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Elastic Helices in Simple Lie Groups

Óscar J. Garay and Lyle Noakes*

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Abstract. Elastic curves (elastica) are classical variational objects with many applications in physics and engineering. Elastica in real space forms are well understood, but in other ambient spaces there are few known explicit examples, except geodesics. The main purpose of the present paper is to construct new families of elastica when the ambient space is a simple Lie group G with a bi-invariant Riemannian metric. Then, using Lie reduction, we give a criterion for a pointwise product of one-parameter subgroups to be an elastic curve. This characterisation is applied first when G is the real space form $\mathrm{SU}(2)$, and comparisons can be made with classical results. We then focus on $G = \mathrm{SU}(3)$, for which very little is known. Analysis of our criterion leads to large families of new elastica in $\mathrm{SU}(3)$ which are *helices*, namely our new examples have constant Frenet curvatures. Elastic helices are also constant-speed tension-cubics, solving a different variational problem.
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1. Introduction

In 1691 J. Bernoulli posed the *flexible rod problem*, to determine the shape of a flexible rod of given length, subject to external forces at the ends, and in 1694 he obtained the differential equation of the *rectangular* plane elastica (when the external force is orthogonal to the ends of the rod). In his 1744 book on the Calculus of Variations [8], L. Euler classified the plane elastica following an idea of D. Bernoulli, who had suggested to apply the least action principle to the mean squared curvature (for details, see [9]). More generally, in modern terms, an elastica is a curve $x : [t_0, t_1] \to M$ of given length in a Riemannian manifold M which is critical for the *bending energy*

$$\int \kappa^2 + \lambda \, ds$$

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where κ denotes the geodesic curvature of x, and λ is a variable Lagrange multiplier associated to the constraint on length, as in §3.

The universal cover of a real space form M of arbitrary dimension is a real space form $\tilde{M}$, where the covering projection is a local isometry. So a lifting $\tilde{x}$ of an elastica x in M is elastic in $\tilde{M}$ and, as noted in [10], [17], we have $\tilde{x}$ contained in a totally geodesic real space form of dimension ≤ 3. So the study of elastica in real space forms M reduces to the case where M is 3-dimensional and simply connected.

The differential equation for plane elastica was solved by L. Euler in Appendix I to [8]. The study of elastica in the other real space forms was initiated in the eighties [10], attracting the interest of many mathematicians. In particular, elastica in simply connected 2-dimensional real space forms were determined in [6], [15], [17]. As for the 3-dimensional case, J. Radon analysed the elastica in $\mathbb{R}^3$ at the beginning of the 20th century [29]. Moreover, following ideas in [17], elastica in $\mathbb{S}^3$ were studied in [1], [2]. Elastica in hyperbolic 3-space can be treated using the arguments in [13], inspired by methods of J. Langer and D. Singer. For a Hamiltonian approach to elastica and the related Kichhoff's elastic rods see [16].

On the other hand, little is known about elastica in spaces with nonconstant curvature, except for some facts about elastic helices. For instance, in [4] any elastic helix in $\mathbb{CP}^2$ is shown to be the image of a translation of a one-parameter subgroup of $SU(3)$ under the natural projection.

The bending energy is the one-dimensional version of the *Willmore energy* [7] for surfaces isometrically immersed in 3-dimensional real space forms. This makes it possible to implement methods which use elastica in real space forms to construct families of Willmore submanifolds (for details, see [11], [12], [18], [27] and references therein).

A second related variational problem concerns *tension-cubics* [25], namely curves that are critical for the functional

$$J_{TC}(x) := \int_{t_0}^{t_1} \langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle + \tau \langle x^{(1)}, x^{(1)} \rangle \, dt$$

where the *tension parameter* $\tau \in \mathbb{R}$ is given and (as distinct from elastica) the squared speed $\langle x^{(1)}, x^{(1)} \rangle$ is not required to be constant. Tension-cubics arise in interpolation problems for mechanical systems [26], and the case where $\tau = 0$ of so-called *Riemannian cubics* is studied in [24], [19], [20], [21], [22], [23]. Except for cases considered below in §3, elastica are usually different from tension-cubics (there are no general formulae for tension-cubics in space forms).

The purpose of the present paper is to construct large families of examples of elastica in spaces M of nonconstant curvature, especially in $\mathbb{SU}(3)$. More generally, M is taken to be a compact simple Lie group G with bi-invariant Riemannian metric. Whereas translations of one-parameter subgroups of G are geodesics and therefore elastic, we seek new examples of the form

$$x(t) = g\gamma_1(t)\gamma_2(t)$$

where γ_1, γ_2 are one-parameter subgroups of G, and $g \in G$. So our new examples will be translations x of pointwise products of one-parameter subgroups, and such curves turn out to be helices. The method of Lie reduction from [28] then gives a criterion for the helix x to be elastic. For $G = \mathbb{SU}(2)$ this criterion is always satisfied. Since elastica in the real space form $\mathbb{SU}(2)$ are well understood, our elastic helices in $\mathbb{SU}(2)$ are related to standard examples as described in §5.

A much more interesting situation occurs in §6, where $G = \mathbb{SU}(3)$ and a careful analysis is required to decide whether a pointwise product x is elastic. Although this usually does not happen when $G = \mathbb{SU}(3)$, we find large families of one-parameter groups γ_1, γ_2 in $\mathbb{SU}(3)$ for which the helices x are elastic. Because of a relationship between certain kinds of elastica and tension-cubics explained in §3, this also gives large families of constant-speed tension-cubics in $\mathbb{SU}(3)$.

In more detail, we begin with a review in §2 of standard results on higher order geodesic curvatures of curves x in Riemannian manifolds M. The maximum number of nonzero curvatures is called the *rank* of x. In §3 we review definitions of elastic curves and tension-cubics, noting that constant geodesic curvature elastica are the same as constant-speed tension cubics. In §4 M is taken to be a compact simple Lie group G with bi-invariant Riemannian metric. Pointwise products x of one-parameter subgroups γ_1, γ_2 of G are shown to have constant geodesic curvatures of all orders, and rank(x) is calculated in terms of the infinitesimal generators V_1, V_2 of γ_1, γ_2. A necessary and sufficient condition (4) is given in terms of V_1, V_2 for x to be elastic. In §5 condition (4) is examined for $G = \mathbb{SU}(2)$ and compared with known results; in particular, pointwise products of one-parameter subgroups are geodesics in Hopf tori. In §7, G is taken to be $\mathbb{SU}(3)$, and condition (4) is applied to exhibit a 6-parameter family of elastic helices with rank typically 6 (the maximum possible). In §8 condition (4) is applied in a different way, resulting in a 7-parameter family of elastic helices also with rank typically 6.

2. Geodesics and Higher Curvatures

For finite m, let M be an m-dimensional Riemannian oriented manifold with metric $\langle\ ,\ \rangle$ and associated Levi-Civita covariant derivative ∇. Let $x : [t_0, t_1] \to M$ be a C^∞ curve satisfying $\langle x^{(1)}, x^{(1)} \rangle \equiv 1$, with

$$x^{(1)},\ \nabla_t x^{(1)},\ \nabla_t^2 x^{(1)}, \ldots, \nabla_t^{n-1} x^{(1)}$$

everywhere linearly independent, where $1 \le n \le m$. Set $e_0 := x^{(1)}$, and define the *unit normal field* e_1 to be the unit vector field along x in the direction of $\nabla_t e_0$. The *geodesic curvature* is

$$\kappa_1(t) \ := \ \langle \nabla_t e_0(t), e_1(t) \rangle.$$

Unit normal fields e_j and curvatures κ_j for $j = 2, \ldots, n-1$ are given inductively by Gramm-Schmidt orthogonalization, as follows. Let $\hat{e}_j(t)$ be the orthogonal projection of $\nabla_t e_{j-1}$ onto the orthogonal complement of $\{e_0(t), e_1(t), e_2(t), \ldots, e_{j-1}(t)\}$. Set

$$e_j(t) \ := \ \frac{\hat{e}_j(t)}{\|\hat{e}_j(t)\|}$$

and $\kappa_j(t) := \langle \nabla_t e_{j-1}, e_j(t) \rangle$. If $n = m$ then the Frenet-Serret formulae are

$$\nabla_t e_{j-1} \;=\; \kappa_j(t) e_j(t) - \kappa_{j-1}(t) e_{j-1}(t) \quad \text{and} \quad \nabla_t e_{m-1} \;=\; -\kappa_{m-1}(t) e_{m-2}(t).$$

Definition 2.1. The *rank* of x is the maximum integer j for which $\kappa_j(t) \neq 0$ for some $t \in [t_0, t_1]$, namely $\text{rank}(x) = n - 1$ where n is the maximum integer for which $x^{(1)}, \nabla_t x^{(1)}, \nabla_t^2 x^{(1)}, \ldots, \nabla_t^{n-1} x^{(1)}$ are linearly independent. ■

So $0 \leq \text{rank}(x) \leq m - 1$, and x is a geodesic if and only if $\text{rank}(x) \leq 1$.

Definition 2.2. When x has constant curvatures κ_k for $1 \leq k \leq \text{rank}(x)$, it is called a *helix* (alternatively a W*-curve* [14]). ■

Remark 2.3. Let $x, y : [t_0, t_1] \to M$ be C^∞ curves of the same rank $m-1$, with respective normal fields e_{j-1} and f_{j-1} where $1 \leq j \leq m$, and the same curvatures κ_j where $1 \leq j \leq m - 1$. Suppose that there is an isometry $g : M \to M$ with the property that $g \circ x(0) = y(0)$ and

$$dg_{x(0)}(e_{j-1}(0)) \;=\; f_{j-1}(0)$$

for all $1 \leq j \leq m$. Then, by uniqueness of solutions of C^∞ systems of ordinary differential equations, $g \circ x(t) = y(t)$ for all $t \in [t_0, t_1]$.

When M is a Riemannian manifold with constant sectional curvature, if we are given $x(0), y(0)$ and the e_{j-1} and f_{j-1}, then a suitable isometry g always exists. Moreover, if the rank of x is $n - 1$, with $2 \leq n < m$, then x maps into an n-dimensional totally geodesic submanifold of M.

On the other hand, this need not be the case for a general Riemannian manifold M. For instance the sectional curvatures of M at $(e_{j-1}(0), e_{k-1}(0))$ and $(f_{j-1}(0), f_{k-1}(0))$ might be different for some j, k.

3. Higher Order Analogues of Geodesics

Two larger classes of curves in M arise as solutions of second order variational problems [6], [17], [28] and [25], [30]. For $i = 0, 1$ let $x_i \in M$ and $v_i \in TM_{x_i}$ be given.

- *Elastic curves.* Suppose $\langle v_0, v_0 \rangle = \langle v_1, v_1 \rangle$. For any C^∞ curve $x : [t_0, t_1] \to M$ satisfying $x(t_i) = x_i$, $x^{(1)}(t_i) = v_i$, and $\langle x^{(1)}(t), x^{(1)}(t) \rangle = 1$ for all t, define
$$J_E(x) \;:=\; \int_{t_0}^{t_1} \langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle \; dt.$$
The curve x is said to be *elastic* when it is a critical point of J_E, namely when
$$\nabla_t^3 x^{(1)} + R(\nabla_t x^{(1)}, x^{(1)}) x^{(1)} + \nabla_t(\phi x^{(1)}) \;=\; \mathbf{0} \tag{1}$$
and $\phi : [t_0, t_1] \to \mathbb{R}$. Because $\langle x^{(1)}, x^{(1)} \rangle$ is constant, $\langle \nabla_t x^{(1)}, x^{(1)} \rangle = 0 \implies$
$$\langle \nabla_t^2 x^{(1)}, x^{(1)} \rangle = -\langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle \implies \langle \nabla_t^3 x^{(1)}, x^{(1)} \rangle = -3 \langle \nabla_t^2 x^{(1)}, \nabla_t x^{(1)} \rangle.$$

Taking inner products of both sides of (1) with $x^{(1)}$ we find that, for some $\tilde{b} \in \mathbb{R}$,

$$\phi(t) \;=\; \frac{3}{2}\kappa_1(t)^2 + \tilde{b}. \tag{2}$$

Remark 3.1. Instead of explicitly requiring $\langle x^{(1)}, x^{(1)} \rangle \equiv 1$, the condition can be enforced by requiring

$$\begin{aligned} \langle x^{(1)}, x^{(1)} \rangle_{x(t_*)} &= 1\,, \\ \langle \nabla_t x^{(1)}, x^{(1)} \rangle_{x(t_*)} &= 0\,, \\ \langle \nabla_t^2 x^{(1)}, x^{(1)} \rangle + \langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle_{x(t_*)} &= 0\,, \end{aligned}$$

for some specific $t_* \in [t_0, t_1]$. Then $\frac{d^k}{dt^k}\langle x^{(1)}, x^{(1)} \rangle$ vanishes at $t = t_*$ for $k = 1, 2$, and

$$\frac{d^3}{dt^3}\langle x^{(1)}, x^{(1)} \rangle \;=\; \langle \nabla_t^3 x^{(1)}, x^{(1)} \rangle + 3\langle \nabla_t^2 x^{(1)}, \nabla_t x^{(1)} \rangle \;=\; \langle \nabla_t^3 x^{(1)}, x^{(1)} \rangle + \phi^{(1)} \;=$$

$$\phi^{(1)}(1 - \langle x^{(1)}, x^{(1)} \rangle) - \phi \langle \nabla_t x^{(1)}, x^{(1)} \rangle$$

on substitution from (1). Then, from uniqueness of solutions of C^∞ differential equations, $\langle x^{(1)}, x^{(1)} \rangle \equiv 1$.

Remark 3.2. When $\kappa_1 \neq 0$ is constant, $\tilde{b}$ is found explicitly from $x(t_*), x^{(1)}(t_*), (\nabla_t x^{(1)})_{t_*}$ and $(\nabla_t^2 x^{(1)})_{t_*}$ as follows. By constant curvature,

$$\langle \nabla_t^2 x^{(1)}, \nabla_t x^{(1)} \rangle = 0 \implies \langle \nabla_t^3 x^{(1)}, \nabla_t x^{(1)} \rangle = -\langle \nabla_t^2 x^{(1)}, \nabla_t^2 x^{(1)} \rangle \implies$$

$$(\frac{3}{2}\kappa_1^2 + \tilde{b})\kappa_1^2 \;=\; \langle \nabla_t^2 x^{(1)}, \nabla_t^2 x^{(1)} \rangle - \langle R(\nabla_t x^{(1)}, x^{(1)})x^{(1)}, \nabla_t x^{(1)} \rangle$$

on taking inner products of both sides of (1) with $\nabla_t x^{(1)}$.

- *Riemannian cubics in tension.* Choose a *tension parameter* $\tau \in \mathbb{R}$. For C^∞ curves $x : [t_0, t_1] \to M$ satisfying $x(t_i) = x_i$ and $x^{(1)}(t_i) = v_i$ for $i = 0, 1$, define

$$J_{TC}(x) \;:=\; \int_{t_0}^{t_1} \langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle + \tau \langle x^{(1)}, x^{(1)} \rangle \; dt.$$

A critical point of J_{TC} is a x is a *tension-cubic*, namely x should satisfy the Euler-Lagrange equation

$$\nabla_t^3 x^{(1)} + R(\nabla_t x^{(1)}, x^{(1)})x^{(1)} - \tau \nabla_t x^{(1)} \;=\; \mathbf{0}. \tag{3}$$

When $\tau = 0$ the tension-cubic x is called a *Riemannian cubic* [24], [19], [20].

So an elastic curve with κ_1 constant is the same as a unit-speed tension-cubic, where $\tau = -\frac{3}{2}\kappa_1^2 - \tilde{b}$.

4. Pointwise Products in Lie Groups

Let M be a compact Lie group G with simple Lie algebra $\mathcal{G} = TG_e$, where $e \in G$ denotes the identity.

Definition 4.1. Given a C^∞ curve $y : [t_0, t_1] \to G$ and a vector field Y defined along y, the *left Lie-reduction* of Y is the curve $\tilde{Y} : [t_0, t_1] \to \mathcal{G}$ given by

$$\tilde{Y}(t) \ := \ dL(y(t)^{-1})_{y(t)} Y(t).$$

Here $L(g) : G \to G$ denotes left-multiplication by $g \in G$.

As usual, for fixed $g \in G$, $\mathrm{Ad}(g) : \mathcal{G} \to \mathcal{G}$ is given by the derivative with respect to $h \in G$ of $h \mapsto ghg^{-1}$. Then $\mathrm{Ad}(g)$ is a Lie isomorphism with respect to the Lie bracket $[\ ,\ ]$ on $\mathcal{G}$. For fixed $v \in \mathcal{G}$, $\mathrm{ad}(v) : \mathcal{G} \to \mathcal{G}$ is the Lie endomorphism given by $\mathrm{ad}(v)(w) := [v, w]$.

A negative multiple of the Killing form is an ad-invariant inner product on $\mathcal{G}$, extending by left-translation to a bi-invariant Riemannian metric $\langle\ ,\ \rangle$ on G. Geodesics are left-translates of one-parameter subgroups $\gamma : \mathbb{R} \to G$. Given $V_i \in \mathcal{G}$ for $i = 1, 2$, let γ_i be one-parameter groups defined over some interval $[t_0, t_1]$ containing 0, with $\gamma_i^{(1)}(0) = V_i$. Define $x : [t_0, t_1] \to G$ by $x(t) = \gamma_1(t)\gamma_2(t)$ and let V be the left Lie-reduction of the derivative $x^{(1)}$ of x. A calculation proves

Lemma 4.2. *$V(t) = \mathrm{Ad}(\gamma_2(t)^{-1})(V_1 + V_2)$ and $V^{(j)} = (-1)^j \mathrm{Ad}(\gamma_2^{-1}) \circ \mathrm{ad}(V_2)^j (V_1)$ for $j = 1, 2, \ldots$.* ■

So $\langle x^{(1)}, x^{(1)} \rangle = \langle V_1 + V_2, V_1 + V_2 \rangle$. Set $V_{(0)} := V_1 + V_2$ and $V_{(j)} := \mathrm{Ad}(\gamma_2)\widetilde{\nabla_t^j x^{(1)}}$ for $j = 1, 2, \ldots$. Then, we have

Lemma 4.3. $V_{(j)} = (\frac{1}{2}\mathrm{ad}(V_1 - V_2))^j V_{(0)}$ *for $j = 1, 2, \ldots$.*

Proof. $V_{(j)} = \mathrm{Ad}(\gamma_2)(\frac{d}{dt}(\mathrm{Ad}(\gamma_2^{-1})V_{(j-1)}) + \frac{1}{2}[\mathrm{Ad}(\gamma_2^{-1})V_{(0)}, \mathrm{Ad}(\gamma_2^{-1})V_{(j-1)}]) =$

$$V_{(j-1)}^{(1)} + [V_{(j-1)}, V_2] + \frac{1}{2}[V_1 + V_2, V_{(j-1)}] \ = \ V_{(j-1)}^{(1)} + \frac{1}{2}[V_1 - V_2, V_{(j-1)}]$$

and the lemma follows by induction. ■

A *Cartan subalgebra* of $\mathcal{G}$ is a maximal abelian subalgebra $\mathcal{H}$ such that $\mathrm{ad}(v)$ is diagonalizable over $\mathbb{C}$ for every $v \in \mathcal{H}$. The dimension r of $\mathcal{H}$ is independent of $\mathcal{H}$ and is called the *rank* of $\mathcal{G}$. Given a Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, there is a splitting

$$\mathcal{H}^\perp \otimes \mathbb{C} \cong \oplus_{1 \le k \le m-r} E_k\,,$$

where the E_k are distinct 1-dimensional subspaces of $\mathcal{G} \otimes C$, with *roots* ρ_k in the dual of $\mathcal{H} \otimes \mathbb{C}$, such that $\mathrm{ad}(v)|E_k$ is multiplication by $\rho_k(v)$ for any $v \in \mathcal{H}$. Choose $W_2 \in \mathcal{G}$ with nontrivial projection w_k onto each E_k. Choose $W_1 \in \mathcal{H}$ with the $\rho_k(W_1)$ distinct and nonzero. Then

Theorem 4.4. *Let G be a simple compact Lie Group. For given $V_i \in \mathcal{G}$ for $i = 1,2$, let γ_i be one-parameter subgroups defined over some interval $[t_0, t_1]$ containing 0, with $\gamma_i^{(1)}(0) = V_i$. Define its pointwise product $x : [t_0, t_1] \to G$ as above, namely $x(t) = \gamma_1(t)\gamma_2(t)$. Then*

i) x is a helix, and $0 \le \mathrm{rank}(x) \le m - r$, where r is the rank of G.

ii) If $V_1 = W_2 - W_1$ and $V_2 = W_2 + W_1$, then $\mathrm{rank}(x) = m - r$.

Proof. From Lemma 4.3 one has that the $V_{(j)}$ are constant and $V_{(1)} = [V_1, V_2]$. So $\langle \nabla_t x^{(1)}, \nabla_t x^{(1)} \rangle = \langle [V_1, V_2], [V_1, V_2] \rangle$. Suppose from now on that

$$\langle V_1 + V_2, V_1 + V_2 \rangle = 1.$$

Then $\kappa_1^2 = \langle [V_1, V_2], [V_1, V_2] \rangle$, and the higher squared curvatures are also constant. So x is a helix, and $\mathrm{rank}(x)$ is the largest integer d for which

$$\{V_{(j)} = \mathrm{ad}(V_1 - V_2)^j (V_1 + V_2) : 0 \le j \le d\}$$

is linearly independent. Because G is compact, every element of $\mathcal{G}$ is contained in some Cartan subalgebra. Choose a Cartan subalgebra $\mathcal{H}$ containing $V_1 - V_2$. For $j \ge 1$ the $V_{(j)}$ lie in $\mathcal{H}^\perp$ which has dimension $m - r$. This proves $i)$.

To prove $ii)$ it suffices to show that $V_{(0)}, V_{(1)}, V_{(2)}, \ldots, V_{(m-r)}$ are linearly independent. For $j \ge 1$, $V_{(j)} = 2(-1)^j \mathrm{ad}(W_1)^j W_2$, and so we must show that the $\mathrm{ad}(W_1)^j W_2$ are linearly independent for $j = 0, 1, 2, \ldots, m - r$. Writing $W_2 = \sum_{1 \le k \le m-r} w_k$,

$$\mathrm{ad}(W_1)^j W_2 = \sum_{1 \le k \le m-r} \rho_k (W_1)^j w_k.$$

Then linear independence follows on computing a Vandermonde determinant. ∎

Next we characterize the pointwise products in G that are elastic.

Theorem 4.5. *Assume that G is a simple compact Lie Group. Then, a pointwise product x is an elastic curve if and only if*

$$[[[V_1, V_2], V_2] - \tau V_2, V_1] = \mathbf{0}. \tag{4}$$

where the tension parameter τ is given by $\tau = -\frac{3}{2}\kappa_1^2 - \tilde{b}$.

Proof. By Lemma 4.2,

$$\begin{aligned} V^{(1)} &= \mathrm{Ad}(\gamma_2^{-1})[V_1, V_2], \\ V^{(2)} &= \mathrm{Ad}(\gamma_2^{-1})[[V_1, V_2], V_2], \\ [V^{(2)}, V] &= \mathrm{Ad}(\gamma_2^{-1})[[[V_1, V_2], V_2], V_1 + V_2]], \\ V^{(3)} &= \mathrm{Ad}(\gamma_2^{-1})[[[V_1, V_2], V_2], V_2], V_2], \\ V^{(3)} - [V^{(2)}, V] &= \mathrm{Ad}(\gamma_2^{-1})[V_1, [[V_1, V_2], V_2]]. \end{aligned}$$

Proceeding as in [25], one has that x is an elastic curve precisely when $V^{(3)} - [V^{(2)}, V] = \tau V^{(1)}$, which is seen to be equivalent to condition (4). ∎

Closed elastic curves are of special geometric significance. Observe that elastic helices given in Theorem 2 are closed precisely when the norm of V_1 is a rational multiple of the norm of V_2.

5. Elastic Pointwise Products in $G = \mathbb{SU}(2)$

Any elastic curve in a sphere of arbitrary dimension $\mathbb{S}^n$ lies in some 3-sphere totally geodesically embedded in $\mathbb{S}^n$, and elastic helices in $\mathbb{S}^3$ are well determined [1], [2]. Hence the most we can hope for when $G = \mathbb{S}^3 \cong \mathbb{SU}(2)$ is to compare Theorems 4.4, 4.5 with previously known results. It is a different story in §6, §7, §8, where Theorems 4.4, 4.5 give new results on elastica in spaces of nonconstant curvature.

For $G = \mathbb{S}^3$, $\mathcal{G}$ is the space $\mathbb{P}$ of pure imaginary quaternions. Define a bi-invariant inner product $\langle\ ,\ \rangle$ on $\mathcal{G}$ by

$$\langle v, w\rangle \ := \ \Re(v\bar{w}) \ = \ -\Re(vw)\,,$$

where the right hand side is the real part of the quaternionic product $v\bar{w}$ and $\bar{w}$ denotes the quaternionic conjugate of w. The Lie bracket on $\mathcal{G}$ is given by $[v,w] = vw - wv = 2\Im(vw) = 2v \times w$, where $\Im$ denotes the pure imaginary part, and $\times$ is the cross-product in $E^3 \cong \mathbb{P}$.

Proposition 5.1. *Let x be a pointwise product of two one-parameter subgroups of $\mathbb{S}^3$ with infinitesimal generators V_1, V_2 where $\|V_1 + V_2\| = 1$. Then x is a helix whose curvatures are given by*

$$\begin{aligned}\kappa_1 &= 2\|V_1 \times V_2\|\,,\\ \kappa_2 &= \sqrt{1 - 4\|V_1 \times V_2\|^2 - 4\langle V_1, V_2\rangle}\,.\end{aligned}$$

Proof. For any C^∞ curve $x : [t_0, t_1] \to \mathbb{S}^3$, the left Lie reduction $V : [t_0, t_1] \to \mathbb{P}$ of $x^{(1)}$ is defined by $x^{(1)}(t) = x(t)V(t)$. Then $\langle x^{(1)}, x\rangle = \langle V, \mathbf{1}\rangle = 0$. Assume x has unit speed, namely $\langle V, V\rangle \equiv 1$. The derivative of the unit tangent $e_0 := xV$ is $x^{(1)}V + xV^{(1)} = x(V^2 + V^{(1)}) = x(-\mathbf{1} + V^{(1)})$. So $\nabla_t e_0 = xV^{(1)}$, and the squared geodesic curvature is

$$\kappa_1^2 \ = \ \langle V^{(1)}, V^{(1)}\rangle\,.$$

Assuming $\kappa_1 \neq 0$, the principal unit normal $e_1 := \kappa_1^{-1} x V^{(1)}$ has derivative

$$-\frac{\kappa_1^{(1)}}{\kappa_1^2} xV^{(1)} + \kappa_1^{-1} x(VV^{(1)} + V^{(2)}) \implies \nabla_t e_1 \ = \ -\frac{\kappa_1^{(1)}}{\kappa_1} e_1 + \kappa_1^{-1} x(V \times V^{(1)} + V^{(2)})\,.$$

So $\hat{e}_2 =$

$$\kappa_1^{-1}(x(V \times V^{(1)} + V^{(2)}) - \langle e_0, x(V \times V^{(1)} + V^{(2)})\rangle e_0 - \langle e_1, x(V \times V^{(1)} + V^{(2)})\rangle e_1) =$$

$$\kappa_1^{-1}(x(V \times V^{(1)} + V^{(2)}) - \langle V, V^{(2)}\rangle e_0 - \kappa_1^{-1}\langle V^{(1)}, V^{(2)}\rangle e_1) =$$

$$\kappa_1^{-1} x(V \times V^{(1)} + V^{(2)}) + \kappa_1 e_0 - \frac{\kappa_1^{(1)}}{\kappa_1} e_1\,.$$

So $\kappa_2^2 = \|\hat{e}_2\|^2 =$

$$\kappa_1^{-2}\|V\times V^{(1)}+V^{(2)}\|^2+\kappa_1^2+(\frac{\kappa_1^{(1)}}{\kappa_1})^2-2\frac{\kappa_1^{(1)}}{\kappa_1^3}\langle V\times V^{(1)}+V^{(2)},V^{(1)}\rangle+2\langle V\times V^{(1)}+V^{(2)},V\rangle=$$

$$\kappa_1^{-2}\|V\times V^{(1)}+V^{(2)}\|^2-\kappa_1^2-(\frac{\kappa_1^{(1)}}{\kappa_1})^2=\kappa_1^{-2}(\|V^{(2)}\|^2+2\langle V^{(2)},V\times V^{(1)}\rangle)+1-\kappa_1^2-(\frac{\kappa_1^{(1)}}{\kappa_1})^2 .$$

Taking x to be a product of one-parameter subgroups with infinitesimal generators V_1, V_2 where $V_1 \times V_2 \neq \mathbf{0}$ and $\|V_1 + V_2\| = 1$, we have that x is a helix in $\mathbb{S}^3$, namely a nongeodesic C^∞ curve with constant curvatures $\kappa_1 > 0$ and $\kappa_2 \geq 0$ given by

$$\begin{aligned}
\kappa_1 &= 2\|V_1 \times V_2\| ,\\
\kappa_2^2 &= 1-\kappa_1^2+16\kappa_1^{-2}\|(V_1\times V_2)\times V_2\|^2+\\
&\quad 16\kappa_1^{-2}\langle(V_1\times V_2)\times V_2,(V_1+V_2)\times(V_1\times V_2)\rangle\\
&= 1-\kappa_1^2+16\kappa_1^{-2}\|V_1\times V_2\|^2\|V_2\|^2-16\kappa_1^{-2}\langle(V_1\times V_2)\times V_2,(V_1\times V_2)\times(V_1+V_2)\rangle\\
&= 1-\kappa_1^2+16\kappa_1^{-2}\|V_1\times V_2\|^2\|V_2\|^2-16\kappa_1^{-2}\|V_1\times V_2\|^2\langle V_2,V_1+V_2\rangle\\
&= 1-\kappa_1^2-16\kappa_1^{-2}\|V_1\times V_2\|^2\langle V_2,V_1\rangle \;=\; 1-4\|V_1\times V_2\|^2-4\langle V_1,V_2\rangle ,
\end{aligned}$$

using Lemma 4.2. ■

So we obtain the following converse to Theorem 4.4, which provides an algebraic characterization of helices in $\mathbb{S}^3$.

Corollary 5.2. *All helices in $\mathbb{S}^3$ are realised as left-translates of pointwise products x of two one-parameter subgroups.*

Proof. By Remark 2.3, it suffices to show that any $\kappa_1 > 0$ and any $\kappa_2 \geq 0$ are realised by some x. Set $V_1 = (r\cos\theta, r\sin\theta, 0)$ and $V_2 = (1 - r\cos\theta, -r\sin\theta, 0)$ where $r > 0$ and $\theta \in (0, \pi/2]$. Then

$$\begin{aligned}
\kappa_1 &= 2r\sin\theta ,\\
\kappa_2 &= |1-2r\cos\theta| . \qquad ■
\end{aligned}$$

On the other hand, the Euler-Lagrange equations (1) and (2) for an arbitrary elastic curve in $\mathbb{S}^3$ reduce to ([1], [2], [28])

$$\begin{aligned}
2\kappa_1^{(2)} + \kappa_1^3 + 2(1+\tilde{b}-\kappa_2^2)\kappa_1 = 0 ,\\
\kappa_2\kappa_1^2 = c ,
\end{aligned}$$

where c is constant. Any helix of $\mathbb{S}^3$ satisfies these equations for some $\tilde{b}$, and is therefore elastic. As we have just seen in Proposition 5.1, a pointwise-product x in $\mathbb{S}^3$ is a helix; so it must be elastic curve. So, by our Theorem 4.5, condition (4) holds for any choice of linearly independent $V_1, V_2 \in E^3$ with $\|V_1 + V_2\| = 1$, for some τ depending on V_1 and V_2. This is verified, and the precise value of τ is computed, as follows.

Proposition 5.3. *The pointwise product x is a tension-cubic with tension parameter $\tau = 4\langle V_1, V_2\rangle$.*

Proof. Condition (4) reads $4((V_1 \times V_2) \times V_2) \times V_1 = \tau V_2 \times V_1$. Taking inner products with $V_1 \times V_2$,

$$-4\kappa_1\langle V_1, V_2\rangle = -16\langle (V_1 \times V_2) \times V_2, (V_1 \times V_2) \times V_1\rangle = -\tau\kappa_1 \implies \tau = 4\langle V_1, V_2\rangle,$$

since $\kappa_1 \neq 0$. Conversely, if $\tau = 4\langle V_1, V_2\rangle$ then $4(V_1 \times V_2) \times V_2 = \tau V_2 - \langle V_2, V_2\rangle V_1$ and (4) follows. ■

Let $\{I, J, K\}$ be the standard basis for the space $\mathbb{P}$ of pure imaginary quaternions. Given a quaternion $q = q_0\mathbf{1} + q_1 I + q_2 J + q_3 K$ with $q_0, q_1, q_2, q_3 \in \mathbb{R}$, we have the usual quaternionic conjugate $\bar{q} := q_0\mathbf{1} - q_1 I - q_2 J - q_3 K$. On the other hand, set[1] $\tilde{q} := q_0\mathbf{1} - q_1 I + q_2 J + q_3 K$.

Definition 5.4. The *Hopf map* $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is given by

$$\pi(q) := I\tilde{q}q = (1 - 2q_2^2 - 2q_3^2, -2q_0q_3 + 2q_1q_2, 2q_0q_2 + 2q_1q_3),$$

where $q = q_0 + q_1 I + q_2 J + q_3 K$ has unit norm, and $\mathbb{S}^2$ is identified with the unit sphere in $\mathbb{P}$.

This definition of the Hopf map differs from that in [27] where $\pi_P(q) = \tilde{q}q$. Our definition has the advantage of mapping into $\mathbb{P}$, simplifying the proofs of Lemma 5.5 and Proposition 5.6 below. Our definition also differs from that in [3], where $\pi_B = \pi_P/2$ maps into the 2-sphere of radius $1/2$.

A *Hopf cylinder*[2] is the preimage $M_C := \pi^{-1}C$ in $\mathbb{S}^3$ of a curve C in $\mathbb{S}^2$. If C is closed and connected then M_C is compact and is known as a *Hopf torus.* From the proof of [3] Theorem 5, any helix in $\mathbb{S}^3$ is a geodesic in an isometric copy of a Hopf torus with generating curve C a circle in $\mathbb{S}^2$. In this case M_C is isometric to a flat torus with constant mean curvature and therefore to a Clifford torus. So, from Corollary 5.2, pointwise products x of one-parameter subgroups of $\mathbb{S}^3$ are geodesics in an isometric copy of a Hopf torus. Before saying more, we need

Lemma 5.5. *Let $W \in \mathbb{S}^2 \subset \mathbb{P}$. Then, for any $\theta \in \mathbb{R}$,*

$$\pi(\mathbf{1}\cos\theta + W\sin\theta) \in C(W) := \{y \in S^2 : \langle y, \tilde{W}\rangle = \langle I, \tilde{W}\rangle\}.$$

Proof. $y := \pi(\mathbf{1}\cos\theta + W\sin\theta) = I(\mathbf{1}\cos^2\theta + I(\tilde{W} + W)\sin\theta\cos\theta + \tilde{W}W\sin^2\theta)$ and so

$$\langle y, \tilde{W}\rangle = \Re(\tilde{W}I\cos^2\theta - I\tilde{W}(\tilde{W} + W)\sin\theta\cos\theta - I\tilde{W}^2W\sin^2\theta) = \langle \tilde{W}, I\rangle \ ■$$

Writing $r_i := \|V_i\|$ for $i = 1, 2$, we have $r_1^2 + r_2^2 = 1 - 2\langle V_1, V_2\rangle = 1 - \kappa_1$ by Proposition 5.1, and $\gamma_i(t) = \cos(r_i t)\mathbf{1} + U_i\sin(r_i t)$ where $U_i := V_i/r_i \in \mathbb{P}$ are unit

[1] There should be no confusion with our notation for the left Lie reduction of a vector field.
[2] This agrees with the definitions in [3], [27].

vectors. Choose rotations A_L and A_R of $\mathbb{S}^3$ that fix $\mathbf{1}$ such that $A_L I = \pm U_1$ and $A_R I = \pm U_2$, and define $B : \mathbb{S}^3 \to \mathbb{S}^3$ by $B(q) = \bar{q}$.

Proposition 5.6. *Let $x : \mathbb{R} \to \mathbb{S}^3$ be a pointwise product, where $V_1 \times V_2 \neq \mathbf{0}$ and $V_1 + V_2$ has unit length. There are unit vectors W_L and W_R such that, for any t,*

$$x(t) \ \in \ A_L M_{C(W_L)} \cap B A_R M_{C(W_R)} \,.$$

Proof. We have $x(t) = (\cos(r_1 t)\mathbf{1} + U_1 \sin(r_1 t))(\cos(r_2 t)\mathbf{1} + U_2 \sin(r_2 t))$. Then $A_L^{-1} x(t) = (\cos(r_1 t)\mathbf{1} \pm I \sin(r_1 t))(\cos(r_2 t)\mathbf{1} + W_L \sin(r_2 t))$ where $W_L := A_L^{-1} U_2$. From the definition of π, $\pi(A_L^{-1} x(t)) = \pi(\cos(r_2 t)\mathbf{1} + W_L \sin(r_2 t)) \in C(W_L)$ by Lemma 5.5. So $x(t) \in A_L M_{C_L}$ where $C_L := C(W_L)$.

We also have $A_R^{-1} \bar{x}(t) = (\cos(r_2 t)\mathbf{1} \mp I \sin(r_2 t))(\cos(r_1 t)\mathbf{1} + W_1 \sin(r_1 t))$ where $W_R := -A_R^{-1} U_1$. By Lemma 5.5, $\pi(A_R^{-1} \bar{x}(t)) = \pi(\cos(r_1 t)\mathbf{1} + W_1 \sin(r_1 t)) \in C(W_1)$. So $\bar{x}(t) \in A_R M_{C(W_R)}$. ∎

So any helix in $\mathbb{S}^3$ lies in an intersection of isometric copies of two distinct Hopf tori. Indeed, since A_L and A_R are defined only up to sign and up to a choice of a point in a torus $\mathbb{S}^1 \times \mathbb{S}^1$, given any helix there are continuously many such pairs of isometries and Hopf tori.

Remark 5.7. There is a method basically due to Bianchi [5] (see [31] pp. $154 - 163$ for details), which constructs families of flat tori in $\mathbb{S}^3$ (*translation tori*) by multiplying two curves $\gamma_i : \mathbb{R} \to \mathbb{S}^3$, $i \in \{1, 2\}$ satisfying certain conditions. A translation torus

$$M_{\gamma_1 \gamma_2} \ := \ \gamma_1(t)\gamma_2(s)$$

is hardly ever a Hopf torus but, when γ_1 and γ_2 are geodesics, M_{γ_1, γ_2} is isometric to a Clifford torus. Our pointwise products x are included in these translation tori, which are congruent to those mentioned in the above proposition by a rotation of $\mathbb{S}^3$ that maps γ_1 to a geodesic whose J, K coordinates are constant.

6. Elastic pointwise products in $G = \mathbb{SU}(3)$

For an arbitrary simple Lie Group G, the situation changes dramatically: there are large and interesting families of elastic pointwise products of one-parameter subgroups, but most pointwise products are not elastic. We illustrate this by taking $G = \mathbb{SU}(3)$.

Let $\langle \ , \ \rangle$ be a negative multiple of the Killing form on the $m = 8$-dimensional real vector space $\mathcal{G} = su(3)$ of 3×3 zero-trace skew Hermitian complex matrices, spanned by

$$h_1 \ = \ \begin{bmatrix} \mathbf{i} & 0 & 0 \\ 0 & -\mathbf{i} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad h_2 \ = \ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{i} & 0 \\ 0 & 0 & -\mathbf{i} \end{bmatrix}$$

and

$$M_1(z_1):=\begin{bmatrix} 0 & z_1 & 0 \\ -\bar{z}_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ M_3(z_3):=\begin{bmatrix} 0 & 0 & z_3 \\ 0 & 0 & 0 \\ -\bar{z}_3 & 0 & 0 \end{bmatrix},\ M_2(z_2):=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & z_2 \\ 0 & -\bar{z}_2 & 0 \end{bmatrix}$$

where $z_1, z_2, z_3 \in \mathbb{C}$. Then $\langle\ ,\ \rangle$ extends by left-translation to a bi-invariant Riemannian metric on $\mathbb{SU}(3)$. The rank r of $su(3)$ is 2.

For any $V_1, V_2 \in su(3)$, $x = \gamma_1\gamma_2 : [t_0, t_1] \to \mathbb{SU}(3)$ is a helix . Since $m = 8$ and $r = 3$, $\mathrm{rank}(x) \leq 6$ by Theorem 4.4. Also by Theorem 4.4, typically $\mathrm{rank}(x) = 6$. By Theorem 4.5, x is elastic precisely when condition (4) holds. Whereas condition (4) always holds in $su(2)$ for some choice of τ, it need not hold in $su(3)$. We illustrate this in some special cases.

7. Products of One-Parameter Subgroups of Different Copies of $\mathbb{SU}(2)$ in $\mathbb{SU}(3)$

Choose $a_i \in \mathbb{R}$ and $z_i \in \mathbb{C}$ for $i = 1, 2$, and set

$$V_i = a_i h_i + M_i(z_i). \tag{5}$$

Theorem 7.1. *Assuming (5), then condition (4) holds with* $\tau = 2a_1a_2$.

Proof. We have

$$\begin{aligned} [V_1, V_2] &= a_2 M_1(\mathbf{i}z_1) - a_1 M_2(\mathbf{i}z_2) + M_3(z_1 z_2)\,, \\ [[V_1, V_2], V_2] &= -(a_2^2 + |z_2|^2) M_1(z_1) - 2a_1a_2 M_2(z_2) + 2a_1|z_2|^2 h_2\,, \\ [[[V_1, V_2], V_2], V_1] &= 2a_1a_2^2 M_1(\mathbf{i}z_1) - 2a_1^2 a_2 M_2(\mathbf{i}z_2) + 2a_1a_2 M_3(z_1z_2) \\ &= 2a_1a_2[V_1, V_2]\,, \end{aligned}$$

as required. ∎

To investigate curvatures of x under condition (5), consider

$$\begin{aligned} V_1 + V_2 &= a_1h_1 + a_2h_2 + M_1(z_1) + M_2(z_2)\,, \\ [V_1, V_2] &= a_2 M_1(\mathbf{i}z_1) - a_1 M_2(\mathbf{i}z_2) + M_3(z_1z_2)\,, \\ [V_1 - V_2, [V_1, V_2]] &= 2a_2|z_1|^2 h_1 + 2a_1|z_2|^2 h_2 - (2a_1a_2 + a_2^2 + |z_2|^2) M_1(z_1) - \\ &\quad (2a_1a_2 + a_1^2 + |z_1|^2) M_2(z_2)\,, \\ [V_1 - V_2, [V_1 - V_2, [V_1, V_2]]] &= -a_2((2a_1 + a_2)^2 + 4|z_1|^2 + |z_2|^2) M_1(\mathbf{i}z_1) \\ &\quad + a_1((a_1 + 2a_2)^2 + |z_1|^2 + 4|z_2|^2) M_2(\mathbf{i}z_2) \\ &\quad - (a_1^2 + 4a_1a_2 + a_2^2 + |z_1|^2 + |z_2|^2) M_3(z_1z_2)\,. \end{aligned}$$

For most $a_1, a_2 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$, the four vectors listed above are linearly independent, implying $\kappa_j > 0$ for $j = 1, 2, 3$. Considering a further 3 vectors, we usually find $\kappa_j > 0$ for $1 \leq j \leq 6$, namely $\mathrm{rank}(x) = 6$. Exceptions include:

- z_1 or z_2 pure imaginary; then $\mathrm{rank}(x) \leq 4$

- z_1 or z_2 real; then $\operatorname{rank}(x) \leq 4$
- z_1 and z_2 pure imaginary; then $\operatorname{rank}(x) \leq 3$
- z_1 and z_2 real; then $\operatorname{rank}(x) \leq 3$.

This means that the elastica we obtained here are not the trivial examples that we would obtain by immersion in $\mathbb{SU}(3)$ of the elastic curves of $\mathbb{S}^3$ found in §5.

8. Generic Cases

Up to conjugation, for any $V_1 \in su(3)$, and some $a_1, a_2 \in \mathbb{R}$,

$$V_1 \;=\; a_1 h_1 + a_2 h_2 \,. \tag{6}$$

Then V_2 can be written in the form $b_1 h_1 + b_2 h_2 + M_1(z_1) + M_2(z_2) + M_3(z_3)$ where $b_1, b_2 \in \mathbb{R}$ and $z_1, z_2, z_3 \in \mathbb{C}$.

Theorem 8.1. *Assuming (6), then condition (4) holds if and only if*

$$(2a_1-a_2)\,(\alpha_1 \mathbf{i} z_1+\beta_1 \bar{z}_2 z_3) = (a_1+a_2)\,(\alpha_3 \mathbf{i} z_3+\beta_3 z_1 z_2) = (2a_2-a_1)\,(\alpha_2 \mathbf{i} z_2+\beta_2 \bar{z}_1 z_3) = 0\,, \tag{7}$$

where $\beta_1 := 3a_2\,,\ \ \beta_3 := 3(a_2 - a_1)\,,\ \ \beta_2 := -3a_1$ *and*

$\alpha_1 := (2a_1-a_2)(2b_1-b_2)+\tau,\ \alpha_3 := (a_1+a_2)(b_1+b_2)+\tau,\ \alpha_2 \;:=\; (2a_2-a_1)(2b_2-b_1)+\tau.$

Proof. After a long computation we find

$$\begin{aligned}
[V_1, V_2] &= (2a_1 - a_2)M_1(\mathbf{i}z_1) + (2a_2 - a_1)M_2(\mathbf{i}z_2) + (a_1 + a_2)M_3(\mathbf{i}z_3)\,,\\
[[V_1, V_2], V_2] &= -2\mathbf{i}((2a_1 - a_2)|z_1|^2 + (a_1 + a_2)|z_3|^2)h_1 - 2\mathbf{i}((2a_2 - a_1)|z_2|^2 + \\
&\quad (a_1 + a_2)|z_3|^2)h_2 + M_1((2a_1 - a_2)(2b_1 - b_2)z_1 - 3a_2\mathbf{i}\bar{z}_2 z_3) + \\
&\quad M_2((2a_2 - a_1)(2b_2 - b_1)z_2 + 3a_1\mathbf{i}\bar{z}_1 z_3)\\
&+ M_3((a_1 + a_2)(b_1 + b_2)z_3 + 3(a_1 - a_2)\mathbf{i}z_1 z_2)\,,\\
[[[V_1, V_2], V_2], V_1] &= -(2a_1 - a_2)M_1((2a_1 - a_2)(2b_1 - b_2)\mathbf{i}z_1 + 3a_2\bar{z}_2 z_3) + \\
&\quad -(2a_2 - a_1)M_2((2a_2 - a_1)(2b_2 - b_1)\mathbf{i}z_2 - 3a_1\bar{z}_1 z_3) + \\
&\quad -(a_1 + a_2)M_3((a_1 + a_2)(b_1 + b_2)\mathbf{i}z_3 + 3(a_2 - a_1)z_1 z_2)\,.
\end{aligned}$$

So (4) is equivalent to

$$\begin{aligned}
(2a_1 - a_2)(((2a_1 - a_2)(2b_1 - b_2) + \tau)\mathbf{i}z_1 + 3a_2\bar{z}_2 z_3) &= 0\,,\\
(2a_2 - a_1)(((2a_2 - a_1)(2b_2 - b_1) + \tau)\mathbf{i}z_2 - 3a_1\bar{z}_1 z_3) &= 0\,,\\
(a_1 + a_2)(((a_1 + a_2)(b_1 + b_2) + \tau)\mathbf{i}z_3 + 3(a_2 - a_1)z_1 z_2) &= 0\,. \quad \blacksquare
\end{aligned}$$

It follows that a pointwise product x is usually not elastic, but we show next that there are large families of examples where x is elastic, in addition to the families given by Theorem 7.1.

For the remainder of this section, let V_1 lie in a unique Cartan subalgebra of $su(3)$, namely suppose that $2a_1 - a_2$, $a_1 + a_2$ and $2a_2 - a_1$ are all nonzero. By Theorem 8.1, condition (4) is equivalent to

$$\begin{aligned} \alpha_1 \mathbf{i} z_1 + \beta_1 \bar{z}_2 z_3 &= 0\,, \\ \alpha_2 \mathbf{i} z_2 + \beta_2 \bar{z}_1 z_3 &= 0\,, \\ \alpha_3 \mathbf{i} z_3 + \beta_3 z_1 z_2 &= 0\,. \end{aligned}$$

Suppose also that the $\alpha_j \in \mathbb{R}$ and $z_j \in \mathbb{C}$ are all nonzero for $j = 1, 2, 3$. Then the β_j are also nonzero, and

$$\theta_3 = \theta_1 + \theta_2 + \varepsilon \frac{\pi}{2} + 2n\pi\,, \tag{8}$$

where $\varepsilon = \pm 1$, n is an integer, and $\theta_j \in [0, 2\pi)$ is the argument of z_j. The moduli r_j satisfy

$$\alpha_1 + \varepsilon \frac{r_2 r_3}{r_1} \beta_1 = \alpha_2 + \varepsilon \frac{r_1 r_3}{r_2} \beta_2 = \alpha_3 - \varepsilon \frac{r_1 r_2}{r_3} \beta_3 = 0\,,$$

which can be written as the following linear system for b_1, b_2 and τ

$$\begin{bmatrix} 4a_1 - 2a_2 & -2a_1 + a_2 & 1 \\ -2a_2 + a_1 & 4a_2 - 2a_1 & 1 \\ a_1 + a_2 & a_1 + a_2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \tau \end{bmatrix} = 3\varepsilon \begin{bmatrix} \frac{-a_2 r_2 r_3}{r_1} \\ \frac{a_1 r_1 r_3}{r_2} \\ \frac{(a_2 - a_1) r_1 r_2}{r_3} \end{bmatrix}$$

with coefficients in terms of a_1, a_2, r_1, r_2, r_3 and ε. If the determinant $-9(a_1^2 - a_1 a_2 + a_2^2)$ is nonzero,

$$\begin{bmatrix} b_1 \\ b_2 \\ \tau \end{bmatrix} = \frac{\varepsilon}{(a_1^2 - a_1 a_2 + a_2^2)} \times \tag{9}$$

$$\begin{bmatrix} \frac{a_2(a_2 - a_1) r_1 r_2}{r_3} - \frac{a_1^2 r_1 r_3}{r_2} + \frac{a_2(a_2 - a_1) r_2 r_3}{r_1} \\ \frac{a_1(a_2 - a_1) r_1 r_2}{r_3} + \frac{a_1(a_2 - a_1) r_1 r_3}{r_2} + \frac{a_2^2 r_2 r_3}{r_1} \\ \frac{(a_2 - a_1)(2a_2 - a_1)(a_2 - 2a_1) r_1 r_2}{r_3} - \frac{a_1(a_1 + a_2)(a_2 - 2a_1) r_1 r_3}{r_2} - \frac{a_2(a_1 + a_2)(2a_2 - a_1) r_2 r_3}{r_1} \end{bmatrix}. \tag{9}$$

So there is a 7-parameter family of pairs V_1, V_2 satisfying condition (4), which can be found as follows.

i) Choose $r_1, r_2, r_3 > 0$, $\varepsilon = \pm 1$, and a_1, a_2 so that none of $2a_1 - a_2$, $2a_2 - a_1$, $a_1 + a_2$, $a_1^2 - a_1 a_2 + a_2^2$ is zero.

ii) Find b_1, b_2, τ from (9).

iii) Choose $\theta_1, \theta_2 \in [0, 2\pi)$ and an integer n, and find θ_3 from (8).

iv) Finally set $V_1 = a_1 h_1 + a_2 h_2$ and $V_2 = b_1 h_1 + b_2 h_2 + M_1(z_1) + M_2(z_2) + M_3(z_3)$.

Example 8.2. Choosing $r_1 = 1.5$, $r_2 = 2.5$, $r_3 = 3.5$, $\varepsilon = -1$, $a_1 = 1.1$, $a_2 = 2.0$, $\theta_1 = \pi/4$, $\theta_2 = \pi/3$, $n = 1$ gives an elastic helix x of rank 6. If a_1 is replaced by 1.0 we still get an elastic helix, but the rank drops to 2.

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Óscar J. Garay
Department of Mathematics
University of the Basque Country
UPV/EHU, Bilbao, Spain
oscarj.garay@ehu.es

Lyle Noakes
School of Mathematics and Statistics
The University of Western Australia
35 Stirling Hwy
Crawley WA 6009, Australia
Lyle.Noakes@uwa.edu.au

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