Ample Parabolic Subalgebras

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Abstract.   Let (L, L_0) be a finite-dimensional transitive pair of Lie algebras. We call the subalgebra L_0 *ample nonlinear* in L if its linear isotropy representation on L/L_0 admits a nontrivial kernel L_1, and the normalizer $N_L(L_1)$ of that kernel is identical to L_0. For semisimple Lie algebras L over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we classify in this paper the ample nonlinear subalgebras L_0. These subalgebras are exactly the *ample parabolic subalgebras* of L.
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1.   Introduction

Let G/G_0 be a (connected) homogeneous space with an almost effective G-action, i.e., G is a Lie group and G_0 is a closed subgroup, which contains only discrete normal subgroups of G. The corresponding Lie algebras are denoted by L and L_0, respectively. The homogeneous space G/G_0 is said to be of first-order if G_0 is not discrete, but the kernel G_1 of the isotropy representation of G_0 on the tangent space $T_{eG_0}(G/G_0) = L/L_0$ is discrete. If the action of the isotropy group G_0 is irreducible, i.e., there exists no nontrivial G_0-invariant proper subspace of the tangent space L/L_0, the homogeneous space G/G_0 is called irreducible. The *irreducible first-order* homogeneous spaces are classified, including all of the irreducible affine symmetric spaces (cf. [Be57]).

The complete list of irreducible homogeneous spaces G/G_0 of higher order over the reals is also well known. The list is due to Kobayashi and Nagano [KN65] and is rather short. Being of higher order (i.e. greater than one) means for a homogeneous space that the isotropy kernel G_1 is no longer discrete. In fact, the classification shows that G is necessarily a simple Lie group and the order of G/G_0 is exactly two (i.e. G/G_1 is a first-order homogeneous space). The *irreducible second-order homogeneous spaces* correspond to simple $|1|$-graded Lie algebras. Supposedly, the most prominent examples of the list are the real projective space $\mathbb{R}P^n = SL(n+1)/G_0$ and the Möbius sphere $S^n = SO(1, n+1)/G_0$ of conformal geometry. In both these cases the kernel L_1 of the isotropy algebra

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L_0 is isomorphic to the Abelian Lie algebra $\mathbb{R}^{n*}$. The complete list is relevant for the classification of the irreducible nonmetric holonomy groups of torsion-free affine connections on smooth manifolds (cf. [Br96]).

In [Oc66] T. Ochiai classifies all the *nonlinear primitive* subalgebras L_0 for an arbitrary finite-dimensional Lie algebra L over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Here *nonlinearity* means that the isotropy kernel L_1 is nontrivial; *primitive* subalgebras are maximal subalgebras of L. The first step of this classifications shows that L is necessarily simple over $\mathbb{K}$. Then it is shown that any nonlinear primitive subalgebra L_0 in a simple L is a parabolic subalgebra, i.e., L_0 contains a Borel subalgebra in the complex case. In fact, any parabolic subalgebra is nonlinear. Via the description of standard parabolic subalgebras by simple roots of L, all the nonlinear primitive subalgebras L_0 can be classified, explicitly. Of course, the underlying isotropy algebra L_0 of any irreducible second-order homogeneous spaces G/G_0 belongs to this class. However, maximal parabolic subalgebras also give rise to simple $|k|$-gradings of L with $k \geq 2$, i.e., L/L_0 is L_0-reducible, in general.

A more general result for the case of primitive group actions is known. Let G/G_0 be an effective homogeneous space with *primitive G-action*, i.e., there exists no completely integrable differential system on G/G_0, which is G-invariant (cf. [Go72]). Note that, for a primitive G-action the isotropy subalgebra L_0 in L need not be maximal (i.e. primitive in the above sense). However, if the isotropy representation of L_0 on L/L_0 is nonlinear and reducible, i.e., there is a (minimal) G-invariant differential system D on G/G_0, which is, of course, not integrable, then L is again a simple Lie algebra and the isotropy L_0 is a maximal parabolic subalgebra of L. In particular, L_0 gives rise to a simple $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with $k \geq 2$ and the grading component $\mathfrak{g}_{-1}$ corresponds to the differential system D in the tangent bundle (cf. [Ya93]).

A nonlinear subalgebra L_0 of L is called *ample* if the normalizer $N_L(L_1)$ of its kernel L_1 is identical to L_0. This notion is introduced in the Appendix of [Oc66] on page 321. However, it seems that this Appendix does not clarify the structure of the ample nonlinear subalgebras of (semi)simple Lie algebras. For *obvious* reasons, any maximal parabolic subalgebra P_0 of a simple L is ample nonlinear. In this paper we show that there exist ample nonlinear subalgebras of simple Lie algebras, which are not maximal. However, they are always parabolic subalgebras. In fact, we give a complete classification of *ample parabolic subalgebras* of semisimple Lie algebras over $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Note that the existence of an ample nonlinear subalgebra of a Lie algebra L does not imply that L is (semi)simple.

We proceed in this paper as follows. In Section 2 we recall basic observations about transitive Lie algebras from [Oc66]. In particular, we introduce the notion of *ample nonlinearity* for a subalgebra (cf. Definition 2.2). In Section 3 we recall basic elements of the structure theory for parabolic subalgebras and $|k|$-gradings of complex semisimple Lie algebras. In Section 4 and 5 we discuss kernels of isotropy algebras in complex simple Lie algebras. The main observation is that any kernel is the highest component of some simple $|k|$-grading (cf. Proposition 5.5). To prove this basic fact, we apply a classical result due to Dynkin, which says that any nonsemisimple Lie subalgebra of a semisimple Lie algebra is a *R-subalgebra*. In Section 6 we study case by case the root systems of complex simple Lie algebras in

order to obtain the complete list of complex ample parabolic subalgebras. Section 7 provides the structure results of ample nonlinear subalgebras and isotropy kernels for complex semisimple Lie algebras (Theorem 7.1, 7.2 and 7.4). Finally, in Section 8, we discuss the ample nonlinear subalgebras of real semisimple Lie algebras. The classification is based on the complex case and structure results for real simple Lie algebras and their standard parabolic subalgebras by dint of Satake diagrams. The classification is given in Theorem 8.2 and Table 1.

2. Nonlinearity and kernels

Let $L \neq \{0\}$ be a finite-dimensional Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with Lie bracket $[\cdot, \cdot]$. We say that a subalgebra $L_0 \leq L$ is effective and (L, L_0) is a *transitive pair* if L_0 contains no nonzero ideal of L. Then $L_0 \lneq L$ is a proper subalgebra and we refer to such an L_0 as *isotropy subalgebra* of L. In case $L_0 \neq \{0\}$ the *derived kernel* of L_0 is

$$L_1 \;=\; \{t \in L_0 |\ [t, L] \subseteq L_0\},$$

which is a proper subalgebra of L_0. More generally, we define

$$L_p \;:=\; \{t \in L_{p-1} |\ [t, L] \subseteq L_{p-1}\}$$

for any $p \geq 1$. In this way, we obtain a sequence of Lie algebras, and since L_p is proper in L_{p-1} for any $p \geq 1$ as long as $L_{p-1} \neq \{0\}$, there exists a minimal $q \in \mathbb{N}_0$ such that $L_q = \{0\}$ and $L_{q-1} \neq \{0\}$ (with convention $L_{-1} = L$). Thus, the transitive pair (L, L_0) defines naturally a sequence

$$L = L_{-1} \geq L_0 \geq \cdots \geq L_{q-1} \geq L_q = \{0\} \tag{1}$$

of subalgebras in L. The following Lemma, which is not difficult to prove, says that (1) is a *filtered Lie algebra*.

Lemma 2.1.   *([Oc66]) $[L_a, L_b] \subseteq L_{a+b}$ for any $b \geq 0$ and $a \in \mathbb{N}_0 \cup \{-1\}$.*

Throughout the text we use the following notions.

Definition 2.2.   (a) Let (L, L_0) be a transitive pair.

1. If $L_1 \neq \{0\}$, then we call L_0 nonlinear in L with (derived) kernel L_1.

2. If L_0 is identical to the normalizer $N_L(L_1)$ of L_1 in L, then we call L_0 an ample nonlinear subalgebra of L.

(b) Let $K \neq \{0\}$ be a subset of L. If there exists an effective $L_0 \leq L$, whose kernel L_1 is K, then we refer to K as an (isotropy) kernel of L.

By construction, L_1 is an ideal of L_0 and $[L, L_1] \subseteq L_0$. This means that any kernel K is a subalgebra in L and satisfies

$$[[L, K], K] \subseteq K \ .$$

In particular, we see that $[L, K]$ is contained in the normalizer $N_L(K)$ of the kernel K in L. Note that $[L, K]$ is a subalgebra of L for any kernel K in L. Obviously, any nonlinear subalgebra $L_0 \leq L$ with kernel $L_1 = K$ is contained in the normalizer $N_L(K)$. Prominent examples for nonlinear subalgebras are all the parabolic subalgebras of simple Lie algebras (cf. Section 4).

The following *monotonicity* Lemma for kernels is a simple, but useful observation.

Lemma 2.3. *Let (L, L_0) be a transitive pair. If L_0 contains a nonlinear subalgebra L_0' of L, then L_0 is nonlinear and $L_1' \leq L_1$.*

If L_0 is nonlinear and a maximal subalgebra of L (i.e. the only subalgebra of L, which properly contains L_0, is L itself), the pair (L, L_0) is called *nonlinear primitive*. Any nonlinear primitive L_0 is ample in L. These subalgebras are well known and classified.

Theorem 2.4. *([Oc66]) A transitive pair (L, L_0) of Lie algebras over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ is nonlinear primitive if and only if*

1. *L is a (noncompact) simple Lie algebra and*

2. *L_0 is a maximal parabolic subalgebra of L.*

Roughly speaking, the proof of this result has two ingredients. First, it is shown that any Lie algebra L, which contains a nonlinear primitive subalgebra L_0, is simple. Then, it is also clear that the kernel L_1 is Abelian. The simplicity of L does not follow in general from the existence of some nonlinear subalgebra. However, the proof that L_1 is Abelian works for any nonlinear subalgebra of a simple Lie algebra. Hence, the term L_2 of any isotropy filtration (1) of a simple Lie algebra is trivial. This fact also follows from our discussions in Section 4.

The second statement of Theorem 2.4 about parabolic subalgebras is based on the following classical observations. In the complex case, it is due to Morozov that any maximal nonsemisimple subalgebra of a semisimple Lie algebra L is parabolic (cf. [OV94]). Similarly, for real semisimple Lie algebras, Mostow shows in [Mo61] that any maximal nonsemisimple subalgebra with noncompact radical is parabolic.

The above result by T. Ochiai poses an obvious question. Which subalgebras of a simple Lie algebra are ample nonlinear (and is it true that any ample nonlinear subalgebra of a semisimple Lie algebra is parabolic; cf. Appendix of [Oc66])? We give in this paper an answer to this question. Note that the proofs for the results of Morozov and Mostow rely strongly on the assumption that L_0 is primitive (in the sense of maximal).

3.   Parabolic subalgebras

We briefly recall here basic notions from the structure theory of complex semisimple Lie algebras. In fact, we mainly discuss parabolic subalgebras and their description

via root systems (cf. e.g. [OV94, CS09]).

Let L be a semisimple Lie algebra over $\mathbb{C}$. A maximal solvable subalgebra of L is called a *Borel subalgebra*. Borel subalgebras can be described in the following way. The choice of a Cartan subalgebra $\mathfrak{h} \leq L$ gives rise to a set of roots Δ and L is the direct sum of $\mathfrak{h}$ and its root spaces,

$$L = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \ .$$

If we choose a set of positive roots Δ^+ in Δ, then

$$B_0(\mathfrak{h}, \Delta^+) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

is a maximal solvable subalgebra of L. This subalgebra is called the standard Borel subalgebra of L to the data $(\mathfrak{h}, \Delta^+)$. It is well known that any two Borel subalgebras are conjugated in L with respect to the automorphism group $Aut(L)$ of the Lie algebra L. In particular, for any given Borel subalgebra B_0 of L, there exists a choice $(\mathfrak{h}, \Delta^+)$ such that B_0 is standard.

A subalgebra P_0 of L which contains some Borel subalgebra B_0 is called a *parabolic subalgebra* of L. (We consider a parabolic subalgebra to be proper in L.) Parabolic subalgebras can be nicely described by root spaces as well. In fact, choose $(\mathfrak{h}, \Delta^+)$ such that B_0 is standard and let $\Delta^0 = \{\alpha_1, \ldots, \alpha_\ell\}$ be the system of simple roots in Δ^+. Then there exists a (nonempty) subset $\Sigma \subseteq \Delta^0$ such that P_0 is given by

$$P_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\beta \in \Pi} \mathfrak{g}_\beta,$$

where Π is the set of negative roots in Δ, which can be written as linear combinations of elements of $\Delta^0 \smallsetminus \Sigma$. We denote this parabolic subalgebra in standard form by $P_0(\Sigma)$. In particular, the standard maximal parabolic subalgebras $P_0(\alpha)$ are given by the choice of a single simple root $\Sigma = \{\alpha\}$.

It is also well known that there exists a correspondence between parabolic subalgebras and $|k|$-*gradings* of L. Recall that a $|k|$-grading of L with $k > 0$ is a family of subspaces $\mathfrak{g}_i \subseteq L$, $i = -k, \ldots, k$, such that

1. $\mathfrak{g}_{-k} \neq \{0\}$ and $\mathfrak{g}_k \neq \{0\}$,

2. $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is a direct sum,

3. $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ for all $i, j \in \mathbb{Z}$ (with the convention $\mathfrak{g}_i = \{0\}$ for $|i| > k$), and

4. $N = \bigoplus_{i=1}^{k} \mathfrak{g}_i$ is generated as Lie algebra by $\mathfrak{g}_1$.

The correspondence works as follows. If $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is a $|k|$-grading of L then $P_0 = \mathfrak{g}_0 \oplus N$ is a parabolic subalgebra. On the other hand, let $P_0 \leq L$ be a parabolic subalgebra. Then there exists a $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $P_0 = \mathfrak{g}_0 \oplus N$ (and N is the nilradical of P_0). Such a $|k|$-grading can be

constructed as follows. Choose $(\mathfrak{h}, \Delta^+, \Sigma)$ such that $P_0 = P_0(\Sigma)$ is standard. The Σ-*height* $ht_\Sigma(\alpha)$ of a root $\alpha = \Sigma_{i=1}^\ell a_i \alpha_i$ is defined by

$$ht_\Sigma(\alpha) := \sum_{i:\alpha_i \in \Sigma} a_i .$$

Then we set $\mathfrak{g}_i(\Sigma) := \bigoplus_{\alpha:ht_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$ for $i \in \mathbb{Z} \smallsetminus \{0\}$ and $\mathfrak{g}_0(\Sigma) := \mathfrak{h} \oplus \bigoplus_{\alpha:ht_\Sigma(\alpha)=0} \mathfrak{g}_\alpha$. These subspaces define a $|k|$-grading

$$L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$$

of L and $P_0 = \bigoplus_{i=0}^k \mathfrak{g}_i$. The number $k \in \mathbb{N}$ is the Σ-height of the highest root.

Note that the family of subspaces $\mathfrak{g}_i \subseteq L$, $i = -k, \ldots, k$, of this construction is not unique for a given P_0. This depends on the choice of $\mathfrak{h}$. However, the nilradical N and the last nontrivial term of its lower central series, which is the highest grading component $\mathfrak{g}_k$, do not depend on this choice. Also note that the 0-component is reductive, i.e., $\mathfrak{g}_0$ is some semisimple factor $\mathfrak{g}_0^s$ plus a center $\mathfrak{c}$.

4.   Kernels in simple Lie algebras

In this and the following Section, we discuss properties of kernels, which clear the way for a classification in (semi)simple Lie algebras.

Let L denote a complex simple Lie algebra and let $P_0 \leq L$ be a (proper) parabolic subalgebra. We have $P_0 = \bigoplus_{i=0}^k \mathfrak{g}_i$ for some $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with $k > 0$. Obviously, the highest component $\mathfrak{g}_k$ is contained in the kernel P_1 of P_0. This shows that any parabolic subalgebra is nonlinear. In fact, $\mathfrak{g}_k$ is the kernel of P_0 as the following Lemma shows.

Lemma 4.1.   *Let $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ be a simple graded Lie algebra and $P_0 = \bigoplus_{i=0}^k \mathfrak{g}_i$ the corresponding parabolic subalgebra.*

1. *The kernel P_1 of P_0 is identical to the highest component $\mathfrak{g}_k$ of the grading.*

2. *The kernel $P_1 = \mathfrak{g}_k$ is an irreducible P_0-module.*

Proof.   (1) We set $S_{-k} := \mathfrak{g}_{-k}$ and, inductively, $S_i := [S_{i-1}, \mathfrak{g}_1]$ for $i > -k$. The direct sum $\hat{S} := \bigoplus_{i=-k}^k S_i \neq \{0\}$ is by construction $\mathfrak{g}_1$- and $\mathfrak{g}_0$-invariant. Moreover, since $[\mathfrak{g}_{-1}, S_{-k}] = 0$, the subspace $\hat{S}$ is $\mathfrak{g}_{-1}$-invariant as well. Since $\mathfrak{g}_{-1}, \mathfrak{g}_0$ and $\mathfrak{g}_1$ span all of L as Lie algebra (through the bracket), this shows that $\hat{S}$ is an ideal in L. Hence, $\hat{S} = L$, which proves $[\mathfrak{g}_{-p-1}, \mathfrak{g}_1] = \mathfrak{g}_{-p}$ for any $0 \leq p \leq k-1$. (For $p < 0$ this property is true by definition.)

Let $X_p \in \mathfrak{g}_p$, $0 \leq p \leq k-1$, be an element such that $[X_p, \mathfrak{g}_{-p-1}] = 0$. For the Killing form of L, we have $0 = B([X_p, \mathfrak{g}_{-p-1}], \mathfrak{g}_1) = B(X_p, \mathfrak{g}_{-p})$. This shows that X_p must be trivial, i.e., for any nontrivial $X_p \neq 0$ there exists $Y_{-p-1} \in \mathfrak{g}_{-p-1}$ such that $[X_p, Y_{-p-1}] \in \mathfrak{g}_{-1}$ does not vanish (cf. [Ta79]).

Now, if $X = \sum_{i=0}^k X_i \in P_0$ is not contained in $\mathfrak{g}_k$, we can choose for a nontrivial term X_p, $p < k$, some $Y_{-p-1} \in \mathfrak{g}_{-p-1}$ with $[Y_{-p-1}, X_p] \neq 0$. Then the

(-1)-component of $[Y_{-p-1}, X]$ does not vanish, i.e., X is not in the kernel of P_0. This proves $\mathfrak{g}_k \supseteq P_1$.

(2) Let us assume that $S \subseteq \mathfrak{g}_k$ is a nontrivial $\mathfrak{g}_0$-invariant subspace, i.e., $[\mathfrak{g}_0, S] \subseteq S$. We define

$$\hat{S} := S \oplus [\mathfrak{g}_{-1}, S] \oplus \cdots \oplus [\mathfrak{g}_{-k}, S] .$$

Again, this subspace is an ideal of L. Hence, $\hat{S} = L$, which is only possible, if $S = \mathfrak{g}_k$. This shows that $\mathfrak{g}_k$ is $\mathfrak{g}_0$-irreducible, hence P_0-irreducible. ∎

Let $P_1 = \mathfrak{g}_k$ be the kernel of a parabolic subalgebra P_0 in a complex simple Lie algebra L. The normalizer $A_0 = N_L(\mathfrak{g}_k)$ contains P_0, hence is a parabolic subalgebra of L as well. In general, we have $P_0 \lneq A_0$. However, we show now that the kernels of these two nonlinear subalgebras of L coincide. This says that A_0 is an *ample parabolic subalgebra* of L. In fact, this statement holds true if we only assume that the normalizer $A_0 = N_L(K)$ of some kernel K is a parabolic subalgebra.

Lemma 4.2. *Let $K \leq L$ be a kernel. If the normalizer $N_L(K)$ contains a maximal solvable subalgebra of L, then $N_L(K)$ is an ample parabolic subalgebra of L and its kernel is K.*

Proof. We assume that $A_0 := N_L(K)$ is a parabolic subalgebra of L. Then $A_0 = \bigoplus_{i=0}^{k} \mathfrak{g}_i$ for some $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ and $A_1 = \mathfrak{g}_k$. Since K is a kernel, there exists some nonlinear subalgebra L_0 in L with kernel $L_1 = K$. Of course, any such L_0 is contained in the normalizer A_0. Thus, by monotonicity, we have $K \leq A_1 = \mathfrak{g}_k$. However, $\mathfrak{g}_k$ is A_0-irreducible and K is A_0-invariant. This shows $A_1 = K$. ∎

A priori it is not clear whether the normalizer $N_L(K)$ of a kernel K in a complex simple Lie algebra is always a parabolic subalgebra. In view of Lemma 4.2, if we knew that this is true, a complete classification of kernels in L is achieved by computing the ample parabolic subalgebras and their kernels. In fact, we will go this way for a classification. However, we have not found a simple argument, which says that $N_L(K)$ always contains a maximal solvable subalgebra.

To explain this, recall that by definition K is a kernel in a simple Lie algebra if and only if the minimal isotropy subalgebra $[L, K]$ of K is contained in the normalizer (or maximal isotropy) $N_L(K)$ of K. The difficulty is that the minimal isotropy $[L, K]$ of a kernel K is in general not a parabolic subalgebra. In fact, the examples of the final classification confirm that neither does $[L, K]$ contain a Cartan subalgebra nor does it contain all positive root spaces (for any choice of a positive root system), in general. This makes it necessary to argue that the normalizer of a kernel K is in general (much) *bigger* then the minimal isotropy $[L, K]$. The definition alone does not yield this, directly. We will close this gap between the minimal and maximal isotropy at the end of the next Section.

For the time being, we introduce the following weaker observation about kernels for further use. Let $K \leq L$ be an arbitrary kernel, i.e., there exists a nonlinear subalgebra L_0 in the simple Lie algebra L with kernel $L_1 = K$, and let

M_0 be a maximal subalgebra of L with $L_0 \leq M_0$. By monotonicity, it is clear that M_0 is nonlinear as well and the corresponding kernel M_1 contains K. Hence, by Theorem 2.4, we know that M_0 is a parabolic subalgebra of L, i.e., we can fix a Cartan subalgebra $\mathfrak{h} \leq L$, a positive root system Δ^+ and a simple root $\alpha_s \in \Delta^0$, $s = 1, \ldots, \ell$, such that $M_0 = P_0(\alpha_s)$ is a standard maximal parabolic subalgebra. The kernel M_1 of M_0 is the highest component $\mathfrak{g}_k(\alpha_s)$ of the corresponding grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with respect to the α_s-height. Obviously, we have $K \leq M_1 = \mathfrak{g}_k$. Thus, the following Lemma is proved.

Lemma 4.3.   *For any kernel $K \leq L$ there exists a simple grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that K is contained in the highest component $\mathfrak{g}_k$.*

Lemma 4.3 confirms that any kernel in a simple Lie algebra is Abelian. In the following Section we will see that any kernel K coincides with the highest component of some simple $|k|$-grading of L.

5.   Regularity

Yet, we do not have an argument, which allows us to conclude that the normalizer of a kernel K contains a maximal solvable subalgebra of L. In fact, we do not even know that $N_L(K)$ contains some Cartan subalgebra of L, which would imply the regularity of K. To solve this problem, we introduce the simple hull for regular kernels in L. Using a classical result of E.B. Dynkin we are then able to prove regularity for arbitrary kernels. In fact, we will see that any kernel K coincides with the highest component of some simple $|k|$-grading, which in turn implies that the subalgebra $N_L(K)$ is parabolic.

In general, let $(L, \mathfrak{h})$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. A subalgebra $A \leq L$ is called $\mathfrak{h}$-*regular* if A is of the form

$$A = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha,$$

where $\Gamma \subseteq \Delta$ is a closed system of roots, i.e., whenever $\alpha, \beta \in \Gamma$ and $\alpha + \beta \in \Delta$ then $\alpha + \beta \in \Gamma$, and $\mathfrak{t}$ is a subspace of $\mathfrak{h}$ (cf. [OV94]). More generally, a subalgebra $A \leq L$ is called *regular*, if A is conjugated to an $\mathfrak{h}$-regular subalgebra (for some choice of $\mathfrak{h}$ in L).

First, let K be a kernel and a regular subalgebra of a complex simple Lie algebra L. We call K a *regular kernel*. The normalizer $N_L(K)$ is regular as well, since it contains some Cartan subalgebra $\mathfrak{h} \leq L$. (We do not know that $N_L(K)$ contains a Borel subalgebra.) Let $M_0 \geq N_L(K)$ be a maximal subalgebra of L. Of course, M_0 is a maximal parabolic subalgebra of L and with an appropriate choice of positive roots Δ^+ and some simple root $\alpha_s \in \Delta^0$ (with respect to $\mathfrak{h} \leq N_L(K)$) we have $M_0 = P_0(\alpha_s)$. Moreover, by monotonicity, we have $K \leq \mathfrak{g}_k(\alpha_s)$, the highest component of the simple $|k|$-grading corresponding to M_0. In particular, there exists a uniquely determined subset $\Gamma(K) \subseteq \Delta$ of $\mathfrak{h}$-roots with α_s-height k such that $K = \bigoplus_{\alpha \in \Gamma(K)} \mathfrak{g}_\alpha$. Recall that the Killing form of L induces an Euclidean scalar product on (the underlying space of) the corresponding root system.

Lemma 5.1.   *If $\Gamma(K) = A \cup C$ is a disjoint union of orthogonal subsets, i.e., $A \perp C$, then either A or C is empty.*

We postpone the proof of Lemma 5.1. The proof will be given below by a case by case study of the root sets $\Gamma(\mathfrak{g}_k) \subseteq \Delta$ of the highest grading components $\mathfrak{g}_k$ in simple Lie algebras. For the time being, let $-K$ denote the regular subalgebra $\bigoplus_{\alpha \in \Gamma(K)} \mathfrak{g}_{-\alpha}$ in $\mathfrak{g}_{-k}$. Obviously, $-K$ is a kernel as well. We define $L_K := -K \oplus [-K, K] \oplus K$, which is a regular subspace of L.

Lemma 5.2. $L_K = -K \oplus [-K, K] \oplus K$ *is a simple $|1|$-graded Lie algebra (and K is a kernel in L_K).*

Proof.   First, note that L_K is a regular subalgebra of L. Indeed, since K and $-K$ are kernels, we have $[[-K, K], K] \subseteq K$, $[[-K, K], -K] \subseteq -K$ and $[[-K, K], [-K, K]] \subseteq [-K, K]$. Moreover, the first two inclusions are equalities of subspaces in L, since $[[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha], \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ for any root α. Furthermore, since the root set $\Gamma(L_K)$ corresponding to L_K is symmetric, i.e., $\Gamma(L_K) = -\Gamma(L_K)$, we know that L_K is reductive (cf. [OV94]). Now we have $[L_K, L_K] = L_K$ and L_K is semisimple.

The simplicity of L_K follows with Lemma 5.1, since K and $-K$ generate L_K as Lie algebra.   ■

In view of Lemma 5.2 we call L_K the *simple hull* of the regular kernel K in L. In particular, Lemma 5.2 says that for any simple grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ the hull $L_{\mathfrak{g}_k} = \mathfrak{g}_{-k} \oplus [\mathfrak{g}_{-k}, \mathfrak{g}_k] \oplus \mathfrak{g}_k$ of the kernel $K = \mathfrak{g}_k$ is a simple regular subalgebra of L.

Now let K be an arbitrary kernel in a complex simple Lie algebra L. We know from Lemma 4.3 that there exists a grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with $k > 0$ such that $K \leq \mathfrak{g}_k$. The 0-part $\mathfrak{g}_0 = \mathfrak{g}_0^s \oplus \mathfrak{c}$ is reductive and, if we choose any Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0^s$, then $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{c}$ is a Cartan subalgebra of L. With respect to $\mathfrak{h}$ we can choose some positive root system Δ^+ and a subset $\Sigma \subseteq \Delta^0$ such that the Σ-height gives rise to the given grading $\mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of L. If we wish we can assume in the following that the only simple regular subalgebra of L, which contains K, is L itself, i.e., L is minimal in this sense. (From Lemma 5.2 we know that $\mathfrak{g}_{-k} \oplus [\mathfrak{g}_{-k}, \mathfrak{g}_k] \oplus \mathfrak{g}_k$ is a simple $|1|$-graded Lie algebra, which is a regular subalgebra of L, and contains K as a kernel. Hence, if L is minimal then $k = 1$.)

Let us define $\mathfrak{f}_K := [\mathfrak{g}_{-k}, K]$. This is a subalgebra of $\mathfrak{g}_0$ in L. In fact, $[[\mathfrak{g}_{-k}, K], [\mathfrak{g}_{-k}, K]] \subseteq [[[\mathfrak{g}_{-k}, K], K], \mathfrak{g}_{-k}] + [[[\mathfrak{g}_{-k}, K], g_{-k}], K] \subseteq [\mathfrak{g}_{-k}, K]$. We denote by $\hat{\mathfrak{f}}_K$ the projection of $\mathfrak{f}_K$ to $\mathfrak{g}_0^s$. The key to our problem is the study of $\mathfrak{f}_K$. We start with the assumption that $\mathfrak{f}_K \leq \mathfrak{g}_0$ is reductive, i.e., $\hat{\mathfrak{f}}_K$ is semisimple. In this case $\mathfrak{f}_K = \hat{\mathfrak{f}}_K \oplus \mathfrak{c}'$ with center $\mathfrak{c}' \subseteq \mathfrak{c}$. Moreover, the $\hat{\mathfrak{f}}_K$-module $\mathfrak{g}_k$ decomposes into $K \oplus S$. Let $S^* = \{X \in \mathfrak{g}_{-k} | \ B_L(X, K) = 0\}$ and $K^* = \{X \in \mathfrak{g}_{-k} | \ B_L(X, S) = 0\}$ be the dual spaces in $\mathfrak{g}_{-k}$ to S and K, respectively. The $\hat{\mathfrak{f}}_K$-module $\mathfrak{g}_{-k}$ decomposes into $K^* \oplus S^*$. Note that the decompositions $K \oplus S$ and $K^* \oplus S^*$ are both $\mathfrak{f}_K$-invariant, since $\mathfrak{c}$ acts by scalars on the $\mathfrak{g}_0$-irreducible modules $\mathfrak{g}_{-k}$ and $\mathfrak{g}_k$.

Lemma 5.3. *The Lie algebra $\mathfrak{f}_K$ is reductive if and only if $K = \mathfrak{g}_k$.*

Proof. If $K = \mathfrak{g}_k$ then Lemma 5.2 says that $\mathfrak{f}_K = [\mathfrak{g}_{-k}, \mathfrak{g}_k]$ is reductive.

On the other hand, let us assume that $K \lneq \mathfrak{g}_k$ is a proper kernel (and $\mathfrak{f}_K$ is reductive). Since S^* is $\mathfrak{f}_K$-invariant, we have $[[S^*, K], [\mathfrak{g}_{-k}, K]] \subseteq [S^*, [[\mathfrak{g}_{-k}, K], K]] + [[[\mathfrak{g}_{-k}, K], S^*], K] \subseteq [S^*, K]$, i.e., $[S^*, K]$ is an ideal in $\mathfrak{f}_K$.

Now we choose $X \in K$ and $Y \in \mathfrak{g}_0^s$ such that $[Y, X] \notin K$. This is possible, since $\mathfrak{g}_k$ is $\mathfrak{g}_0^s$-irreducible, and we have the decomposition $[Y, X] = [Y, X]_K + [Y, X]_S$ with respect to $K \oplus S$ and $[Y, X]_S \neq 0$. Then we can choose $Z \in S^*$ with $B_L(Z, [Y, X]_S) \neq 0$. This implies $0 \neq B_L(Z, [Y, X]) = B_L([Z, X], Y)$. Hence, $[Z, X] \neq 0$. In fact, we have $[Z, X] \notin \mathfrak{c}'$, since $B_L(\mathfrak{c}, \mathfrak{g}_0^s) = 0$.

Furthermore, we have $B_L([S^*, K], [\mathfrak{g}_{-k}, K]) = B_L(S^*, [[\mathfrak{g}_{-k}, K], K]) = B_L(S^*, K) = 0$. *Cartan's criterion* shows that $[S^*, K]$ is a solvable ideal of $\mathfrak{f}_K$. However, $[S^*, K]$ is not in $\mathfrak{c}'$. This is a contradiction and we conclude $K = \mathfrak{g}_k$. $\blacksquare$

Let us consider for a moment an arbitrary subalgebra $\mathfrak{a} \leq \mathfrak{g}_0$. We set $K_{\mathfrak{a}} := \{X \in \mathfrak{g}_k | [\mathfrak{g}_{-k}, X] \subseteq \mathfrak{a}\}$. We call $\mathfrak{a}$ regular in $\mathfrak{g}_0$ if its projection is a regular subalgebra in $\mathfrak{g}_0^s$.

Lemma 5.4. *If $K_{\mathfrak{a}} \neq \{0\}$, then $K_{\mathfrak{a}}$ is a kernel. If, in addition, $\mathfrak{a}$ is regular, then $K_{\mathfrak{a}}$ is a regular kernel.*

Proof. First, we show that $K_{\mathfrak{a}}$ is $\mathfrak{a}$-invariant. In fact, $[\mathfrak{g}_{-k}, [\mathfrak{a}, X]] \subseteq [\mathfrak{g}_{-k}, X] + [\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$ for any $X \in K_{\mathfrak{a}}$. Hence, we have $[[L, K_{\mathfrak{a}}], K_{\mathfrak{a}}] = [[\mathfrak{g}_{-k}, K_{\mathfrak{a}}], K_{\mathfrak{a}}] \subseteq [\mathfrak{a}, K_{\mathfrak{a}}] \subseteq K_{\mathfrak{a}}$, i.e., any nontrivial $K_{\mathfrak{a}}$ is a kernel.

Now let $\mathfrak{a}$ be regular with respect to $\mathfrak{h}_0 \leq \mathfrak{g}_0^s$ and let $\Gamma(\mathfrak{g}_k) \subseteq \Delta$ be the set of roots of Σ-height k. An arbitrary element of $K_{\mathfrak{a}}$ can be written as $X = \sum_{\alpha \in \Gamma(\mathfrak{g}_k)} X_\alpha$ for certain $X_\alpha \in \mathfrak{g}_\alpha$. Then we have $[X_{-\beta}, X] = \sum_\alpha [X_{-\beta}, X_\alpha] \in \mathfrak{a}$ for any $X_{-\beta} \in \mathfrak{g}_{-\beta}$ with $\beta \in \Gamma(\mathfrak{g}_k)$. Since $\mathfrak{a}$ is $\mathfrak{h}_0$-regular, every term of $\sum_\alpha [X_{-\beta}, X_\alpha]$ is in $\mathfrak{a}$. Hence, every term X_α of X is in $K_{\mathfrak{a}}$ as well. $\blacksquare$

It is a well known result in [Dy52] that any nonsemisimple subalgebra of a semisimple Lie algebra is a R-*subalgebra*. In our situation this implies for any kernel K either $K = \mathfrak{g}_k$ (by Lemma 5.3) or otherwise there exists a regular proper subalgebra $\overline{\mathfrak{f}_K}$ with $\mathfrak{f}_K \leq \overline{\mathfrak{f}_K} \lneq \mathfrak{g}_0$. The latter case is not possible, as we prove now. Thus, we obtain the following improvement of Lemma 4.3.

Proposition 5.5. *Let $K \leq L$ be a kernel in a complex simple Lie algebra. Then there exists some simple grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with $k > 0$ such that $K = \mathfrak{g}_k$.*

Proof. Without loss of generality, we can assume that L is the only regular simple subalgebra of L, which contains K. Hence, $k = 1$ and $[\mathfrak{g}_{-k}, \mathfrak{g}_k] = \mathfrak{g}_0$. This shows that, if $\mathfrak{f}_K \leq \overline{\mathfrak{f}_K} \lneq \mathfrak{g}_0$, then $K_{\overline{\mathfrak{f}_K}}$ must be a proper regular subalgebra in $\mathfrak{g}_k$. However, with $K \leq K_{\overline{\mathfrak{f}_K}}$ and the simple hull of $K_{\overline{\mathfrak{f}_K}}$ in L, we have a contradiction to the minimality of L. Hence, only $K = \mathfrak{g}_k$ is possible (no matter if L is minimal or not). $\blacksquare$

Proposition 5.5 immediately implies the regularity and the fact that the normalizer $N_L(K)$ is a parabolic subalgebra for any kernel $K \leq L$.

Proposition 5.6. *Let $K \leq L$ be a kernel in a complex simple Lie algebra. Then the normalizer $N_L(K)$ is an ample parabolic subalgebra of L.*

6. Root Computations

From our discussion so far we know that any kernel in a complex simple Lie algebra is the kernel of an ample parabolic subalgebra. We derive in this Section all standard ample parabolic subalgebras by computations in the corresponding root systems, case by case. This gives rise to a complete classification of ample nonlinear subalgebras and kernels in complex simple Lie algebras, up to conjugation. We still have to prove Lemma 5.1. This problem is also solved by explicit computations with roots.

Before we start, recall that any maximal parabolic subalgebra is ample. Thus we have to clarify in practice which of the standard parabolic subalgebras $P_0(\Sigma)$ are ample, when $\Sigma \subseteq \Delta^0$ contains more than one simple root. The parabolic subalgebras, which are not ample, have the kernel of an ample parabolic subalgebra. It is only the height of the highest component of the corresponding $|k|$-grading that increases.

For some of the exceptional Lie algebras we do not give every root computation in detail and we do not present every kernel explicitly as a sum of root spaces. However, in each case of our discussion we provide the highest root in terms of fundamental weights. The highest root and $\Sigma \subseteq \Delta^0$ directly determine, to which irreducible $\mathfrak{g}_0(\Sigma)$-module the kernel $P_1(\Sigma) = \mathfrak{g}_k(\Sigma)$ of $P_0(\Sigma)$ is isomorphic.

Recall that the list of simple complex Lie algebras consists of the 4 families $\mathfrak{sl}(\ell+1, \mathbb{C})$, denoted by A_ℓ, $\ell \geq 1$, $\mathfrak{so}(2\ell+1, \mathbb{C})$ with letter B_ℓ, $\ell \geq 2$, $\mathfrak{sp}(2\ell, \mathbb{C})$ with letter C_ℓ, $\ell \geq 3$, and $\mathfrak{so}(2\ell, \mathbb{C})$, denoted by D_ℓ, $\ell \geq 4$. Moreover, there are the exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$.

According to the description of parabolic subalgebras $P_0(\Sigma)$ in simple Lie algebras by *Dynkin diagrams*, we refer to elements in Σ often as the *crossed roots*. We summarize our results in Section 7.

The family A_ℓ. Let $L = \mathfrak{sl}(\ell+1, \mathbb{C})$, $\ell \geq 1$, be the special linear algebra of traceless $(\ell+1) \times (\ell+1)$-matrices with complex entries. A Cartan subalgebra $\mathfrak{h} \leq L$ is given by the diagonal matrices. We denote by e_i, $i = 1, \ldots, \ell+1$, the function which evaluates the ith diagonal entry of a matrix in L. Then the roots of $(L, \mathfrak{h})$ are

$$\Delta = \{\pm(e_i - e_{j+1}) \mid 1 \leq i \leq j \leq \ell\},$$

and a system of positive roots is given by $\Delta^+ = \{e_i - e_{j+1} \mid 1 \leq i \leq j \leq \ell\}$ with simple roots $\Delta^0 = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i \leq \ell\}$. The highest root in terms of fundamental weights is $\lambda_1 + \lambda_\ell$.

Let $\Sigma = \{\alpha_{a_1}, \ldots, \alpha_{a_k}\} \subseteq \Delta^0$ be a nontrivial subset with $1 \leq a_1 < \cdots < a_k \leq \ell$. Then the highest grading component for $P_0(\Sigma)$ is

$$P_1 = \mathfrak{g}_k = \bigoplus_{i \leq a_1, a_k \leq j} \mathfrak{g}_{e_i - e_{j+1}} .$$

Obviously, the normalizer $N_L(P_1)$ of P_1 in L is the parabolic subalgebra $P_0(\{\alpha_{a_1}, \alpha_{a_k}\})$, which has one ($k = 1$) or two crossed roots ($k > 1$). This shows that the (standard) ample parabolic subalgebras of $L = \mathfrak{sl}(\ell + 1, \mathbb{C})$ are exactly those with one or two crossed roots.

In order to demonstrate how a kernel $P_1 = \mathfrak{g}_k$ can be described as $\mathfrak{g}_0$-module, let us cross here one root, i.e., $\Sigma = \{\alpha_s\}$. Then we have $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}(s, \mathbb{C}) \oplus \mathfrak{sl}(\ell - s + 1)$ and the kernel $P_1(\alpha_s) = \mathfrak{g}_1(\alpha_s)$ of the maximal parabolic subalgebra $P_0(\alpha_s)$ is the irreducible $\mathfrak{g}_0$-module with highest weight $\lambda_1 + \lambda_\ell$, i.e., $\mathfrak{g}_1(\alpha_s)$ is isomorphic to the tensor product of the standard $\mathfrak{sl}$-module $\mathbb{C}^s$ times the dual of the standard $\mathfrak{sl}$-module $\mathbb{C}^{\ell - s + 1}$. The dimension of $P_1(\alpha_s) = \mathfrak{g}_1(\alpha_s)$ is $s(\ell - s + 1)$.

Recall that any (regular) kernel K of L is a regular subspace of some highest grading component $\mathfrak{g}_k$. So let $\alpha = e_i - e_{j+1}$ and $\beta = e_u - e_{v+1}$ be an arbitrary pair of orthogonal roots in the root set $\Gamma(\mathfrak{g}_k)$ of $\mathfrak{g}_k$. In this case the four indices of α, β are pairwise different. We compute $[[\mathfrak{g}_{e_{j+1}-e_u}, \mathfrak{g}_\alpha], \mathfrak{g}_\beta] = \mathfrak{g}_\gamma$ with $\gamma = e_i - e_{v+1}$ and γ is neither orthogonal to α nor to β. This is sufficient to see that Lemma 5.1 is true for the special linear algebras, i.e., the hull L_K of any regular kernel $K \leq L$ is simple.

The family B_ℓ. Let $L = \mathfrak{so}(2\ell + 1, \mathbb{C})$, $\ell \geq 2$, be an odd-dimensional orthogonal Lie algebra. The positive roots of $\mathfrak{so}(2\ell + 1, \mathbb{C})$ can be presented in the form

$$\Delta^+ = \{e_i \pm e_j |\ i < j\} \cup \{e_i\}$$

with simple roots $\Delta^0 = \{\alpha_j = e_j - e_{j+1} |\ j = 1, \ldots, \ell - 1\} \cup \{\alpha_\ell = e_\ell\}$. Note that $e_i = \sum_{s=i}^{\ell} \alpha_s$, $e_i + e_j = \sum_{s=i}^{\ell} \alpha_s + \sum_{s=j}^{\ell} \alpha_s$ and $e_i - e_{j+1} = \sum_{s=i}^{j} \alpha_s$. The heighest root is λ_2.

Let $\Sigma = \{\alpha_{a_1}, \ldots, \alpha_{a_k}\} \subseteq \Delta^0$ be nontrival with $a_1 < \ldots < a_k$. First, we assume $a_1 > 1$. Then the kernel of $P_0(\Sigma)$ is

$$P_1 = \mathfrak{g}_{2k} = \bigoplus_{i < j \leq a_1} \mathfrak{g}_{e_i + e_j}$$

with normalizer $N_L(P_1) = P_0(\alpha_{a_1})$. For $a_1 = 1$, $a_2 = 2$ we have $P_1 = \mathfrak{g}_{2k-1} = \mathfrak{g}_{e_1 + e_2}$ and $N_L(P_1) = P_0(\alpha_2)$. For $a_1 = 1$, $a_2 > 2$ we have $P_1 = \mathfrak{g}_{2k-1} = \bigoplus_{i=2}^{a_2} \mathfrak{g}_{e_1 + e_i}$ and $N_L(P_1) = P_0(\alpha_1, \alpha_{a_2})$. Thus, for ample parabolic subalgebras, the set Σ has at most two crossed roots. If two roots are crossed, then $\alpha_1 \in \Sigma$ and $\alpha_2 \notin \Sigma$.

Obviously, the highest component of any simple $|k|$-grading of $L = \mathfrak{so}(2\ell + 1, \mathbb{C})$ sits (up to conjugation) either in $\mathfrak{g}_1(\alpha_1) = \mathfrak{g}_{e_1} \oplus \bigoplus_{j=2}^{\ell}(\mathfrak{g}_{e_1-e_j} \oplus \mathfrak{g}_{e_1+e_j})$ or otherwise in $\mathfrak{g}_2(\alpha_\ell) = \bigoplus_{i<j\leq\ell} \mathfrak{g}_{e_i+e_j}$. First, let $\alpha, \beta \in \Gamma(\mathfrak{g}_1(\alpha_1))$ be orthogonal roots, whose root spaces are contained in $\mathfrak{g}_1(\alpha_1)$. Up to the ordering, we have $\alpha = e_1 + e_j$ and $\beta = e_1 - e_j$ for some $2 \leq j \leq \ell$. For $k \neq 1, j$ we compute $[[\mathfrak{g}_{-e_1+e_k}, \mathfrak{g}_\alpha], \mathfrak{g}_\beta] = \mathfrak{g}_\gamma$ with $\gamma = e_1 + e_k$ and γ is neither orthogonal to α nor to β. Thus, if a kernel contains the root spaces to α, β, then this kernel also contains the roots space to γ, which shows that any orthogonal splitting of the roots of a (regular) kernel in $\mathfrak{g}_1(\alpha_1)$ is trivial. Further, if $\alpha, \beta \in \Gamma(\mathfrak{g}_2(\alpha_\ell))$ are orthogonal, then $\alpha = e_i + e_j$ and $\beta = e_u + e_v$ with four pairwise different indices and we have $[[\mathfrak{g}_{-e_i-e_u}, \mathfrak{g}_\alpha], \mathfrak{g}_\beta] = \mathfrak{g}_\gamma$

with $\gamma = e_j + e_v$. Since γ is neither orthogonal to α nor to β, Lemma 5.1 holds true for the case B_ℓ.

The family C_ℓ. Let $L = \mathfrak{sp}(2\ell, \mathbb{C})$, $\ell \geq 3$, be a symplectic algebra. A positive root set of $\mathfrak{sp}(2\ell, \mathbb{C})$ is given by

$$\Delta^+ \;=\; \{e_i \pm e_j|\ i < j\} \cup \{2e_i\}$$

with simple roots $\Delta^0 = \{\alpha_j = e_j - e_{j+1}|\ j = 1, \ldots, \ell - 1\} \cup \{\alpha_\ell = 2e_\ell\}$. We have $e_i + e_j = \sum_{s=i}^{\ell} \alpha_s + \sum_{s=j}^{\ell-1} \alpha_s$ and $e_i - e_{j+1} = \sum_{s=i}^{j} \alpha_s$ for $i \leq j < \ell$. The highest root is $2\lambda_1$.

Let $\Sigma = \{\alpha_{a_1}, \ldots, \alpha_{a_k}\} \subseteq \Delta^0$ be a nontrivial subset with $a_1 < \cdots < a_k$. The kernel of $P_0(\Sigma)$ is

$$P_1 \;=\; \bigoplus_{i \leq j \leq a_1} \mathfrak{g}_{e_i + e_j} \ .$$

Its normalizer $N_L(P_1)$ is the parabolic subalgebra $P_0(\alpha_{a_1})$. This shows that the (standard) ample parabolic subalgebras of $L = \mathfrak{sp}(2\ell, \mathbb{C})$ are exactly the maximal ones (with one crossed root).

This time any (regular) kernel of L is conjugated to a regular subspace of $\mathfrak{g}_1(\alpha_\ell)$. Let $\alpha = e_i + e_j, \beta = e_u + e_v \in \Gamma(\mathfrak{g}_1(\alpha_\ell))$ be an orthogonal pair of roots, i.e., all indices are different. Then we have $(-e_i - e_u) + \alpha = -e_u + e_j$, $(-e_u + e_j) + \beta = e_j + e_v$ and $\gamma = e_j + e_v$ is not orthogonal to α, β. This proves Lemma 5.1 for the case C_ℓ.

The family D_ℓ. Let $L = \mathfrak{so}(2\ell, \mathbb{C})$, $\ell \geq 4$, be an orthogonal algebra in even dimension. A positive root system of $\mathfrak{so}(2\ell, \mathbb{C})$ is

$$\Delta^+ = \{e_i \pm e_j|\ i < j\}$$

with simple roots $\Delta^0 = \{\alpha_j = e_j - e_{j+1}|\ j = 1, \ldots, \ell - 1\} \cup \{\alpha_\ell = e_{\ell-1} + e_\ell\}$. We have $e_i + e_j = \sum_{s=i}^{\ell} \alpha_s + \sum_{s=j}^{\ell-2} \alpha_s$ for $i < j \leq n - 2$, $e_i + e_{\ell-1} = \sum_{s=i}^{\ell} \alpha_s$, $e_i + e_\ell = \alpha_\ell + \sum_{s=i}^{\ell-2} \alpha_s$ and $e_i - e_{j+1} = \sum_{s=i}^{j} \alpha_s$. The highest root is λ_2.

Of course, the family D_ℓ behaves similarly as B_ℓ. What is slightly different here is the case when $\alpha_2, \ldots, \alpha_{\ell-2}$ are not crossed, i.e., in the nonmaximal case either α_ℓ or $\alpha_{\ell-1}$ is crossed, at least. In fact, up to an outer automorphism of $L = \mathfrak{so}(2\ell, \mathbb{C})$, we can assume that α_ℓ is crossed. Then we have $P_1(\alpha_{\ell-1}, \alpha_\ell) = \bigoplus_{i<j<\ell} \mathfrak{g}_{e_i+e_j}$ and $N_L(P_1) = P_0(\alpha_{\ell-1}, \alpha_\ell)$. Futher, $P_1(\alpha_1, \alpha_\ell) = \bigoplus_{j=1}^{\ell} \mathfrak{g}_{e_1+e_j}$ and $N_L(P_1) = P_0(\alpha_1, \alpha_\ell)$, and finally, $P_1(\alpha_1, \alpha_{\ell-1}, \alpha_\ell) = \bigoplus_{j=1}^{\ell-1} \mathfrak{g}_{e_1+e_j}$ and $N_L(P_1) = P_0(\alpha_1, \alpha_{\ell-1}, \alpha_\ell)$. All these cases describe ample parabolic subalgebras.

The same computations as for B_ℓ show that Lemma 5.1 is true for the family D_ℓ.

The exceptional case E_6. Let $\mathfrak{e}_6$ be the exceptional Lie algebra of rank 6 and dimension 78 with positive roots $\Delta^+ = \{2e_7\} \cup \{e_i - e_j : 1 \leq i < j \leq 6\} \cup \{e_i + e_j + e_k + e_7 : 1 \leq i < j < k \leq 6\}$ and simple roots $\Delta^0 = \{\alpha_i = e_i - e_{i+1}|i = 1, \ldots, 5\} \cup \{\alpha_6 = e_4 + e_5 + e_6 + e_7\}$. The fundamental weight $\lambda_6 = 2e_7$ is the highest root. Note that there is an outer automorphism of L, which corresponds to the automorphism of the Dynkin diagram, exchanging the order of α_1 to α_5. The simple root α_6 remains fixed.

Writing the positive roots in terms of simple roots, we easily observe that, if $\alpha_6 \in \Sigma$, then $P_1(\Sigma) = \mathfrak{g}_{\lambda_6}$. With $N_L(\mathfrak{g}_{\lambda_6}) = P_0(\alpha_6)$ we see that, if $P_0(\Sigma)$ is ample and α_6 is crossed, then $\Sigma = \{\alpha_6\}$. The analogous statement for α_3 is true as well. In this case we have $P_1(\alpha_3) = \mathfrak{g}_3 = \mathfrak{g}_{\lambda_6} \oplus \mathfrak{g}_{\lambda_6 - \alpha_6}$, where $\lambda_6 - \alpha_6 = e_1 + e_2 + e_3 + e_7$.

So let us assume that Σ has at least two roots and $\alpha_3, \alpha_6 \notin \Sigma$. Then, up to an outer automorphism, we only have to check whether $\Sigma = \{\alpha_2, \alpha_4\}, \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_1, \alpha_2, \alpha_4\}$ or $\{\alpha_1, \alpha_2, \alpha_5\}$ give rise to ample parabolic subalgebras.

In fact, we find $P_1(\alpha_2, \alpha_4) = \mathfrak{g}_4 = \mathfrak{g}_{\lambda_6} \oplus \mathfrak{g}_{\lambda_6 - \alpha_6} \oplus \mathfrak{g}_{\lambda_6 - \alpha_6 - \alpha_3}$ with $\lambda_6 - \alpha_6 - \alpha_3 = e_1 + e_2 + e_4 + e_7$ and $N_L(\mathfrak{g}_4) = P_0(\alpha_2, \alpha_4)$. Thus, $P_0(\alpha_2, \alpha_4)$ is ample. Adding α_1 to $\Sigma = \{\alpha_2, \alpha_4\}$ only increases the height of $\mathfrak{g}_4$ by one, but the kernel remains unchanged.

Furthermore, we have $P_1(\alpha_1, \alpha_4) = \mathfrak{g}_3 = P_1(\alpha_2, \alpha_4) \oplus \mathfrak{g}_{\lambda_6 - \alpha_6 - \alpha_3 - \alpha_2}$ with $\lambda_6 - \alpha_6 - \alpha_3 - \alpha_2 = e_1 + e_3 + e_4 + e_7$ and $N_L(P_1(\alpha_1, \alpha_4)) = P_0(\alpha_1, \alpha_4)$. The kernel of $P_0(\alpha_1, \alpha_4, \alpha_5)$ is the same. Hence, by an outer automorphism, we see that $P_0(\alpha_1, \alpha_2, \alpha_5)$ is not ample.

It remains to check $P_0(\alpha_1, \alpha_5)$. Here we find $P_1(\alpha_1, \alpha_5) = \mathfrak{g}_2 = \mathfrak{g}_{\lambda_6} \oplus \mathfrak{g}_{e_1 - e_6} \oplus \bigoplus_{2 \leq i < j \leq 4} \mathfrak{g}_{e_7 - e_6 - e_i - e_j}$ has dimension 8. And we see that the root spaces to $-\alpha_2, -\alpha_3, -\alpha_4$ and $-\alpha_6$ normalize P_1, whereas $-\alpha_1$ and $-\alpha_5$ do not normalize P_1. Hence, $P_0(\alpha_1, \alpha_5)$ is ample.

Note that any (regular) kernel of $\mathfrak{e}_6$ is conjugated (up to an outer automorphism) to a regular subspace of $\mathfrak{g}_1(\alpha_1)$. So let α, β be an arbitrary pair of orthogonal roots of $\mathfrak{g}_1(\alpha_1)$. Applying the subgroup of the Weyl group of $\mathfrak{e}_6$, which stabilizes $\mathfrak{g}_1(\alpha_1)$, we can easily see that α is conjugated to λ_6 (holding $\mathfrak{g}_1(\alpha_1)$ fixed). The λ_6-orthogonal roots in $\mathfrak{g}_1(\alpha_1)$ are $e_1 - e_i$ with $i = 2, \ldots, 5$. Again, applying the Weyl group, we can assume $\alpha = \lambda_6$ and $\beta = \alpha_1 + \alpha_2 + \alpha_3 = e_1 - e_4$. However, $[[\mathfrak{g}_{-\lambda_6 + \alpha_6}, \mathfrak{g}_{\lambda_6}], \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3}] = \mathfrak{g}_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6}$, and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_6$ is neither orthogonal to λ_6 nor to $\alpha_1 + \alpha_2 + \alpha_3$. This is sufficient to see that Lemma 5.1 is true for $\mathfrak{e}_6$.

The exceptional case E_7. Let $\mathfrak{e}_7$ be the exceptional Lie algebra of rank 7 with positive roots
$$\Delta^+ = \{e_i - e_j : 1 \leq i < j \leq 7\} \cup \{e_8 - e_i : i = 1, \ldots, 7\}$$
$$\cup \{e_8 + e_i + e_j + e_k : 1 \leq i < j < k \leq 7\}$$
and simple roots $\Delta^0 = \{\alpha_i = e_i - e_{i+1} | i = 1, \ldots, 6\} \cup \{\alpha_7 = e_8 + e_5 + e_6 + e_7\}$. The fundamental weight $\lambda_6 = e_8 - e_7$ is the highest root.

Expressing the positive roots by simple roots, we easily see that, if Σ has at least two crosses and $P_0(\Sigma)$ is ample, then α_7 must be crossed. In fact, it turns out that Σ has at most two crosses and the following combinations give rise to ample parabolic subalgebras: $\Sigma = \{\alpha_1, \alpha_7\}, \{\alpha_2, \alpha_7\} \{\alpha_3, \alpha_7\}$.

Moreover, we observe that the highest component of any simple $|k|$-grading of $\mathfrak{e}_7$ sits up to conjugation either in the highest component $\mathfrak{g}_2(\alpha_7)$ or in $\mathfrak{g}_1(\alpha_1)$. We have $\mathfrak{g}_2(\alpha_7) = \bigoplus_{i=1}^7 \mathfrak{g}_{e_8 - e_i}$. Obviously, there are no pairs of orthogonal roots for $\mathfrak{g}_2(\alpha_7)$. Furthermore, we have $\mathfrak{g}_1(\alpha_1) = \bigoplus_{i=2}^7 (\mathfrak{g}_{e_8 - e_i} \oplus \mathfrak{g}_{e_1 - e_i}) \oplus \bigoplus_{2 \leq i < j \leq 7} \mathfrak{g}_{e_8 + e_1 + e_i + e_j}$, which is 27-dimensional. Let $\alpha, \beta \in \Gamma(\mathfrak{g}_2(\alpha_7))$ be a pair of orthogonal roots. Applying the Weyl group we can assume $\alpha = \lambda_6 = e_8 - e_7$ and $\beta = e_1 - e_6$ or $e_8 + e_1 + e_6 + e_7$. We compute $[[\mathfrak{g}_{e_7 - e_1}, \mathfrak{g}_{e_8 - e_7}], \mathfrak{g}_{e_1 - e_6}] = \mathfrak{g}_{e_8 - e_6}$ and

$[[\mathfrak{g}_{e_5-e_8}, \mathfrak{g}_{e_8-e_7}], \mathfrak{g}_{e_8+e_1+e_6+e_7}] = \mathfrak{g}_{e_8+e_1+e_5+e_6}$. The roots $e_8 - e_6$ and $e_8 + e_1 + e_5 + e_6$ are neither orthogonal to α nor to β in each case. This proves Lemma 5.1.

For example, the simple hull L_K of the kernel $K = \mathfrak{g}_2(\alpha_7)$ is $\mathfrak{sl}(8, \mathbb{C}) \leq \mathfrak{e}_7$, which has the same rank as $\mathfrak{e}_7$.

The exceptional case E_8. Let $\mathfrak{e}_8$ be the exceptional Lie algebra of rank 8 with positive roots

$$\Delta^+ = \{e_i - e_j : 1 \leq i < j \leq 9\} \cup \{e_i + e_j + e_k : 1 \leq i < j < k \leq 8\}$$
$$\cup \{-e_i - e_j - e_9 : 1 \leq i < j \leq 8\}$$

and simple roots $\Delta^0 = \{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, 7\} \cup \{\alpha_8 = e_6 + e_7 + e_8\}$. The fundamental weight $\lambda_1 = e_1 - e_9$ is the highest root.

Expressing the positive roots in terms of simple roots, it is not difficult to see that there are exactly two ample parabolic subalgebras $P_0(\Sigma)$, which are not maximal. These ample parabolic subalgebras are given by $\Sigma = \{\alpha_6, \alpha_8\}$ and $\{\alpha_7, \alpha_8\}$.

This time any (regular) kernel of $\mathfrak{e}_8$ sits up to conjugation either in the highest component $\mathfrak{g}_3(\alpha_8)$ or otherwise in $\mathfrak{g}_2(\alpha_7)$. We have $\mathfrak{g}_3(\alpha_8) = \bigoplus_{i=1}^8 \mathfrak{g}_{e_i-e_9}$. Obviously, any two roots of this component are not orthogonal. Moreover, we have $\mathfrak{g}_2(\alpha_7) = \bigoplus_{i=1}^7 (\mathfrak{g}_{e_i-e_9} \oplus \mathfrak{g}_{-e_i-e_8-e_9})$. Let $\alpha, \beta \in \Gamma(\mathfrak{g}_2(\alpha_7))$ be two orthogonal roots. Applying the Weyl group we can assume $\alpha = e_1 - e_9$ and $\beta = -e_1 - e_8 - e_9$. Then $[[\mathfrak{g}_{e_9-e_i}, \mathfrak{g}_{e_1-e_9}], \mathfrak{g}_{e_1-e_8-e_9}] = \mathfrak{g}_{-e_i-e_8-e_9}$ for $1 < i < 8$. This proves Lemma 5.1.

For example, the simple hull of $\mathfrak{g}_3(\alpha_8)$ is $\mathfrak{sl}(9, \mathbb{C})$. The simple hull of $\mathfrak{g}_2(\alpha_7)$ is $\mathfrak{so}(16, \mathbb{C})$. These are the two proper simple regular subalgebras of $\mathfrak{e}_8$ of rank 8.

The exceptional case F_4. Let $\mathfrak{f}_4$ be the exceptional Lie algebra of rank 4 with positive roots

$$\Delta^+ = \{e_i : i = 1, \ldots, 4\} \cup \{e_i \pm e_j : 1 \leq i < j \leq 4\} \cup \{\tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$$

and simple roots $\Delta^0 = \{\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4), \alpha_2 = e_4, \alpha_3 = e_3 - e_4, \alpha_4 = e_2 - e_3\}$. The highest root is $\lambda_4 = e_1 + e_2$.

Expressing the positive roots by simple roots, we see that the kernel of $P_0(\alpha_4)$ is the root space of $\lambda_4 = e_1 + e_2$, i.e., one-dimensional. We also see easily that the kernel of the parabolic subalgebra $P_0(\Sigma)$ for any set $\Sigma \subseteq \Delta^0$ with $\alpha_4 \in \Sigma$ is the highest root space $\mathfrak{g}_{\lambda_4}$. This shows that, if $\alpha_4 \in \Sigma$, then $P_0(\Sigma)$ is only ample for $\Sigma = \{\alpha_4\}$.

For $\Sigma = \{\alpha_3\}$ we have $P_1 = \mathfrak{g}_{\lambda_4} \oplus \mathfrak{g}_{\lambda_4-\alpha_4}$. If we cross an additional root α_1, α_2 or both, then the kernel P_1 remains unchanged. Furthermore, we have $P_1(\alpha_2) = P_1(\alpha_1, \alpha_2) = \mathfrak{g}_{\lambda_4} \oplus \mathfrak{g}_{\lambda_4-\alpha_4} \oplus \mathfrak{g}_{\lambda_4-\alpha_4-\alpha_3}$. We conclude that the ample parabolic subalgebras are exactly the maximal ones. Finally, we see that $P_1(\alpha_1) = \mathfrak{g}_2(\alpha_1)$ is isomorphic to the standard $\mathfrak{co}(7)$-module, i.e., the dimension is 7. In fact, we have $P_1(\alpha_1) = \mathfrak{g}_{e_1} \oplus \bigoplus_{i=2}^4 (\mathfrak{g}_{e_1+e_i} \oplus \mathfrak{g}_{e_1-e_i})$.

From the above results, it is clear that the root sets of the 1-, 2- and 3-dimensional kernels admit only trivial orthogonal splittings. For the case $\mathfrak{g}_2(\alpha_1)$ we have the bracket $[[\mathfrak{g}_{-e_1+e_j}, \mathfrak{g}_{e_1+e_i}], \mathfrak{g}_{e_1-e_i}] = \mathfrak{g}_{e_1+e_j}$ for $i > 1$ and $j \neq 1, i$, and $e_1 + e_j$ is not orthogonal to $e_1 \pm e_i$. This proves Lemma 5.1.

The simple hull L_K of the kernel $K = \mathfrak{g}_2(\alpha_1)$ is $\mathfrak{so}(9, \mathbb{C}) \leq \mathfrak{f}_4$ with rank 4 as well.

The exceptional case G_2. Let $\mathfrak{g}_2$ be the exceptional Lie algebra of rank 2 and dimension 14 with simple root system $\Delta^0 = \{\alpha_1, \alpha_2\}$, where α_1 denotes

the short simple root. The highest root is $\lambda_2 = e_1 - e_3$.

Looking at the G_2-root system in the Euclidean plane, it is obvious that there are exactly two kernels (up to conjugation). One kernel has dimension 1 and is the root space of the highest root $3\alpha_1 + 2\alpha_2$. This kernel K_1 is normalized by $\mathfrak{g}_{-\alpha_1}$, but not by $\mathfrak{g}_{-\alpha_2}$. The normalizer $N_L(K_1) = P_0(\alpha_2)$ is maximal.

The other kernel has dimension 2. In fact, we have $K_2 = \mathfrak{g}_{3\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2}$ and $N_L(K_2) = P_0(\alpha_1)$ is again maximal. Thus, the ample parabolic subalgebras are exactly the two maximal ones.

Lemma 5.1 is certainly true for $\mathfrak{g}_2$, since the two roots of K_2 are not orthogonal. Note that the simple hull of K_2 is $\mathfrak{sl}(3, \mathbb{C})$ of rank 2.

7.   Classification results

We state now the classification results for (ample) nonlinear subalgebras and kernels in (complex) semisimple Lie algebras L. We begin with the classification of ample parabolic subalgebras in simple Lie algebras, due to our discussion from the previous Section. With our convention any parabolic subalgebra is a proper subalgebra of L.

Theorem 7.1.   *Let L be a complex simple Lie algebra. Then any maximal parabolic subalgebra is ample. The standard nonmaximal ample parabolic subalgebras $P_0(\Sigma)$ are given by the following crossed root sets Σ (with conventions for simple roots as in Section 6):*

A_ℓ. *Σ consists of exactly two crossed simple roots.*

B_ℓ. *$\Sigma = \{\alpha_1, \alpha_a\}$ with $a \in \{3, \ldots, \ell\}$.*

D_ℓ. *$\Sigma = \{\alpha_1, \alpha_a\}$ with $a \in \{3, \ldots, \ell\}$ and $\Sigma = \{\alpha_{\ell-1}, \alpha_\ell\}, \{\alpha_1, \alpha_{\ell-1}, \alpha_\ell\}$.*

E_6. *$\Sigma = \{\alpha_2, \alpha_4\}, \{\alpha_2, \alpha_5\}, \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}$.*

E_7. *$\Sigma = \{\alpha_1, \alpha_7\}, \{\alpha_2, \alpha_7\}, \{\alpha_3, \alpha_7\}$.*

E_8. *$\Sigma = \{\alpha_6, \alpha_8\}, \{\alpha_7, \alpha_8\}$.*

The proof of the following classification Theorem in the complex case follows mainly from the discussions so far. In Section 8 we will see that the result is also true for real simple Lie algebras.

Theorem 7.2.   *Let L be a simple Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.*

(a) For some nontrival subalgebra $K \lneq L$ the following conditions are equivalent:

1. $[[L, K], K] = K$,

2. $[[L, K], K] \subseteq K$,

3. K is the highest component $\mathfrak{g}_k$ of some simple $|k|$-grading of L,

4. K is the kernel of some ample parabolic subalgebra in L,

5. K is the kernel of some parabolic subalgebra in L,

6. K is a kernel of L, i.e., there exists some nonlinear subalgebra $L_0 \leq L$ with kernel $L_1 = K$.

In particular, the ample nonlinear subalgebras of L are exactly the ample parabolic subalgebras (which are listed in standard form for the complex case in Theorem 7.1; see Theorem 8.2 and Table 1 for the real case). There is a natural one-to-one-correspondence between kernels K and ample parabolic subalgebras A_0 of L. This correspondence is given by $A_0 = N_L(K)$ and $K = A_1$.

(b) Moreover, $L_0 \leq L$ is nonlinear if and only if there exists a nontrivial subalgebra $K \lneq L$ such that

$$[L, K] \subseteq L_0 \subseteq N_L(K) , \qquad (2)$$

i.e., L_0 lies between the ample parabolic subalgebra $N_L(K)$ and the minimal isotropy algebra $[L, K]$ of K. In this case the kernel L_1 of L_0 is identical to K.

Proof. For part (a) we only remark again that $[[\mathfrak{g}_{-\alpha}, \mathfrak{g}_\alpha], \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha$ for any root α. This implies the equivalence of the first two conditions.

It only remains to argue for part (b). Here, of course, if L_0 is nonlinear then $K = L_1$ satisfies (2). On the other hand, if K satisfies (2), then K is a kernel of L. Hence, $K = [[L, K], K] \leq L_0$ is in the kernel L_1 of L_0. By monotonicity, we obtain $K = L_1$. ∎

Part (b) of Theorem 7.2 allows in principle to write down any regular nonlinear subalgebra of L, including those which are not parabolic. However, since the minimal isotropy does not contain a Cartan subalgebra of L, in general, there exist nonregular nonlinear subalgebras of L.

Now let $L = \bigoplus_{i=1}^m L^i$ be a semisimple Lie algebra with simple ideals L^i and let K be a kernel in L. We denote the projections of K to the simple ideals by K^i, $i = 1, \ldots, m$.

Lemma 7.3. The projections K^i are contained in K for all $i = 1, \ldots, m$. In particular, $K = \bigoplus_{i=1}^m K^i$.

Proof. Let $k = \sum_i k^i$ and $k' = \sum_i k'^i$ be arbitrary elements of the kernel K. We have $[[L^i, k^i], k'^i] = [[L^i, k], k'] \subseteq K \cap L^i \subseteq K^i$ for all i, i.e., any nontrivial projection K^i is a kernel in L^i. Hence, $K^i = [[L^i, K^i], K^i] = [[L^i, K], K] \subseteq K$ for all i. ∎

Theorem 7.4. Let $L = \bigoplus_{i=1}^m L^i$ be a semisimple Lie algebra. The ample nonlinear subalgebras of L are exactly the ample parabolic subalgebras of L, which contain no simple ideal L^i.

Proof.   Let $A_0 \leq L$ be ample nonlinear. By Lemma 7.3, the kernel A_1 is a direct sum $\bigoplus_{i=1}^m A_1^i$, where the nontrivial A_1^i's are kernels in L^i. However, if only one of the projections of the kernel A_1 vanishes, then the normalizer of $N_L(A_1)$ contains a simple ideal of L, i.e., the normalizer is not effective and $A_0 \neq N_L(A_1)$. This shows that $A_1 = \bigoplus_{i=1}^m A_1^i$ is a direct sum of only nontrivial A_1^i's. Accordingly, the normalizer $A_0 = N_L(A_1)$ is a direct sum of ample parabolic subalgebras, i.e., A_0 is an effective ample parabolic subalgebra of L.   ■

The proof of Theorem 7.4 shows that for any ample nonlinear subalgebra A_0 of $L = \bigoplus_{i=1}^m L^i$ there exist ample parabolic subalgebras $A_0^i \leq L^i$ for every simple ideal such that $A_0 = \bigoplus_{i=1}^m A_0^i$. Thus, with the classifciation results for the simple case, we can describe the ample nonlinear subalgebras and their kernels for any complex semisimple Lie algebra L in explicit form. Note that the proof of Lemma 7.3 and Theorem 7.4 works over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (once we know that any ample nonlinear subalgebra of a real simple Lie algebra is parabolic).

8.   The real case

We prove here Theorem 7.2 for the case of real (semi)simple Lie algebras. Moreover, we go through the description of parabolic subalgebras in the real simple case, in order to obtain an explicit classification of the standard ample parabolic subalgebras. Up to conjugation, these are exactly the ample nonlinear subalgebras in a real simple Lie algebra.

In this Section we denote by L a real Lie algebra and by $L_\mathbb{C} = L \otimes \mathbb{C}$ its complexification. As a subspace, L is a *real form* of $L_\mathbb{C}$.

Lemma 8.1.   1. (L, L_0) *is a transitive real pair if and only if* $(L_\mathbb{C}, (L_0)_\mathbb{C})$ *is complex transitive.*

2. $L_0 \leq L$ *is nonlinear if and only if* $(L_0)_\mathbb{C} \leq L_\mathbb{C}$ *is nonlinear. In this case the kernel of* $(L_0)_\mathbb{C}$ *is* $(L_1)_\mathbb{C}$.

3. $K \leq L$ *is a real kernel if and only if* $K_\mathbb{C} \leq L_\mathbb{C}$ *is a complex kernel.*

4. $A_0 \leq L$ *is ample nonlinear if and only if* $(A_0)_\mathbb{C} \leq L_\mathbb{C}$ *is ample nonlinear.*

Proof.   (1) If I is an ideal in L, then $I_\mathbb{C}$ also in $L_\mathbb{C}$. Hence, if $(L_\mathbb{C}, (L_0)_\mathbb{C})$ is transitive, then L_0 cannot contain an ideal of L.

On the other hand, let us assume that $I \leq (L_0)_\mathbb{C}$ is a complex ideal of $L_\mathbb{C}$ and let σ denote the involution of $L_\mathbb{C}$ with respect to the real form L. Then $I + \sigma(I) \leq (L_0)_\mathbb{C}$ is a complex ideal of $L_\mathbb{C}$, which is σ-stable, i.e., $I + \sigma(I)$ is the complexification of some ideal $Q \leq L_0$ of L. Since (L, L_0) is transitive, Q must be trivial. Hence, I is trivial.

(2) Assume L_0 to be nonlinear. The complexification of the derived kernel L_1 is certainly contained in the derived kernel of $(L_0)_\mathbb{C}$. Hence, $(L_0)_\mathbb{C}$ is nonlinear. On the other hand, if $l + il' \in (L_0)_\mathbb{C}$ is an element with $[L_\mathbb{C}, l + il'] \subseteq (L_0)_\mathbb{C}$, then $[L, l] + i[L, l'] \subseteq (L_0)_\mathbb{C}$, which implies $l, l' \in L_1$. Hence, nonlinearity of L_0 and

$(L_0)_{\mathbb{C}}$ are equivalent and the complexification of the kernel L_1 is the kernel of $(L_0)_{\mathbb{C}}$. This also proves (3).

(4) The complexification of the normalizer $N_L(K)$ is $N_{L_{\mathbb{C}}}(K_{\mathbb{C}})$ for any $K \leq L$. This shows that ample nonlinearity for A_0 and $(A_0)_{\mathbb{C}}$ are equivalent. ∎

By definition, a subalgebra P_0 of a real semisimple Lie algebra L is called *parabolic* if its complexification $(P_0)_{\mathbb{C}}$ is a parabolic subalgebra of the complex semisimple Lie algebra $L_{\mathbb{C}}$. Any real parabolic subalgebra is conjugated in L to a standard parabolic subalgebra. We explain roughly how to describe the standard parabolic subalgebras via the root decomposition of real semisimple Lie algebras. For details we refer to [OV94, On04, CS09].

Let L be a real semisimple Lie algebra. We choose a *Cartan involution* θ to obtain a decomposition $L = \mathfrak{k} \oplus \mathfrak{q}$ and we choose a maximally noncompact and θ-stable Cartan subalgebra $\mathfrak{h} \leq L$. For the root system $\Delta(L_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of the complexification, we choose a set of positive roots Δ^+ such that the conjugation (with respect to the real form L) of any positive root $\alpha \in \Delta^+$ is either $-\alpha$ or again some positive root. The roots $\alpha \in \Delta$, whose conjugation gives $-\alpha$, are the *compact roots* with respect to the real form L. We denote the set of compact roots by Δ_c and $\Delta_c^0 = \Delta_c \cap \Delta^0$ are the simple compact roots. Moreover, we have the *restricted root system* of $(L, \mathfrak{a})$ for $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{q}$, which is denoted by Δ_r. The choice of Δ^+ determines positiv and simple subsystems in Δ_r as well.

Now we set $B_0 = \mathfrak{a} \oplus C_{\mathfrak{k}}(\mathfrak{a}) \oplus N$, where $C_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and N is the direct sum of the positive restriced root spaces. This is a subalgebra of L and any subalgebra of L, which contains B_0 is a standard parabolic subalgebra. For such a standard parabolic subalgebra P_0 there exists a uniquely determined (nontrivial) subset $\Sigma \subseteq \Delta_r^0$ of simple restricted roots such that P_0 is the direct sum of B_0 and the restricted root spaces to those negative restricted roots which can be written as linear combinations of simple restricted roots from $\Delta_r^0 \smallsetminus \Sigma$. We denote this parabolic subalgebra of L by $P_0(\Sigma)$. Note that the subset Σ corresponds uniquely to a subset of $\Delta^0 \smallsetminus \Delta_c^0$, which is stable under conjugation (with respect to the real form L). We denote this subset of the simple *noncompact* roots $\Delta^0 \smallsetminus \Delta_c^0$ of $L_{\mathbb{C}}$ with respect to L again by Σ.

As in the complex case, there is a close relationship between parabolic subalgebras and $|k|$-gradings of L. The definition of real $|k|$-gradings works as in the complex case; see Section 3. Obviously, $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ is a real $|k|$-grading if and only if $L_{\mathbb{C}} = (\mathfrak{g}_{-k})_{\mathbb{C}} \oplus \cdots \oplus (\mathfrak{g}_k)_{\mathbb{C}}$ is a complex $|k|$-grading. To explain the correspondence, let $P_0(\Sigma) \leq L$, $\Sigma \subseteq \Delta_r^0$ nontrivial, be a standard parabolic subalgebra. The Σ-height of the restricted root spaces gives rise to a $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ with $k > 0$. On the other hand, for any grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ of a real semisimple Lie algebra, the direct sum $P_0 = \bigoplus_{i=0}^{k} \mathfrak{g}_i$ is a parabolic subalgebra of L (and with appropriate choices this grading is determined by some Σ-height).

Proof of Theorem 7.2 in the real case.   Let $K \leq L$ be a kernel of the real simple Lie algebra L. Then $K_{\mathbb{C}}$ is a kernel of $L_{\mathbb{C}}$ and $N_{L_{\mathbb{C}}}(K_{\mathbb{C}})$ is an ample parabolic subalgebra of $L_{\mathbb{C}}$ with kernel $K_{\mathbb{C}}$. Since the complexification of $N_L(K)$ is $N_{L_{\mathbb{C}}}(K_{\mathbb{C}})$, we conclude that $A_0 = N_L(K)$ is an ample parabolic subalgebra of

L with kernel $A_1 = K$. This parabolic subalgebra A_0 gives rise to a $|k|$-grading $L = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that $A_0 = \bigoplus_{i=0}^{k} \mathfrak{g}_i$. Of course, $\mathfrak{g}_k$ is contained in the kernel $A_1 = K$. Lemma 4.1 applies also in the real case and we see that $K = A_1 = \mathfrak{g}_k$.

The equivalence of the other conditions in Theorem 7.2 follows now in a rather obvious way. ∎

The upshot of the real version of Theorem 7.2 is that there is also a natural one-to-one-correspondence between ample parabolic subalgebras A_0, kernels A_1 and highest grading components $\mathfrak{g}_k$ in any real simple Lie algebra. The ample nonlinear subalgebras are exactly the ample parabolic subalgebras. Theorem 7.4 for real semisimple Lie algebras follows with the same proof as in the complex case of Section 7.

It remains to list the ample parabolic subalgebras of real simple Lie algebras L in standard form. To achieve this we go through the list of real simple Lie algebras and determine the simple root sets $\Sigma \subseteq \Delta^0$, which give rise to standard ample parabolic subalgebras in L. For this purpose, we have to pay attention to the following rules. First, the root set Σ has to give rise to a complex ample parabolic subalgebra in $L_{\mathbb{C}}$. Secondly, $\Sigma \cap \Delta_c$ has to be empty with respect to the real form L. And, finally, Σ has to be a root set, which is invariant under conjugation with respect to the real form L.

The classification of real simple Lie algebras can be visualized by *Satake diagrams* (cf. [OV94, On04, CS09]). The compact roots Δ_c of a real simple Lie algebra are indicated by *black nodes* in the Satake diagram, whereas the other nodes are white. The *white nodes* for conjugated simple roots are related by an arrow. The elements of Σ are than indicated in a Satake diagram by *crossed nodes*, again. Only white nodes can be crossed and conjugated white nodes are either both crossed or both not crossed.

We omit drawing the Satake diagrams. Instead, Table 1 lists the real simple Lie algebras, identifies the compact roots Δ_c and lists the possible crossed root sets Σ in each case. We also state whether the ample parabolic subalgebra in question is maximal or not. In fact, there are maximal parabolic subalgebars in a real simple Lie algebra, whose complexifications are not maximal in $L_{\mathbb{C}}$. Note that the semisimple part of $\mathfrak{g}_0$ is determined by those simple roots, which are not crossed.

The *compact real forms* never admit parabolic subalgebras. The *split real forms* without arrows in the Satake diagram behave like its complexified versions, i.e., parabolic subalgebras in L and $L_{\mathbb{C}}$ correspond uniquely to each other. We omit these cases in Table 1. Furthermore, the underlying real Lie algebra $L_{\mathbb{R}}$ of a complex simple Lie algebra L is simple as well. The Satake diagram of $L_{\mathbb{R}}$ consists of two copies of the Dynkin diagram of L with white nodes only. The obviously related simple roots of these two copies are conjugated and thus joined by an arrow. Any complex (ample) parabolic subalgebra of L is a real (ample) parabolic subalgebra of $L_{\mathbb{R}}$. Thus, the *realification* of the classification of Theorem 7.1 applies in this case. We omit these case in Table 1 as well. It remains to list the possible crossed roots Σ for 10 families of classical real simple Lie algebras and 7 cases of noncompact real forms of exceptional algebras.

Real form L, $\quad \Delta_c^0$	$\Sigma \subseteq \Delta^0 \smallsetminus \Delta_c$	maximal ?
$\mathfrak{sl}(\frac{\ell+1}{2}, \mathbb{H})$, $\ell \geq 3$ odd, $\{\alpha_p \mid p = \text{odd}\}$	$\{\alpha_a\}$, a even, $\{\alpha_a, \alpha_b\}$, $a \neq b$ even	yes / no
$\mathfrak{su}(p, \ell+1-p), 1 \leq p \leq \frac{\ell}{2}$, ℓ even, $\{\alpha_{p+1}, \ldots, \alpha_{\ell+1-p}\}$	$\{\alpha_a, \alpha_{\ell-a}\}$, $1 \leq a \leq p$	yes
$\mathfrak{su}(p,p)$, $p = \frac{\ell+1}{2}$, $\ell \geq 3$ odd, $\emptyset$	$\{\alpha_a, \alpha_{\ell-a}\}, 1 \leq a < p$, $\{\alpha_p\}$	yes / yes
$\mathfrak{so}(p, 2\ell+1-p), 1 \leq p \leq \ell$, $\{\alpha_{p+1}, \ldots, \alpha_\ell\}$	$\{\alpha_a\}$, $1 \leq a \leq p$, $\{\alpha_1, \alpha_a\}$, $3 \leq a \leq p$	yes / no
$\mathfrak{sp}(p, \ell-p)$, $1 \leq p \leq \frac{\ell-1}{2}$, $\{\alpha_{2i-1} \mid 1 \leq i \leq p\} \cup \{\alpha_{2p+1}, \ldots, \alpha_\ell\}$	$\{\alpha_a\}$, even $a \leq 2p$	yes
$\mathfrak{sp}(p,p)$, $p = \frac{\ell}{2}$, $\{\alpha_{2i-1} \mid 1 \leq i \leq p\}$	$\{\alpha_a\}$, a even	yes
$\mathfrak{so}(p, 2\ell-p)$, $1 \leq p \leq \ell-2$, $\{\alpha_{p+1}, \ldots, \alpha_\ell\}$	$\{\alpha_a\}$, $1 \leq a \leq p$, $\{\alpha_1, \alpha_a\}$, $3 \leq a \leq p$	yes / no
$\mathfrak{so}(\ell-1, \ell+1)$, $\emptyset$	$\{\alpha_a\}$, $1 \leq a \leq \ell-2$, $\{\alpha_1, \alpha_a\}$, $3 \leq a \leq \ell-2$ $\{\alpha_{\ell-1}, \alpha_\ell\}, \{\alpha_1, \alpha_{\ell-1}, \alpha_\ell\}$	yes / no / yes, no
$\mathfrak{so}^*(2\ell)$, ℓ even, $\{\alpha_p \mid p = \text{odd}\}$	$\{\alpha_a\}$, $a = $ even	yes
$\mathfrak{so}^*(2\ell)$, ℓ odd, $\{\alpha_p \mid p = 1, 3, \ldots, \ell-2\}$	$\{\alpha_a\}$, even $a \leq \ell-2$, $\{\alpha_{\ell-1}, \alpha_\ell\}$	yes / yes
EII of $\mathfrak{e}_6$, $\quad \emptyset$	$\{\alpha_3\}$, $\{\alpha_6\}$, $\{\alpha_1, \alpha_5\}$, $\{\alpha_2, \alpha_4\}$	yes / yes
EIII of $\mathfrak{e}_6$, $\{\alpha_2, \alpha_3, \alpha_4\}$	$\{\alpha_6\}$, $\{\alpha_1, \alpha_5\}$	yes
EIV of $\mathfrak{e}_6$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_6\}$	$\{\alpha_1\}$, $\{\alpha_5\}$, $\{\alpha_1, \alpha_5\}$	yes / no
EVI of $\mathfrak{e}_7$, $\{\alpha_1, \alpha_3, \alpha_7\}$	$\{\alpha_2\}$, $\{\alpha_4\}$, $\{\alpha_5\}$, $\{\alpha_6\}$	yes
EVII of $\mathfrak{e}_7$, $\{\alpha_3, \alpha_4, \alpha_5, \alpha_7\}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_6\}$	yes
E$VIII$ of $\mathfrak{e}_8$, $\{\alpha_4, \alpha_5, \alpha_6, \alpha_8\}$	$\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_3\}$, $\{\alpha_7\}$	yes
FII of $\mathfrak{f}_4$, $\{\alpha_2, \alpha_3, \alpha_4\}$	$\{\alpha_1\}$	yes

Table 1

Theorem 8.2. *Let L be a real (noncompact) simple Lie algebra and let $P_0(\Sigma) \leq L$ be a standard ample parabolic subalgebra.*

1. *If L admits a compact root or at least one pair of conjugated restricted roots, then all pairs (L, Σ) are listed in Table 1.*

2. *In case L is a split form (without any pair of conjugated restricted roots) the $(L, P_0(\Sigma))$ correspond by complexification to the cases in Theorem 7.1.*

3. *If L is the realification of some complex simple Lie algebra, then the $P_0(\Sigma)$ are exactly the realifications of the examples in Theorem 7.1.*

In cases (2) and (3) of Theorem 8.2 a real ample parabolic subalgebra is maximal if and only if the complex version is maximal. The classification of Theorem 8.2 comprises all the case of the primitive classification in [Oc66]. Moreover, ample nonlinear subalgebras, which are not maximal, do exist in real simple Lie algebras.

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