Journal of Lie Theory
Volume **25** (2015) 257–270

A Generalized Weil Representation for the Finite Split Orthogonal Group $O_q(2n, 2n)$, q odd and > 3.

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Communicated by A. Pasquale

Abstract. We construct via generators and relations a generalized Weil representation for the split orthogonal group $O_q(2n, 2n)$ over a finite field of q elements. Besides, we give an initial decomposition of the representation found. We also show that the constructed representation is equal to the restriction of the Weil representation to $O_q(2n, 2n)$ for the reductive dual pair $(Sp_2(\mathbb{F}_q), O_q(2n, 2n))$ and that the initial decomposition is the same as the decomposition with respect to the action of $Sp_2(\mathbb{F}_q)$.
Mathematics Subject Classification 2010: 20C33.
Key Words and Phrases: Weil representation, split orthogonal group, involutive analogues of classical groups.

1. Introduction

Weil representations have proven to be a powerful tool in the theory of group representations. They originate from a very general construction of A. Weil ([12]), which has as a consequence the existence of a projective representation of the group $Sp(2n, K)$, K a locally compact field. Weil built this representation taking advantage of the representation theory of the related Heisenberg group, as described by the Stone-von Neumann theorem in the real case ([7]). In particular, these representations have allowed to build in a universal and uniform way all irreducible complex linear representations of the general linear group of rank 2 over a finite field ([11]), and later over a local field, except in residual characteristic two ([6]).

Weil representations can be constructed in various ways. For instance, they can be constructed via Heisenberg groups, via constructions of equivariant vector bundles ([4]), via presentations or via dual pairs ([1]), as we shall see later. The method using presentations is accomplished by having a simple presentation of the group, and then defining linear operators on a suitable vector space which preserve the relations among the generators of the presentation. This idea was originally suggested by Cartier, and used successfully by Soto-Andrade, for symplectic groups $Sp(2n, \mathbb{F}_q)$ ([11]) and complex irreducible representations of $SL(2, \mathbb{F}_q)$. In

*The author was partially supported by Fondecyt Grant 1120578.

ISSN 0949–5932 / \$2.50

the case of $\mathrm{Sp}(2n, \mathbb{F}_q)$ he considered this group as a group "SL(2)" but with entries in the matrix ring $M_n(\mathbb{F}_q)$, and thus he obtained a suitable presentation for the group. In this way, he constructed the Weil representation for the symplectic groups $\mathrm{Sp}(2n, \mathbb{F}_q)$. In the case of $\mathrm{Sp}(4, \mathbb{F}_q)$, Soto-Andrade obtained all the irreducible representations of this group by decomposing the two Weil representations associated to the two isomorphy types of quadratic forms of rank 4 over $\mathbb{F}_q$.

This point of view was generalized and gave rise to the groups $\mathrm{SL}_*^{\varepsilon}(2, A)$ for $(A, *)$ an involution ring and $\varepsilon = \pm 1 \in A$. These groups are a generalization of special linear groups $\mathrm{SL}(2, K)$, where K is a field. They were defined for $\varepsilon = -1$ by Pantoja and Soto-Andrade in [9] and generalized to $\varepsilon = 1$ in [10].

The groups $\mathrm{SL}_*^{\varepsilon}(2, A)$ include, among others, symplectic and split orthogonal groups. For instance, if K is a field, $A = M_n(K)$, $\diamond$ the transposition of matrices then $\mathrm{SL}_{\diamond}^{\varepsilon=-1}(2, A)$ is the symplectic group $\mathrm{Sp}(2n, K)$ and $\mathrm{SL}_{\diamond}^{\varepsilon=1}(2, A)$ is the split orthogonal group $\mathrm{O}_K(2n, 2n)$. Thus, these groups allow us to look at higher rank classical groups as rank two groups, considering them with coefficients in a new ring.

The procedure used by Soto-Andrade in [11] was approached even in the case of a nonsemi-simple involutive ring $(A, *)$ with nontrivial nilpotent Jacobson radical. In fact, in [2] Gutiérrez found a Bruhat presentation and a generalized Weil representation for $\mathrm{SL}_*^{\varepsilon=-1}(2, A_m)$, where $A_m = \mathbb{F}_q[x]/\langle x^m \rangle$.

In [3] Gutiérrez, Pantoja and Soto-Andrade build Weil representations in a very general way via generators and relations, for the groups $\mathrm{SL}_*^{\varepsilon}(2, A)$ for which a "Bruhat" presentation analogue to the classical one holds. In this way, in order to use this method it is very important to have an adequate presentation of the group. In [8], Pantoja generalized the classical Bruhat presentation of $\mathrm{SL}(2, K)$ to $\mathrm{SL}_*^{\varepsilon=-1}(2, A)$, when A is simple artinian ring with involution.

In this work, we construct a generalized Weil representation of finite split orthogonal groups $\mathrm{O}_q(2n, 2n)$, using the method described in [3]. As mentioned above, this group is naturally the group $\mathrm{SL}_{\diamond}^{\varepsilon=1}(2, M_{2n}(\mathbb{F}_q))$. However one of the results of this paper is to realize this group as a $\mathrm{SL}_{\sim}^{\varepsilon=-1}(2, M_{2n}(\mathbb{F}_q))$ group, where $\sim$ is a certain specific involution in $M_{2n}(\mathbb{F}_q)$, different from $\diamond$. This allows to use the Bruhat-like presentation that is exhibited in [8] and facilitate significantly technical aspects of the construction. In fact, the result is more general, and provides an isomorphism between the groups $\mathrm{SL}_*^{+}(2, M_2(A_0))$ and $\mathrm{SL}_{\sim}^{-}(2, M_2(A_0))$, where $(A_0, *)$ is a unitary involutive ring and $\sim$ is another involution in A_0 obtained from $*$.

Also, we study the structure of the associated unitary group and using this group we get an initial decomposition of the representation.

In section 6 we show compatibility of the method of Gutiérrez, Pantoja and Soto-Andrade with theory of dual pairs. For this, we prove that the representation of $\mathrm{O}_q(2n, 2n)$ constructed using this method is equal to the restriction of the Weil representation to $\mathrm{O}_q(2n, 2n)$ for the dual pair $(\mathrm{Sp}(2, k), \mathrm{O}_q(2n, 2n))$. Also, we prove that the initial decomposition mentioned above is the same as decomposition with respect to the action of $\mathrm{Sp}(2, k)$ via the Weil representation.

The author would like to thank the referee for the valuable comments on this paper.

2. The groups $\mathrm{SL}^{\varepsilon}_{*}(2, A)$.

Let $(A, *)$ be a unitary ring with an involution $*$. We can extend the involution $*$ in A to the ring $M_2(A)$ putting

$$T^* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

We consider $\varepsilon = \pm 1 \in A$ and $J_\varepsilon = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \in M_2(A)$. Let us denote by H_ε the associated ε-hermitian form defined by the matrix J_ε.

Definition 2.1. The group $\mathrm{SL}^{\varepsilon}_{*}(2, A)$ is the set of all automorphisms g of the A-module $M = A \times A$ such that $H_\varepsilon \circ (g \times g) = H_\varepsilon$. In matrix form:

$$\mathrm{SL}^{\varepsilon}_{*}(2, A) = \{T \in M_2(A) \mid TJ_\varepsilon T^* = J_\varepsilon\}.$$

Remark 2.1. In [10] it is shown that a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$ is in $\mathrm{SL}^{\varepsilon}_{*}(2, A)$ if and only if the following equalities hold:

1. $ab^* = -\varepsilon ba^*$;
2. $cd^* = -\varepsilon dc^*$;
3. $a^*c = -\varepsilon c^*a$;
4. $b^*d = -\varepsilon d^*b$;
5. $ad^* + \varepsilon bc^* = a^*d + \varepsilon c^*b = 1$.

We note that if $(A, \diamond)$ is the matrix ring $M_m(\mathbb{F}_q)$ with the transpose involution $\diamond$, then $\mathrm{SL}^{-1}_{\diamond}(2, A)$ is the symplectic group $\mathrm{Sp}(2m, \mathbb{F}_q)$ defined over $\mathbb{F}_q$. On the other hand $\mathrm{SL}^{1}_{\diamond}(2, A)$ gives the split orthogonal group $\mathrm{O}_q(m, m)$.

In what follows we put $\mathrm{SL}^{+}_{*}(2, A) = \mathrm{SL}^{1}_{*}(2, A)$, $\mathrm{SL}^{-}_{*}(2, A) = \mathrm{SL}^{-1}_{*}(2, A)$ and $\diamond$ always will be the transpose involution in a matrix ring.

Let $(A_0, *)$ be a unitary ring with involution $*$ and $A = M_2(A_0)$. Let us consider the matrix $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in A^\times$, which satisfies $J^{-1} = J^* = -J$. Using the matrix J we can define a new involution in A, namely, $a^\sim = Ja^*J^{-1}$. Let us consider $M_2(A)$ provided with the involutions $*$ and $\sim$ inherited from A.

Theorem 2.2. *The groups* $\mathrm{SL}^{+}_{*}(2, A)$ *and* $\mathrm{SL}^{-}_{\sim}(2, A)$ *are isomorphic.*

Proof. Let $U = \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \in \mathrm{GL}_2(A)$. A direct computation proves that $T^\sim U = UT^*$ for all $T \in M_2(A)$. It is clear that $TJ_-T^\sim = J_-$ if and only if $TJ_-UT^* = J_-U$. Then we must show that J_-U and J_+ are equivalent. In fact, the (orthogonal) matrix $P = \begin{pmatrix} 0 & J \\ 1 & 0 \end{pmatrix} \in M_2(A)$ satisfies $PJ_+P^* = J_-U$. ■

Although the split orthogonal group is naturally a "SL^+"- group, in practical terms it is better to look at it as a "SL^-"-group, because this fact will greatly facilitate technical aspects.

Corollary 2.3. *The split orthogonal group* $\mathrm{O}_q(2n,2n)$ *is isomorphic to the group* $\mathrm{SL}^-_\sim(2, M_{2n}(\mathbb{F}_q))$, *where the involution* $\sim$ *in* $M_{2n}(\mathbb{F}_q)$ *is given by* $a^\sim = J_{2n}a^\diamond J_{2n}^{-1}$. ($J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n}(\mathbb{F}_q)$).

Proof. Taking $(A_0, *)$ as the involutive ring $(M_n(\mathbb{F}_q), \diamond)$ and using Theorem 2.2, we get that the groups $\mathrm{SL}^+_\diamond(2, M_{2n}(\mathbb{F}_q))$ and $\mathrm{SL}^-_\sim(2, M_{2n}(\mathbb{F}_q))$ are isomorphic. ∎

3. A Bruhat presentation for $\mathrm{SL}^\varepsilon_*(2, A)$.

Let A be a unitary ring with involution $*$ .We will write $A^s_{\varepsilon,*}$ to denote the set of all ε- symmetric elements in A respect to the involution $*$. Namely;

$$A^s_{\varepsilon,*} = \{a \in A \mid a^* = -\varepsilon a\}.$$

In order to facilitate the notation, we put $A^s_{+,*} = A^s_{1,*}$ and $A^s_{-,*} = A^s_{-1,*}$.
Let us consider

$$h_t = \begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix} (t \in A^\times), \quad w = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}, \quad u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} (s \in A^s_{\varepsilon,*}).$$

Definition 3.1. We will say that $\mathrm{SL}^\varepsilon_*(2, A)$ has a Bruhat presentation if it is generated by the above elements with defining relations:

1. $h_t h_{t'} = h_{tt'}$, $u_s u_{s'} = u_{s+s'}$;
2. $w^2 = h_\varepsilon$;
3. $h_t u_s = u_{tst^*} h_t$;
4. $w h_t = h_{t^{*-1}} w$;
5. $w u_{t^{-1}} w u_{-\varepsilon t} w u_{t^{-1}} = h_{-\varepsilon t}, \quad t \in A^\times \cap A^s_{\varepsilon,*}$.

Remark 3.2. Observing the last relation, we note that in order to have a Bruhat presentation for $\mathrm{SL}^\varepsilon_*(2, A)$ is necessary that $A^\times \cap A^s_{\varepsilon,*} \neq \emptyset$.

In [8] it is proved that if A is a simple artinian ring with involution $*$ that either is infinite or isomorphic to the full matrix ring over $\mathbb{F}_q$ with $q > 3$, then the group $\mathrm{SL}^-_*(2, A)$ has a Bruhat presentation.

Thus, the group $\mathrm{SL}^-_\sim(2, M_{2n}(\mathbb{F}_q))$ mentioned in Corollary 2.3 has a Bruhat presentation if $q > 3$. We will use this fact to construct the desired representation.

4. A generalized Weil representation for the split orthogonal group $\mathbf{O}_q(2n,2n)$.

In this section, our aim is to construct a Weil representation for the split orthogonal group seen as the group $\mathrm{SL}^{-}_{\sim}(2, M_{2n}(\mathbb{F}_q))$. One way to this goal is to construct the representation using the Bruhat presentation. For this purpose we will use the result that follows ([3]).

Let A be a ring with an involution $*$. Let us suppose that the ring A is finite and that the group $G = \mathrm{SL}^{\varepsilon}_{*}(2, A)$ has a Bruhat presentation. Let M be a finite right A-module and let us consider the following data:

1. A bi-additive function $\chi : M \times M \longrightarrow \mathbb{C}^{\times}$ and a character $\alpha \in \widehat{A}^{\times}$ such that for all $x, y \in M$, $t \in A^{\times}$:
   (a) $\chi(xt, y) = \alpha(tt^*)\chi(x, yt^*)$
   (b) $\chi(y, x) = \chi(-\varepsilon x, y)$
   (c) $\chi(x, y) = 1$ for all $x \in M \Rightarrow y = 0$.
2. A function $\gamma : A^s_{\varepsilon,*} \times M \longrightarrow \mathbb{C}^{\times}$ such that for all $s, s' \in A^s_{\varepsilon,*}$, $x, z \in M$, $r \in A^{\times}$, $t \in A^s_{\varepsilon,*} \cap A^{\times}$:
   (a) $\gamma(s + s', x) = \gamma(s, x)\gamma(s', x)$
   (b) $\gamma(s, xr) = \gamma(rsr^*, x)$
   (c) $\gamma(t, x + z) = \gamma(t, x)\gamma(t, z)\chi(x, zt)$.
3. $c \in \mathbb{C}^{\times}$ such that $c^2|M| = \alpha(\varepsilon)$, and for all $t \in A^s_{\varepsilon,*} \cap A^{\times}$ the following equality holds:
$$\sum_{y\in M} \gamma(t, y) = \frac{\alpha(\varepsilon t)}{c}\,.$$

Theorem 4.1. (Gutiérrez, Pantoja and Soto-Andrade, [3]) *Let M be a finite right A-module. Denote $L^2(M)$ the vector space of all complex-valued functions on M, endowed with the inner product with respect to the counting measure on M. Set:*

1. $\rho(h_t)(f)(x) = \overline{\alpha}(t)f(xt)$, $\quad f \in L^2(M), t \in A^{\times}, x \in M$;
2. $\rho(u_s)(f)(x) = \gamma(s, x)f(x)$, $\quad f \in L^2(M), b \in A^s_{\varepsilon,*}, x \in M$;
3. $\rho(w)(f)(x) = c\sum_{y\in M} \chi(-\varepsilon x, y)f(y)$, $\quad f \in L^2(M), x \in M$

(where $\overline{\alpha}$ denotes the complex conjugate of the character α). These formulas define a unitary linear representation $(L^2(M), \rho)$ of G, called the generalized Weil representation of G associated to the data $(M, \alpha, \gamma, \chi)$.

Remark 4.2. Let us note that this definition contains the classic Weil representation of $\mathrm{SL}_2(K)$, where K is a field (see [11], for instance).

In what follows, we will focus on finding the necessary data to construct a generalized Weil representation for $\mathrm{O}_q(2n, 2n)$.

From now on we put $k = \mathbb{F}_q$, $A = M_{2n}(k)$ and we consider q odd greater than 3.

We will apply Theorem 4.1 to the group $\mathrm{SL}^-_\sim(2, A) \cong \mathrm{O}_q(2n, 2n)$. To do this, we recall the following fact. Let E be a finite dimensional vector space over a field K. In [5], the authors describe a correspondence between the linear anti-automorphisms of $\mathrm{End}_K(E)$ and the equivalence classes of nondegenerate bilinear forms on E modulo multiplication by a factor in $K^\times$. Under this correspondence, K-linear involutions on $\mathrm{End}_K(E)$ correspond to nondegenerate bilinear forms which are either symmetric or skew-symmetric. Let B be a nondegenerate bilinear form. The aforementioned correspondence associates B with σ_B, where σ_B is a linear anti-automorphism in $\mathrm{End}_K(E)$ defined by the following equality:

$$B(f(x), y) = B(x, \sigma_B(f)y), \qquad f \in \mathrm{End}_K(E),\ x, y \in E. \tag{1}$$

Now, let V a vector space of k-dimension $2n$. We fix a basis for V in order to identify $M_{2n}(k) \simeq \mathrm{End}_k(V)$.
Let $\langle\ ,\ \rangle : V \times V \longrightarrow k$ be the nondegenerate symmetric bilinear form given by the standard dot product. We consider the nondegenerate skew-symmetric bilinear form $[\ ,\ ] : V \times V \longrightarrow k$, given by $[x, y] = \langle x, yJ_{2n}\rangle$. According to the correspondence between involutions and nondegenerate bilinear forms described above, the symmetric bilinear form $\langle\ ,\ \rangle$ corresponds to the transpose involution $\diamond$. Similarly, the skew-symmetric bilinear form $[\ ,\ ]$ corresponds to the new involution $\sim$. That is, for all $x, y \in V$ and $a \in A$:

$$\langle xa, y\rangle = \langle x, ya^\diamond\rangle\,, \tag{2}$$
$$[xa, y] = [x, ya^\sim]\,. \tag{3}$$

Now, let ψ be a nontrivial character of k^+. Using the notation above let us consider:

1. M the right $A-$module V^2 action: $(x, y)a = (xa, ya)$, $a \in A$, $x, y \in V$.
2. $\chi : M \times M \longrightarrow \mathbb{C}^\times$, $\chi((x, y), (v, z)) = \psi([x, z] - [y, v])$.
3. α the trivial character of $A^\times$.
4. $\gamma : A^s_{-,\sim} \times M \longrightarrow \mathbb{C}^\times$, $\gamma(u, (x, y)) = \psi([xu, y])$.

Lemma 4.3. *For all $u \in A^\times \cap A^s_{-,\sim}$, the map $Q_u : V^2 \longrightarrow k$ given by $Q_u((x, y)) = [xu, y]$ is a nondegenerate split quadratic form. Furthermore, for $u, u' \in A^\times \cap A^s_{-,\sim}$ the quadratic forms Q_u and $Q_{u'}$ are equivalent.*

Proof. Let $\lambda \in k, (x, y), (v, z) \in V^2$. Clearly $Q_u(\lambda(x, y)) = \lambda^2 Q_u((x, y))$. We will prove that $B((x, y), (v, z)) = Q_u((x + v, y + z)) - Q_u((x, y)) - Q_u((v, z))$ is a symmetric nondegenerate bilinear form. We have:

$$B((x, y), (v, z)) = [xu + vu, y + z] - [xu, y] - [vu, z] = [xu, z] + [vu, y].$$

Now:

$$\begin{aligned}B((x,y)+(r,t),(v,z)) &= [(x+r)u,z]+[vu,(y+t)]\\ &= [xu,z]+[ru,z]+[vu,y]+[vu,t]\\ &= B((x,y),(v,z))+B((r,t),(v,z))\,,\\ B(\lambda(x,y),(v,z)) &= [\lambda xu,z]+[vu,\lambda y]\\ &= \lambda[xu,z]+\lambda[vu,y]\\ &= \lambda B((x,y),(v,z)).\end{aligned}$$

Then, B is a symmetric bilinear form.

Let us suppose that $B((x,y),(v,z))=0$ for all $(v,z)\in V^2$. If we choose $v=0$, then $[xu,z]=0$ for all $z\in V$. Since $[\ ,\ ]$ is nondegenerate and u is invertible, we get $x=0$. Similarly $y=0$. Therefore, B is nondegenerate.

Now, if $u,u'\in A^\times\cap A^s_{-,\sim}$ then $uJ_{2n}^\diamond$ and $u'J_{2n}^\diamond$ are invertible skew symmetric matrices. In fact, if $u\in A^s_{-,\sim}$ then $u^\sim=J_{2n}u^\diamond J_{2n}^{-1}=u$. Also $J_{2n}^\diamond=-J_{2n}$, so we get that $(uJ_{2n}^\diamond)^\diamond=J_{2n}u^\diamond=uJ_{2n}=-uJ_{2n}^\diamond$. Thus $uJ_{2n}^\diamond$ and $u'J_{2n}^\diamond$ represent a nondegenerate skew symmetric bilinear form, therefore they are equivalent. So, there exists $j\in A^\times$ such that $uJ_{2n}^\diamond=ju'J_{2n}^\diamond j^\diamond$. Thus, $Q_{u'}((xj,yj))=[xju',yj]=Q_u((x,y))$.

If we choose $u=I_{2n}$, the quadratic form Q_u is represented by the matrix $\begin{pmatrix}0 & -J_{2n}\\ J_{2n} & 0\end{pmatrix}$. Thus, Q_u is split. ■

Theorem 4.4. *The data (M,α,γ,χ) describe a Generalized Weil Representation for $G=\mathrm{O}_q(2n,2n)$. Furthermore, this representation is independent of the choice of the character ψ.*

Proof. We check that χ satisfies the corresponding conditions. Let $(x,y),(v,z)\in M$, $a\in A$. Then:

(a)
$$\begin{aligned}\chi((x,y)a,(v,z)) &= \psi([xa,z]-[ya,v])\\ &= \psi([x,za^\sim]-[y,va^\sim])\,, \qquad \text{(by (3))}\\ &= \chi((x,y),(v,z)a^\sim).\end{aligned}$$

(b)
$$\begin{aligned}\chi((v,z),(x,y)) &= \psi([v,y]-[z,x])\\ &= \psi([x,z]-[y,v])\\ &= \chi((x,y),(v,z)).\end{aligned}$$

(c) Let us suppose that $\chi((x,y),(v,z))=1$ for all $(v,z)\in M$. If $v=0$, then $\psi([x,z])=1$ for all $z\in V$. If $x\neq 0$, then $[x,\ \cdot\ ]:V\longrightarrow k$ is a nontrivial linear functional. Therefore it is surjective. Let $\lambda\in k$ such that $\psi(\lambda)\neq 1$, and $t=t(\lambda)\in V$ such that $\lambda=[x,t]$, then we get the following contradiction:

$$1=\psi([x,t])=\psi(\lambda).$$

Therefore $x=0$, and similarly $y=0$.

Now, we will prove that γ satisfies the corresponding properties. Let $u, u' \in A^s_{-,\sim}$, $a \in A^\times$, $(x,y),(v,z) \in M$:

$$\begin{aligned}
&\text{(a)} \quad \gamma(u+u',(x,y)) = \psi([xu+xu',y]) \\
&\qquad = \psi([xu,y])\psi([xu',y]) \\
&\qquad = \gamma(u,(x,y))\gamma(u',(x,y)). \\
&\text{(b)} \quad \gamma(u,(x,y)a) = \psi([xau,ya]) \\
&\qquad = \psi([xaua^\sim,y]) \\
&\qquad = \gamma(aua^\sim,(x,y)). \\
&\text{(c)} \quad \gamma(u,(x,y)+(v,z)) = \psi([(x+v)u,y+z]) \\
&\qquad = \psi([xu,y])\psi([xu,z])\psi([vu,y])\psi([vu,z]) \\
&\qquad = \gamma(u,(x,y))\gamma(u,(v,z))\psi([x,zu]-[y,vu]) \\
&\qquad = \gamma(u,(x,y))\gamma(u,(v,z))\chi((x,y),(v,z)u).
\end{aligned}$$

Now, we must choose $c \in \mathbb{C}^\times$ satisfying $c^2|M| = 1$ and show that for $u \in A^\times \cap A^s_{-,\sim}$ the following equality holds:

$$\sum_{(x,y)\in M} \gamma(u,(x,y)) = \sum_{x,y\in V} \psi([xu,y]) = \frac{1}{c}.$$

According to Lemma 4.3, we know that $\sum_{(x,y)\in M} \gamma(u,(x,y))$ is a Gauss sum associated to a split quadratic form in a vector space of even dimension $4n$. This sum is calculated, for instance, in [11]. In fact, $\sum_{(x,y)\in M} \gamma(u,(x,y)) = q^{2n}$. We choose $c = \frac{1}{q^{2n}}$. Thus, $\sum_{(x,y)\in M} \gamma(u,(x,y)) = \frac{1}{c}$.
Now, let ψ_1 and ψ_2 be two nontrivial characters of k^+. Let us prove that the corresponding representations are isomorphic.

Let $\lambda \in k^\times$ such that $\psi_2(r) = \psi_1(\lambda r)$ for all $r \in k$. Let $(L^2(M),\rho_1)$ and $(L^2(M),\rho_2)$ the Weil representations obtained from ψ_1 and ψ_2 respectively. Then, the linear automorphism $\Psi : L^2(M) \longrightarrow L^2(M)$ given by $(\Psi f)(x,y) = f(x,\lambda y)$ is a isomorphism between the representations $(L^2(M),\rho_1)$ and $(L^2(M),\rho_2)$. ■

5. An initial decomposition.

Definition 5.1. The group $\mathrm{U}(\gamma,\chi)$ is the group of all A-linear automorphisms β of M such that:

1. $\gamma(u,\beta(x,y)) = \gamma(u,(x,y))$ for all $u \in A^s_{\varepsilon,*}, (x,y) \in M$.
2. $\chi(\beta(x,y),\beta(v,z)) = \chi((x,y),(v,z))$ for all $(x,y),(v,z) \in M$.

In what follows we will denote $\mathrm{U}(\gamma,\chi)$ simply by U.

Following the idea of [3], if we know the structure of the group U and the set of its irreducible representations, we can find an *initial* decomposition of the Weil Representation in the sense that we do not know if the components obtained are irreducible. In what follows, we make this decomposition explicit.

For $\beta \in \mathrm{U}$ and $x \in M$ we put $\beta.x = \beta(x)$. The group U acts naturally on $L^2(M)$. That is to say the action is given by:

$$\sigma : \mathrm{U} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(L^2(M)), \sigma_\beta(f)(x) = f(\beta^{-1}.x).$$

In [3] it is shown that the natural action of U on $L^2(M)$ commutes with the action of the Weil Representation.

Let $\widehat{\mathrm{U}}$ be the set of the irreducible representations of U. We consider the isotypic decomposition of $L^2(M)$ with respect to U:

$$L^2(M) \cong \bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} n_\pi V_\pi.$$

Since $n_\pi = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{U}}(V_\pi, L^2(M))) = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi))$, we can write this decomposition in the following way:

$$L^2(M) \cong \bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} (\mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi) \otimes_{\mathbb{C}} V_\pi).$$

If we put $m_\pi = \dim_{\mathbb{C}}(V_\pi)$, we get:

$$L^2(M) \cong \bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} m_\pi \mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi).$$

If $(V_\pi, \pi) \in \widehat{\mathrm{U}}$ and $\beta \in \mathrm{U}$, we denote by π_β the map $\pi(\beta) : V_\pi \longrightarrow V_\pi$. The space $\mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi)$ is formed by linear functions $\Theta : L^2(M) \longrightarrow V_\pi$ such that for any $\beta \in \mathrm{U}$

$$\Theta \circ \sigma_\beta = \pi_\beta \circ \Theta. \tag{4}$$

Let us consider the Delta functions $\{e_x \mid x \in M\}$ and the map $\theta : M \longrightarrow V_\pi$ such that $\theta(x) = \Theta(e_x)$ for all $x \in M$. Since $\sigma_\beta(e_x) = e_{\beta.x}$, condition (4) becomes:

$$\theta(\beta.x) = \pi_\beta \circ \theta(x). \tag{5}$$

Conversely, let $\theta : M \longrightarrow V_\pi$ satisfying (5). We extend linerarly and we get a map $\Theta : L^2(M) \longrightarrow V_\pi$ such that (4) holds.

Thus, we can see the space $\mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi)$ as the function space formed by maps $\theta : M \longrightarrow V_\pi$ such that $\theta(\beta.x) = \pi_\beta \circ \theta(x)$ for all $\beta \in \mathrm{U}, x \in M$. The group $G = \mathrm{SL}_*^\varepsilon(2, A)$ acts on this space via the Weil representation, using the same formulas as defined in Theorem 4.1. Similarly, it is possible to define the natural action of the group U in this space, because–like $L^2(M)$–it is formed for functions with domain M.

Let ρ denote the Weil action of G on $L^2(M)$ and $\widehat{\rho}$ the Weil action of G on $\bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} m_\pi \mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi)$. Because of how we define the Weil representation, there exist scalars $K_g(x, y) \in \mathbb{C}$, depending only on $g \in G$ and $x, y \in M$, such that for all $f \in L^2(M)$ and $\Lambda \in \bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} m_\pi \mathrm{Hom}_{\mathrm{U}}(L^2(M), V_\pi)$ the following statements hold:

$$\rho_g(f) = \sum_{y\in M} K_g(\cdot, y) f(y); \tag{6}$$

$$\widehat{\rho}_g(\Lambda) = \sum_{y\in M} K_g(\cdot, y) \Lambda(y). \tag{7}$$

In this way, we get:

Lemma 5.2. *$(L^2(M),\rho)$ and $(\bigoplus_{(V_\pi,\pi)\in\widehat{\mathrm{U}}} m_\pi \mathrm{Hom}_{\mathrm{U}}(L^2(M),V_\pi),\widehat{\rho})$ are isomorphic representations of G.*

Finally, we have:

Proposition 5.3. *The space $\mathrm{Hom}_{\mathrm{U}}(L^2(M),V_\pi)$ is invariant under the Weil action of G.*

Proof. Let $g\in G$, $\theta\in\mathrm{Hom}_{\mathrm{U}}(L^2(M),V_\pi)$, $\beta\in\mathrm{U}$, $x\in M$. Then:

$$\begin{aligned}(\widehat{\rho}_g\theta)(\beta.x) &= \sigma_{\beta^{-1}}(\widehat{\rho}_g\theta)(x), && \text{(definition of } \sigma_\beta)\\ &= \widehat{\rho}_g(\sigma_{\beta^{-1}}\theta)(x)\\ &= \widehat{\rho}_g(\pi_\beta\circ\theta)(x), && \text{(since } \sigma_{\beta^{-1}}(\theta)=\pi_\beta\circ\theta)\\ &= \pi_\beta(\widehat{\rho}_g\theta(x)).\end{aligned}$$

The last equality holds because of (7). ■

Now, having made the decomposition above explicit, our purpose is to obtain an initial decomposition for our case $G=\mathrm{O}_q(2n,2n)\cong\mathrm{SL}^-_\sim(2,M_{2n}(k))$. For this, it is enough to know the structure of the group U and the set of irreducible representations.

Remark 5.4. We note that since in our case $A=M_{2n}(k)$ and $A^s_{-,\sim}\cap A^\times\neq\emptyset$, the first condition in definition (5.1) implies the second one (see [3]).

Theorem 5.5. *Let γ and χ be the functions defined above. Then $\mathrm{U}(\gamma,\chi)\cong\mathrm{SL}_2(k)$.*

Proof. Let $\beta\in\mathrm{U}(\gamma,\chi)$. In particular β is k-linear, therefore we can suppose that $\beta\in M_{4n}(k)$. We can write the action of A on M in matrix language as follows:

$$\begin{pmatrix}x & y\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}=\begin{pmatrix}xa & ya\end{pmatrix}, \qquad x,y\in V,\ a\in A.$$

Since β is A-linear we have that $\beta(x,y)a=\beta(xa,ya)$. In matrix form:

$$\beta\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}=\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\beta. \tag{8}$$

Let $\beta_1,\beta_2,\beta_3,\beta_4\in A$ such that $\beta=\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix}$. Then, using (8), we get that each of these blocks must be scalar. Thus, there are $b_1,b_2,b_3,b_4\in k$ such that $\beta=\begin{pmatrix}b_1I_{2n} & b_2I_{2n}\\ b_3I_{2n} & b_4I_{2n}\end{pmatrix}$ and hence $\gamma(u,\beta(x,y))=\psi([(b_1x+b_3y)u,b_2x+b_4y])$. Let us note that the bilinear form $(x,y)\mapsto[xu,y]$ is skew symmetric for all $u\in A^s_{-,\sim}$, hence $[xu,x]=0$ for all $x\in V,u\in A^{sym}$. Thus for all $x,y\in V,u\in A^s_{-,\sim}$

$$\gamma(u,\beta(x,y))=\psi((b_1b_4-b_2b_3)[xu,y])=\psi([xu,y])=\gamma(u,(x,y)).$$

Consequently $\psi((b_1b_4-b_2b_3-1)[xu,y])=1$ for all $x,y\in V,u\in A^s_{-,\sim}$. From this last equality it follows that $b_1b_4-b_2b_3=1$. In fact, let $u\in A^s_{-,\sim}\cap A^\times$,

$x \neq 0$ and let us suppose $b_1b_4 - b_2b_3 - 1 \neq 0$. The map $F_{x,u} : V \longrightarrow k$ given by $F_{x,u}(z) = [(b_1b_4 - b_2b_3 - 1)xu, z]$ is a nontrivial linear functional and therefore is surjective. Let $\lambda \in k$ such that $\psi(\lambda) \neq 1$ and $z = z(\lambda) \in V$ such that $\lambda = [(b_1b_4 - b_2b_3 - 1)xu, z]$. Then $\psi(\lambda) = \psi([(b_1b_4 - b_2b_3 - 1)xu, z]) = 1$. This contradicts our assumption and therefore our result follows. ■

Thus, for our case, we get an initial decomposition of the Weil Representation $(L^2(M), \rho)$. We expect to address the question about irreducibility elsewhere.

6. Dual Pairs.

In this section we will prove that the representation $(L^2(M), \rho)$ of $O_q(2n, 2n)$ constructed in section 4 is equal to the restriction of the Weil representation to $O_q(2n, 2n)$ for the dual pair $(Sp(2, k), O_q(2n, 2n))$. Also, we will prove that the initial decomposition described above is the same as decomposition with respect to the action of $Sp(2, k)$ via the Weil representation.

Let $J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in M_{2n}(k)$ and $F = \begin{pmatrix} 0 & J_{2n} \\ -J_{2n} & 0 \end{pmatrix} \in M_{4n}(k)$. The matrix F defines the following nondegenerate split symmetric bilinear form in $V_1 = k^{4n}$:

$$(u, v) = v^t F u\,, \qquad u, v \in V_1.$$

The group G of isometries of this form is isomorphic to the split orthogonal group $O_q(2n, 2n)$. As before, set

$$a^{\sim} = J_{2n} a^t J_{2n}, \qquad a \in M_{2n}(k).$$

A direct calculation shows that the following matrices belong to the group G:

$$h_a = \begin{pmatrix} a & 0 \\ 0 & (a^{\sim})^{-1} \end{pmatrix}, \quad w = \begin{pmatrix} 0 & I_{2n} \\ -I_{2n} & 0 \end{pmatrix}, \quad u_s = \begin{pmatrix} I_{2n} & s \\ 0 & I_{2n} \end{pmatrix}$$

$(a \in M_{2n}(k)^{\times}, s = s^{\sim} \in M_{2n}(k))$.

Therefore $G = SL_{\sim}^{-}(2, M_{2n}(k))$.

Let $V_2 = k^2$ and $W = \mathrm{Hom}(V_1, V_2)$. The following formula defines a nondegenerate symplectic form on W:

$$\langle\!\langle w_1, w_2 \rangle\!\rangle = \mathrm{tr}(w_1 F w_2^t J_2)\,, \qquad w_1, w_2 \in W\,.$$

The group G acts on W by

$$g(w_1) = w_1 g^{-1}\,, \qquad g \in G, w_1 \in W\,.$$

This action preserves the symplectic form $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. In fact, since $g \in G$,

$$\langle\!\langle w_1 g^{-1}, w_2 g^{-1} \rangle\!\rangle = tr(w_1 g^{-1} F (g^{-1})^t w_2^t J_2) = \mathrm{tr}(w_1 F w_2^t J_2) = \langle\!\langle w_1, w_2 \rangle\!\rangle.$$

Let

$$X = \{(x, 0) \mid x \in M_{2,2n}(k)\}, \qquad Y = \{(0, y) \mid y \in M_{2,2n}(k)\}.$$

Then $W = X \oplus Y$ is a complete polarization. We will consider the Schrödinger model of the Weil representation of $\mathrm{Sp}(W)$ attached to the above complete polarization realized on $L^2(X)$ as in [1]. Let $(L^2(X), \omega)$ such representation.

We identify X with $M_{2,2n}(k)$ in the canonical way

$$X \ni (x, 0) \leftrightsquigarrow x \in M_{2,2n}(k).$$

Remark 6.1. Let us note that the module M in section 4 is canonically isomorphic to X. Consequently the spaces $L^2(M)$ and $L^2(X)$ are also isomorphic.

Let ψ a nontrivial character of the additive group k^+. For all $x \in X$, $y \in Y$ it is clear that $h_a(x) = xh_a^{-1} \in X$ and $h_a(y) = yh_a^{-1} \in Y$, then the matrix h_a preserves X and Y. Also, $\det(h_a|_X) \in k^{\times 2}$. Thus, Proposition 34 in [1] shows that

$$\omega(h_a)f(x) = f(xa)\,, \qquad f \in L^2(X)\,.$$

Thus, $\omega(h_a) = \rho(h_a)$.

Now, let us see the action of ω on u_s. The matrix u_s acts trivially on Y and on W/Y. Therefore, Proposition 35 in [1] shows that

$$\omega(u_s)f(x) = \psi(\langle\!\langle xc(-u_s), x\rangle\!\rangle)f(x),$$

where $c(-u_s) = \begin{pmatrix} 0 & -s/2 \\ 0 & 0 \end{pmatrix} \in M_{4n}(k)$ is the Cayley transform for $-u_s$.

Let $x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix} \in X$, $x_1, x_2 \in k^{2n}$. Then,

$$\langle\!\langle xc(-u_s), x\rangle\!\rangle = x_2 s J_{2n} x_1^t.$$

In order to prove that $\omega(u_s) = \rho(u_s)$ we have to check that

$$[x_1 s, x_2] = \langle\!\langle xc(-u_s), x\rangle\!\rangle,$$

where $[\ ,\ ]$ is the symplectic form defined in section 4. In fact,

$$[x_1 s, x_2] = [x_1, x_2 s] = -[x_2 s, x_1] = x_2 s J_{2n} x_1^t$$

It is clear that the matrix w maps X bijectively onto Y and Y onto X, and $w^2 = -1$. Then, using Proposition 36 of [1], we get:

$$\omega(w)f(x) = \frac{1}{\sqrt{|X|}} \sum_{x' \in X} \psi(\langle\!\langle w(x), x'\rangle\!\rangle)f(x')\,.$$

Thus, in order to prove that $\omega(w) = \rho(w)$ we have to check that

$$\chi(x, x') = \psi(\langle\!\langle xw^{-1}, x'\rangle\!\rangle).$$

Let $x = \begin{pmatrix} x_1 & 0 \\ x_2 & 0 \end{pmatrix}$, $x' = \begin{pmatrix} x_1' & 0 \\ x_2' & 0 \end{pmatrix}$, $x_1, x_2, x_1', x_2' \in k^{2n}$. So,

$$\psi(\langle\!\langle xw^{-1}, x'\rangle\!\rangle) = \psi(x_2 J_{2n}(x_1')^t - x_1 J_{2n}(x_2')^t).$$

On the other hand,

$$\begin{aligned}\chi(x,x') &= \psi([x_1,x_2'] + [x_1',x_2]) \\ &= \psi((x_1 \quad x_2)\begin{pmatrix} 0 & -J_{2n} \\ J_{2n} & 0 \end{pmatrix}\begin{pmatrix} (x_1')^t \\ (x_2')^t \end{pmatrix}) \\ &= \psi(x_2 J_{2n}(x_1')^t - x_1 J_{2n}(x_2')^t)).\end{aligned}$$

Thus, we have shown that the representation constructed in section 4 is equal to the restriction of the Weil representation to $\mathrm{O}_q(2n,2n)$ for the dual pair $(\mathrm{Sp}(2,k),\mathrm{O}_q(2n,2n))$.

Furthermore, since an element $g \in \mathrm{Sp}(2,k) = \mathrm{SL}_2(k)$ preserves X and Y and $\det(g|_X) \in k^{\times 2}$, using Proposition 34 in [1], we get that the group $\mathrm{SL}_2(k)$ acts on $L^2(X)$ as follows:

$$\omega(g)f(x) = f(g^{-1}x), \qquad g \in \mathrm{SL}_2(k),\ f \in L^2(X),\ x \in X.$$

Therefore, the initial decomposition in section 5 is the same as the decomposition with respect to the action of $\mathrm{SL}_2(k)$ via the Weil representation.

Acknowledgments. The author would like to thank José Pantoja, Jorge Soto, Luis Gutiérrez and Anne-Marie Aubert for their never ending support.

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Received November 3, 2013
and in final form June 30, 2014