see Theorem 1.2).

Now we shall describe more precisely the principal results of the paper. Put

$$\nu = |\det|_F,$$

where $|\cdot|_F$ denotes the normalized absolute value on a local non-archimedean field F . For $u, v \in \mathbb{R}$ such that $v - u \in \mathbb{Z}_{\geq 0}$, and for an irreducible cuspidal representation ρ of $GL(p, F)$, the set

$$[\nu^u \rho, \nu^v \rho] = \{\nu^u \rho, \nu^{u+1} \rho, \dots, \nu^{v-1} \rho, \nu^v \rho\}$$

is called a segment in cuspidal representations (of general linear groups). The representation $\nu^u \rho$ is denoted by $b([\nu^u \rho, \nu^v \rho])$, and called the beginning of the segment $[\nu^u \rho, \nu^v \rho]$. We say that such a segment Δ_1 precedes another segment Δ_2 , and write

$$\Delta_1 \rightarrow \Delta_2,$$

if $\Delta_1 \cup \Delta_2$ is a segment different from Δ_1 and Δ_2 , and if the beginnings of Δ_1 and $\Delta_1 \cup \Delta_2$ are the same.

Let $\Delta = [\nu^u \rho, \nu^v \rho]$ be a segment in cuspidal representations. For $z \in \mathbb{R}$, we denote

$$\nu^z \Delta = \{\nu^z \rho'; \rho' \in \Delta\}.$$

Consider the representation

$$\text{Ind}^{GL((v-u+1)p, F)}(\nu^v \rho \otimes \nu^{v-1} \rho \otimes \dots \otimes \nu^u \rho),$$

parabolically induced from the appropriate parabolic subgroup containing regular upper triangular matrices (see the second section). Then the above representation has a unique irreducible subrepresentation. This subrepresentation is essentially square integrable. It is denoted by

$$\delta(\Delta).$$

Let $a = (\Delta_1, \dots, \Delta_k)$ be a finite multiset of segments in cuspidal representations. Write $\Delta_i = [\nu^{u_i} \rho_i, \nu^{v_i} \rho_i]$, where $\delta(\Delta_i)$ is a representation of $GL(n_i, F)$ and ρ_i

¹This is the lowest rank in which we get length 1000 at some reducibility point beyond the complementary series starting with a single Speh representation.

are unitarizable irreducible cuspidal representations. Take a permutation σ of $\{1, \dots, k\}$ such that

$$u_{\sigma(1)} + v_{\sigma(1)} \geq \dots \geq u_{\sigma(k)} + v_{\sigma(k)}. \tag{1}$$

Then the representation

$$\text{Ind}^{GL(n_1+\dots+n_k, F)}(\delta(\Delta_{\sigma(1)}) \otimes \dots \otimes \delta(\Delta_{\sigma(k)})),$$

parabolically induced from the appropriate parabolic subgroup containing regular upper triangular matrices, has a unique irreducible quotient (whose equivalence class does not depend on the permutation σ which satisfy the above condition²). We denote it by

$$L(a).$$

Then attaching $a \mapsto L(a)$ is one possible description of the Langlands classification of the non-unitary duals of groups $GL(n, F)$'s (by multisets of segments in cuspidal representations). We shall use this version of Langlands classification for general linear groups in this paper.

We add finite multisets of segments in obvious way:

$$a_1 + a_2 = (\Delta_1^{(1)}, \dots, \Delta_{k(1)}^{(1)}, \Delta_1^{(2)}, \dots, \Delta_{k(2)}^{(2)}),$$

where $a_i = (\Delta_1^{(i)}, \dots, \Delta_{k(i)}^{(i)})$, $i = 1, 2$.

Let $\Delta = [\nu^u \rho, \nu^v \rho]$ be a segment in cuspidal representations such that ρ is unitarizable and $u + v = 0$. Fix $n \in \mathbb{Z}_{\geq 1}$. Then the representation

$$u(\delta(\Delta), n) := L(\nu^{(n-1)/2} \Delta, \nu^{(n-1)/2-1} \Delta, \dots, \nu^{-(n-1)/2} \Delta)$$

is called a Speh representation. Such a representation is unitarizable, and each irreducible unitary representation of a general linear group is constructed from several such representations in a simple way ([14]). If an irreducible representation become a Speh representations after a twists by a characters, then it will be called essentially Speh representation. In other words, essentially Speh representations are the representations of a form $\nu^\alpha u(\delta(\Delta), n)$, with $\alpha \in \mathbb{R}$.

Let $k \in \mathbb{Z}$ and denote $d = \text{card}(\Delta)$. We shall consider representations

$$\mathbf{R}^t(n, d)_{(k)}^{(\rho)} = \text{Ind}(\nu^{-k/2} u(\delta(\Delta), n) \otimes \nu^{k/2} u(\delta(\Delta), n)).$$

For $k = 1$, this is the end of complementary series. Denote

$$\Delta_i = \nu^{-k/2}(\nu^{-(n-1)/2+i-1} \Delta), \quad \Gamma_i = \nu^{k/2}(\nu^{-(n-1)/2+i-1} \Delta), \quad i = 1, \dots, n.$$

Then

$$\nu^{-k/2} u(\delta(\Delta), n) = L(\Delta_1, \dots, \Delta_n), \quad \nu^{k/2} u(\delta(\Delta), n) = L(\Gamma_1, \dots, \Gamma_n),$$

and thus

$$\mathbf{R}^t(n, d)_{(k)}^{(\rho)} = \text{Ind}(L(\Delta_1, \dots, \Delta_n) \otimes L(\Gamma_1, \dots, \Gamma_n)).$$

²The multiset does not change if we make a permutation of elements in it. Nevertheless, when we define multisets, we always fix some ordering on the segments that determine it (often in this paper completely opposite to (1))

Definition 1.1. For $j = 0$ and for $1 \leq j \leq n$ for which $\Delta_n \rightarrow \Gamma_j$, denote

$$r_j(n, d)_{(k)}^{(\rho)} = \sum_{i=1}^j (\Delta_{i+n-j} \cup \Gamma_i, \Delta_{i+n-j} \cap \Gamma_i) + \sum_{i=1}^{n-j} (\Delta_i, \Gamma_{i+j}). \quad (2)$$

In other words, $r_0(n, d)_{(k)}^{(\rho)} = (\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_n)$, and for $1 \leq j \leq n$ for which $\Delta_n \rightarrow \Gamma_j$ we get $r_j(n, d)_{(k)}^{(\rho)}$ by replacing in $r_0(n, d)_{(k)}^{(\rho)}$ the part

$$\Delta_{n-j+1}, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_j$$

with

$$\Delta_{n-j+1} \cup \Gamma_1, \Delta_{n-j+1} \cap \Gamma_1, \dots, \Delta_n \cup \Gamma_j, \Delta_n \cap \Gamma_j.$$

One gets easily that $\Delta_n \rightarrow \Gamma_j$ if and only if

$$\max(n - k + 1, 1) \leq j \leq \min(n - k + d, n).$$

Theorem 1.2. Let $k \in \mathbb{Z}_{\geq 0}$. Then:

1. The representation $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation. It has a unique irreducible subrepresentation and unique irreducible quotient. The irreducible subrepresentation is isomorphic to $L(r_0(n, d)_{(k)}^{(\rho)})$. Further, the representations $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$ and $\mathbf{R}^{\mathbf{t}}(n, d)_{(-k)}^{(\rho)}$ have the same composition series.
2. For $n + d \leq k$, and for $k = 0$, $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$ is irreducible. Then $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)} \cong L(r_0(n, d)_{(k)}^{(\rho)})$.
3. For $0 < k < n + d$, the composition series of $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$ consists of

$$L(r_i(n, d)_{(k)}^{(\rho)}), \quad \max(n - k + 1, 1) \leq i \leq \min(n - k + d, n),$$

together with $L(r_0(n, d)_{(k)}^{(\rho)})$. The irreducible quotient of $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$ is isomorphic to $L(r_{\min(n-k+d, n)}(n, d)_{(k)}^{(\rho)})$.

Besides the composition series, in this paper we also completely determine the lattices of subrepresentations of representations $\mathbf{R}^{\mathbf{t}}(n, d)_{(k)}^{(\rho)}$.

The main tool in our handling of the composition series that we consider in this paper, are derivatives ([6], [4], [19]). A very simple and nice formula for the derivatives of Speh representations (described in 2.10) is crucial for our applications of derivatives. The formula was obtained in [7] (it was conjectured much earlier in [15]). Actually, the formula of E. Lapid and A. Mínguez is much more general - it is for ladder representations (defined in [7]³). Another tool that we use in this paper is the Mœglin-Waldspurger algorithm from [10] for the Zelevinsky involution.

³Let $\pi \cong L(a)$, where $a = (\Delta_1, \dots, \Delta_k)$ is a finite multiset of segments in cuspidal representations. Suppose that $\Delta_i \not\subseteq \Delta_j$ whenever $i \neq j$. If π is supported by one cuspidal \mathbb{Z} -line (this is not an essential condition), then π is called a ladder representation.

Although the main results of the paper are presented in the introduction in terms of the Langlands classification, the methods by which we have obtained them in the paper are based on the Zelevinsky classification (which is dual to the Langlands classification). We deal in the most of the paper with the representations $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ which are defined in terms of the Zelevinsky classification. The relation with the representations $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ which we describe in the introduction is very simple

$$\mathbf{R}^t(n, d)_{(k)}^{(\rho)} = \mathcal{R}(d, n)_{(k)}^{(\rho)}.$$

The main reason why the Zelevinsky classification is much more convenient for working with the derivatives, is that there is a very simple explicit formula for the highest derivative of an irreducible representation in terms of this classification (Theorem 8.1 of [19]; in the case of Langlands classification we have an algorithm).

We are very thankful to B. Leclerc who has informed us that our result about composition series in the case when the cuspidal representation ρ is the trivial character of F^\times can be deduced from Theorem 2 of [8], which addresses Hecke algebra representations (his result is more general in this case - it gives combinatorial rule for calculating the composition factors there). The theory of types for general linear groups from [5] opens a possibility of approach to get the case of general ρ using B. Leclerc result. We have not used this possibility. This way of proving the general case would be technically more complicated, relaying on types, attached Hecke algebras etc. (and already in the unramified case, it is not simple since [8] is based on two previous papers, one of which uses a very non-trivial positivity result of G. Lusztig).

The main reason for our approach is that the statement of our principal result does not include types, and therefore it is natural to (try to) have a proof of it which does not use them. The second reason are derivatives which we use in this paper (and develop further methods for applying them). They are a very natural tool in the study of questions related to the irreducible unitary representations. Namely, recall that already the main result of the first crucial paper [3] on the unitarizability in the p -adic case, relates unitarizability and derivatives for general linear groups. This J. Bernstein paper was followed by the second paper [14] where the unitarizability was solved completely, with essential use of derivatives (and soon realized in [13] that the solution can be extended to the archimedean situation, avoiding derivatives in this case). At the end, let us mention that our experience with the problems related to the unitary representations, is that it is very important to have as simple (and direct) understanding of them as possible.

We are very thankful to the referee for a number of corrections and very useful suggestions.

The content of the paper is as follows. The second section recalls the notation that we use in the paper. The third section contains preparatory technical results, while in the fourth section we define the multisegments $r_i(n, d)_{(k)}^{(\rho)}$. The fifth section is devoted to the calculation of the composition series in the case of disjoint beginnings of segments, while the sixth section gives a description of the lattice of subrepresentations in this case. In the seventh section we deal with the composition series in the remaining case (of non-disjoint beginnings of segments) and the eighth section brings a description of the lattice of subrepresentations for this case. The

ninth section brings an interpretation of the main results of the paper in terms of the Langlands classification. At the end of this section, we present a conjectural description of the composition series of the representation parabolically induced with a tensor product of two arbitrary essentially Speh representations.

2. Notation and preliminaries

We recall in this section very briefly some notation for general linear groups in the non-archimedean case (one can find more details in [19] and [11]).

2.1. Finite multisets.

Let X be a set. The set of all finite multisets in X is denoted by $M(X)$ (we can view each multiset as a functions $X \rightarrow \mathbb{Z}_{\geq 0}$ with finite support; here finite subsets correspond to all functions $\mathcal{S}(\mathcal{C}) \rightarrow \{0, 1\}$ with finite support). Elements of $M(X)$ are denoted by (x_1, \dots, x_n) (repetitions of elements can occur; the multiset does not change if we permute x_i 's). The number n is called the cardinality of (x_1, \dots, x_n) , and it is denoted by

$$\text{card}(x_1, \dots, x_n).$$

On $M(X)$ we have a natural structure of a commutative associative semi group with zero:

$$(x_1, \dots, x_n) + (y_1, \dots, y_m) = (x_1, \dots, x_n, y_1, \dots, y_m).$$

For $x, y \in M(X)$ we write

$$x \subseteq y$$

if there exists $z \in M(X)$ such that $x + z = y$.

2.2. Segments in \mathcal{C} .

Let F be a non-archimedean locally compact non-discrete field and $|\cdot|_F$ its modulus character. Denote

$$G_n = GL(n, F), n \geq 0$$

(we take G_0 to be the trivial group; we consider it formally as the group of 0×0 matrices). The set of all equivalence classes of irreducible representations of all groups $G_n, n \geq 0$, is denoted by

$$Irr.$$

The subset of all cuspidal classes of representations $G_n, n \geq 1$, is denoted by

$$\mathcal{C}.$$

Unitarizable classes in \mathcal{C} are denoted by \mathcal{C}^u . Put

$$\nu = |\det|_F.$$

Fix $u, v \in \mathbb{R}$ such that $v - u \in \mathbb{Z}_{\geq 0}$, and $\rho \in \mathcal{C}$. Then the set

$$[\nu^u \rho, \nu^v \rho] = \{\nu^u \rho, \nu^{u+1} \rho, \dots, \nu^{v-1} \rho, \nu^v \rho\}$$

is called a segment in \mathcal{C} . The set of all segments in cuspidal representations of general linear groups is denoted by

$$\mathcal{S}(\mathcal{C}).$$

Let $\Delta = [\nu^u \rho, \nu^v \rho] \in \mathcal{S}(\mathcal{C})$. The representation $\nu^u \rho$ is called the beginning of the segment Δ , and $\nu^v \rho$ is called the end of the segment Δ . We denote the beginning and the end by

$$b(\Delta) \quad \text{and} \quad e(\Delta)$$

respectively.

For $z \in \mathbb{R}$, denote

$$\nu^z \Delta = \{\nu^z \rho'; \rho' \in \Delta\}.$$

We define Δ^- and ${}^-\Delta$ by

$$\Delta^- = [\nu^u \rho, \nu^{v-1} \rho] \quad \text{and} \quad {}^-\Delta = [\nu^{u+1} \rho, \nu^v \rho]$$

if $u < v$. Otherwise we take ${}^-\Delta = \emptyset$.

Segments $\Delta_1, \Delta_2 \in \mathcal{S}(\mathcal{C})$ are called linked if $\Delta_1 \cup \Delta_2 \in \mathcal{S}(\mathcal{C})$ and $\Delta_1 \cup \Delta_2 \notin \{\Delta_1, \Delta_2\}$. If the segments Δ_1 and Δ_2 are linked and if Δ_1 and $\Delta_1 \cup \Delta_2$ have the same beginnings, we say that Δ_1 precedes Δ_2 . In this case we write

$$\Delta_1 \rightarrow \Delta_2.$$

2.3. Multisegments.

Let $a = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S}(\mathcal{C}))$. Suppose that Δ_i and Δ_j are linked for some $1 \leq i < j \leq k$. Let c be the multiset that we get by replacing segments Δ_i and Δ_j by segments $\Delta_i \cup \Delta_j$ and $\Delta_i \cap \Delta_j$ in a (we omit $\Delta_i \cap \Delta_j$ if $\Delta_i \cap \Delta_j = \emptyset$). In this case we write

$$c \prec a.$$

Using \prec , we generate in a natural way an ordering \leq on $M(\mathcal{S}(\mathcal{C}))$.

For $a = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S}(\mathcal{C}))$, denote

$$a^- = (\Delta_1^-, \dots, \Delta_k^-), \quad {}^-a = ({}^-\Delta_1, \dots, {}^-\Delta_k) \in M(\mathcal{S}(\mathcal{C}))$$

(again, we omit Δ_i^- and ${}^-\Delta_i$ if they are empty sets).

Further, for $a = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S}(\mathcal{C}))$ define

$$\text{supp}(a) = \sum_{i=0}^k \Delta_i \in M(\mathcal{C}),$$

where we consider in the above formula Δ_i 's as elements of $M(\mathcal{C})$.

The multiset of all beginnings (reps. ends) of segments from $a \in M(\mathcal{S}(\mathcal{C}))$ is denoted by $\mathcal{B}(a)$ (resp. $\mathcal{E}(a)$). Clearly,

$$\mathcal{B}(a), \mathcal{E}(a) \in M(\mathcal{C}).$$

Take positive integers n and d and let $\rho \in \mathcal{C}$. Denote

$$a(n, d)^{(\rho)} = (\nu^{-\frac{n-1}{2}} \Delta, \nu^{-\frac{n-1}{2}+1} \Delta, \dots, \nu^{\frac{n-1}{2}} \Delta) \in M(\mathcal{S}(\mathcal{C})), \quad (3)$$

where

$$\Delta = [\nu^{-(d-1)/2} \rho, \nu^{(d-1)/2} \rho].$$

2.4. Algebra of representations.

The category of all smooth representations of G_n is denoted by $\text{Alg}(G_n)$. The set of all equivalence classes of irreducible smooth representations of G_n is denoted by

$$\tilde{G}_n.$$

The subset of unitarizable classes in \tilde{G}_n is denoted by

$$\hat{G}_n.$$

The Grothendieck group of the category $\text{Alg}_{\text{f.l.}}(G_n)$ of all smooth representations of G_n of finite length is denoted by R_n . It is a free \mathbb{Z} -module with basis \tilde{G}_n . We have the canonical mapping

$$\text{s.s.} : \text{Alg}_{\text{f.l.}}(G_n) \rightarrow R_n.$$

The set of all finite sums in R_n of elements of the basis \tilde{G}_n is denoted by $(R_n)_+$. Set

$$R = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} R_n,$$

$$R_+ = \sum_{n \in \mathbb{Z}_{\geq 0}} (R_n)_+.$$

The ordering on R is defined by $r_1 \leq r_2 \iff r_2 - r_1 \in R_+$.

An additive mapping $\varphi : R \rightarrow R$ is called positive if

$$r_1 \leq r_2 \implies \varphi(r_1) \leq \varphi(r_2).$$

For two finite length representations π_1 and π_2 of G_n we shall write $\text{s.s.}(\pi_1) \leq \text{s.s.}(\pi_2)$ shorter

$$\pi_1 \leq \pi_2.$$

2.5. Parabolic induction.

Let

$$M_{(n_1, n_2)} := \left\{ \begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix}; g_i \in G_i \right\} \subseteq G_{n_1+n_2},$$

and let σ_1 and σ_2 be smooth representations of G_{n_1} and G_{n_2} , respectively. We consider $\sigma_1 \otimes \sigma_2$ as the representation

$$\begin{bmatrix} g_1 & * \\ 0 & g_2 \end{bmatrix} \mapsto \sigma_1(g_1) \otimes \sigma_2(g_2)$$

of $M_{(n_1, n_2)}$. By

$$\sigma_1 \times \sigma_2$$

is denoted the representation of $G_{n_1+n_2}$ parabolically induced by $\sigma_1 \otimes \sigma_2$ from $M_{(n_1, n_2)}$ (the induction that we consider is smooth and normalized). For three representations, we have

$$(\sigma_1 \times \sigma_2) \times \sigma_3 \cong \sigma_1 \times (\sigma_2 \times \sigma_3). \quad (4)$$

The induction functor is exact and we can lift it in a natural way to a \mathbb{Z} -bilinear mapping $\times : R_{n_1} \times R_{n_2} \rightarrow R_{n_1+n_2}$, and further to $\times : R \times R \rightarrow R$. In this way, R becomes graded commutative ring. The commutativity implies that if $\pi_1 \times \pi_2$ is irreducible for $\pi_i \in \tilde{G}_{n_i}$, then

$$\pi_1 \times \pi_2 \cong \pi_2 \times \pi_1.$$

2.6. Classifications of non-unitary duals. Let $\Delta = [\nu^u \rho, \nu^v \rho] \in \mathcal{S}(\mathcal{C})$. The representation

$$\nu^v \rho \times \nu^{v-1} \rho \times \dots \times \nu^u \rho.$$

has a unique irreducible subrepresentation, which is denoted by

$$\delta(\Delta),$$

and a unique irreducible quotient, which is denoted by

$$\mathfrak{z}(\Delta).$$

The irreducible subrepresentation is essentially square integrable, i.e. it becomes square integrable (modulo center) after twisting with a suitable character of the group.

Let $a = (\Delta_1, \dots, \Delta_n) \in M(\mathcal{S}(\mathcal{C}))$ be non-empty (i.e. $n \geq 1$). Choose an enumeration of Δ_i 's such that for all $i, j \in \{1, 2, \dots, n\}$ the following holds:

$$\text{if } \Delta_i \rightarrow \Delta_j, \text{ then } j < i.$$

Then the representations

$$\begin{aligned} \zeta(a) &:= \mathfrak{z}(\Delta_1) \times \mathfrak{z}(\Delta_2) \times \dots \times \mathfrak{z}(\Delta_n), \\ \lambda(a) &:= \delta(\Delta_1) \times \delta(\Delta_2) \times \dots \times \delta(\Delta_n) \end{aligned}$$

are determined by a up to an isomorphism (i.e., their isomorphism classes do not depend on the enumerations which satisfies the above condition). The representation $\zeta(a)$ has a unique irreducible subrepresentation, which is denoted by

$$Z(a),$$

while the representation $\lambda(a)$ has a unique irreducible quotient, which is denoted by

$$L(a).$$

For the empty multisegment \emptyset , we take $Z(\emptyset) = L(\emptyset)$ to be the trivial (one-dimensional) representation of the trivial group G_0 . This is the identity of the ring R (and it is very often denoted simply by 1).

In this way we obtain two classifications of Irr by $M(\mathcal{S}(\mathcal{C}))$. Here, Z is called Zelevinsky classification of Irr , while L is called Langlands classification of Irr .

It is well known (see [19]) that for $a, b \in M(\mathcal{S}(\mathcal{C}))$ holds

$$Z(b) \leq \zeta(a) \iff b \leq a. \tag{5}$$

The contragredient representation of π is denoted by $\tilde{\pi}$. For $\Delta \in \mathcal{S}(\mathcal{C})$, set $\tilde{\Delta} := \{\tilde{\rho}; \rho \in \Delta\}$. If $a = (\Delta_1, \dots, \Delta_k) \in M(\mathcal{S}(\mathcal{C}))$, then we put $\tilde{a} = (\tilde{\Delta}_1, \dots, \tilde{\Delta}_k)$. Then

$$L(a)^\sim = L(\tilde{a}) \quad \text{and} \quad Z(a)^\sim = Z(\tilde{a}).$$

Analogous relations hold for Hermitian contragredients. The Hermitian contragredient of a representation π is denoted by π^+ .

2.7. Classification of the unitary dual. Denote by

$$B_{\text{rigid}} = \{Z(a(n, d)^{(\rho)}); n, d \in \mathbb{Z}_{\geq 1}, \rho \in \mathcal{C}^u\}.$$

and

$$B = B_{\text{rigid}} \cup \{\nu^\alpha \sigma \times \nu^{-\alpha} \sigma; \sigma \in B_{\text{rigid}}, 0 < \alpha < 1/2\}.$$

Then the unitary dual is described by the following:

Theorem 2.1. ([14]) *The map*

$$(\tau_1, \dots, \tau_r) \mapsto \tau_1 \times \dots \times \tau_r$$

is a bijection between $M(B)$ and the set of all equivalence classes of irreducible unitary representations of groups $GL(n, F)$, $n \geq 0$.

2.8. Duality - Zelevinsky involution. Define a mapping ${}^t : Irr \rightarrow Irr$ by $Z(a)^t = L(a)$, $a \in M(\mathcal{S}(\mathcal{C}))$. Obviously,

$$\mathfrak{z}(\Delta)^t = \delta(\Delta), \quad \Delta \in \mathcal{S}(\mathcal{C}). \tag{6}$$

We lift t to an additive homomorphism ${}^t : R \rightarrow R$. Clearly, t is a positive mapping, i.e., satisfies: $r_1 \leq r_2 \implies r_1^t \leq r_2^t$. A non-trivial fact is that t is also multiplicative, i.e., a ring homomorphism (see [1] and [12]⁴). Further, t is an involution. Define $a^t \in M(\mathcal{S}(\mathcal{C}))$ for $a \in M(\mathcal{S}(\mathcal{C}))$ by the requirement

$$(L(a))^t = L(a^t).$$

We could also use the Zelevinsky classification to define ${}^t : M(\mathcal{S}(\mathcal{C})) \rightarrow M(\mathcal{S}(\mathcal{C}))$, and we would get the same involutive mapping. Recall that

$$Z(a_1 + a_2) \leq Z(a_1) \times Z(a_2)$$

⁴More precisely, A.V. Zelevinsky used (6) to define the involution on R , and [1] and [12] prove the positivity of this involution. Our definition of t is equivalent to that of A.V. Zelevinsky (see the beginning of section 9).

(Proposition 8.4 of [19]). From this follows directly

$$Z((a_1^t + a_2^t)^t) \leq Z(a_1) \times Z(a_2). \tag{7}$$

One can find more information about the involution in [11].

2.9. Algorithm of C. Mœglin and J.-L. Waldspurger. Let $a \in M(\mathcal{S}(\mathcal{C}))$ be non-empty. Fix $\rho \in \mathcal{C}$ and denote by $X_\rho(a)$ the set of all $x \in \mathbb{R}$ such that there exists a segment Δ in a satisfying $e(\Delta) \cong \nu^x \rho$.

Now fix ρ such that $X_\rho(a) \neq \emptyset$. Let $x = \max(X_\rho(a))$, and consider segments Δ in a such that $e(\Delta) \cong \nu^x \rho$. Among these segments, choose one of minimal cardinality. Denote it by Δ_1 . This will be called the first stage of the algorithm.

Consider now segments Δ in a such that $e(\Delta) \cong \nu^{x-1} \rho$, and which are linked with Δ_1 . Among them, if this set is non-empty, choose one with minimal cardinality. Denote it by Δ_2 .

One continues this procedure with ends $x - 2, x - 3$, etc., as long as it is possible. The segments considered in this procedure are $\Delta_1, \dots, \Delta_k$ ($k \geq 1$). Let

$$\Gamma_1 = [e(\Delta_k), e(\Delta_1)] = [\nu^{x-k+1} \rho, \nu^x \rho] \in \mathcal{S}(\mathcal{C}).$$

Let a^\leftarrow be the multiset of $M(\mathcal{C})$ which we get from a by replacing each Δ_i by Δ_i^- , $i = 1, \dots, k$ (we omit those Δ_i^- for which $\Delta_i^- = \emptyset$).

If a^\leftarrow is non-empty, we now repeat the above procedure with a^\leftarrow as long as possible. In this way we get a segment Γ_2 and $(a^\leftarrow)^\leftarrow \in M(\mathcal{S}(\mathcal{C}))$.

Continuing this procedure as long as possible, until we reach the empty set, we get $\Gamma_1, \dots, \Gamma_m \in \mathcal{S}(\mathcal{C})$. Then by [10] we have

$$a^t = (\Gamma_1, \dots, \Gamma_m).$$

This algorithm will be denoted by MWA^\leftarrow .

There is also a dual (or "left") version of this algorithm, denoted by $\rightarrow\text{MWA}$ (see [18]). With this, it is easy to show that

$$Z(a(n, d)^{(\rho)}) \cong L(a(d, n)^{(\rho)}) \tag{8}$$

for $n, d \in \mathbb{Z}_{\geq 1}$ and $\rho \in \mathcal{C}$ (in [14] is obtained this relation in a different way).

2.10. Derivatives on the level of R . The algebra R is a \mathbb{Z} -polynomial algebra over $\{\mathfrak{z}(\Delta); \Delta \in \mathcal{S}(\mathcal{C})\}$ (Corollary 7.5 of [19]). Therefore, there exists a unique ring homomorphism $\mathcal{D} : R \rightarrow R$ satisfying

$$\mathcal{D}(\mathfrak{z}(\Delta)) = \mathfrak{z}(\Delta) + \mathfrak{z}(\Delta^-), \quad \forall \Delta \in \mathcal{S}(\mathcal{C}). \tag{9}$$

Let $r \in R_n$, $r > 0$. Write $\mathcal{D}(r) = \sum_i r^{(i)}$, where $r^{(i)} \in R_i$. Then obviously $r^{(i)} = 0$ for $i > n$, and $r^{(n)} = r$. Denote by k the maximal index satisfying $r^{(i)} = 0$ for all $i < k$. Then we define the highest derivative $\text{h.d.}(r)$ of r by

$$\text{h.d.}(r) = r^{(k)}.$$

Let $r_i \in R_{n_i}$, $r_i > 0$, for $i = 1, 2$. Then obviously we have

$$\text{h.d.}(r_1 \times r_2) = \text{h.d.}(r_1) \times \text{h.d.}(r_2)$$

since R is a graded integral domain.

We shall use the derivatives on the level of R in most of the paper. Only in the sections 6 and 8 we shall use derivatives on the level of representations (where we study the lattices of subrepresentations). The following two fundamental very non-trivial facts about \mathcal{D} play an important role in our paper⁵:

(D1) \mathcal{D} is a positive homomorphism (i.e. $r > 0 \implies \mathcal{D}(r) > 0$).

(D2) Let $\pi = Z(a)$ be an irreducible representation of G_n , and consider it as an element of R . Then $\text{h.d.}(Z(a)) = Z(a^-)$.

Observe that using the above formula for the highest derivative of $Z(a)$, one easily reconstructs $Z(a)$ from its highest derivative and the cuspidal support of a . This implies that using the formula

$$\text{h.d.}(Z(a_1) \times \dots \times Z(a_k)) = Z(a_1^-) \times \dots \times Z(a_k^-),$$

we can reconstruct from the composition series of the above highest derivative all the irreducible subquotients $Z(a)$ of $Z(a_1) \times \dots \times Z(a_k)$ which satisfy

$$\text{card}(a) = \text{card}(a_1 + \dots + a_k).$$

Moreover, the corresponding multiplicities also coincide.

E. Lapid and A. Mínguez have obtained in [7] the formula for the derivative of the ladder representations (ladder representations are defined in [7]). Representations $Z(a(n, d)^\rho)$ are very special case of ladder representations. We shall now explain this formula in the case of representations $Z(a(n, d)^\rho)$ (this formula will play crucial role in our paper). Write $a(n, d)^\rho = (\Delta_1, \dots, \Delta_n)$ in a way that $\Delta_1 \rightarrow \dots \rightarrow \Delta_n$. Then

$$\begin{aligned} \mathcal{D}(Z(\Delta_1, \dots, \Delta_n)) &= Z(\Delta_1, \dots, \Delta_n) \\ &+ Z(\Delta_1^-, \Delta_2, \dots, \Delta_n) + \dots + Z(\Delta_1^-, \dots, \Delta_{n-1}^-, \Delta_n) + Z(\Delta_1^-, \dots, \Delta_n^-). \end{aligned}$$

Remark 2.2. (i) In a similar way as R was constructed from representations of groups $GL(n, F)$, one can construct such an algebra $R_{\mathcal{A}}$ for general linear groups over a local division algebra \mathcal{A} over F (see [16]). Then one can see (in several ways) that $R_{\mathcal{A}}$ is a \mathbb{Z} -polynomial algebra over all segment representations $\mathfrak{z}(\Delta)$ ($\mathfrak{z}(\Delta)$ in this setting are defined for example in [16]; see also [9]). Therefore, there exists a unique ring homomorphism

$$\mathcal{D}_{\mathcal{A}} : R_{\mathcal{A}} \rightarrow R_{\mathcal{A}}$$

which satisfies (9). This ring homomorphism will be called a formal derivative on $R_{\mathcal{A}}$. E. Lapid and A. Mínguez observe in [7] that their character formula for the

⁵One can find at the beginning of section 6 the references for them.

ladder representations holds also in the setting of general linear groups over \mathcal{A} (after minor changes). This implies that the above formula for the derivative also holds for the formal derivative of an essentially Speh representation of a general linear group over \mathcal{A} (one can find a definition of the Speh representations in this setting in [16], while a Zelevinsky classification necessary for this case is established in [9]).

Now if one would be able to prove (D1) and (D2) for $\mathcal{D}_{\mathcal{A}}$, then our computation of the composition series done in this paper would automatically carry to the case of general linear groups over \mathcal{A} .

(ii) Sometimes it is useful to consider the positive ring homomorphism

$$\tilde{\mathcal{D}} : R \rightarrow R, \pi \mapsto (\mathcal{D}(\tilde{\pi}))^\sim.$$

This homomorphism has analogous properties as \mathcal{D} : it is positive and it sends

$$\mathfrak{z}(\Delta) \mapsto \mathfrak{z}(\Delta) + \mathfrak{z}({}^-\Delta).$$

Here the highest derivative of $Z(a)$ for this homomorphism is $Z({}^-a)$. Further one has

$$\begin{aligned} \tilde{\mathcal{D}}^\sim(Z(a(n, d)^{(\rho)})) &= Z(\Delta_1, \dots, \Delta_n) \\ &+ Z(\Delta_1, \dots, \Delta_{n-1}, {}^-\Delta_n) + \dots + Z(\Delta_1, {}^-\Delta_2, \dots, {}^-\Delta_n) + Z({}^-\Delta_1, \dots, {}^-\Delta_n). \end{aligned}$$

3. Some general technical lemmas

3.1. Representations. We shall consider the representations

$$\begin{aligned} \mathcal{R}(n, d)_{(l)}^{(\rho)} &:= \nu^{-l/2} Z(a(n, d)^{(\rho)}) \times \nu^{l/2} Z(a(n, d)^{(\rho)}) \\ &= Z(a(n, d)^{(\nu^{-l/2}\rho)}) \times Z(a(n, d)^{(\nu^{l/2}\rho)}), \end{aligned}$$

where $n, d \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}$ and $\rho \in \mathcal{C}$. Observe that in R we have

$$(\mathcal{R}(n, d)_{(l)}^{(\rho)})^t = \mathcal{R}(d, n)_{(l)}^{(\rho)} \tag{10}$$

and

$$\text{s.s.}(\mathcal{R}(n, d)_{(l)}^{(\rho)}) = \text{s.s.}(\mathcal{R}(n, d)_{(-l)}^{(\rho)}).$$

The formula for the highest derivative is

$$\text{h.d.}(\mathcal{R}(n, d)_{(k)}^{(\rho)}) = \mathcal{R}(n, d-1)_{(k)}^{(\nu^{-1/2}\rho)}$$

(we take formally $\mathcal{R}(n, 0)_{(k)}^{(\rho)}$ to be $Z(\emptyset)$).

In what follows, we shall always assume $k \in \mathbb{Z}_{\geq 0}$. When n, d, k and ρ are fixed, we denote

$$a_- = a(n, d)^{(\nu^{-k/2}\rho)}, \quad a_+ = a(n, d)^{(\nu^{k/2}\rho)}.$$

Write segments $\Delta_1, \dots, \Delta_n$ of a_- in a way that

$$\Delta_1 \rightarrow \Delta_2 \rightarrow \dots \rightarrow \Delta_n,$$

and segments $\Gamma_1, \dots, \Gamma_n$ of a_+ also in a way that

$$\Gamma_1 \rightarrow \Gamma_2 \rightarrow \dots \rightarrow \Gamma_n.$$

Now we introduce the following numbers (in $\frac{1}{2}\mathbb{Z}$):

$$\begin{aligned} A_- &= -\frac{d-1}{2} - \frac{n-1}{2} - \frac{k}{2}, & B_- &= \frac{d-1}{2} - \frac{n-1}{2} - \frac{k}{2}, \\ C_- &= -\frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2}, & D_- &= \frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2}, \\ A_+ &= -\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2}, & B_+ &= \frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2}, \\ C_+ &= -\frac{d-1}{2} + \frac{n-1}{2} + \frac{k}{2}, & D_+ &= \frac{d-1}{2} + \frac{n-1}{2} + \frac{k}{2}. \end{aligned}$$

Observe that

$$\Delta_1 = [\nu^{A_-} \rho, \nu^{B_-} \rho], \quad \Delta_2 = [\nu^{A_-+1} \rho, \nu^{B_-+1} \rho], \quad \dots, \quad \Delta_n = [\nu^{C_-} \rho, \nu^{D_-} \rho],$$

$$\Gamma_1 = [\nu^{A_+} \rho, \nu^{B_+} \rho], \quad \Gamma_2 = [\nu^{A_++1} \rho, \nu^{B_++1} \rho], \quad \dots, \quad \Gamma_n = [\nu^{C_+} \rho, \nu^{D_+} \rho].$$

Obviously, $A_- \leq B_-, C_- \leq D_-, A_- \leq C_-, B_- \leq D_-$ and $B_- - A_- = D_- - C_-$. Analogous relations hold for A_+, B_+, C_+ and D_+ .

It is well known that $\mathcal{R}(n, d)_{(0)}^{(\rho)}$ is irreducible (see [2], and also [14]).

Observe that for $D_- + 2 \leq A_+$, $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible by Proposition 8.5 of [19]. In other words, for

$$n + d \leq k$$

$\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible. Therefore, we can have reducibility (for $k \geq 0$) only if

$$1 \leq k \leq n + d - 1. \tag{11}$$

We shall assume in the rest of this section that (11) holds.

Consider exponents that show up in the cuspidal supports of both a_- and a_+ . The cardinality of this set is $D_- - A_+ + 1$, which is

$$n + d - 1 - k. \tag{12}$$

3.2. Unique irreducible subrepresentation and quotient.

Proposition 3.1. *Suppose $1 \leq k \leq n + d - 1$. The representation $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$ has a unique irreducible subrepresentation, and it is isomorphic to*

$$Z(a(n, d)^{(\nu^{-k/2}\rho)} + a(n, d)^{(\nu^{k/2}\rho)}). \tag{13}$$

Further, it has a unique irreducible quotient, and it is isomorphic to

$$Z((a(d, n)^{(\nu^{-k/2}\rho)} + a(d, n)^{(\nu^{k/2}\rho)})^t). \tag{14}$$

Both irreducible representations have multiplicity one in the whole representation.

The representation $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has (14) as unique irreducible subrepresentation and (13) as unique irreducible quotient. Their position is opposite to the position in $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$.

Proof. Let t be any integer satisfying $0 \leq t \leq n$. We have $Z(a_+) \hookrightarrow \zeta(a_+)$ and $Z(\Gamma_{n-t+1}, \dots, \Gamma_n) \times Z(\Gamma_1, \dots, \Gamma_{n-t}) \hookrightarrow \zeta(a_+)$. Since $\zeta(a_+)$ has a unique irreducible subrepresentation, and it is $Z(a_+)$, we get that there exists an embedding $Z(a_+) \hookrightarrow Z(\Gamma_{n-t+1}, \dots, \Gamma_n) \times Z(\Gamma_1, \dots, \Gamma_{n-t})$. Analogously, we get $Z(a_-) \hookrightarrow Z(\Delta_{t+1}, \dots, \Delta_n) \times Z(\Delta_1, \dots, \Delta_t)$. Now we shall specify t . If $C_- + D_- \leq A_+ + B_+$ (i.e. $n-1 \leq k$), we take $t = n$. In the opposite case $A_+ + B_+ < C_- + D_-$ (i.e. $k < n-1$), $t = k$. Observe that this implies that $\Gamma_1 = \Delta_{t+1}, \dots, \Gamma_{n-t} = \Delta_n$. Now we have $Z(a_+) \times Z(a_-) \hookrightarrow$

$$\begin{aligned} & Z(\Gamma_{n-t+1}, \dots, \Gamma_n) \times Z(\Gamma_1, \dots, \Gamma_{n-t}) \times Z(\Delta_{t+1}, \dots, \Delta_n) \times Z(\Delta_1, \dots, \Delta_t) \\ & \cong Z(\Gamma_{n-t+1}, \dots, \Gamma_n) \times Z(\Gamma_1, \dots, \Gamma_{n-t}, \Delta_{t+1}, \dots, \Delta_n) \times Z(\Delta_1, \dots, \Delta_t) \end{aligned}$$

since $Z(\Gamma_1, \dots, \Gamma_{n-t}) \times Z(\Delta_{t+1}, \dots, \Delta_n)$ is irreducible (we are in the essentially unitary situation, from which we get irreducibility of the induced representation). Now $Z(\Gamma_{n-t+1}, \dots, \Gamma_n) \times Z(\Gamma_1, \dots, \Gamma_{n-t}, \Delta_{t+1}, \dots, \Delta_n) \times Z(\Delta_1, \dots, \Delta_t) \hookrightarrow$

$$\zeta(\Gamma_{n-t+1}, \dots, \Gamma_n) \times \zeta(\Gamma_1, \dots, \Gamma_{n-t}, \Delta_{t+1}, \dots, \Delta_n) \times \zeta(\Delta_1, \dots, \Delta_t) \cong \zeta(a_+ + a_-).$$

Therefore, $Z(a_+) \times Z(a_-)$ has a unique irreducible subrepresentation, and it is $Z(a_+ + a_-)$.

For the quotient setting, observe that

$$\mathcal{R}(n, d)_{(-k)}^{(\rho)} \cong L(a(d, n)^{(\nu^{k/2}\rho)}) \times L(a(d, n)^{(\nu^{-k/2}\rho)}).$$

Now applying similar arguments as above (dealing with quotients instead of subrepresentations), we get that $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$ has a unique irreducible quotient, and that this quotient is

$$L(a(d, n)^{(\nu^{k/2}\rho)} + a(d, n)^{(\nu^{-k/2}\rho)}) = Z((a(d, n)^{(\nu^{k/2}\rho)} + a(d, n)^{(\nu^{-k/2}\rho)})^t).$$

To get the description of the irreducible quotient and the irreducible subrepresentation of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$, apply the contragredient to $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$.

The proof of the proposition is now complete. \blacksquare

3.3. Highest derivatives of irreducible subquotients.

Lemma 3.2. *Let $Z(b) \leq Z(a_-) \times Z(a_+)$. Denote by $c(b)$ the cardinality of b . Then $n \leq c(b) \leq 2n$, and we get the cuspidal support of the highest derivative of b from the cuspidal support of $a_- + a_+$ removing ends of all segments in a_+ and removing ends of the first $c(b) - n$ segments in a_- . Therefore, the multiset of ends of b consists of all ends of segments Γ_i , and the ends of the first $c(b) - n$ ends of segments Δ_i .*

Proof. Recall that

$$\begin{aligned} \mathcal{D}(Z(a_-) \times Z(a_+)) &= \left(Z(\Delta_1, \dots, \Delta_n) + Z(\Delta_1^-, \Delta_2, \dots, \Delta_n) + \dots + Z(\Delta_1^-, \dots, \Delta_n^-) \right) \\ &\quad \times \left(Z(\Gamma_1, \dots, \Gamma_n) + Z(\Gamma_1^-, \Gamma_2, \dots, \Gamma_n) + \dots + Z(\Gamma_1^-, \dots, \Gamma_n^-) \right). \end{aligned}$$

Observe that the end of the last segment in a_+ is not in the support of the highest derivative of b . Therefore, the highest derivative of $Z(b)$ must be a subquotient of the product, in which the second factor is $Z(\Gamma_1^-, \dots, \Gamma_n^-)$. Now the claim of the lemma follows directly (use (5)). ■

3.4. Symmetry.

Lemma 3.3. *Let $Z(b) \leq Z(a_-) \times Z(a_+)$. Suppose that ρ is unitarizable. Then $\tau \mapsto \tau^+$ defines a bijection from the multiset of all ends of segments in b onto the multiset of all beginnings of segments in b , i.e. $\mathcal{E}(b)^+ = \mathcal{B}(b)$.*

Proof. Observe that $Z(b^+) \leq Z(a_-) \times Z(a_+)$ since ρ is unitarizable (the unitarizability implies that the representation on the right hand side is a Hermitian element of the Grothendieck group since $Z(a_-)^+ = Z(a_+)$).

Denote by $c(b)$ the cardinality of b (we know $n \leq c(b) \leq 2n$ from the previous proposition). Then the last proposition tells us that the multiset of all ends of b depends only on $c(b)$.

Obviously, b and b^+ have the same cardinality. Therefore, the multisets of the ends of b^+ and of b are the same, i.e. $\mathcal{E}(b) = \mathcal{E}(b^+)$. Observe that one gets the multiset of the beginnings of b from the multiset of the ends of b^+ applying $\tau \mapsto \tau^+$, i.e. $\mathcal{B}(b) = \mathcal{E}(b^+)^+$. Therefore $\mathcal{E}(b)^+ = \mathcal{E}(b^+)^+ = \mathcal{B}(b)$, i.e. $\tau \mapsto \tau^+$ is a bijection from the multiset of all ends of segments in b onto the multiset of all beginnings of segments in b (it preserves the multisets). ■

3.5. Key lemma.

The following lemma will be crucial for exhaustion of composition series.

Lemma 3.4. *Let $Z(b) \leq Z(a_-) \times Z(a_+)$. Suppose that ρ is unitarizable.*

1. *There exists a (multi)set $b_{\text{left}} = (\Delta'_1, \dots, \Delta'_n)$ such that $b(\Delta'_i) = b(\Delta_i)$, $i = 1, \dots, n$, $e(\Delta_1) \leq e(\Delta'_1) < \dots < e(\Delta'_n)$ and that each segment of b_{left} is contained in b , i.e. $b_{\text{left}} \subseteq b$.*
2. *There exists a (multi)set $b_{\text{right}} = (\Gamma'_1, \dots, \Gamma'_n)$ such that $e(\Gamma'_i) = e(\Gamma_i)$, $i = 1, \dots, n$, $b(\Gamma'_1) < \dots < b(\Gamma'_n) \leq b(\Gamma_n)$ and that each segment of b_{right} is contained in b , i.e. $b_{\text{right}} \subseteq b$.*
3. *Suppose that $\mathcal{B}(a_-)$ and $\mathcal{B}(a_+)$ are disjoint and that the cardinality $c(b)$ of b is strictly smaller than $2n$. Denote $l = 2n - c(b)$ (i.e. $c(b) = 2n - l$). Then*

$$(a) \quad \Delta'_i = \Delta_i, \quad i = 1, \dots, n - l.$$

$$(b) \quad \Gamma'_i = \Gamma_i, \quad i = l + 1, \dots, n.$$

(c) *For $i = 1, \dots, l$, Δ_{n-l+i} and Γ_i are disjoint, $\Delta_{n-l+i} \cup \Gamma_i$ is a segment, and $\Delta'_{n-l+i} = \Delta_{n-l+i} \cup \Gamma_i = \Gamma'_i$.*

$$(d) \quad b = (\Delta_1, \dots, \Delta_{n-l}, \Delta'_{n-l+1}, \dots, \Delta'_n, \Gamma_{l+1}, \dots, \Gamma_n).$$

$$(e) \quad d \leq k \text{ (i.e. } B_- < A_+) \text{ and } k + 1 \leq n + d.$$

(f) The multiset b as above is unique. Further, l is the maximal index such that $\Delta_n \cup \Gamma_l \in \mathcal{S}(\mathcal{C})$.

(g) In this situation we have $(a_-^t + a_+^t)^t = b$.

(h) Further, $Z(b)$ is the unique irreducible subrepresentation of $Z(a_-) \times Z(a_+)$ in this situation.

4. Suppose that $\mathcal{B}(a_-)$ and $\mathcal{B}(a_+)$ are not disjoint (this is equivalent to $A_+ \leq C_-$). Further, assume that $n \leq d$ (this is equivalent to $C_- \leq B_-$). Then the cardinality $c(b)$ of b is equal $2n$.

Proof. Note that $Z(b^t) \leq Z(a_-^t) \times Z(a_+^t) = Z(a(d, n)^{(\nu^{-k/2}\rho)}) \times Z(a(d, n)^{(\nu^{k/2}\rho)})$, which implies $b^t \leq a_-^t + a_+^t = a(d, n)^{(\nu^{-k/2}\rho)} + a(d, n)^{(\nu^{k/2}\rho)}$. This implies that in b^t there is a unique segment Δ such that $b(\Delta) = b(\Delta_1)$. This segment must have obviously at least n elements. Now $\rightarrow\text{MWA}$ implies (1). Analogously follows (2) using the dual version of MWA^{\leftarrow} (see [18]).

Now we shall prove (3). We assume that conditions of (3) on a_- and a_+ hold (in particular, disjointness of beginning of segments is equivalent to $C_- < A_+$). Recall that by Lemma 3.2 the multiset of ends of b consists of all ends of segments Γ_i , and the ends of the first $n - l$ ends of segments Δ_i . Now Lemma 3.3 implies that the multiset of beginnings of b consists of all beginnings of segments Δ_i , and the beginnings of the last $n - l$ ends of segments Γ_i .

Let ρ' be an element of some Δ_i or Γ_j . Then we can write $\rho' = \nu^x \rho$ for unique $x \in \frac{1}{2}\mathbb{Z}$. In this case we write $\exp_\rho(\rho') = x$, and we shall say that x is the exponent of ρ' with respect to ρ .

Denote by X the multiset of all exponents with respect to ρ of beginnings of segments in b . Write elements of X as $b_1 < \dots < b_{2n-l}$. Then by Lemma 3.3, $-X$ is the multiset of all exponents with respect to ρ of ends of segments in b . Write elements of $-X$ as $e_1 < \dots < e_{2n-l}$. Therefore, there exists a permutation σ of $\{1, \dots, 2n - l\}$ satisfying $b_i \leq e_{\sigma(i)}$, $i = 1, \dots, 2n - l$ such that $b =$

$$([\nu^{b_1}\rho, \nu^{e_{\sigma(1)}}\rho], \dots, [\nu^{b_{2n-l}}\rho, \nu^{e_{\sigma(2n-l)}}\rho]) = ([\nu^{b_{\sigma^{-1}(1)}}\rho, \nu^{e_1}\rho], \dots, [\nu^{b_{\sigma^{-1}(2n-l)}}\rho, \nu^{e_{2n-l}}\rho]).$$

Now by (1) and (2) we must have $\sigma(1) < \dots < \sigma(n)$ and $\sigma^{-1}(n - l + 1) < \dots < \sigma^{-1}(2n - l)$. This implies $i \leq \sigma(i)$, $i = 1, \dots, n$ and $\sigma^{-1}(j) \leq j$, $j = n - l + 1, \dots, 2n - l$.

To shorten the formulas below, for $\Delta \in \{\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_n\}$ we use the following notation:

$$b_\rho(\Delta) := \exp_\rho(b(\Delta)) \quad \text{and} \quad e_\rho(\Delta) := \exp_\rho(e(\Delta)).$$

Since the cuspidal supports of $a_- + a_+$ and b must be the same, their cardinalities must be the same. The first cardinality is

$$2nd = \sum_{i=1}^n (e_\rho(\Delta_i) - b_\rho(\Delta_i) + 1) + \sum_{i=1}^n (e_\rho(\Gamma_i) - b_\rho(\Gamma_i) + 1)$$

Our next aim will be to prove that the permutation σ is the identity permutation.

In the following considerations, we shall use the fact that the cuspidal supports of b and $a_- + a_+$ must coincide (in the previous considerations we have used only the fact that their cardinalities must be the same).

Consider the case $B_- + 1 < A_+$. Suppose $\sigma(1) \neq 1$. Then $1 < \sigma(1)$. Recall $i \leq \sigma(i)$, $1 \leq i \leq n$. This implies that the multiplicity of $\nu^{B_-+1}\rho$ in the cuspidal support of b is strictly bigger than the multiplicity in the cuspidal support of $a_- + a_+$. Thus, $\sigma(1) = 1$. In the same way we get $\sigma(2) = 2$ if $B_- + 2 < A_+$. Continuing in this way, we get that

$$\sigma(i) = i, \quad \text{if } B_- + i < A_+.$$

We shall now find the maximal i such that $B_- + i < A_+$. In n, d, k -notation, this condition becomes $\frac{d-1}{2} - \frac{n-1}{2} - \frac{k}{2} + i < -\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2}$, i.e. $d - 1 + i < k$, which is equivalent to $d + i \leq k$. Therefore

$$\sigma(i) = i \quad \text{for all } 1 \leq i \leq k - d = n - l, \tag{17}$$

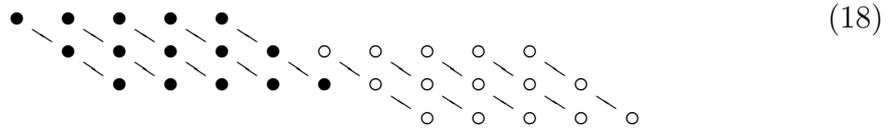
since we know $l = n + d - k$.

This implies that σ carries $\{n - l + 1, \dots, 2n - l\}$ into itself. Since σ^{-1} is monotone on this set, we get that σ is the identity permutation. This completes the proof of (a) - (d) in (3).

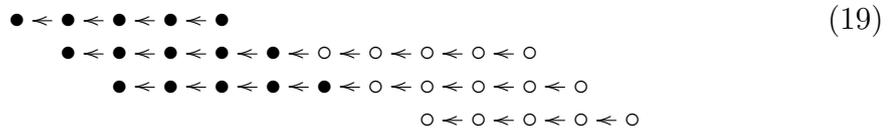
It remains to prove (g). Because of (8), it is enough to prove

$$(a(d, n)^{(\nu^{-k/2}\rho)} + a(d, n)^{(\nu^{k/2}\rho)})^t = b.$$

The proof of this relation is a simple straight-forward application of MWA^{\leftarrow} , and we shall not present in detail here. Rather we shall illustrate the proof with an example (16). We illustrate $a_-^t + a_+^t$ by the drawing



Now MWA^{\leftarrow} applied to $a_-^t + a_+^t$ can be illustrated by:



The general proof goes in the same way.

The claim (h) follows from Proposition 3.1 and (g) (we use this in the proof of Theorem 5.2).

It remains to prove (4). The proof is a simple modification of the proof of (3). We assume that conditions of (3) on a_- and a_+ hold (in particular, $A_+ \leq C_-$, which means that the beginnings of segments are not disjoint). Suppose $c(b) < n$. Write $c(b) = 2n - l$. Then $1 \leq l \leq n$. Write $b = (\Delta'_1, \dots, \Delta'_{2n-l})$.

Since the cuspidal supports of $a_- + a_+$ and b must be the same, their cardinalities must be the same. The first cardinality is

$$2nd = \sum_{i=1}^n (e_\rho(\Delta_i) - b_\rho(\Delta_i) + 1) + \sum_{i=1}^n (e_\rho(\Gamma_i) - b_\rho(\Gamma_i) + 1)$$

From the other side, using the fact that $b_\rho(\Delta'_i) \leq e_\rho(\Delta'_i)$, $i = 1, \dots, 2n - l$, the second cardinality is

$$\sum_{i=1}^{2n-l} (e_\rho(\Delta'_i) - b_\rho(\Delta'_i) + 1) = \sum_{i=1}^{2n-l} (e_\rho(\Delta'_i) + 1) - \sum_{i=1}^{2n-l} b_\rho(\Delta'_i). \quad (20)$$

The multiset of ends of b consists of all ends of segments Γ_i , and the ends of the first $n - l$ ends of segments Δ_i . Further, the multiset of beginnings of b consists of all beginnings of segments Δ_i , and the beginnings of the last $n - l$ ends of segments Γ_i . Using this fact, and permuting elements in the sums, we easily get that we can write (20) as

$$\sum_{i=1}^{n-l} (e_\rho(\Delta_i) - b_\rho(\Delta_i) + 1) + \sum_{i=l+1}^n (e_\rho(\Gamma_i) - b_\rho(\Gamma_i) + 1) + \sum_{i=n-l+1}^n (e_\rho(\Gamma_{i-(n-l)}) - b_\rho(\Delta_i) + 1).$$

The above two cardinalities must be the same, which implies

$$\begin{aligned} \sum_{i=n-l+1}^n (e_\rho(\Delta_i) - b_\rho(\Delta_i) + 1) + \sum_{i=1}^l (e_\rho(\Gamma_i) - b_\rho(\Gamma_i) + 1) \\ = \sum_{i=n-l+1}^n (e_\rho(\Gamma_{i-(n-l)}) - b_\rho(\Delta_i) + 1). \end{aligned}$$

This further implies

$$\sum_{i=n-l+1}^n e_\rho(\Delta_i) + \sum_{i=1}^l (e_\rho(\Gamma_i) - b_\rho(\Gamma_i) + 1) = \sum_{i=n-l+1}^n e_\rho(\Gamma_{i-(n-l)}) = \sum_{i=1}^l e_\rho(\Gamma_i).$$

Thus $\sum_{i=n-l+1}^n e(\Delta_i) + \sum_{i=1}^l (-b_\rho(\Gamma_i) + 1) = 0$, i.e.,

$$\sum_{i=1}^l e_\rho(\Delta_{n-l+i}) + \sum_{i=1}^l (-b_\rho(\Gamma_i) + 1) = \sum_{i=1}^l (e_\rho(\Delta_{n-l+i}) - b_\rho(\Gamma_i) + 1) = 0.$$

Since all the numbers $e_\rho(\Delta_{n-l+i}) - b_\rho(\Gamma_i) + 1$, $i = 1, \dots, l$, are the same, we get that

$$e_\rho(\Delta_{n-l+i}) + 1 = b_\rho(\Gamma_i), \quad i = 1, \dots, l. \quad (21)$$

For $i = l$ we get $e_\rho(\Delta_n) + 1 = b_\rho(\Gamma_l)$, i.e. $D_- + 1 = A_+ + l - 1$. Going to n, d, k -notation, we get $\frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2} + 1 = -\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2} + l - 1$, which gives $d+n = k+l$. This implies $l = n+d-k$. Recall $1 \leq l \leq n$. Therefore $k+1 \leq n+d$, $d \leq k$.

The last inequality is equivalent to $B_- < A_+$.

On the other side, we suppose $A_+ \leq C_-$ and $C_- \leq B_-$, which implies $A_+ \leq B_-$. This contradicts to $B_- < A_+$.

The proof is now complete. ■

4. Definition of multisegments representing composition series

Denote $r_0(n, d)_{(k)}^{(\rho)} = (\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_n)$. For $1 \leq j \leq n$ we define multisegments $r_j(n, d)_{(k)}^{(\rho)}$ whenever $\Delta_n \rightarrow \Gamma_j$. Observe that $\Delta_n \rightarrow \Gamma_j$ if and only if $\Delta_{n-j+1} \rightarrow \Gamma_1$. This is the case if and only if $1 \leq b_\rho(\Gamma_1) - b_\rho(\Delta_{n-j+1}) \leq d$. The last condition becomes $1 \leq (-\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2}) - (-\frac{d-1}{2} - \frac{n-1}{2} - \frac{k}{2} + n - j) \leq d$. Therefore if $j \neq 0$, the multisegments $r_j(n, d)_{(k)}^{(\rho)}$ are defined for indices

$$\max(n - k + 1, 1) \leq j \leq \min(n - k + d, n).$$

Then we get $r_j(n, d)_{(k)}^{(\rho)}$ by replacing in $r_0(n, d)_{(k)}^{(\rho)} = (\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_n)$ the part $\Delta_{n-j+1}, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_j$ with $\Delta_{n-j+1} \cup \Gamma_1, \Delta_{n-j+1} \cap \Gamma_1, \dots, \Delta_n \cup \Gamma_j, \Delta_n \cap \Gamma_j$ (we omit \emptyset if it shows up in the above formula and the formulas below).

In other words we have $r_j(n, d)_{(k)}^{(\rho)} =$

$$\begin{aligned} & \sum_{i=1}^j (\Delta_{i+n-j} \cup \Gamma_i, \Delta_{i+n-j} \cap \Gamma_i) + \sum_{i=1}^{n-j} (\Delta_i, \Gamma_{n-i+1}) \\ & = \sum_{i=1}^j (\Delta_{i+n-j} \cup \Gamma_i, \Delta_{i+n-j} \cap \Gamma_i) + \sum_{i=1}^{n-j} (\Delta_i, \Gamma_{i+j}) \end{aligned}$$

for $j = 0$ and for $1 \leq j \leq n$ for which $\Delta_n \rightarrow \Gamma_j$.

Suppose that Δ and Γ are segments such that their intersection is non-empty. Then obviously $((\Delta \cup \Gamma)^-, (\Delta \cap \Gamma)^-) = (\Delta^- \cup \Gamma^-, \Delta^- \cap \Gamma^-)$ (as above, we omit \emptyset if it shows up). This implies that

$$\text{h.d.}(Z(r_j(n, d)_{(k)}^{(\rho)})) = \nu^{-1/2} Z(r_j(n, d-1)_{(k)}^{(\rho)}) \tag{22}$$

if $d \geq 2$ and $\Delta_n \rightarrow \Gamma_j$ such that $\Delta_n \cap \Gamma_j \neq \emptyset$.

5. Composition series - disjoint beginnings of segments

We continue with the notation of the previous section. In this section we shall assume that $C_- < A_+$, i.e., $\{b(\Delta_1), \dots, b(\Delta_n)\} \cap \{b(\Gamma_1), \dots, b(\Gamma_n)\} = \emptyset$. We also assume $A_+ \leq D_- + 1$ (recall that for $A_+ > D_- + 1$ the representation $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible).

In the n, d, k -notation these two conditions become

$$\begin{aligned} -\frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2} &< -\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2} \text{ and} \\ -\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2} &\leq \frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2} + 1, \end{aligned}$$

i.e., $n \leq k$, $k \leq n + d - 1$ (i.e. $k < n + d$).

Let us fix j such that $\Delta_n \rightarrow \Gamma_j$. This is the case if and only if $A_+ + j - 1 \leq D_- + 1$. The last condition becomes $-\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2} + j - 1 \leq \frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2} + 1$, i.e. $k + j \leq d + n$. Therefore, the multisegments $r_j(n, d)_{(k)}^{(\rho)}$ are defined for indexes

$$0 \leq j \leq \min(n + d - k, n)$$

in the case that we consider in this section.

Observe that in the case $k \leq d$, the $r_j(n, d)_{(k)}^{(\rho)}$ are defined for all $0 \leq j \leq n$, and then we have

$$\begin{aligned}
 r_0(n, d)_{(k)}^{(\rho)} &= \sum_{i=1}^n (\Delta_i, \Gamma_{n-i+1}). \\
 r_1(n, d)_{(k)}^{(\rho)} &= (\Delta_n \cup \Gamma_1, \Delta_n \cap \Gamma_1) + \sum_{i=1}^{n-1} (\Delta_i, \Gamma_{i+1}) \\
 &\vdots \\
 r_j(n, d)_{(k)}^{(\rho)} &= \sum_{i=1}^j (\Delta_{i+n-j} \cup \Gamma_i, \Delta_{i+n-j} \cap \Gamma_i) + \sum_{i=1}^{n-j} (\Delta_i, \Gamma_{i+j}). \\
 &\vdots \\
 r_n(n, d)_{(k)}^{(\rho)} &= \sum_{i=1}^n (\Delta_i \cup \Gamma_i, \Delta_i \cap \Gamma_i).
 \end{aligned}$$

In the case $d < k$, let j be the greatest index for which $\Delta_n \cup \Gamma_j$ is still a segment (with the assumptions of the present section, this implies $\Delta_n \rightarrow \Gamma_j$). Then only multisegments $r_0(n, d)_{(k)}^{(\rho)}, \dots, r_j(n, d)_{(k)}^{(\rho)}$ are defined (i.e. the first $j+1$ terms above are defined).

Proposition 5.1. *Let $A_+ \leq D_- + 2$, i.e. $k \leq n + d$ (otherwise, $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible⁶). Suppose $C_- < A_+$, i.e. $n \leq k$. Then*

1. $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation.
2. Let $B_- < A_+$. In that case $d \leq k \leq n + d$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has length $n + d + 1 - k$, and its composition series consists of $Z(r_i(n, d)_{(k)}^{(\rho)})$, $0 \leq i \leq n + d - k$.
3. Suppose $A_+ \leq B_-$, i.e. $k < d$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has length $n + 1$, and its composition series consists of $Z(r_i(n, d)_{(k)}^{(\rho)})$, $0 \leq i \leq n$.

Proof. One can easily see that if any of the claims of the above proposition holds for a unitarizable ρ , then it holds for any $\nu^\alpha \rho, \alpha \in \mathbb{R}$ (for that n, d and k). This follows from the fact that in general holds $\nu^\alpha Z(a) \cong Z(\nu^\alpha a)$, $\nu^\alpha(\pi_1 \times \pi_2) \cong (\nu^\alpha \pi_1) \times (\nu^\alpha \pi_2)$, $\nu^\alpha(\mathcal{R}(n, d)_{(k)}^{(\rho)}) \cong \mathcal{R}(n, d)_{(k)}^{(\nu^\alpha \rho)}$ and $\nu^\alpha(r_i(n, d)_{(k)}^{(\rho)}) \cong r_i(n, d)_{(k)}^{(\nu^\alpha \rho)}$.

We shall first prove (2). While proving (2), we shall prove also that all the multiplicities are one.

We fix n and k . The proof will go by induction with respect to d . Observe that $k - n \leq d \leq k$. For $d = k - n$ (i.e. $k = n + d$), we know that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible, and isomorphic to $Z(r_0(n, d)_{(k)}^{(\rho)})$. Therefore, we have the basis of the induction.

Suppose $d = 1$. Then $k \leq n + 1$ and $n \leq k$. If $k = n + 1$, we have observed above that (2) holds. Let $k = n$. Then easily follows from [19] that (2), together with multiplicity one, holds. Now in the rest of the proof, it is enough to consider the case of $d > 1$.

Let $k - n < d \leq k$, and suppose that (2) holds for $d - 1$, together with the multiplicity one claim (in (1)). We shall now show that (2) holds also for d

⁶Observe that also for $k = n + d$ we have irreducibility.

(together with the multiplicity one claim). Recall

$$\text{h.d.}(\mathcal{R}(n, d)_{(k)}^{(\rho)}) = \nu^{-1/2} \mathcal{R}(n, d-1)_{(k)}^{(\rho)} \tag{23}$$

for $d \geq 2$. Also

$$\text{h.d.}(Z(r_j(n, d)_{(k)}^{(\rho)})) = \nu^{-1/2} Z(r_j(n, d-1)_{(k)}^{(\rho)}) \tag{24}$$

for $0 \leq j \leq n + d - k - 1$. From the first relation and the inductive assumption follows that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has precisely $n + d - k$ irreducible subquotients $\pi = Z(a)$ such that the cardinality of a is $2n$. It also implies that all these subquotients have multiplicity one. The inductive assumption tells us that the irreducible subquotients of the highest derivative are $\nu^{-1/2} Z(r_j(n, d-1)_{(k)}^{(\rho)})$, $0 \leq j \leq n + (d-1) - k$. Now the second relation above implies that these subquotients $Z(a)$ are representations $Z(r_j(n, d)_{(k)}^{(\rho)})$, $0 \leq j \leq n + (d-1) - k$. Now Proposition 3.1 and Lemma 3.4 (3) (g) imply also that $Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$ is irreducible subquotient, and that the multiplicity of this subquotient is one. It remains to prove that these are all the irreducible subquotients. Let $\pi = Z(a)$ be an arbitrary irreducible subquotient of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$. If the cardinality of a is $2n$, then we have seen that it must be one of $Z(r_j(n, d)_{(k)}^{(\rho)})$, $0 \leq j \leq n + (d-1) - k$. Suppose that the cardinality of a is strictly smaller than $2n$. Now we shall find an index l such that $e_\rho(\Delta_n) + 1 = b_\rho(\Gamma_l)$, i.e. $\frac{n-1}{2} + \frac{d-1}{2} - \frac{k}{2} + 1 = -\frac{n-1}{2} - \frac{d-1}{2} + \frac{k}{2} + l - 1$, which implies $l = n + d - k$. Then (3) of Lemma 3.4 implies that $\pi \cong Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$. This completes the proof of (2).

Now we shall prove (3). At the same time we shall prove also that all the multiplicities are one. This and the first part of the proof, will then give (1).

We prove (3) by induction with respect to d . Observe that we need to prove for $k < d$. Actually, we shall prove the claim of (3) (together with multiplicity one property) by induction for $k \leq d$. For $d = k$, claim (3) holds by (2) (observe that in this case $n + d - k$ is n , and we have $n + 1$ irreducible subquotients). Also the multiplicity one holds in this case.

Let $k < d$, and suppose that our claim holds for $d-1$. Then in the same way as in the first part of the proof, looking at the highest derivative of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$, the inductive assumption implies that $Z(r_j(n, d)_{(k)}^{(\rho)})$, $0 \leq j \leq n$ are subquotients, and all have multiplicity one. Further, (4) of Lemma 3.4 and the condition $A_+ \leq B_-$ (which we assume in (3)) imply that these are all the irreducible subquotients.

The proof of the proposition is now complete. ■

Summing up, we get the following:

Theorem 5.2. *Let $k \in \mathbb{Z}_{\geq 0}$.*

1. *The representations $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ and $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$ have the same composition series.*
2. *For $n + d \leq k$, $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible, and $\mathcal{R}(n, d)_{(k)}^{(\rho)} \cong Z(r_0(n, d)_{(k)}^{(\rho)})$.*

- 3. For $n \leq k \leq n + d$, $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation. Its composition series consists of $Z(r_i(n, d)_{(k)}^{(\rho)})$, $0 \leq i \leq \min(n, n + d - k)$.
- 4. The representation $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has a unique irreducible subrepresentation as well as a unique irreducible quotient. The irreducible quotient is isomorphic to $Z(r_0(n, d)_{(k)}^{(\rho)})$, and the irreducible subrepresentation is isomorphic to $Z(r_{\min(n, n+d-k)}(n, d)_{(k)}^{(\rho)})$.

For $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$, we have opposite situation regarding irreducible subrepresentation and quotient.

Proof. It remains to prove only the claim regarding the irreducible quotient in (4). For $d \leq k \leq d + n - 1$, this follows from (g) in (3) of Lemma 3.4. Therefore, we shall suppose $k < d$. For this, one needs to prove $(r_0(d, n)_{(k)}^{(\rho)})^t = r_n(n, d)_{(k)}^{(\rho)}$. One can get this directly by MWA[←] and Proposition 3.1. We shall give here a different argument. Observe that $r_0(d, n)_{(k)}^{(\rho)} = a(d + k, n)^{(\rho)} + a(d - k, n)^{(\rho)}$. Now we have

$$Z(a(d + k, n)^{(\rho)} + a(d - k, n)^{(\rho)})^t = Z(a(d + k, n)^{(\rho)})^t \times Z(a(d - k, n)^{(\rho)})^t = Z(a(n, d + k)^{(\rho)}) \times Z(a(n, d - k)^{(\rho)}) = Z(a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}),$$

since the unitary parabolic induction is irreducible for general linear groups. This implies that $(r_0(d, n)_{(k)}^{(\rho)})^t = a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}$. One gets directly that $r_n(n, d)_{(k)}^{(\rho)} = a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}$. This completes the proof that $(r_0(d, n)_{(k)}^{(\rho)})^t = r_n(n, d)_{(k)}^{(\rho)}$.

Considering the Hermitian contragredients, we get that $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$ has opposite irreducible quotient and subrepresentation from their position in $\mathcal{R}(n, d)_{(k)}^{(\rho)}$. The proof is now complete. ■

6. Derivatives and the lattice of subrepresentations - disjoint beginnings of segments

6.1. On the lattice of subrepresentations of a multiplicity one representation of finite length. We shall first present several simple and well known observations about the lattice of subrepresentations of a multiplicity free representation $(\pi, V) \in \text{Alg}_{f.l.}(G_n)$ (actually, the general discussion holds on the level of modules).

Let π be as above (i.e. a multiplicity one representation of finite length), and denote by $J.H.(\pi)$ the set of all irreducible subquotients of π .

Fix any $\sigma \in J.H.(\pi)$, and let V_1 and V_2 be two subrepresentations of V such that both of them have σ for a subquotient. The multiplicity one implies that their intersection has also σ for subquotient. Define $\pi_{\text{smallest}}(\sigma)$ to be the smallest (with respect to the inclusion) subrepresentation of π which has σ for a subquotient (it is equal to the intersection of all subrepresentations of π which have σ for a subquotient).

Now one directly sees that for a multiplicity one representation of finite length π and $\sigma, \sigma_1, \sigma_2 \in J.H.(\pi)$ holds the following:

1. $\pi_{\text{smallest}}(\sigma)$ is a cyclic representation.
2. $\pi_{\text{smallest}}(\sigma)$ has minimal length among all the subrepresentations of π which have σ for a sub quotient (this property characterizes $\pi_{\text{smallest}}(\sigma)$).
3. $\pi_{\text{smallest}}(\sigma)$ has a unique irreducible quotient, which is σ (this property also characterizes $\pi_{\text{smallest}}(\sigma)$).
4. $\pi_{\text{smallest}}(\sigma_1) = \pi_{\text{smallest}}(\sigma_2) \iff \sigma_1 = \sigma_2$.
5. The map $V \mapsto J.H.(V)$ is an injective map from the set of all subrepresentations of V into the partitive set $\mathcal{P}(J.H.(\pi))$ of $J.H.(\pi)$.
6. For two subrepresentations V_1 and V_2 of V we have $V_1 \subseteq V_2 \iff J.H.(V_1) \subseteq J.H.(V_2)$.

Definition 6.1. We shall say that π has a minimal lattice of subrepresentations if $\pi_{\text{smallest}}(\sigma), \sigma \in J.H.(\pi)$ is a complete lattice of the non-zero subrepresentations of π

Let now π be a multiplicity one representation of finite length n . Suppose that π has a minimal lattice of subrepresentations. Let σ_1, σ_2 be different members of $J.H.(\pi)$. Then one proves easily

- (i) Either $\sigma_1 \in J.H.(\pi_{\text{smallest}}(\sigma_2))$ or $\sigma_2 \in J.H.(\pi_{\text{smallest}}(\sigma_1))$
- (ii) Either $\pi_{\text{smallest}}(\sigma_1) \subseteq \pi_{\text{smallest}}(\sigma_2)$ or $\pi_{\text{smallest}}(\sigma_2) \subseteq \pi_{\text{smallest}}(\sigma_1)$ (this property characterizes representations with a minimal lattice of subrepresentations).
- (iii) There exists an enumeration $\sigma_1, \sigma_2, \dots, \sigma_n$ of the members of $J.H.(\pi)$, such that the map $V' \mapsto J.H.(V')$ is an isomorphism of the lattice of subrepresentations of π and $\{\{0\}, \{\sigma_1\}, \{\sigma_1, \sigma_2\}, \dots, \{\sigma_1, \sigma_2, \dots, \sigma_n\}\}$. The above enumeration is uniquely determined by the requirement $i < j \iff \sigma_i \in J.H.(\pi_{\text{smallest}}(\sigma_j))$.
- (iv) The cardinal number of the set of all non-zero subrepresentations of π is n , i.e. the length of π (this property characterizes representations with a minimal lattice of subrepresentations).

Therefore, a multiplicity one representations of finite length with a minimal lattice of subrepresentations has a very simple lattice of subrepresentations. To determine explicitly this lattice, one needs only to determine which of the inclusions in (i) hold.

6.2. Derivatives of representations.

Now we shall recall the definition of the derivatives of representations. We shall follow the notation of the third section of [4]. The group G_{n-1} is imbedded into G_n in a usual way: $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. The subgroup of G_n consisting of all the

matrices which have the bottom row equal to $(0, \dots, 0, 1)$ is denoted by P_n . Its unipotent radical is denoted by V_n .

Let (π, W) be a smooth representation of P_n . Denote by $\Psi^-(\pi)$ the normalized Jacquet module of π , i.e. $W/W(V_n, 1_{V_n})$, where $W(V_n, 1_{V_n}) = \text{span}_{\mathbb{C}}\{\pi(v)w - w; v \in V_n, w \in W\}$ (the action of G_{n-1} is the quotient action, twisted by the square root of the modular character of P_n ; see [4]). One extends Ψ^- to a functor $\Psi^- : \text{Alg}(P_n) \rightarrow \text{Alg}(G_{n-1})$ in a standard way. Observe that $\Psi^- : W \mapsto W/W(V_n, 1_{V_n})$ carries the invariant subspaces in one category to the invariant subspaces in the other one (although Ψ^- is not an intertwining of representations of G_{n-1}).

We fix a non-trivial character ψ of the additive group of the field, and define a character θ of V_n by $v \mapsto \psi(v_{n-1, n})$. This character is normalized by P_{n-1} . Let now (π, W) be a smooth representation of P_n . Denote $W(V_n, \theta) = \text{span}_{\mathbb{C}}\{\pi(v)w - \theta(v)w; v \in V_n, w \in W\}$. Define now $\Phi^-(\pi)$ to be the quotient representation of P_{n-1} on $W/W(V_n, \theta)$, again twisted by a square root of the modular character (see [4] for details). One extends Φ^- to a functor

$$\Phi^- : \text{Alg}(P_n) \rightarrow \text{Alg}(P_{n-1}).$$

Again this functor carries invariant subspaces in one category to the invariant subspaces in the other category (although it is not intertwining of objects).

Both of the above functors are exact. We have a natural functor $\text{Alg}(G_n) \rightarrow \text{Alg}(P_n)$.

Now for $1 \leq k \leq n$ consider the functor

$$\Psi^- \circ (\Phi^-)^{k-1} : \text{Alg}(P_n) \rightarrow \text{Alg}(G_{n-k}).$$

Actually, we shall consider this functor only on $\text{Alg}(G_n)$. This functor is called k -th derivative, and it is denoted by $\pi \mapsto \pi^{(k)}$. One takes $\pi^{(0)}$ to be just π . Observe that again the k -th derivative functor carries the invariant subspaces to invariant subspaces. One defines the highest derivative of a representation in the same way as we did in section 3.

Lemma 4.5 of [4] tells us that $(\pi_1 \times \pi_2)^{(k)}$ is glued from the representations

$$\pi_1^{(i)} \times \pi_2^{(k-i)}, \quad 0 \leq i \leq k. \quad (25)$$

In other words, there exists a filtration $\{0\} = U_0 \subseteq U_1 \subseteq \dots \subseteq U_k$ of the representation space of $(\pi_1 \times \pi_2)^{(k)}$, such that the sequence of representations U_i/U_{i-1} coincide (up to isomorphisms) to the sequence (25), after a suitable reenumeration.

Remark 6.2. Observe that in the case of the highest derivatives, the above result gives the complete description of the highest derivative of the product of representations. It implies the following. Let $\pi_i^{(k_i)}$ be the highest derivatives of π_i , for $i = 1, 2$. Then the highest derivative of the parabolically induced representation $\pi_1 \times \pi_2$ is $(\pi_1 \times \pi_2)^{(k_1+k_2)}$, and we have

$$(\pi_1 \times \pi_2)^{(k_1+k_2)} \cong \pi_1^{(k_1)} \times \pi_2^{(k_2)}.$$

6.3. Connection with the derivatives on the level of R .

If ρ is an irreducible cuspidal representation of G_n , then $\rho^{(i)} = 0$ for $0 < i < n$ and $\rho^{(0)} = Z(\emptyset)$ (Theorem 4.4 of [4]). From this and (25) easily follows that $\pi^{(i)}$ is a finite length representation, for any $0 \leq i \leq n$ and any $\pi \in \tilde{G}_n$. Therefore, one can define

$$\mathcal{D}'(\pi) = \sum_{i=0}^n \text{s.s.}(\pi^{(i)}) \in R.$$

Extend \mathcal{D}' to an additive endomorphism of R . Obviously, \mathcal{D}' is positive. Then (25) implies that the additive endomorphism \mathcal{D}' is actually a ring homomorphism.

Theorem 8.1 of [19] tells us that the highest derivative of a representation $Z(a)$ is $Z(a^-)$, for $a \in M(\mathcal{S}(\mathcal{C}))$. This implies $\mathcal{D} = \mathcal{D}'$. Therefore, \mathcal{D} is positive and the highest derivative of $Z(a)$ on the level of R is $Z(a^-)$ (as we noted in 2.10).

6.4. Lattice of subrepresentations.

Proposition 6.3. *Let $k \in \mathbb{Z}_{\geq 0}$ and $n \leq k \leq n+d$. Denote $\ell = \min(n, n+d-k)$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation of length $\ell + 1$, and it is a representation with the minimal lattice of subrepresentations. Further, for $0 \leq i \leq \ell$, we have*

$$J.H. \left(\left(\mathcal{R}(n, d)_{(k)}^{(\rho)} \right)_{\text{smallest}} (Z(r_i(n, d)_{(k)}^{(\rho)})) \right) = \{Z(r_j(n, d)_{(k)}^{(\rho)}), i \leq j \leq \ell\}. \quad (26)$$

Denote by \mathcal{L} the set of all non-zero subrepresentations of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$. Then the mapping $J : V \mapsto \{i; Z(r_i(n, d)_{(k)}^{(\rho)}) \leq V\}$, is a an isomorphism of the partially ordered sets \mathcal{L} and $\{\{\ell\}, \{\ell - 1, \ell\}, \dots, \{0, 1, \dots, \ell\}\}$.

Proof. We shall first prove the claim of the proposition that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ are multiplicity one representations of length $\ell + 1$ with the minimal lattice of subrepresentations, and that the formula (26) holds. The proof of this claim will go by induction with respect to d (with n and k fixed). We break the induction into two parts. These parts follow the proofs of claims (3) and (4) of Proposition 5.1.

The first part of the induction is for indexes d which satisfy $d \leq k$. In this situation $n + d - k = \min(n, n + d - k)$.

If $d = k - n$, then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is irreducible and isomorphic to $Z(r_0(n, d)_{(k)}^{(\rho)})$. Therefore, the claim on the composition series holds and we have the basis of the induction.

Suppose $d = 1$ and $d > k - n$ (i.e. $1 > k - n$). Then the assumptions on the indexes imply $k = n$. Now it is well known fact that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a length two representation which is not semi simple. Further, $Z(r_0(n, d)_{(k)}^{(\rho)})$ is an irreducible quotient of it. This implies that the claim hold also for $d = 1$. Therefore, it is enough to consider the case $d > 1$.

Remark 6.2 implies that for $d \geq 2$, the highest derivative of the representation $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is isomorphic to

$$\nu^{-1/2} \mathcal{R}(n, d - 1)_{(k)}^{(\rho)}.$$

Let $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ be a representation of G_m , and let its highest derivative be a representation of G_{m-p}

Recall that $\text{h.d.}(Z(r_j(n, d)_{(k)}^{(\rho)})) = \nu^{-1/2}Z(r_j(n, d - 1)_{(k)}^{(\rho)})$ for $0 \leq j \leq n + d - k - 1$.

Observe that $\Psi^- \circ (\Phi^-)^{p-1}$ is surjective. Further, if we consider the action of this functor on the irreducible subquotients of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$, it sends only $Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$ to 0. We can factor

$$\varphi : \mathcal{R}(n, d)_{(k)}^{(\rho)} / Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)}) \rightarrow \nu^{-1/2}\mathcal{R}(n, d - 1)_{(k)}^{(\rho)}.$$

Recall that $\Psi^- \circ (\Phi^-)^{p-1}$ is an exact functor. This (together with the fact that the highest derivative carries irreducible representations to the irreducible ones) implies that φ carries the composition series to the composition series (i.e. if $\sigma_1, \dots, \sigma_m$ is a composition series of a subrepresentation π , then $\varphi(\sigma_1), \dots, \varphi(\sigma_m)$ is a composition series of $\varphi(\pi)$). This (together with the multiplicity one) implies that φ is an injective mapping considered on the lattice of subrepresentations of $\mathcal{R}(n, d)_{(k)}^{(\rho)} / Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$. The inductive assumption implies that the quotient $\mathcal{R}(n, d)_{(k)}^{(\rho)} / Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$ has at most $n + d - k$ non-zero subrepresentations (since $\nu^{-1/2}\mathcal{R}(n, d - 1)_{(k)}^{(\rho)}$ satisfies this property). This immediately implies that $\mathcal{R}(n, d)_{(k)}^{(\rho)} / Z(r_{n+d-k}(n, d)_{(k)}^{(\rho)})$ is a representation with a minimal lattice of subrepresentations.

Further, the uniqueness of an irreducible subrepresentation of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ implies that this representation has at most $n + d - k + 1$ non-zero subrepresentations. Since $n + d - k + 1$ is its length, we conclude that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has the minimal lattice of subrepresentations.

Applying $\Psi^- \circ (\Phi^-)^{p-1}$ (or φ) and using the inductive assumption, we get that for a non-zero subrepresentation V of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ holds

$$Z(r_j(n, d)_{(k)}^{(\rho)}) \in J.H.(V) \implies Z(r_i(n, d)_{(k)}^{(\rho)}) \in J.H.(V) \text{ for all } j \leq i \leq n + d - k \quad (27)$$

This finishes the proof of the inductive step for the first part.

For the second part of the induction, i.e. when $k < d$, we proceed similarly. For $d = k$ we know that the claim holds from the first part of the induction which we have proved. Observe that in this case we have $n = \min(n, n + d - k)$ (since $k \leq d$). The proof of the inductive step for this case follows the proof of the first part (actually, it is simpler, since we do not need to go to a quotient of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$). One concludes directly the claim inductively, using the formula for the highest derivative and the minimality of the lattice of subrepresentations of the highest derivative (which is the inductive assumption). We get also (27) from the inductive assumption. This finishes the proof of the claim in this case.

The claim of the proposition that we have proved, directly implies the claim of the proposition on the lattice \mathcal{L} of subrepresentations. This finishes the proof of the proposition. ■

7. Composition series - non-disjoint beginnings of segments

We continue with the notation of the previous sections. In this section we shall assume that $A_+ \leq C_-$. Passing to the n, d, k -notation, this becomes $-\frac{d-1}{2} - \frac{n-1}{2} + \frac{k}{2} \leq -\frac{d-1}{2} + \frac{n-1}{2} - \frac{k}{2}$, i.e. $k < n$.

Recall that for $j \neq 0$, $r_j(n, d)_{(k)}^{(\rho)}$ is defined whenever $\Delta_n \rightarrow \Gamma_j$. Then we get $r_j(n, d)_{(k)}^{(\rho)}$ by replacing in $r_0(n, d)_{(k)}^{(\rho)} = (\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_n)$ the part $\Delta_{n-j+1}, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_j$ with $\Delta_{n-j+1} \cup \Gamma_1, \Delta_{n-j+1} \cap \Gamma_1, \dots, \Delta_n \cup \Gamma_j, \Delta_n \cap \Gamma_j$.

Recall that $\Delta_n \rightarrow \Gamma_j$ if and only $\max(n - k + 1, 1) \leq j \leq \min(n, n + d - k)$.

Proposition 7.1. *Let $A_+ \leq C_-$, i.e. $k < n$. Then*

1. $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation.
2. Suppose $B_- < A^+$. In that case $d \leq k < n$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has length $d+1$, and its composition series consists of $Z(r_i(n, d)_{(k)}^{(\rho)})$, $n - k + 1 \leq i \leq n - k + d$, together with $Z(r_0(n, d)_{(k)}^{(\rho)})$.
3. Suppose $A^+ \leq B_-$, i.e. $k < d$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has length $k + 1$, and its composition series consists of $Z(r_i(n, d)_{(k)}^{(\rho)})$, $n - k + 1 \leq i \leq n$, together with $Z(r_0(n, d)_{(k)}^{(\rho)})$.

Proof. We shall first prove (2), proving in the same time also that all the multiplicities are one. This statement (with the multiplicity one claim) will be denoted by $(2)^+$. We fix n and k and prove $(2)^+$ by induction with respect to d .

For $d = 1$, $1 \leq k < n$ implies that $\mathcal{R}(n, 1)_{(k)}^{(\rho)}$ is a multiplicity one representation of length two. In its composition series there is obviously $Z(r_0(n, 1)_{(k)}^{(\rho)})$. Further, a composition factor is also $Z((a_-^t + a_+^t)^t)$, which one directly computes using MWA[←], and the result is $r_{n-k+1}(n, 1)_{(k)}^{(\rho)}$. This completes the proof for $d = 1$. Also we have the basis of the induction.

Suppose $1 < d \leq k$, and suppose that $(2)^+$ holds for $d-1$. Recall $\text{h.d.}(\mathcal{R}(n, d)_{(k)}^{(\rho)}) = \nu^{-1/2} \mathcal{R}(n, d-1)_{(k)}^{(\rho)}$ for $d \geq 2$. Also

$$\text{h.d.}(Z(r_j(n, d)_{(k)}^{(\rho)})) = \nu^{-1/2} Z(r_j(n, d-1)_{(k)}^{(\rho)}) \tag{28}$$

for $\max(n - k + 1, 1) \leq j \leq \min(n, n + d - k - 1)$, and also for $j = 0$ holds (28). Observe that $k < n$ implies $2 \leq n - k + 1$. Also $d \leq k$ implies $n + d - k \leq n$.

From the first relation and the inductive assumption follows that $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has precisely d irreducible subquotients $\pi = Z(a)$ such that the cardinality of a is $2n$, and that all of them have multiplicity one in $\mathcal{R}(n, d)_{(k)}^{(\rho)}$. Now (28) and the inductive assumption imply that these subquotients are the representations $Z(r_j(n, d)_{(k)}^{(\rho)})$, $n - k + 1 \leq j \leq n - k + d - 1$. Now Proposition 3.1 imply also that $Z((r_0(n, d)_{(k)}^{(\rho)})^t)$ is an irreducible subquotient, and that the multiplicity of this subquotient is one. We need to show that $(r_0(n, d)_{(k)}^{(\rho)})^t = r_{n-k+d}(n, d)_{(k)}^{(\rho)}$. This

Now consider $\mathcal{R}(d, n)_{(k)}^{(\rho)}$ and denote $n' = d$, $d' = n$. In the same way as we have introduced numbers $A_{\pm}, B_{\pm}, C_{\pm}, D_{\pm}$ for the triple n, d, k , we introduce $A'_{\pm}, B'_{\pm}, C'_{\pm}, D'_{\pm}$ for the triple n', d', k . Observe that $A'_{\pm} = A_{\pm}, B'_{\pm} = C_{\pm}, C'_{\pm} = B_{\pm}$ and $D'_{\pm} = D_{\pm}$. Then $n' < d'$ (i.e. $C'_- < B'_-$). Since $A_+ \leq C_-$, $A^+ \leq B_-$ and $B_- < C_-$, we have $A'_+ \leq B'_-, A'_+ \leq C'_-$ and $C'_- < B'_-$. Therefore, $\mathcal{R}(n', d')_{(k)}^{(\rho)} = \mathcal{R}(d, n)_{(k)}^{(\rho)}$ is covered by (3), and the inductive assumption implies that the length of this representation is $k + 1$ (recall $n' = d < n$). The proof of the proposition is now complete. ■

Theorem 7.2. *Let $k \in \mathbb{Z}_{\geq 0}$.*

1. *Representations $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ and $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$ have the same composition series. They are multiplicity one representations.*
2. *For $k < n$, the composition series of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ are given by $Z(r_i(n, d)_{(k)}^{(\rho)})$, $n - k + 1 \leq i \leq \min(n - k + d, n)$, together with $Z(r_0(n, d)_{(k)}^{(\rho)})$.*
3. *The representation $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ has a unique irreducible subrepresentation and also a unique irreducible quotient. The irreducible quotient is isomorphic to $Z(r_0(n, d)_{(k)}^{(\rho)})$. The irreducible subrepresentation is isomorphic to $Z(r_{\min(n-k+d, n)}(n, d)_{(k)}^{(\rho)})$.*

For $\mathcal{R}(n, d)_{(-k)}^{(\rho)}$, we have opposite situation regarding irreducible subrepresentation and quotient.

Proof. It remains to prove only the claim regarding the irreducible subrepresentation in (3). For $d \leq k (< n)$, this is proved in the proof of the previous proposition. Therefore, we shall suppose $k < d$. For this, one needs to prove $(r_0(d, n)_{(k)}^{(\rho)})^t = r_n(n, d)_{(k)}^{(\rho)}$. One can get this directly from MWA[←]. We can give here also a different argument, like in the proof of Theorem 5.2. Again $r_0(d, n)_{(k)}^{(\rho)} = a(d + k, n)^{(\rho)} + a(d - k, n)^{(\rho)}$. Further, as before we have

$$Z(a(d + k, n)^{(\rho)} + a(d - k, n)^{(\rho)})^t = Z(a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}).$$

This implies

$$(r_0(d, n)_{(k)}^{(\rho)})^t = a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}.$$

Direct checking gives $r_n(n, d)_{(k)}^{(\rho)} = a(n, d + k)^{(\rho)} + a(n, d - k)^{(\rho)}$. The proof is now complete. ■

8. Lattice of subrepresentations - non-disjoint beginnings of segments

One proves the following proposition in a similar way as Proposition 6.3. Therefore, we omit the proof here (the proof proceeds in two parts, following claims (2) and (3) of Proposition 7.1, similarly as the proof of Proposition 6.3 followed claims (2) and (3) of Proposition 5.1).

Proposition 8.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and $k < n$. Denote $\ell = \min(n, n + d - k)$. Then $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation of length $\ell - n + k + 1$, and it is a representation with the minimal lattice of subrepresentations. Further, for $n - k + 1 \leq i \leq \ell$, holds*

$$J.H. \left(\left(\mathcal{R}(n, d)_{(k)}^{(\rho)} \right)_{\text{smallest}} \left(Z(r_i(n, d)_{(k)}^{(\rho)}) \right) \right) = \{ Z(r_j(n, d)_{(k)}^{(\rho)}), i \leq j \leq \ell \}.$$

Denote by $\mathcal{L}_{\text{proper}}$ the set of all non-zero proper subrepresentations of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$. Then the mapping $J : V \mapsto \{i; Z(r_i(n, d)_{(k)}^{(\rho)}) \leq V\}$, is a an isomorphism of the partially ordered sets $\mathcal{L}_{\text{proper}}$ and $\{\{\ell\}, \{\ell - 1, \ell\}, \dots, \{n - k + 1, \dots, \ell - 1, \ell\}\}$.

Since the lattice \mathcal{L} of all the non-zero subrepresentations of $\mathcal{R}(n, d)_{(k)}^{(\rho)}$ is an union of $\mathcal{L}_{\text{proper}}$ and $\{\mathcal{R}(n, d)_{(k)}^{(\rho)}\}$, the above description of $\mathcal{L}_{\text{proper}}$ completely describe the whole lattice \mathcal{L} .

9. Translation to the setting of the Langlands classification

9.1. Connection between Zelevinsky and Langlands classifications. In the second section we have recalled the definition of the Zelevinsky involution, and some very basic properties of it (this was enough for getting the composition series in terms of the Zelevinsky classification). To translate our main results from the Zelevinsky classification to the Langlands classification, we shall first recall how the Zelevinsky involution relates these two classifications (following F. Rodier’s paper [11]).

First recall that originally A.V. Zelevinsky has defined a ring homomorphism ${}^t : R \rightarrow R$ determined by the requirement that $\mathfrak{z}(\Delta)^t = \delta(\Delta)$, for all $\Delta \in \mathcal{S}(\mathcal{C})$ (recall that R is a polynomial algebra over $\mathfrak{z}(\Delta), \Delta \in \mathcal{S}(\mathcal{C})$). He has also shown that t is an involution. As we already noted, the fundamental result is that this involution is positive, i.e. that $r \geq 0$ implies $r^t \geq 0$ (this is proved in [1], and in [12]). This implies that the restriction of t to the irreducible representations is a bijection. We recall of the following simple result of F. Rodier ([11]):

Proposition 9.1. *With t defined in a such way, for $a \in M(\mathcal{S}(\mathcal{C}))$ we have $Z(a)^t = L(a)$ and $L(a)^t = Z(a)$.*

This proposition implies that Zelevinsky’s original definition of the involution agrees with the one that we have used (from the second section). For the convenience of the reader, we shall recall of a very simple argument of F. Rodier showing the above relation.

Proof. Write $a = (\Delta_1, \dots, \Delta_n)$. The proof goes by induction with respect to the standard ordering on $M(\mathcal{S}(\mathcal{C}))$. Suppose that a is minimal with respect to this ordering. Then $\mathfrak{z}(\Delta_1) \times \dots \times \mathfrak{z}(\Delta_n)$ is irreducible, and from the definition of t it follows that $Z(a)^t = (\mathfrak{z}(\Delta_1) \times \dots \times \mathfrak{z}(\Delta_n))^t = \delta(\Delta_1) \times \dots \times \delta(\Delta_n)$. Since $L(a)$ is a subquotient of the right hand side, we get $Z(a)^t = L(a)$ in this case.

Suppose now that a is arbitrary, and that the formula holds for all $a' < a$. By Theorem 7.1 of [19], there exist positive integers $m_{a,a'}$ such that $\zeta(a) = Z(a) + \sum_{a' < a} m_{a,a'} Z(a')$. Now the inductive assumption implies $\lambda(a) = \mathfrak{z}(a)^t = Z(a)^t + \sum_{a' < a} m_{a,a'} L(a')$. Since $L(a) \leq \lambda(a)$, we conclude $L(a) = Z(a)^t$. This proves the first relation in the proposition. The second relation follows immediately from the first one. ■

Corollary 9.2. *Let $P \in \mathbb{Z}[X_1, \dots, X_n]$ be a polynomial and $a_1, \dots, a_n \in M(\mathcal{S}(\mathcal{C}))$. Then $P(Z(a_1), \dots, Z(a_n)) = 0$ iff $P(L(a_1), \dots, L(a_n)) = 0$ in R .*

Proof. Write $P = \sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} X_1^{i_1} \dots X_n^{i_n}$. Recall that t is a bijection (since it is an involution). Thus $P(Z(a_1), \dots, Z(a_n)) = 0$ iff

$$\begin{aligned} &P(Z(a_1), \dots, Z(a_n))^t = 0 \text{ iff} \\ &(\sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} Z(a_1)^{i_1} \dots Z(a_n)^{i_n})^t = 0 \text{ iff} \\ &\sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} (Z(a_1)^t)^{i_1} \dots (Z(a_n)^t)^{i_n} = 0 \text{ iff} \\ &\sum_{i_1, \dots, i_n \geq 0} c_{i_1, \dots, i_n} L(a_1)^{i_1} \dots L(a_n)^{i_n} = 0 \text{ iff} \\ &P(L(a_1), \dots, L(a_n)) = 0. \end{aligned}$$

Definition 9.3. We denote

$$\begin{aligned} \mathbf{R}^t(n, d)_{(l)}^{(\rho)} &:= \nu^{-l/2} L(a(n, d)^{(\rho)}) \times \nu^{l/2} L(a(n, d)^{(\rho)}) \\ &= L(a(n, d)^{(\nu^{-l/2}\rho)}) \times L(a(n, d)^{(\nu^{l/2}\rho)}), \end{aligned}$$

where $n, d \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}$ and $\rho \in \mathcal{C}$.

Observe that (8) implies

$$\mathbf{R}^t(n, d)_{(l)}^{(\rho)} = Z(a(d, n)^{(\nu^{-l/2}\rho)}) \times Z(a(d, n)^{(\nu^{l/2}\rho)}) = \mathcal{R}(d, n)_{(l)}^{(\rho)}. \tag{32}$$

Now from Theorem 5.2 easily follows the following completely analogous theorem in the setting of the Langlands classification⁷:

Theorem 9.4. *Let $k \in \mathbb{Z}_{\geq 0}$.*

1. *The representations $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ and $\mathbf{R}^t(n, d)_{(-k)}^{(\rho)}$ have the same composition series. They are multiplicity one representations.*
2. *For $n + d \leq k$, $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ is irreducible, and $\mathbf{R}^t(n, d)_{(k)}^{(\rho)} \cong L(r_0(n, d)_{(k)}^{(\rho)})$.*
3. *For $n \leq k \leq n + d$, $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ is a multiplicity one representation. Its composition series consists of $L(r_i(n, d)_{(k)}^{(\rho)})$, $0 \leq i \leq \min(n, n + d - k)$.*
4. *The representation $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ has a unique irreducible subrepresentation as well as a unique irreducible quotient. The irreducible subrepresentation is*

⁷The only significant difference in the formulation of the theorem below is the description of the parameters of the unique irreducible subrepresentation and the irreducible quotient.

isomorphic to $L(r_0(n, d)_{(k)}^{(\rho)})$, and the irreducible quotient is isomorphic to $L(r_{\min(n, n+d-k)}(n, d)_{(k)}^{(\rho)})$.

For $\mathbf{R}^t(n, d)_{(-k)}^{(\rho)}$, we have opposite situation regarding irreducible subrepresentation and quotient.

Proof. Theorem 5.2 and (32) directly imply (1).

To prove (2)⁸, let P be the polynomial $X_2X_3 - X_1$ and take

$$X_1 = r_0(n, d)_{(k)}^{(\rho)}, \quad X_2 = a(n, d)^{(\nu^{-1/2}\rho)}, \quad X_3 = a(n, d)^{(\nu^{1/2}\rho)}.$$

Now (2) of Theorem (5.2) implies

$$P(Z(r_0(n, d)_{(k)}^{(\rho)}), Z(a(n, d)^{(\nu^{-1/2}\rho)}), Z(a(n, d)^{(\nu^{1/2}\rho)})) = 0.$$

The above corollary implies

$$P(L(r_0(n, d)_{(k)}^{(\rho)}), L(a(n, d)^{(\nu^{-1/2}\rho)}), L(a(n, d)^{(\nu^{1/2}\rho)})) = 0,$$

i.e. $L(r_0(n, d)_{(k)}^{(\rho)}) = L(a(n, d)^{(\nu^{-1/2}\rho)}) \times L(a(n, d)^{(\nu^{1/2}\rho)}) = \mathbf{R}^t(n, d)_{(k)}^{(\rho)}$, which is the claim of (2)

We shall now comment the proof of (3). Denote $\ell = \min(n, n + d - k)$. Take now $P = X_{\ell+1}X_{\ell+2} - X_0 - \dots - X_\ell$, $X_i = r_i(n, d)_{(k)}^{(\rho)}$, $0 \leq i \leq \ell$, $X_{\ell+1} = a(n, d)^{(\nu^{-1/2}\rho)}$, $X_{\ell+2} = a(n, d)^{(\nu^{1/2}\rho)}$. Now (3) of Theorem 5.2 implies

$$P(Z(r_0(n, d)_{(k)}^{(\rho)}), \dots, Z(r_\ell(n, d)_{(k)}^{(\rho)}), Z(a(n, d)^{(\nu^{-1/2}\rho)}), Z(a(n, d)^{(\nu^{1/2}\rho)})) = 0.$$

The above corollary implies

$$P(L(r_0(n, d)_{(k)}^{(\rho)}), \dots, L(r_\ell(n, d)_{(k)}^{(\rho)}), L(a(n, d)^{(\nu^{-1/2}\rho)}), L(a(n, d)^{(\nu^{1/2}\rho)})) = 0,$$

which tells us that that (in R) we have

$$L(r_0(n, d)_{(k)}^{(\rho)}) + \dots + L(r_\ell(n, d)_{(k)}^{(\rho)}) = L(a(n, d)^{(\nu^{-1/2}\rho)}) \times L(a(n, d)^{(\nu^{1/2}\rho)}),$$

which is equal to $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$. This is the claim of (3).

Recall $\mathbf{R}^t(n, d)_{(l)}^{(\rho)} = \mathcal{R}(d, n)_{(l)}^{(\rho)}$. Therefore, $\mathbf{R}^t(n, d)_{(k)}^{(\rho)}$ has a unique irreducible quotient and a unique irreducible subrepresentation. These two irreducible representations are $Z(r_0(d, n)_{(k)}^{(\rho)})$ and $Z(r_{\min(n, n+d-k)}(d, n)_{(k)}^{(\rho)})$ respectively. Further, they are equal $L((r_0(d, n)_{(k)}^{(\rho)})^t)$ and $L((r_{\min(n, n+d-k)}(d, n)_{(k)}^{(\rho)})^t)$ respectively. We have seen that they are respectively $L(r_{\min(n, n+d-k)}(n, d)_{(k)}^{(\rho)})$ and $L(r_0(n, d)_{(k)}^{(\rho)})$. The proof is now complete. \blacksquare

⁸We could prove this statement easily using the principal properties of the Langlands classification. Rather, we present here the proof which follows general principle how one lifts a result from one classification to the other classification.

In completely the same way we prove Theorem 7.2 in the setting of the Langlands classification:

Theorem 9.5. *Let $k \in \mathbb{Z}_{\geq 0}$.*

1. *Representations $\mathbf{R}^{\mathfrak{t}}(n, d)_{(k)}^{(\rho)}$ and $\mathbf{R}^{\mathfrak{t}}(n, d)_{(-k)}^{(\rho)}$ have the same composition series, and they are multiplicity one representations.*
2. *For $k < n$, the composition series of $\mathbf{R}^{\mathfrak{t}}(n, d)_{(k)}^{(\rho)}$ are given by $L(r_i(n, d)_{(k)}^{(\rho)})$, $n - k + 1 \leq i \leq \min(n - k + d, n)$, together with $L(r_0(n, d)_{(k)}^{(\rho)})$.*
3. *The representation $\mathbf{R}^{\mathfrak{t}}(n, d)_{(k)}^{(\rho)}$ has a unique irreducible subrepresentation and also a unique irreducible quotient. The irreducible subrepresentation is isomorphic to $L(r_0(n, d)_{(k)}^{(\rho)})$. The irreducible quotient is isomorphic to $L(r_{\min(n-k+d, n)}(n, d)_{(k)}^{(\rho)})$.*

For $\mathbf{R}^{\mathfrak{t}}(n, d)_{(-k)}^{(\rho)}$, we have opposite situation regarding irreducible subrepresentation and quotient.

9.2. Expectation. A following natural question after this paper (and Theorem 1.2), is the question of determining the composition series of a product of two arbitrary essentially Speh representations⁹. We expect the multiplicity one to hold also here. Further, we expect the description of the composition factors that we give in this paper to hold here also in essentially the same form. More precisely, let $\pi_1 = L(\Delta_1, \dots, \Delta_n)$ and $\pi_2 = L(\Gamma_1, \dots, \Gamma_m)$ be two essentially Speh representations (segments Δ_1 and Γ_1 do not need to be of the same length anymore). Choose numerations of segments which satisfy $\Delta_1 \rightarrow \Delta_2 \rightarrow \dots \rightarrow \Delta_n$ and $\Gamma_1 \rightarrow \Gamma_2 \rightarrow \dots \rightarrow \Gamma_m$. Suppose that $\pi_1 \times \pi_2$ reduces (there is a simple criterion describing the reducibility in [18]). Choose a numeration of π_1 and π_2 such that $\nu^\alpha b(\Delta_1) = b(\Gamma_1)$ for some $\alpha \geq 0$ (actually, then α is a positive integer). Denote by I_{π_1, π_2} the set of all indexes $j \in \{1, 2, \dots, m\}$ for which hold $\Delta_n \rightarrow \Gamma_j$ and $1 \leq n - j - 1$ ¹⁰. Let $a_{\pi_1, \pi_2}^{(0)} = (\Delta_1, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_m)$. For each $j \in I_{\pi_1, \pi_2}$ denote by $a_{\pi_1, \pi_2}^{(j)}$ a multisegment which we obtain when we replace in $a_{\pi_1, \pi_2}^{(0)}$ the part $\Delta_{n-j+1}, \dots, \Delta_n, \Gamma_1, \dots, \Gamma_j$ with

$$\Delta_{n-j+1} \cup \Gamma_1, \Delta_{n-j+1} \cap \Gamma_1, \dots, \Delta_n \cup \Gamma_j, \Delta_n \cap \Gamma_j.$$

Then we expect $J.H.(\pi_1 \times \pi_2) = \{L(a_{\pi_1, \pi_2}^{(j)}); j \in I_{\pi_1, \pi_2} \cup \{0\}\}$.

Regarding the proofs, we expect that the same strategy can be used, except that we can not use the "symmetry", which is a topic of Lemma 3.3. This symmetry is used to transfer an information that we get about the ends of segments defining irreducible subquotients of products of essentially Speh representations using the

⁹The case of the "first" reducibility point may be interesting for the problem of unitarizability for classical groups. Such reducibility is usually not complicated (with the representation of length two).

¹⁰This condition is automatically satisfied if π_1 is a twist of π_2 , which is the case for the representations that we studied in this paper.

derivatives, to get the corresponding information about the beginnings of these segments. In the general case, there is no such a symmetry. Instead, we expect that the derivatives considered in Remark (ii) of 2.2 will provide with the corresponding information (this is equivalent to passing to the contragredient setting).

References

- [1] Aubert, A. M., *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif p -adique*, Trans. Amer. Math. Soc. **347** (1995), 2179–2189. *Erratum*, Trans. Amer. Math. Soc. **348** (1996), 4687–4690.
- [2] Badulescu, A. I., *On p -adic Speh representations*, Bulletin de la Soc. Math. France, to appear.
<http://arxiv.org/pdf/1110.5080v1.pdf>
- [3] Bernstein, J., *P -invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-archimedean case)*, “Lie Group Representations II,” Lecture Notes in Math. **1041**, Springer-Verlag, Berlin, 1984, 50–102.
- [4] Bernstein, J., and A. V. Zelevinsky, *Induced representations of reductive p -adic groups. I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 441–472.
- [5] Bushnell, C. J., and P. C. Kutzko, “The admissible dual of $GL(N)$ via compact open subgroups,” Annals of Mathematics Studies **129**, Princeton University Press, Princeton, NJ, 1993.
- [6] Gelfand, I. M., and D. A. Kajdan, *Representations of the group $GL(n, K)$ where K is a local field*, “Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971),” Halsted, New York, 1975, 95–118.
- [7] Lapid, E., and A. Mínguez, *On a determinantal formula of Tadić*, Amer. J. Math. **136** (2014), 111–142.
- [8] Leclerc, B. A., *Littlewood-Richardson rule for evaluation representations of $U_q(\hat{\mathfrak{sl}}_n)$* , Sémin. Lothar. Combin. **50** (2003/04), Art. B50e, 12 pp., (electronic).
- [9] Mínguez, A., and V. Sécherre, *Représentations banales de $GL(m, D)$* , Compos. Math. **149** (2013), 679–704.
- [10] Mœglin, C., and J.-L. Waldspurger, *Sur l'involution de Zelevinski*, J. Reine Angew. Math. **372** (1986), 136–177.
- [11] Rodier, F., *Représentations de $GL(n, k)$ où k est un corps p -adique*, Séminaire Bourbaki no. 587, 1982, Astérisque **92–93** (1982), 201–218.
- [12] Schneider, P., and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. IHES **85** (1997), 97–191.

- [13] Tadić, M., *Unitary representations of general linear group over real and complex field*, Preprint MPI/SFB **85–22** Bonn (1985)
<http://www.mpim-bonn.mpg.de/preblob/5395>
- [14] —, *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Sci. École Norm. Sup. **19** (1986), 335–382.
- [15] —, *Unitary representations of $GL(n)$, derivatives in the non-archimedean case*, Berichte der Mathematisch-Statistischen Sektion in der Forschungsgesellschaft Joaneum, Graz **281** (1987), 1–19.
- [16] —, *Induced representations of $GL(n, A)$ for p -adic division algebras A* , J. Reine Angew. Math. **405** (1990), 48–77.
- [17] —, *On characters of irreducible unitary representations of general linear groups*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 341–363.
- [18] —, *Irreducibility criterion for representations induced by essentially unitary ones (case of non-archimedean $GL(n, A)$)*, Glasnik Mat. **49(69)** (2014), 123–161.
- [19] Zelevinsky, A. V., *Induced representations of reductive p -adic groups II. On irreducible representations of $GL(n)$* , Ann. Sci. École Norm. Sup. **13** (1980), 165–210.

Marko Tadić
Department of Mathematics
University of Zagreb
Bijenička 30
10000 Zagreb, Croatia
tadic@math.hr

Received January 13, 2014
and in final form June 20, 2014