Journal of Lie Theory
Volume **25** (2015) 271–306


# Matrix Coefficients of Discrete Series Representations of $SU(3,1)$

**Takahiro Hayata, Harutaka Koseki, Tadashi Miyazaki, and Takayuki Oda**

Communicated by G. Olshanski

**Abstract.** In this paper, for large discrete series representations of $SU(3,1)$, we give expressions of the radial parts of their matrix coefficients in terms of the generalized hypergeometric series, and describe their asymptotic behavior, explicitly. Geometrically speaking, this is to obtain an explicit formula for some Hilbert space of non-holomorphic harmonic $L^2$-sections in an $SU(3,1)$-equivariant vector bundle over the complex hyperball of dimension 3.
*Mathematics Subject Classification 2010:* 22E30, 22E45, 43A90.
*Key Words and Phrases:* Matrix coefficients, discrete series.

## 1. Introduction

Matrix coefficients play important roles in the representation theory of Lie groups as well as in the theory of automorphic forms. When the associated Riemannian symmetric space $G/K$ is of Hermitian type, a semisimple Lie group $G$ has holomorphic discrete series representations, and matrix coefficients of these representations at the minimal $K$-type are known explicitly as the Bergman kernel. This expression played the crucial role in the calculation of the dimensions of the spaces of holomorphic automorphic forms on $G/K$ via the Godement-Selberg expression of the dimensions of them. (See Ibukiyama [5] for the introduction of the extensive studies. See Kato [6], [7] and Koseki [9], specifically for the $SU(n,1)$-case.) On the contrary, we know little about automorphic forms belonging to non-holomorphic discrete series representations. We want to have some case study of the counterpart of the Bergman kernel in a case of a non-holomorphic discrete series representation. In this paper, we give explicit formulas of matrix coefficients of each discrete series representation of $SU(3,1)$ at the minimal $K$-type.

Let us explain our problem in a precise form. Let $G = SU(n,1)$ be the special unitary group of signature $(n,1)$ for an integer $n \geq 2$. We fix a Cartan decomposition $G = KAK$, where $K$ is a maximal compact subgroup of $G$ and $A$ is the identity component of a maximal split torus of $G$. Let $(\Pi, H_\Pi)$ be a discrete series representation of $G$, and we denote by $(\Pi^\vee, H_\Pi^\vee)$

ISSN 0949–5932 / \$2.50 

the contragradient representation of $\Pi$. Let $(\tau, V_\tau)$ be the minimal $K$-type of $\Pi$, and we denote by $(\tau^\vee, V_\tau^\vee)$ the contragradient representation of $\tau$. We fix $K$-embeddings $\iota\colon V_\tau \to H_\Pi$ and $\iota^\vee\colon V_\tau^\vee \to H_\Pi^\vee$. We define the $K\times K$-homomorphism $\phi\colon V_\tau^\vee \boxtimes_{\mathbb{C}} V_\tau \to C^\infty(G)$ by $\phi(v^\vee \boxtimes v)(g) = \langle \iota^\vee(v^\vee), \Pi(g)\iota(v)\rangle$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing. For $v^\vee \in V_\tau^\vee$ and $v \in V_\tau$, we call $\phi(v^\vee \boxtimes v)$ a matrix coefficient of $\Pi$ at the minimal $K$-type. Because of the Cartan decomposition $G = KAK$ and the equality $\phi(v^\vee \boxtimes v)(k_1 g k_2) = \phi(\tau^\vee(k_1^{-1})v^\vee \boxtimes \tau(k_2)v)(g)$ $(k_1, k_2 \in K)$, the $K\times K$-homomorphism $\phi$ is characterized by $\phi(v^\vee \boxtimes v)|_A$ $(v^\vee \in V_\tau^\vee,\ v \in V_\tau)$. Tsuzuki [13, Appendix 1] gives the explicit formulas of $\phi(v^\vee \boxtimes v)|_A$ in the case of $n = 2$. The purpose of this paper is to give explicit formulas of $\phi(v^\vee \boxtimes v)|_A$ in the case of $n = 3$.

Since the real rank of $G$ is $1$, one may imagine that the explicit evaluation of spherical functions on $G$ is not difficult. However, there seems to be little explicit results for non-scalar $K$-types in the literature. In order to obtain explicit formulas of matrix coefficients, we should treat both the analytical nature of functions on $G$ and the combinatorial nature stemming from the representations $\tau^\vee$ and $\tau$. Here we introduce a new notion, which works effectively to handle the combinatorial nature. Let $M = Z_K(A)$ be the centralizer of $A$ in $K$, and take the irreducible decomposition $V_\tau = \bigoplus_\nu V_{\tau,\nu}$ as an $M$-module. Then, we also have the irreducible decomposition $V_\tau^\vee = \bigoplus_\nu V_{\tau,\nu}^\vee$ as an $M$-module. Here we identify the dual space $V_{\tau,\nu}^\vee$ of $V_{\tau,\nu}$ with a subspace $\{v^\vee \in V_\tau^\vee \mid \langle v^\vee, v\rangle = 0\ (v \in V_{\tau,\nu'},\ \nu' \neq \nu)\}$ of $V_\tau^\vee$. In §2.6, we show that there exist functions $\{\phi[\nu]\}_\nu$ on $A$ such that

$$\phi(v^\vee \boxtimes v)|_A = \begin{cases} \langle v^\vee, v\rangle \phi[\nu] & \text{if } \nu = \nu', \\ 0 & \text{otherwise} \end{cases}$$

for $v^\vee \in V_{\tau,\nu'}^\vee$ and $v \in V_{\tau,\nu}$. Hence, it suffices to give an explicit formula of $\phi[\nu]$ for each $\nu$. We call $\phi[\nu]$ the $\nu$-component of $\phi$.

In §3, we give explicit formulas of $\{\phi[\nu]\}_\nu$ for each discrete series representation $\Pi$ of $G = SU(3,1)$. This is the main results of this paper. Since the holomorphic and anti-holomorphic discrete series cases are well-known and easily obtained as in Proposition 3.1, our main problem is to give explicit formulas of $\{\phi[\nu]\}_\nu$ when $\Pi$ is large in the sense of Vogan [16, Definition 6.1]. For an index $\nu = (\nu_1, \nu_2)$, we describe the $\nu$-component $\phi[\nu]$ for a large discrete series representation $\Pi$ of $G = SU(3,1)$ in terms of the generalized hypergeometric series (Theorem 3.3):

$$\phi[\nu](a) = a_1^{-\mu_1-\mu_2-\mu_3+\nu_1+\nu_2} {}_3F_2\left( \begin{matrix} \nu_1-\mu_3+2,\ \nu_2-\mu_3+1,\ -\mu_3+2 \\ \mu_1-\mu_3+3,\ \mu_2-\mu_3+2 \end{matrix} ; 1-a_1^2 \right),$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ is the Blattner parameter of $\Pi$ and

$$a = \left(\begin{array}{c|c|c} a_1 & & a_2 \\ \hline & 1_{n-1} & \\ \hline a_2 & & a_1 \end{array}\right) \in A.$$

Here we note that the Blattner parameter $\mu$ is a highest weight of a finite dimensional representation of $K$. Hence it defines a homogeneous vector bundle $E_\mu$ over

$G/K$ with the canonical $G$-action. Then we can define certain Dirac operator on $E_\mu$ called the Schmid operator, and $\Pi$ is realized on the $L^2$-kernel of this operator (cf. Schmid [11]). Using this fact, we write down a holonomic system of differential equations which characterize $\{\phi[\nu]\}_\nu$. This holonomic system is found to be a generalized hypergeometric system of one variable.

In Theorem 3.2, we also give explicit formulas of $\{\phi[\nu]\}_\nu$ for large discrete series representations $\Pi$ of $G = SU(2,1)$, which are found in Appendix 1 of Tsuzuki's paper [13] without a detailed proof. For the sake of completeness, we give the proof of these formulas in §5 of this paper.

Because the asymptotic behaviors of the Bergman kernels of bounded domains have been studied by some people, we are interested in the asymptotic behaviors of our matrix coefficients. Utilizing an advantage that we have explicit formulas in terms of the generalized hypergeometric series, we give a formula of the form

$$\phi[\nu](a) = R_1(a_1)\log(a_1) + R_2(a_1) \qquad (a \in A)$$

with some rational functions $R_1(z)$ and $R_2(z)$ (Corollaries 3.5 and 3.6). This formula is analogous to Fefferman's formula (p.45 of Fefferman [2]). The new point is that our $L^2$-space consists of non-holomorphic harmonic forms.

We have to remark that the same method in this paper can be applicable to the explicit computation of matrix coefficients of principal series representations of $SU(2,1)$ and $SU(3,1)$. In this case, the corresponding formulas should resemble the formulas of Shintani functions for $U(n,1)$ in Tsuzuki [14, Theorem 8.2.1]. We also remark that Taniguchi [12, Proposition 3.2.3] gives explicit formulas for parts of Whittaker functions belonging to large discrete series representations of $SU(n,1)$. We hope that the same method in this paper would give explicit formulas of matrix coefficients of discrete series representations of general $SU(n,1)$. In view of the explicit description of the Langlands lattice of $SU(n,1)$ (See Collingwood [1, figure 4.4]), one might even have the reason of this lattice structure by analyzing the nature of the parameters of generalized hypergeometric series in the formulas of matrix coefficients.

We give the contents of this paper. In §2, we introduce some notation and basic objects of this paper. In §3, we state our main results. In §4, we recall the Gelfand-Tsetlin basis of simple $U(n)$-modules, and give the proof of the formulas for holomorphic discrete series representations of $SU(n,1)$ for an integer $n \geq 2$. In §5 and §6, we give the proof of the formulas for large discrete series representations of $SU(2,1)$ and $SU(3,1)$, respectively.

## 2. Notation and basic objects

In this section, we make a preparation to describe our main results. We introduce some notation and basic objects of this paper. We also review the Cartan-Weyl theory, the Blattner parameterization of discrete series representations and the theory of the generalized hypergeometric series.

**2.1. Notation.** We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. For $l \in \mathbb{Z}$, let $\mathbb{Z}_{\geq l}$ be the set of integers which are no less than $l$, and $\mathbb{Z}_{\leq l}$ be the set of integers which are no more than $l$. For $z \in \mathbb{C}$, we denote the real part and the imaginary part of $z$ by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. We denote by $\delta_{i,j}$ the Kronecker delta, that is,

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let $GL(m,\mathbb{C})$, $SL(m,\mathbb{C})$, $U(m)$ and $SU(m)$ be the complex general linear group, the complex special linear group, the unitary group and the special unitary group of degree $m$, respectively. We denote by $\mathfrak{gl}(m,\mathbb{C})$, $\mathfrak{sl}(m,\mathbb{C})$, $\mathfrak{u}(m)$ and $\mathfrak{su}(m)$ the associated Lie algebras of $GL(m,\mathbb{C})$, $SL(m,\mathbb{C})$, $U(m)$ and $SU(m)$, respectively. For a Lie algebra $\mathfrak{h}$, we denote by $\mathfrak{h}_{\mathbb{C}}$ the complexification $\mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathfrak{h}$. For a Lie group $H$, let $C^\infty(H)$ be the space of complex valued smooth functions on $H$. For a unimodular Lie group $H$, we denote by $L^2(H)$ the space of complex valued measurable functions on $H$ which are square-integrable with respect to the Haar measure.

We denote by $1_m$ the unit matrix of size $m$. Let $E^{(m)}_{i,j}$ be the matrix unit of size $m$ with 1 at the $(i,j)$-th entry and 0 at other entries. For a matrix $X$, we denote by ${}^t X$ and $\overline{X}$ the transpose and the elementwise complex conjugation of $X$, respectively. We use the convention that unwritten entries of a matrix are zero.

**2.2. Basic objects.** The special unitary group $G = SU(n,1)$ of signature $(n,1)$ is realized as

$$G = \{g \in SL(n+1,\mathbb{C}) \mid {}^t\overline{g} 1_{n,1} g = 1_{n,1}\}, \quad 1_{n,1} = \left(\begin{array}{c|c} 1_n & \\ \hline & -1 \end{array}\right).$$

We define a Cartan involution $\theta$ of $G$ by $G \ni g \mapsto {}^t\overline{g}^{-1} \in G$. Then

$$K = \{g \in G \mid \theta(g) = g\} = \left\{ \left(\begin{array}{c|c} u & \\ \hline & (\det u)^{-1} \end{array}\right) \,\middle|\, u \in U(n) \right\} \simeq U(n)$$

is a maximal compact subgroup of $G$.

The associated Lie algebra $\mathfrak{g}$ of $G$ is given by

$$\mathfrak{g} = \mathfrak{su}(n,1) = \{X \in \mathfrak{sl}(n+1,\mathbb{C}) \mid {}^t\overline{X} 1_{n,1} + 1_{n,1} X = 0\}.$$

If we denote the differential of $\theta$ again by $\theta$, then we have $\theta(X) = -{}^t\overline{X}$ for $X \in \mathfrak{g}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and the $-1$ eigenspaces of $\theta$ in $\mathfrak{g}$, respectively. Then $\mathfrak{k}$ is the associated Lie algebra of $K$, and $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Take a compact Cartan subalgebra

$$\mathfrak{t} = \bigoplus_{i=1}^{n} \mathbb{R}\left(\sqrt{-1}\tilde{E}^{(n+1)}_{i,i}\right), \qquad \tilde{E}^{(n+1)}_{i,i} = E^{(n+1)}_{i,i} - E^{(n+1)}_{n+1,n+1}.$$

For $1 \leq i \leq n+1$, define a linear form $e_i$ on $\mathfrak{t}_{\mathbb{C}}$ by

$$e_i(t) = t_i \qquad \text{for } t = \sum_{i=1}^{n+1} t_i E^{(n+1)}_{i,i} \in \mathfrak{t}_{\mathbb{C}}.$$

In this paper, we identify a linear form $\gamma$ on $\mathfrak{t}_{\mathbb{C}}$ with the complex vector

$$(\gamma(\tilde{E}^{(n+1)}_{1,1}),\ \gamma(\tilde{E}^{(n+1)}_{2,2}),\ \cdots,\ \gamma(\tilde{E}^{(n+1)}_{n,n})) \in \mathbb{C}^n.$$

Note that $e_i$ is identified with $(\overbrace{0,\cdots,0}^{i-1},1,\overbrace{0,\cdots,0}^{n-i}) \in \mathbb{C}^n$ for $1 \le i \le n$, and $e_{n+1}$ is identified with $(-1,-1,\cdots,-1) \in \mathbb{C}^n$.

The root system $\Sigma$ of $(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is given by

$$\Sigma = \Sigma(\mathfrak{t}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}) = \{e_i - e_j \mid 1 \le i \ne j \le n+1\},$$

and the root space for $e_i - e_j$ is given by $\mathfrak{g}_{e_i-e_j} = \mathbb{C}E^{(n+1)}_{i,j}$. The subset $\Sigma^+ = \{e_i - e_j \mid 1 \le i < j \le n+1\}$ of $\Sigma$ forms a positive root system, and the set of compact and non-compact positive roots are given by

$$\Sigma^+_{\mathrm{c}} = \{\alpha \in \Sigma^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{k}_{\mathbb{C}}\} = \{e_i - e_j \mid 1 \le i < j \le n\},$$
$$\Sigma^+_{\mathrm{n}} = \{\alpha \in \Sigma^+ \mid \mathfrak{g}_\alpha \subset \mathfrak{p}_{\mathbb{C}}\} = \{e_i - e_{n+1} \mid 1 \le i \le n\},$$

respectively. Then $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ have the following decompositions:

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Sigma_{\mathrm{c}}} \mathfrak{g}_\alpha \qquad \left(\Sigma_{\mathrm{c}} = \Sigma^+_{\mathrm{c}} \cup (-\Sigma^+_{\mathrm{c}})\right),$$

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_- \qquad \left(\mathfrak{p}_+ = \bigoplus_{\alpha \in \Sigma^+_{\mathrm{n}}} \mathfrak{g}_\alpha,\ \ \mathfrak{p}_- = \bigoplus_{\alpha \in \Sigma^+_{\mathrm{n}}} \mathfrak{g}_{-\alpha}\right).$$

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, which is defined by $\mathfrak{a} = \mathbb{R}H_1$ with $H_1 = E^{(n+1)}_{1,n+1} + E^{(n+1)}_{n+1,1}$. We set

$$A = \exp(\mathfrak{a}) = \{\mathrm{a}[t] \mid t \in \mathbb{R}\}$$

with

$$\mathrm{a}[t] = \left(\begin{array}{c|c|c} \mathrm{ch}(t) & & \mathrm{sh}(t) \\ \hline & 1_{n-1} & \\ \hline \mathrm{sh}(t) & & \mathrm{ch}(t) \end{array}\right),$$
$$\mathrm{ch}(t) = \frac{\exp(t)+\exp(-t)}{2}, \qquad \mathrm{sh}(t) = \frac{\exp(t)-\exp(-t)}{2}.$$

Then $G$ admits the Cartan decomposition $G = KAK$. Let $M$ be the centralizer of $A$ in $K$, that is,

$$M = \left\{ \left(\begin{array}{c|c|c} u_1 & & \\ \hline & u_2 & \\ \hline & & u_1 \end{array}\right) \;\middle|\; \begin{array}{l} u_1 \in U(1),\ u_2 \in U(n-1), \\ (u_1)^2 \det u_2 = 1 \end{array} \right\}.$$

**2.3. The Cartan-Weyl theory for $U(n)$.** The Cartan-Weyl theory parametrizes irreducible (continuous) representations of a compact connected Lie group in terms of their highest weights. In this subsection, we recall this theory for $U(n)$. (See Knapp [8, Chapter IV] for details.)

We denote the set of non-increasing $n$-tuples of integers by $\Lambda_n$, that is,

$$\Lambda_n = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}.$$

It is well-known that every irreducible representation $(\tau, V_\tau)$ of $U(n)$ is finite dimensional and has a weight space decomposition

$$V_\tau = \bigoplus_{\gamma=(\gamma_1,\gamma_2,\cdots,\gamma_n)\in\mathbb{Z}^n} V_\tau(\gamma), \qquad V_\tau(\gamma) = \{v \in V_\tau \mid \tau(E_{i,i}^{(n)})v = \gamma_i v,\ 1 \leq i \leq n\}.$$

Here we denote the differential of $\tau$ again by $\tau$. We call $\gamma \in \mathbb{Z}^n$ a weight of $\tau$ if the corresponding weight space $V_\tau(\gamma)$ has nonzero elements. Nonzero vectors in weight spaces are called weight vectors. There exists the weight $\lambda_\tau$ of $\tau$ which satisfies $\lambda_\tau \geq_{\mathrm{lex}} \gamma$ for any weight $\gamma$ of $\tau$ in the lexicographical order. Such $\lambda_\tau$ is contained in $\Lambda_n$ and is called the highest weight of $\tau$. Here the lexicographical order is defined as follows: $(\gamma_1, \gamma_2, \cdots, \gamma_n) >_{\mathrm{lex}} (\gamma_1', \gamma_2', \cdots, \gamma_n')$ if and only if

$$\begin{aligned} & \gamma_1 > \gamma_1' \\ \text{or} \quad & (\gamma_1 = \gamma_1' \text{ and } \gamma_2 > \gamma_2') \\ \text{or} \quad & (\gamma_1 = \gamma_1',\ \gamma_2 = \gamma_2' \text{ and } \gamma_3 > \gamma_3') \\ \text{or} \quad & (\gamma_1 = \gamma_1',\ \gamma_2 = \gamma_2',\ \gamma_3 = \gamma_3' \text{ and } \gamma_4 > \gamma_4') \\ & \quad \vdots \\ \text{or} \quad & (\gamma_1 = \gamma_1',\ \gamma_2 = \gamma_2',\ \gamma_3 = \gamma_3',\ \cdots, \gamma_{n-1} = \gamma_{n-1}' \text{ and } \gamma_n > \gamma_n'). \end{aligned}$$

It is known that $V_\tau(\lambda_\tau)$ is one dimensional and

$$V_\tau(\lambda_\tau) = \{v \in V_\tau \mid \tau(E_{i,j}^{(n)})v = 0\ (1 \leq i < j \leq n)\}.$$

A nonzero vector in $V_\tau(\lambda_\tau)$ is called a highest weight vector.

An irreducible representation of $U(n)$ is characterized by its highest weight, that is, $\tau \simeq \tau'$ if and only if $\lambda_\tau = \lambda_{\tau'}$ for two irreducible representations $\tau$ and $\tau'$ of $U(n)$. Moreover, for each $\lambda \in \Lambda_n$, there exists an irreducible representation of $U(n)$ whose highest weight is $\lambda$. Therefore, the map $\tau \mapsto \lambda_\tau$ induces a bijection from the set of the equivalence classes of irreducible representations of $U(n)$ to the set $\Lambda_n$.

We denote by $(\tau_\lambda^{(n)}, V_\lambda^{(n)})$ the irreducible representation of $U(n)$ whose highest weight is $\lambda \in \Lambda_n$. In this paper, via the isomorphism

$$\kappa\colon K \ni \left(\begin{array}{c|c} u & \\ \hline & (\det u)^{-1} \end{array}\right) \mapsto u \in U(n), \tag{2.1}$$

we also regard $(\tau_\lambda^{(n)}, V_\lambda^{(n)})$ as a representation of $K$.

**2.4. Matrix coefficients.** We regard $C^\infty(G)$ as a $G \times G$-bimodule by the regular action

$$(L(g_1)R(g_2)\varphi)(x) = \varphi(g_1^{-1} x g_2) \qquad (\varphi \in C^\infty(G),\ x \in G,\ (g_1, g_2) \in G \times G).$$

We denote the differentials of $L$ and $R$ again by $L$ and $R$, respectively.

Let $(\Pi, H_\Pi)$ be an irreducible admissible Banach representation of $G$. We denote by $H_{\Pi,K}$ the subspace of $H_\Pi$ consisting of $K$-finite vectors. Let $(\Pi^\vee, H_\Pi^\vee)$ be the contragradient representation of $\Pi$. We define a natural homomorphism $\Phi_\Pi\colon H_{\Pi,K}^\vee \boxtimes_{\mathbb{C}} H_{\Pi,K} \to C^\infty(G)$ of $K\times K$-modules by $\Phi_\Pi(f^\vee \boxtimes f)(g) = \langle f^\vee, \Pi(g)f\rangle$, where $\langle\cdot,\cdot\rangle$ is the canonical pairing on $H_\Pi^\vee \times H_\Pi$. For $f^\vee \in H_{\Pi,K}^\vee$ and $f \in H_{\Pi,K}$, the function $\Phi_\Pi(f^\vee \boxtimes f)$ is called a ($K$-finite) matrix coefficient of $\Pi$. Then we have the equalities

$$\begin{aligned} &L(X)\Phi_\Pi(f^\vee \boxtimes f) = \Phi_\Pi((\Pi^\vee(X)f^\vee) \boxtimes f),\\ &R(X)\Phi_\Pi(f^\vee \boxtimes f) = \Phi_\Pi(f^\vee \boxtimes (\Pi(X)f)) \qquad (X \in \mathfrak{g}_{\mathbb{C}}) \end{aligned}$$

by definition, and we also have the equality

$$\Phi_\Pi(f^\vee \boxtimes f)(g) = \Phi_{\Pi^\vee}(f \boxtimes f^\vee)(g^{-1}) \qquad (g \in G) \tag{2.2}$$

by the duality.

**2.5. The irreducible decompositions of $\tau_\lambda^{(n)}|_M$ and $\tau_\lambda^{(n)\vee}|_M$.** Until the end of this section, we assume $n \geq 2$. Via the embedding

$$\iota_{n-1,n}\colon U(n-1) \ni u \mapsto \left(\begin{array}{c|c} 1 & \\ \hline & u \end{array}\right) \in U(n), \tag{2.3}$$

we regard $U(n-1)$ as a subgroup of $U(n)$. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$. By Proposition A.3 in Zhelobenko [19], we have the irreducible decomposition $\tau_\lambda^{(n)}|_{U(n-1)} \simeq \bigoplus_{\nu \in B(\lambda)} \tau_\nu^{(n-1)}$ with

$$B(\lambda) = \{\nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in \Lambda_{n-1} \mid \lambda_i \geq \nu_i \geq \lambda_{i+1}\ (1 \leq i \leq n-1)\}.$$

Let

$$V_\lambda^{(n)} = \bigoplus_{\nu \in B(\lambda)} V_{\lambda,\nu}^{(n)} \tag{2.4}$$

be the corresponding decomposition of $V_\lambda^{(n)}$, that is, $V_{\lambda,\nu}^{(n)}$ is the subspace of $V_\lambda^{(n)}$ such that $V_{\lambda,\nu}^{(n)} \simeq V_\nu^{(n-1)}$ as $U(n-1)$-modules.

**Lemma 2.1.** *Retain the notation. Then (2.4) is the irreducible decomposition of $V_\lambda^{(n)}$ as an $M$-module. Moreover, for $\nu, \nu' \in B(\lambda)$, it holds that $V_{\lambda,\nu}^{(n)} \simeq V_{\lambda,\nu'}^{(n)}$ as $M$-modules if and only if $\nu = \nu'$.*

**Proof.** Let $Z_{\kappa(M)}$ and $Z_{U(n-1)}$ be the centers of $\kappa(M)$ and $U(n-1)$, respectively. It is easy to see that $Z_{\kappa(M)} = \{\mathrm{z}[u] \mid u \in U(1)\}$ with

$$\mathrm{z}[u] = u^{n-1}\iota_{n-1,n}(u^{-n-1}1_{n-1}) = \mathrm{diag}(u^{n-1}, u^{-2}, u^{-2}, \cdots, u^{-2}).$$

Since an element $\mathrm{z}[u]$ of $Z_{\kappa(M)}$ acts on $V_{\lambda,\nu}^{(n)}$ by

$$\tau_\lambda^{(n)}(\mathrm{z}[u])v = u^{(n-1)(\lambda_1+\lambda_2+\cdots+\lambda_n)-(n+1)(\nu_1+\nu_2+\cdots+\nu_{n-1})}v \qquad (v \in V_{\lambda,\nu}^{(n)}),$$

it holds that $V_{\lambda,\nu}^{(n)} \simeq V_{\lambda,\nu'}^{(n)}$ as $Z_{\kappa(M)}$-modules if and only if $\nu_1+\nu_2+\cdots+\nu_{n-1} = \nu_1'+\nu_2'+\cdots+\nu_{n-1}'$ for $\nu=(\nu_1,\nu_2,\cdots,\nu_{n-1}), \nu'=(\nu_1',\nu_2',\cdots,\nu_{n-1}') \in B(\lambda)$.

Since $U(n-1) = SU(n-1)Z_{U(n-1)}$, the space $V_{\lambda,\nu}^{(n)}$ is a simple $SU(n-1)$-module. Moreover, we see that $V_{\lambda,\nu}^{(n)} \simeq V_{\lambda,\nu'}^{(n)}$ as $SU(n-1)$-modules if and only if $\nu'$ is of the form $\nu' = \nu + (i,i,\cdots,i)$ $(i \in \mathbb{Z})$.

Since $\kappa(M) = \iota_{n-1,n}(SU(n-1))Z_{\kappa(M)}$, we obtain the assertion from the above arguments. ∎

We denote by $(\tau_\lambda^{(n)\vee}, V_\lambda^{(n)\vee})$ the contragradient representation of $\tau_\lambda^{(n)}$. Then $\tau_\lambda^{(n)\vee}$ is the irreducible representation of $U(n)$ whose highest weight is $\widehat{\lambda} = (-\lambda_n, -\lambda_{n-1}, \cdots, -\lambda_1)$. For each $\nu \in B(\lambda)$, let $d(\nu) = \dim_{\mathbb{C}} V_\nu^{(n-1)}$ and fix a basis $\{v_{\nu,i}\}_{i=1}^{d(\nu)}$ of $V_{\lambda,\nu}^{(n)}$. We note that $\{v_{\nu,i}\}_{\nu\in B(\lambda),\ 1\le i\le d(\nu)}$ is a basis of $V_\lambda^{(n)}$, and take its dual basis $\{v_{\nu,i}^\vee\}_{\nu\in B(\lambda),\ 1\le i\le d(\nu)}$ of $V_\lambda^{(n)\vee}$. Define

$$V_{\lambda,\nu}^{(n)\vee} = \bigoplus_{i=1}^{d(\nu)} \mathbb{C} v_{\nu,i}^\vee$$

as a subspace of $V_\lambda^{(n)\vee}$, then $V_\lambda^{(n)\vee}$ has the following irreducible decomposition as an $M$-module:

$$V_\lambda^{(n)\vee} = \bigoplus_{\nu\in B(\lambda)} V_{\lambda,\nu}^{(n)\vee}.$$

**2.6. The $\nu$-components of matrix coefficients.** Let $(\Pi, H_\Pi)$ be an irreducible admissible Banach representation of $G$, and take a $K$-type $(\tau_\lambda^{(n)}, V_\lambda^{(n)})$ of $\Pi$. Let $\iota^\vee \colon V_\lambda^{(n)\vee} \to H_{\Pi,K}^\vee$ and $\iota \colon V_\lambda^{(n)} \to H_{\Pi,K}$ be $K$-embeddings and set $\phi = \Phi_\Pi \circ (\iota^\vee \boxtimes \iota)$.

For $v^\vee \in V_\lambda^{(n)\vee}$ and $v \in V_\lambda^{(n)}$, we have

$$\phi(v^\vee \boxtimes v)(k_1 g k_2) = \phi(\tau_\lambda^{(n)\vee}(k_1^{-1})v^\vee \boxtimes \tau_\lambda^{(n)}(k_2)v)(g) \qquad (g\in G,\ k_1,k_2\in K).$$

From this equality and the Cartan decomposition $G = KAK$, we know that $\phi$ is characterized by the radial part $\phi(v^\vee \boxtimes v)|_A$ $(v^\vee \in V_\lambda^{(n)\vee},\ v \in V_\lambda^{(n)})$.

We take $v_{\nu,i},\ v_{\nu,i}^\vee$ $(\nu \in B(\lambda),\ 1 \le i \le d(\nu))$ as in §2.5. For $\nu \in B(\lambda)$, we define the $\nu$-component $\phi[\nu]$ of $\phi$ by

$$\phi[\nu] = \frac{1}{d(\nu)} \sum_{i=1}^{d(\nu)} \phi(v_{\nu,i}^\vee \boxtimes v_{\nu,i})|_A. \tag{2.5}$$

Note that the definition of the $\nu$-component $\phi[\nu]$ of $\phi$ does not depend on the choice of $\{v_{\nu,i}\}_{1\le i\le d(\nu)}$. The next proposition shows that $\phi$ is characterized by $\{\phi[\nu]\}_{\nu\in B(\lambda)}$.

**Proposition 2.2.** *Retain the notation and let $\nu, \nu' \in B(\lambda)$. For $v^\vee \in V_{\lambda,\nu}^{(n)\vee}$ and $v \in V_{\lambda,\nu'}^{(n)}$, we have*

$$\phi(v^\vee \boxtimes v)|_A = \begin{cases} \langle v^\vee, v\rangle \phi[\nu] & \text{if } \nu = \nu', \\ 0 & \text{otherwise.} \end{cases}$$

*Here we denote by $\langle \cdot, \cdot \rangle$ the canonical pairing on $V_\lambda^{(n)\vee} \times V_\lambda^{(n)}$.*

**Proof.** Let $a \in A$. Since

$$\begin{aligned} \phi(v^\vee \boxtimes v)(a) &= \phi(v^\vee \boxtimes v)(m^{-1}am) \\ &= \phi(\tau_\lambda^{(n)\vee}(m)v^\vee \boxtimes \tau_\lambda^{(n)}(m)v)(a) \qquad (m \in M), \end{aligned}$$

we have

$$\begin{aligned} \phi(v^\vee \boxtimes v)(a) &= \int_M \phi(v^\vee \boxtimes v)(a)\ dm \\ &= \int_M \phi(\tau_\lambda^{(n)\vee}(m)v^\vee \boxtimes \tau_\lambda^{(n)}(m)v)(a)\ dm. \end{aligned}$$

Here $dm$ is the Haar measure on $M$ normalized by $\int_M dm = 1$. Decomposing

$$\tau_\lambda^{(n)\vee}(m)v^\vee = \sum_{i=1}^{d(\nu)} \langle \tau_\lambda^{(n)\vee}(m)v^\vee, v_{\nu,i}\rangle v_{\nu,i}^\vee,$$
$$\tau_\lambda^{(n)}(m)v = \sum_{j=1}^{d(\nu')} \langle v_{\nu',j}^\vee, \tau_\lambda^{(n)}(m)v\rangle v_{\nu',j},$$

we have

$$\begin{aligned} &\phi(v^\vee \boxtimes v)(a) \\ &= \sum_{i=1}^{d(\nu)} \sum_{j=1}^{d(\nu')} \left\{ \int_M \langle \tau_\lambda^{(n)\vee}(m)v^\vee, v_{\nu,i}\rangle \langle v_{\nu',j}^\vee, \tau_\lambda^{(n)}(m)v\rangle\ dm \right\} \phi(v_{\nu,i}^\vee \boxtimes v_{\nu',j})(a). \end{aligned}$$

Applying Schur's orthogonality relations ([8, Corollary 1.10])

$$\int_M \langle \tau_\lambda^{(n)\vee}(m)v^\vee, w\rangle \langle w^\vee, \tau_\lambda^{(n)}(m)v\rangle\ dm = \begin{cases} \dfrac{\langle v^\vee, v\rangle \langle w^\vee, w\rangle}{d(\nu)} & \text{if } \nu = \nu', \\ 0 & \text{otherwise,} \end{cases}$$
$$(v^\vee \in V_{\lambda,\nu}^{(n)\vee},\ w \in V_{\lambda,\nu}^{(n)},\ w^\vee \in V_{\lambda,\nu'}^{(n)\vee},\ v \in V_{\lambda,\nu'}^{(n)})$$

to the right hand side of this equality, we obtain the assertion. ■

**2.7. The parameterization of discrete series representations.** We call $(\Pi, H_\Pi)$ a discrete series representation of $G$ if $\Pi$ is an irreducible unitary representation of $G$ whose matrix coefficients are contained in $L^2(G)$. In this subsection, we recall the Blattner parameter of discrete series representations of $G$. See [8, Theorems 9.20, 12.21] for details.

There are $n+1$ positive systems $\Sigma^{+,J}$ $(0 \le J \le n)$ of $\Sigma$ which contain $\Sigma_{\mathrm{c}}^{+}$ (cf. §2.2). The positive root system $\Sigma^{+,J}$ is specified by the simple root system $\Delta_J$ defined as follows:

$$\begin{aligned}
\Delta_0 =& \{e_i - e_{i+1} \mid 1 \le i \le n\}, \\
\Delta_J =& \{e_i - e_{i+1} \mid 1 \le i \le n-1,\ i \ne n-J\} \\
& \cup \{e_{n-J} - e_{n+1}, e_{n+1} - e_{n-J+1}\} \qquad\qquad (1 \le J \le n-1), \\
\Delta_n =& \{e_i - e_{i+1} \mid 1 \le i \le n-1\} \cup \{e_{n+1} - e_1\}.
\end{aligned}$$

For $0 \le J \le n$, the set $\Sigma_{\mathrm{n}}^{+,J}$ of noncompact positive roots of $\Sigma^{+,J}$ is given by

$$\Sigma_{\mathrm{n}}^{+,J} = \{e_i - e_{n+1} \mid 1 \le i \le n-J\} \cup \{e_{n+1} - e_i \mid n-J+1 \le i \le n\},$$

and the half sum $\rho_J$ of the roots in $\Sigma^{+,J}$ is given as follows:

$$\begin{aligned}
\rho_0 &= (n, n-1, \cdots, 2, 1), \\
\rho_J &= (n-J, n-J-1, \cdots, 2, 1, -1, -2, \cdots, -J) \qquad (1 \le J \le n-1), \\
\rho_n &= (-1, -2, \cdots, -n+1, -n).
\end{aligned}$$

Moreover, the half sum $\rho_{[K]}$ of the positive compact roots is given by

$$2\rho_{[K]} = (n-1, n-3, \cdots, n+1-2i, \cdots, -n+1).$$

We set $\Xi^{(n)} = \bigcup_{J=0}^{n} \Xi_J^{(n)}$ with

$$\Xi_J^{(n)} = \{\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n \mid \lambda_{n-J} > n-2J,\ n-2J > \lambda_{n-J+1}\}.$$

Here we omit the condition $\lambda_{n-J} > n-2J$ if $J = n$, and omit the condition $n-2J > \lambda_{n-J+1}$ if $J=0$. For each $0 \le J \le n$ and $\mu \in \Xi_J^{(n)}$, there is a discrete series representation $(\Pi, H_\Pi)$ of $G$ which has the following properties:

(a) The infinitesimal character of $\Pi$ is given by $\mu - \rho_J + 2\rho_{[K]}$.

(b) If $\dim_{\mathbb{C}} \mathrm{Hom}_K(V_\lambda^{(n)}, H_{\Pi,K}) > 0$, then the highest weight $\lambda$ is of the form

$$\mu + \sum_{\alpha \in \Delta_J} m_\alpha \alpha \qquad\qquad (m_\alpha \in \mathbb{Z}_{\ge 0}).$$

Moreover, $\dim_{\mathbb{C}} \mathrm{Hom}_K(V_\mu^{(n)}, H_{\Pi,K}) = 1$.

Such $\Pi$ is unique up to equivalence, and the parameter $\mu$ is called the Blattner parameter of $\Pi$. The $K$-type $\tau_\mu^{(n)}$ is called the minimal $K$-type of $\Pi$. We denote by $(\Pi_\mu, H_\mu)$ the discrete series representation of $G$ with the Blattner parameter $\mu$. It is known that the set $\{(\Pi_\mu, H_\mu) \mid \mu \in \Xi^{(n)}\}$ exhausts the equivalence classes of discrete series representations of $G$.

When $\mu \in \Xi_0^{(n)}$, the representation $\Pi_\mu$ is a lowest weight module, and is usually called a holomorphic discrete series representation. When $\mu \in \Xi_J^{(n)}$ with

$0 < J < n$, the representation $\Pi_\mu$ is large in the sense of Vogan [16, Definition 6.1], and is called a large discrete series representation.

For $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \Xi_J^{(n)}$, we note that the contragradient representation $\Pi_\mu^\vee$ of $\Pi_\mu$ is a discrete series representation of $G$ with the Blattner parameter $\widehat{\mu} = (-\mu_n, -\mu_{n-1}, \cdots, -\mu_1) \in \Xi_{n-J}^{(n)}$, that is, $\Pi_\mu^\vee \simeq \Pi_{\widehat{\mu}}$. Hence, because of (2.2), it suffices to consider matrix coefficients of $\Pi_\mu$ for each $0 \le J \le n/2$ and $\mu \in \Xi_J^{(n)}$.

**2.8. The generalized hypergeometric series.** For later use, we recall the basic facts of the Gaussian hypergeometric series and the generalized hypergeometric series. See Whittaker–Watson [18] for details.

For $a \in \mathbb{C}$ and $i \in \mathbb{Z}_{\ge 0}$, we define the Pochhammer symbol $(a)_i$ by

$$(a)_i = \frac{\Gamma(a+i)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+i-1),$$

where $\Gamma(z)$ is the gamma function. We set

$$(\mathbb{C}^2)_{\mathrm{P}} = \{(a,b) \in \mathbb{C}^2 \mid b \notin \mathbb{Z}_{\le 0}\} \cup \{(a,b) \in \mathbb{Z}^2 \mid b \le a \le 0\}.$$

Then $\dfrac{(a)_i}{(b)_i}$ is well-defined for $(a,b) \in (\mathbb{C}^2)_{\mathrm{P}}$ and $i \in \mathbb{Z}_{\ge 0}$.

The Gaussian hypergeometric series is defined by

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{z^i}{i!} \tag{2.6}$$

where $a, b, c$ are complex numbers such that $(b,c) \in (\mathbb{C}^2)_{\mathrm{P}}$. Here this series converges absolutely for $|z| < 1$. If $a \in \mathbb{Z}_{\le 0}$, this series is a polynomial function of $z \in \mathbb{C}$.

The Gaussian hypergeometric series $F(z) = {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right)$ is characterized by the following two conditions:

(i) $F(z)$ satisfies the Gaussian hypergeometric differential equation

$$\left\{ z(1-z)\frac{d^2}{dz^2} + (c-(a+b+1)z)\frac{d}{dz} - ab \right\} F(z) = 0. \tag{2.7}$$

(ii) $F(z)$ is regular at $z = 0$ and $F(0) = 1$.

Let us recall basic equalities of the Gaussian hypergeometric series. It holds that

$$(1-z)^{a+b-c}\, {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; z\right) = {}_2F_1\left(\begin{matrix} c-a,\ c-b \\ c \end{matrix}; z\right). \tag{2.8}$$

If $\mathrm{Re}(c-a-b) > 0$, the Gaussian hypergeometric series (2.6) converges absolutely for $|z| \le 1$ and its value at $z = 1$ is given by

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.9}$$

When $a \in \mathbb{Z}_{\leq 0}$, the both sides of (2.9) are rational functions of $(b, c) \in (\mathbb{C}^2)_{\mathrm{P}}$. Hence, (2.9) holds for $a \in \mathbb{Z}_{\leq 0}$ and $(b, c) \in (\mathbb{C}^2)_{\mathrm{P}}$ without the condition $\mathrm{Re}(c - a - b) > 0$.

The generalized hypergeometric series of type $(3, 2)$ is defined by

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} ; z\right) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i (c)_i}{(d)_i (e)_i} \frac{z^i}{i!} $$

where $a, b, c, d, e$ are complex numbers such that $(b, d), (c, e) \in (\mathbb{C}^2)_{\mathrm{P}}$. This series converges absolutely for $|z| < 1$. If $a \in \mathbb{Z}_{\leq 0}$, this series is a polynomial function of $z \in \mathbb{C}$.

By definition, we have the following equalities:

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ c \end{matrix} ; z\right) = {}_2F_1\left(\begin{matrix} a,\ b \\ d \end{matrix} ; z\right), \tag{2.10} $$

$$ \left(z\frac{d}{dz} + a\right) {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix} ; z\right) = a\, {}_3F_2\left(\begin{matrix} a+1,\ b,\ c \\ d,\ e \end{matrix} ; z\right). \tag{2.11} $$

We introduce the recent result of Vidūnas.

**Theorem 2.3** ([15, Theorem 9.1]). *For $p, q, r \in \mathbb{Z}_{\geq 0}$, we have*

$$ {}_2F_1\left(\begin{matrix} p+r+1,\ p+q+r+1 \\ p+q+2r+2 \end{matrix} ; z\right) = Q_{[p,q,r]}(z) \log(1-z) + R_{[p,q,r]}(1-z), $$

*where $Q_{[p,q,r]}(z)$ and $R_{[p,q,r]}(z)$ are rational functions defined by*

$$ \begin{aligned} Q_{[p,q,r]}(z) =& (-1)^{p+1} z^{-p-q-2r-1} \\ & \times \frac{(p+q+2r)!(p+q+2r+1)!}{r!(p+r)!(q+r)!(p+q+r)!} {}_2F_1\left(\begin{matrix} -r,\ -q-r \\ -p-q-2r \end{matrix} ; z\right), \\ R_{[p,q,r]}(z) =& (-1)^{p+1}(p+q+2r+1)!(1-z)^{-p-q-2r-1} \\ & \times \left\{ \sum_{k=0}^{r} \frac{\psi(q+r-k+1) + \psi(r-k+1) - \psi(p+k+1) - \psi(k+1)}{(p+k)!(q+r-k)!(r-k)!k!} z^k \right. \\ & \left. - \sum_{k=0}^{p-1} \frac{(-1)^{p+k}(p-k-1)! z^{-p+k}}{(p+q+r-k)!(p+r-k)!k!} + \sum_{k=0}^{q-1} \frac{(-1)^{q+k}(q-k-1)! z^{q+r-k}}{(p+q+r-k)!(q+r-k)!k!} \right\}. \end{aligned} $$

*Here $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function.*

From this theorem, we obtain the following proposition.

**Proposition 2.4.** *For $b, c, d, m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}$ such that*

$$ d > b \geq c \geq 1, \qquad a \notin \mathbb{Z}_{\leq 0}, \qquad (a-d+1+m, a-d+1) \in (\mathbb{C}^2)_{\mathrm{P}}, $$

*we have*

$$ {}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix} ; z\right) = \tilde{Q}_{[a,m;b,c,d]}(z) \log(1-z) + \tilde{R}_{[a,m;b,c,d]}(1-z), $$

*where* $\tilde{Q}_{[a,m;b,c,d]}(z)$ *and* $\tilde{R}_{[a,m;b,c,d]}(z)$ *are rational functions defined by*

$$\begin{aligned}\tilde{Q}_{[a,m;b,c,d]}(z) =& \frac{(-1)^{b+c-d+1}(d-2)!(d-1)!(a-d+1)_m}{(b-1)!(d-b-1)!(c-1)!(d-c-1)!(a)_m} z^{-d+1} \\ & \times {}_3F_2\left(\begin{matrix} 1-d+b,\ 1-d+c,\ a-d+1+m \\ -d+2,\ a-d+1 \end{matrix};z\right), \\ \tilde{R}_{[a,m;b,c,d]}(z) =& \sum_{\substack{0\le j\le m \\ d-b-c\le j}} \binom{m}{j} \frac{(b)_j(c)_j}{(d)_j(a)_j}(1-z)^j R_{[b+c-d+j,b-c,d-b-1]}(z) \\ & + \sum_{\substack{0\le j\le m \\ j<d-b-c}} \binom{m}{j} \frac{(b)_j(c)_j}{(d)_j(a)_j}(1-z)^j z^{d-b-c-j} R_{[d-b-c-j,b-c,c+j-1]}(z).\end{aligned}$$

*Here* $\binom{m}{j} = \dfrac{m(m-1)\cdots(m-j+1)}{j!}$ *is the binomial coefficient, and the function* $R_{[p,q,r]}(z)$ *is the rational function defined in Theorem 2.3.*

We give the proof of this proposition in §6.4.

## 3. The main results

In this section, we state the main results of this paper.

Let $n \in \mathbb{Z}_{\ge 2}$. For $\mu \in \Xi^{(n)}$, let $\iota_\mu^\vee \colon V_\mu^{(n)\vee} \to H_{\mu,K}^\vee$ and $\iota_\mu \colon V_\mu^{(n)} \to H_{\mu,K}$ be $K$-embeddings such that

$$\langle \iota_\mu^\vee(v^\vee), \iota_\mu(v)\rangle = \langle v^\vee, v\rangle \qquad (v^\vee \in V_\mu^{(n)\vee},\ v \in V_\mu^{(n)}), \tag{3.1}$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing. We set $\phi_\mu = \Phi_{\Pi_\mu} \circ (\iota_\mu^\vee \boxtimes \iota_\mu)$, that is,

$$\phi_\mu(v^\vee \boxtimes v)(g) = \langle \iota_\mu^\vee(v^\vee), \Pi_\mu(g)\iota_\mu(v)\rangle \qquad (g \in G,\ v^\vee \in V_\mu^{(n)\vee},\ v \in V_\mu^{(n)}).$$

For $\nu \in B(\mu)$, we denote by $\phi_\mu[\nu]$ the $\nu$-component of $\phi_\mu$ (cf. (2.5)). By the normalization (3.1), we have

$$\phi_\mu[\nu](1_{n+1}) = 1 \qquad (\nu \in B(\mu)). \tag{3.2}$$

Matrix coefficients of a holomorphic discrete series representation are well-known as the Bergman kernel. The Bergman kernel is a kernel function of the holomorphic section of the symmetric domain. (See Kato [6] in the case of the complex hyperball, and Godement [3] in the case of Siegel's upper half space.) On another aspect of the kernel function of the representation space of a discrete series representation, it is also obtained by the matrix coefficient. (See Wallach and Wolf [17].) For the sake of completeness, we settle explicit formulas of matrix coefficients of a holomorphic discrete series representation in our context.

**Proposition 3.1.** *Let* $n \in \mathbb{Z}_{\ge 2}$ *and* $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \Xi_0^{(n)}$. *We take* $\phi_\mu[\nu]$ $(\nu \in B(\mu))$ *as above. For* $\nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in B(\mu)$, *we have*

$$\phi_\mu[\nu](\mathrm{a}[t]) = \mathrm{ch}(t)^{-\mu_1-\mu_2-\cdots-\mu_n+\nu_1+\nu_2+\cdots+\nu_{n-1}}.$$

The proof of this proposition is given in §4.3. Because of Proposition 3.1 and the argument in §2.7, it suffices to consider matrix coefficients of $\Pi_\mu$ for $\mu \in \Xi_1^{(n)}$ when $n = 2, 3$. For $\mu \in \Xi_1^{(2)}$, Tsuzuki gives the explicit formulas of matrix coefficients of $\Pi_\mu$ at the minimal $K$-type, without a detailed proof. We rewrite those formulas in our context.

**Theorem 3.2** ([13, Theorem A.1.1]). *Let $n = 2$. Let $\mu = (\mu_1, \mu_2) \in \Xi_1^{(2)}$. We take $\phi_\mu[\nu]$ ($\nu \in B(\mu)$) as above. For $\nu = \nu_1 \in B(\mu)$, we have*

$$\begin{aligned}\phi_\mu[\nu](\mathrm{a}[t]) &= \mathrm{ch}(t)^{-\mu_1-\mu_2+\nu}{}_2F_1\left(\begin{matrix}\nu-\mu_2+1,\ -\mu_2+1\\ \mu_1-\mu_2+2\end{matrix};1-\mathrm{ch}(t)^2\right)\\ &= \mathrm{ch}(t)^{\mu_1+\mu_2-\nu}{}_2F_1\left(\begin{matrix}\mu_1-\nu+1,\ \mu_1+1\\ \mu_1-\mu_2+2\end{matrix};1-\mathrm{ch}(t)^2\right).\end{aligned}$$

For the sake of completeness, we give the proof of this theorem in §5.3. For $\mu \in \Xi_1^{(3)}$, we give the explicit formulas of matrix coefficients of $\Pi_\mu$ at the minimal $K$-type. This is our main result.

**Theorem 3.3.** *Let $n = 3$. Let $\mu = (\mu_1, \mu_2, \mu_3) \in \Xi_1^{(3)}$. We take $\phi_\mu[\nu]$ ($\nu \in B(\mu)$) as above. For $\nu = (\nu_1, \nu_2) \in B(\mu)$, we have*

$$\begin{aligned}\phi_\mu[\nu](\mathrm{a}[t]) =& \,\mathrm{ch}(t)^{-\mu_1-\mu_2-\mu_3+\nu_1+\nu_2}\\ &\times {}_3F_2\left(\begin{matrix}\nu_1-\mu_3+2,\ \nu_2-\mu_3+1,\ -\mu_3+2\\ \mu_1-\mu_3+3,\ \mu_2-\mu_3+2\end{matrix};1-\mathrm{ch}(t)^2\right).\end{aligned}$$

The proof of this theorem is given in §6.3.

**Remark 3.4.** Following the suggestion of the referee, we make some comments for the explicit formulas of matrix coefficients of discrete series representations of $SU(n,1)$ ($n \geq 2$). The formulas in Theorems 3.2 and 3.3 show a certain pattern, and we expect that $\phi_\mu[\nu]$ ($\nu \in B(\mu)$) are described in terms of the generalized hypergeometric series of type $(n, n-1)$ for $\mu \in \Xi_1^{(n)}$ in general. The explicit formulas of $\phi_\mu[\nu]$ ($\nu \in B(\mu)$) for $\mu \in \Xi_J^{(n)}$ ($2 \leq J \leq n-2$) might be more complicated.

By Theorem 2.3 and Proposition 2.4, we know that the functions in the above theorems are elementary functions of the form

$$R_1(\mathrm{ch}(t))\log(\mathrm{ch}(t)) + R_2(\mathrm{ch}(t)).$$

where $R_1(z)$ and $R_2(z)$ are some rational functions of $z \in \mathbb{C}$.

**Corollary 3.5.** *Let $n = 2$. Keeping the notation in Theorem 3.2, we have*

$$\phi_\mu[\nu](\mathrm{a}[t]) = \mathrm{ch}(t)^p\Big\{2Q_{[p,q,r]}(1-\mathrm{ch}(t)^2)\log(\mathrm{ch}(t)) + R_{[p,q,r]}(\mathrm{ch}(t)^2)\Big\}$$

*with $p = |\mu_1 + \mu_2 - \nu|$, $q = |\nu|$ and $r = \min\{\mu_1, -\mu_2, \mu_1 - \nu, \nu - \mu_2\}$. Here $Q_{[p,q,r]}(z)$ and $R_{[p,q,r]}(z)$ are the rational functions defined in Theorem 2.3.*

**Proof.** The statement follows immediately from Theorems 2.3 and 3.2. ∎

**Corollary 3.6.** *Let $n = 3$. Keeping the notation in Theorem 3.3, we have*

$$\begin{aligned}\phi_\mu[\nu](\mathrm{a}[t]) =& \mathrm{ch}(t)^{-\mu_1-\mu_2-\mu_3+\nu_1+\nu_2} \\ &\times \left\{ 2\tilde{Q}_{[a,m;b,c,d]}(1-\mathrm{ch}(t)^2)\log(\mathrm{ch}(t)) + \tilde{R}_{[a,m;b,c,d]}(\mathrm{ch}(t)^2) \right\}\end{aligned}$$

*with $a = \mu_2 - \mu_3 + 1$, $d = \mu_1 - \mu_3 + 3$, $m = \nu_1 - \mu_2$ and $b, c \in \mathbb{Z}$ such that*

$$b \geq c, \qquad \{b, c\} = \{\nu_2 - \mu_3 + 1,\ -\mu_3 + 2\}.$$

*Here $\tilde{Q}_{[a,m;b,c,d]}(z)$ and $\tilde{R}_{[a,m;b,c,d]}(z)$ are the rational functions defined in Proposition 2.4.*

**Proof.** The statement follows immediately from Proposition 2.4 and Theorem 3.3. ∎

**Remark 3.7.** The behavior of $\phi_\mu[\nu](\mathrm{a}[t])$ when $e^t \to 0$ or $e^t \to \infty$ is called "asymptotic behavior". Corollaries 3.5 and 3.6 describe them for large discrete representations of $SU(2,1)$ and $SU(3,1)$, respectively. There is a similar result for middle discrete series representations of $SU(2,2)$ in [4].

## 4. Matrix coefficients of holomorphic discrete series representations of $SU(n,1)$

**4.1. The Gelfand-Tsetlin basis.** Here we briefly review the Gelfand-Tsetlin basis of irreducible representations of $U(n)$. We also introduce a twisted version of the Gelfand-Tsetlin basis. The twisted version has some properties, which are convenient for our computation. We also give a description about its dual basis.

Let $n \in \mathbb{Z}_{\geq 1}$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \Lambda_n$. A triangular array

$$L = \begin{pmatrix} l_{n,1} & l_{n,2} & \cdots & l_{n,n} \\ & l_{n-1,1} & \cdots & l_{n-1,n-1} \\ & \cdots & \cdots & \cdots \\ & & l_{2,1} \quad l_{2,2} & \\ & & l_{1,1} & \end{pmatrix}$$

of integers is called a Gelfand-Tsetlin pattern (G-pattern) of type $\lambda$ if

$$l_{j,i} \geq l_{j-1,i} \geq l_{j,i+1} \qquad (1 \leq i < j \leq n),$$

and if $(l_{n,1}, l_{n,2}, \cdots, l_{n,n}) = \lambda$. Let $\mathbf{G}(\lambda)$ be the set of G-patterns of type $\lambda$. For $1 \leq i \leq j \leq n$ and $L \in \mathbf{G}(\lambda)$, we define the symbols $\gamma_j^L$ and $\Delta_{j,i}$ by

$$\gamma_j^L = \sum_{h=1}^{j} l_{j,h} - \sum_{h=1}^{j-1} l_{j-1,h},$$

$$\Delta_{j,i} = \begin{pmatrix} \delta_{(j,i),(n,1)} & \delta_{(j,i),(n,2)} & \cdots & \delta_{(j,i),(n,n)} \\ & \delta_{(j,i),(n-1,1)} & \cdots & \delta_{(j,i),(n-1,n-1)} \\ & \cdots & \cdots & \cdots \\ & \delta_{(j,i),(2,1)} & \delta_{(j,i),(2,2)} & \\ & & \delta_{(j,i),(1,1)} & \end{pmatrix},$$

where $\delta_{(j,i),(j',i')} = \delta_{j,j'}\delta_{i,i'}$ is the product of the Kronecker deltas.

We recall the following theorem, which is originally proved by Gelfand and Tsetlin. See Molev [10, Theorem 2.3] for details.

**Theorem 4.1.** *There exists a basis $\{\xi(L)\}_{L\in\mathbf{G}(\lambda)}$ of $V_\lambda^{(n)}$, on which the generators $E_{j,j}^{(n)}$, $E_{j,j+1}^{(n)}$ and $E_{j+1,j}^{(n)}$ of $\mathfrak{u}(n)_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$ act by the following formulas:*

$$\tau_\lambda^{(n)}(E_{j,j}^{(n)})\xi(L) = \gamma_j^L\xi(L),$$

$$\tau_\lambda^{(n)}(E_{j,j+1}^{(n)})\xi(L) = -\sum_{i=1}^{j} \frac{\prod_{h=1}^{j+1}(l_{j,i} - l_{j+1,h} - i + h)}{\prod_{1\le h\le j,\ h\ne i}(l_{j,i} - l_{j,h} - i + h)}\xi(L + \Delta_{j,i}),$$

$$\tau_\lambda^{(n)}(E_{j+1,j}^{(n)})\xi(L) = \sum_{i=1}^{j} \frac{\prod_{h=1}^{j-1}(l_{j,i} - l_{j-1,h} - i + h)}{\prod_{1\le h\le j,\ h\ne i}(l_{j,i} - l_{j,h} - i + h)}\xi(L - \Delta_{j,i}).$$

*Here it is supposed that $\xi(L\pm\Delta_{j,i}) = 0$ if $L\pm\Delta_{j,i}$ is not a G-pattern.*

For $1\le i\le j<n$ and $L\in\mathbf{G}(\lambda)$, we define the symbol $P_{j,i}(L)$ by

$$P_{j,i}(L) = \begin{cases} \dfrac{\prod_{h=1}^{i}(l_{j+1,h} - l_{j,i} - h + i)}{\prod_{h=1}^{i-1}(l_{j,h} - l_{j,i} - h + i)} \displaystyle\prod_{h=2}^{i} \frac{l_{j-1,h-1} - l_{j,i} - h + i}{l_{j,h-1} - l_{j,i} - h + i} & \text{if } i = 1 \text{ or } l_{j,i-1} > l_{j,i}, \\ 0 & \text{otherwise.} \end{cases}$$

For $L\in\mathbf{G}(\lambda)$, we define the dual pattern $\widehat{L}$ of $L$ by

$$\widehat{L} = \begin{pmatrix} -l_{n,n} & -l_{n,n-1} & \cdots & -l_{n,1} \\ & -l_{n-1,n-1} & \cdots & -l_{n-1,1} \\ & \cdots & \cdots & \cdots \\ & -l_{2,2} & -l_{2,1} & \\ & & -l_{1,1} & \end{pmatrix}.$$

Here $\widehat{L}$ is a G-pattern of type $\widehat{\lambda} = (-\lambda_n, -\lambda_{n-1}, \cdots, -\lambda_1)$. Though $\Delta_{j,i}$ itself is not a G-pattern, we use the same convention that $\widehat{\Delta_{j,i}} = -\Delta_{j,j+1-i}$ for $1\le i\le j\le n$.

For computation, we use the twisted version $\{\zeta(L)\}_{L\in\mathbf{G}(\lambda)}$ of the Gelfand-Tsetlin basis $\{\xi(L)\}_{L\in\mathbf{G}(\lambda)}$, which is in the following lemma.

**Lemma 4.2.** *There exists a basis $\{\zeta(L)\}_{L\in\mathbf{G}(\lambda)}$ of $V_\lambda^{(n)}$, on which the generators $E_{j,j}^{(n)}$, $E_{j,j+1}^{(n)}$ and $E_{j+1,j}^{(n)}$ of $\mathfrak{u}(n)_{\mathbb{C}}$ act by the following formulas:*

$$\tau_\lambda^{(n)}(E_{j,j}^{(n)})\zeta(L) = \gamma_{n+1-j}^L\zeta(L),$$

$$\tau_\lambda^{(n)}(E_{j,j+1}^{(n)})\zeta(L) = \sum_{i=1}^{n-j} P_{n-j,i}(\widehat{L})\zeta(L+\widehat{\Delta_{n-j,i}}),$$
$$\tau_\lambda^{(n)}(E_{j+1,j}^{(n)})\zeta(L) = \sum_{i=1}^{n-j} P_{n-j,i}(L)\zeta(L+\Delta_{n-j,i}).$$

*Here it is supposed that $\zeta(L')=0$ if $L'$ is not a G-pattern.*

**Proof.** We set $\zeta(L)=c(L)\tau_\lambda(w_n)\xi(L)$ with

$$c(L) = \prod_{h=2}^{n}\prod_{1\le i\le j<h}\frac{(l_{h-1,i}-l_{h,j+1}-i+j)!}{(l_{h,i}-l_{h,j+1}-i+j)!},$$
$$w_n = \begin{pmatrix} & & & & 1\\ & & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix} \in U(n).$$

Since $w_n^{-1}E_{i,j}^{(n)}w_n = E_{n+1-i,n+1-j}^{(n)}$, we have

$$\tau_\lambda^{(n)}(E_{j,j}^{(n)})\zeta(L) = c(L)\tau_\lambda(w_n)\tau_\lambda^{(n)}(E_{n+1-j,n+1-j}^{(n)})\xi(L) = \gamma_{n+1-j}^L\zeta(L),$$
$$\begin{aligned}\tau_\lambda^{(n)}(E_{j,j+1}^{(n)})\zeta(L) &= c(L)\tau_\lambda(w_n)\tau_\lambda^{(n)}(E_{n-j+1,n-j}^{(n)})\xi(L)\\ &= \sum_{i=1}^{n-j}\frac{\prod_{h=1}^{n-j-1}(l_{n-j,i}-l_{n-j-1,h}-i+h)}{\prod_{1\le h\le n-j,\ h\ne i}(l_{n-j,i}-l_{n-j,h}-i+h)}c(L)\tau_\lambda(w_n)\xi(L-\Delta_{n-j,i})\\ &= \sum_{i=1}^{n-j}P_{n-j,i}(\widehat{L})\zeta(L+\widehat{\Delta_{n-j,i}}),\end{aligned}$$
$$\begin{aligned}\tau_\lambda^{(n)}(E_{j+1,j}^{(n)})\zeta(L) &= c(L)\tau_\lambda(w_n)\tau_\lambda^{(n)}(E_{n-j,n-j+1}^{(n)})\xi(L)\\ &= -\sum_{i=1}^{n-j}\frac{\prod_{h=1}^{n-j+1}(l_{n-j,i}-l_{n-j+1,h}-i+h)}{\prod_{1\le h\le n-j,\ h\ne i}(l_{n-j,i}-l_{n-j,h}-i+h)}c(L)\tau_\lambda(w_n)\xi(L+\Delta_{n-j,i})\\ &= \sum_{i=1}^{n-j}P_{n-j,i}(L)\zeta(L+\Delta_{n-j,i}).\end{aligned}$$
■

**Lemma 4.3.** *We take the basis $\{\zeta(L)\}_{L\in\mathbf{G}(\lambda)}$ of $V_\lambda^{(n)}$ as in Lemma 4.2. Let $\{\zeta(L)^\vee\}_{L\in\mathbf{G}(\lambda)}$ be its dual basis of $V_\lambda^{(n)\vee}$. Then we have the following:*
*(i) Assume $n\ge 2$. For $\nu\in B(\lambda)$, let $\mathbf{G}(\lambda,\nu)$ be the subset of $\mathbf{G}(\lambda)$ consisting of G-patterns $L$ of type $\lambda$ such that $(l_{n-1,1},l_{n-1,2},\cdots,l_{n-1,n-1})=\nu$. Then we have $V_{\lambda,\nu}^{(n)}=\bigoplus_{L\in\mathbf{G}(\lambda,\nu)}\mathbb{C}\zeta(L)$ and the linear map $V_\nu^{(n-1)}\to V_{\lambda,\nu}^{(n)}$ defined by*

$$\zeta(L')\mapsto\zeta\begin{pmatrix}\lambda\\ L'\end{pmatrix}\qquad\qquad (L'\in\mathbf{G}(\nu))$$

*is an isomorphism of* $U(n-1)$*-modules.*
*(ii) We define an involution* $\omega$ *on* $U(n)$ *by* $\omega(u)=d\overline{u}d$ *with*

$$d=\mathrm{diag}(-1,1,-1,\cdots,(-1)^n)\in U(n).$$

*We also define an isomorphism* $T_\lambda\colon V_\lambda^{(n)}\to V_{\widehat{\lambda}}^{(n)}$ *of* $\mathbb{C}$*-vector spaces by*

$$T_\lambda(\zeta(L))=\zeta(\widehat{L}) \qquad\qquad (L\in\mathbf{G}(\lambda)).$$

*Then we have* $T_\lambda\circ\tau_\lambda^{(n)}(u)=\tau_{\widehat{\lambda}}^{(n)}(\omega(u))\circ T_\lambda$ *for* $u\in U(n)$.
*(iii) The generators* $E_{j,j}^{(n)}$, $E_{j,j+1}^{(n)}$ *and* $E_{j+1,j}^{(n)}$ *of* $\mathfrak{u}(n)_\mathbb{C}$ *act on* $\{\zeta(L)^\vee\}_{L\in\mathbf{G}(\lambda)}$ *by the following formulas:*

$$\begin{aligned}
\tau_\lambda^{(n)\vee}(E_{j,j}^{(n)})\zeta(L)^\vee&=-\gamma_{n+1-j}^{L}\zeta(L)^\vee,\\
\tau_\lambda^{(n)\vee}(E_{j,j+1}^{(n)})\zeta(L)^\vee&=-\sum_{i=1}^{n-j}P_{n-j,i}(\widehat{L}-\Delta_{n-j,i})\zeta(L-\widehat{\Delta_{n-j,i}})^\vee,\\
\tau_\lambda^{(n)\vee}(E_{j+1,j}^{(n)})\zeta(L)^\vee&=-\sum_{i=1}^{n-j}P_{n-j,i}(L-\Delta_{n-j,i})\zeta(L-\Delta_{n-j,i})^\vee.
\end{aligned}$$

*Here it is supposed that* $\zeta(L')^\vee=0$ *if* $L'$ *is not a G-pattern.*

**Proof.** The statement (i) follows immediately from the formulas in Lemma 4.2. The statement (ii) follows from

$$\omega(E_{j,j}^{(n)})=-E_{j,j}^{(n)},\quad \omega(E_{j,j+1}^{(n)})=E_{j+1,j}^{(n)},\quad \omega(E_{j+1,j}^{(n)})=E_{j,j+1}^{(n)},\quad \gamma_j^{\widehat{L}}=-\gamma_j^L,$$

and the formulas in Lemma 4.2. Here we denote the differential of $\omega$ again by $\omega$. The statement (iii) follows from

$$\begin{aligned}
\tau_\lambda^{(n)\vee}(X)\zeta(L)^\vee&=\sum_{L'\in\mathbf{G}(\lambda)}\langle\tau_\lambda^{(n)\vee}(X)\zeta(L)^\vee,\,\zeta(L')\rangle\,\zeta(L')^\vee\\
&=-\sum_{L'\in\mathbf{G}(\lambda)}\langle\zeta(L)^\vee,\,\tau_\lambda^{(n)}(X)\zeta(L')\rangle\,\zeta(L')^\vee
\end{aligned}$$

for $X\in\mathfrak{u}(n)_\mathbb{C}=\mathfrak{gl}(n,\mathbb{C})$, and the formulas in Lemma 4.2. ∎

**4.2. The action of** $\mathfrak{p}_\mathbb{C}$**.** In order to obtain explicit formulas of matrix coefficients, we construct the system of differential equations derived from the action of $\mathfrak{p}_\mathbb{C}$. As a preparation, we write down the action of $\mathfrak{p}_\mathbb{C}$ on the radial parts of matrix coefficients and the $K$-module structure of $\mathfrak{p}_\mathbb{C}$ in this subsection.

**Lemma 4.4.** *(i) Let* $\phi\colon V_\lambda^{(n)\vee}\boxtimes_\mathbb{C}V_\lambda^{(n)}\to C^\infty(G)$ *be a* $K\times K$*-embedding. For* $v^\vee\in V_\lambda^{(n)\vee}$, $v\in V_\lambda^{(n)}$, $t\in\mathbb{R}$ *and* $X\in\mathfrak{k}_\mathbb{C}$, *we have*

$$R(H_1)\phi(v^\vee\boxtimes v)(\mathrm{a}[t])=\frac{d}{dt}\phi(v^\vee\boxtimes v)(\mathrm{a}[t]),$$

$$R(\mathrm{Ad}(\mathrm{a}[t]^{-1})X)\phi(v^\vee \boxtimes v)(\mathrm{a}[t]) = -\phi(\tau_\lambda^{(n)\vee}(X)v^\vee \boxtimes v)(\mathrm{a}[t]),$$
$$R(X)\phi(v^\vee \boxtimes v)(\mathrm{a}[t]) = \phi(v^\vee \boxtimes \tau_\lambda^{(n)}(X)v)(\mathrm{a}[t]).$$

*(ii) The basis* $\{E_{i,n+1}^{(n+1)},\ E_{n+1,i}^{(n+1)}\ (1 \le i \le n)\}$ *of* $\mathfrak{p}_{\mathbb{C}}$ *has the following expressions according to the decomposition* $\mathfrak{g}_{\mathbb{C}} = \mathrm{Ad}(\mathrm{a}[t]^{-1})\mathfrak{k}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{k}_{\mathbb{C}}$ $(t \ne 0)$*:*

$$\begin{aligned}
E_{1,n+1}^{(n+1)} &= \frac{1}{2}\left\{\frac{1}{\mathrm{sh}(2t)}\,\mathrm{Ad}(\mathrm{a}[t]^{-1})\tilde{E}_{1,1}^{(n+1)} + H_1 - \frac{\mathrm{ch}(2t)}{\mathrm{sh}(2t)}\tilde{E}_{1,1}^{(n+1)}\right\},\\
E_{n+1,1}^{(n+1)} &= \frac{1}{2}\left\{-\frac{1}{\mathrm{sh}(2t)}\,\mathrm{Ad}(\mathrm{a}[t]^{-1})\tilde{E}_{1,1}^{(n+1)} + H_1 + \frac{\mathrm{ch}(2t)}{\mathrm{sh}(2t)}\tilde{E}_{1,1}^{(n+1)}\right\},\\
E_{i,n+1}^{(n+1)} &= \frac{1}{\mathrm{sh}(t)}\,\mathrm{Ad}(\mathrm{a}[t]^{-1})E_{i,1}^{(n+1)} - \frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}E_{i,1}^{(n+1)},\\
E_{n+1,i}^{(n+1)} &= -\frac{1}{\mathrm{sh}(t)}\,\mathrm{Ad}(\mathrm{a}[t]^{-1})E_{1,i}^{(n+1)} + \frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}E_{1,i}^{(n+1)},
\end{aligned}$$

*for* $2 \le i \le n$*.*

**Proof.** This lemma follows immediately from the definition. ∎

We regard $\mathfrak{p}_{\mathbb{C}}$ as a $K$-module by the adjoint action Ad of $K$. Then $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ is an irreducible decomposition as a $K$-module, and the highest weights of $\mathfrak{p}_+$ and $\mathfrak{p}_-$ are given by $e_1 - e_{n+1} = (2, 1, 1, \cdots, 1)$ and $-e_n + e_{n+1} = (-1, -1, \cdots, -1, -2)$, respectively.

**Lemma 4.5.** *For* $1 \le h \le n$*, we set*

$$D_h = \sum_{j=1}^{h} \Delta_{n+1-j,1} + \sum_{1 \le i \le j \le n} \Delta_{j,i} \in \mathbf{G}(e_1 - e_{n+1}).$$

*We define linear maps* $I_{\mathfrak{p}}^+ : V_{e_1-e_{n+1}}^{(n)} \to \mathfrak{p}_+$ *and* $I_{\mathfrak{p}}^- : V_{-e_n+e_{n+1}}^{(n)} \to \mathfrak{p}_-$ *by*

$$I_{\mathfrak{p}}^+(\zeta(D_h)) = E_{h,n+1}^{(n+1)} \qquad I_{\mathfrak{p}}^-(\zeta(\widehat{D_h})) = (-1)^h E_{n+1,h}^{(n+1)} \qquad (1 \le h \le n).$$

*Then* $I_{\mathfrak{p}}^+$ *and* $I_{\mathfrak{p}}^-$ *are* $K$*-homomorphisms.*

**Proof.** By direct computation, we have

$$\begin{aligned}
\mathrm{ad}(\tilde{E}_{i,i}^{(n+1)})E_{h,n+1}^{(n+1)} &= (\delta_{i,h} + 1)E_{h,n+1}^{(n+1)},\\
\mathrm{ad}(E_{j,j+1}^{(n+1)})E_{h,n+1}^{(n+1)} &= \delta_{j+1,h}E_{h-1,n+1}^{(n+1)},\\
\mathrm{ad}(E_{j+1,j}^{(n+1)})E_{h,n+1}^{(n+1)} &= \delta_{j,h}E_{h+1,n+1}^{(n+1)},\\
\mathrm{ad}(\tilde{E}_{i,i}^{(n+1)})E_{n+1,h}^{(n+1)} &= -(\delta_{i,h} + 1)E_{n+1,h}^{(n+1)},\\
\mathrm{ad}(E_{j,j+1}^{(n+1)})E_{n+1,h}^{(n+1)} &= -\delta_{j,h}E_{n+1,h+1}^{(n+1)},\\
\mathrm{ad}(E_{j+1,j}^{(n+1)})E_{n+1,h}^{(n+1)} &= -\delta_{j+1,h}E_{n+1,h-1}^{(n+1)}
\end{aligned}$$

for $1 \le i \le n$ and $1 \le j \le n-1$. Comparing these equalities with the formulas in Lemma 4.2, we obtain the assertion. ∎

### 4.3. Proof of Proposition 3.1.

**Proof of Proposition 3.1.** We define the $K$-homomorphism $\tilde{\iota}_\mu \colon \mathfrak{p}_\mathbb{C} \otimes_\mathbb{C} V_\mu^{(n)} \to H_{\mu,K}$ by $\tilde{\iota}_\mu(X \otimes v) = \Pi(X)\iota_\mu(v)$. Since the highest weight of an irreducible subrepresentation of $\mathfrak{p}_- \otimes_\mathbb{C} V_\mu^{(n)}$ is of the form

$$\mu - \alpha \qquad\qquad (\alpha \in \Sigma_\mathfrak{n}^+),$$

we have $\mathrm{Hom}_K(\mathfrak{p}_- \otimes_\mathbb{C} V_\mu^{(n)}, H_{\mu,K}) = \{0\}$. In particular, the image of $\mathfrak{p}_- \otimes_\mathbb{C} V_\mu^{(n)}$ under $\tilde{\iota}_\mu$ is $\{0\}$. Hence, we have

$$R(X)\phi_\mu(v^\vee \boxtimes v) = (\Phi_{\Pi_\mu} \circ (\iota_\mu^\vee \boxtimes \tilde{\iota}_\mu))(v^\vee \boxtimes (X \otimes v)) = 0$$

for $X \in \mathfrak{p}_-$, $v^\vee \in V_\mu^{(n)\vee}$ and $v \in V_\mu^{(n)}$. Let $\nu \in B(\mu)$ and take $L^{(\nu)} \in \mathbf{G}(\mu, \nu)$. Then we have $\gamma_n^{L^{(\nu)}} = \mu_1 + \mu_2 + \cdots + \mu_n - \nu_1 - \nu_2 - \cdots - \nu_{n-1}$. Applying Lemmas 4.2, 4.3, 4.4 and Proposition 2.2 to the equality

$$R(E_{n+1,1}^{(n+1)})\phi_\mu(\zeta(L^{(\nu)})^\vee \boxtimes \zeta(L^{(\nu)}))(\mathrm{a}[t]) = 0,$$

we have

$$\frac{1}{2}\left(\frac{d}{dt} + \gamma_n^{L^{(\nu)}} \frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right)\phi_\mu[\nu](\mathrm{a}[t]) = 0.$$

The solution space of this differential equation is 1-dimensional. By the normalization (3.2), we conclude the assertion. ■

## 5. Matrix coefficients of large discrete series representations of $SU(2,1)$

Throughout this section, we set $n = 2$.

**5.1. Decomposition of tensor products and injectors for $n = 2$.** Let $\lambda = (\lambda_1, \lambda_2) \in \Lambda_2$. The set of G-patterns of type $\lambda$ is given by

$$\mathbf{G}(\lambda) = \left\{ L = \begin{pmatrix} \lambda_1 \ \lambda_2 \\ \nu \end{pmatrix} \middle| \, \nu \in \mathbb{Z},\ \lambda_1 \geq \nu \geq \lambda_2 \right\}.$$

Let $\{\zeta(L)\}_{L \in \mathbf{G}(\lambda)}$ be the basis of $V_\lambda^{(2)}$ in Lemma 4.2, and take its dual basis $\{\zeta(L)^\vee\}_{L \in \mathbf{G}(\lambda)}$ of $V_\lambda^{(2)\vee}$. We set $\zeta(L) = 0$ and $\zeta(L)^\vee = 0$ if $L$ is not a G-pattern. Then the basis $E_{i,j}^{(2)}$ $(1 \leq i, j \leq 2)$ of $\mathfrak{u}(2)_\mathbb{C}$ act on these bases by the following formulas:

$$\tau_\lambda^{(2)}(E_{1,1}^{(2)})\zeta(L) = (\lambda_1 + \lambda_2 - \nu)\zeta(L),$$
$$\tau_\lambda^{(2)}(E_{2,2}^{(2)})\zeta(L) = \nu\zeta(L),$$
$$\tau_\lambda^{(2)}(E_{1,2}^{(2)})\zeta(L) = (\nu - \lambda_2)\zeta\Big(L - \left(\begin{smallmatrix} 0 \ 0 \\ 1 \end{smallmatrix}\right)\Big),$$
$$\tau_\lambda^{(2)}(E_{2,1}^{(2)})\zeta(L) = (\lambda_1 - \nu)\zeta\Big(L + \left(\begin{smallmatrix} 0 \ 0 \\ 1 \end{smallmatrix}\right)\Big),$$

and

$$\begin{aligned}
\tau_\lambda^{(2)\vee}(E_{1,1}^{(2)})\zeta(L)^\vee &= -(\lambda_1+\lambda_2-\nu)\zeta(L)^\vee,\\
\tau_\lambda^{(2)\vee}(E_{2,2}^{(2)})\zeta(L)^\vee &= -\nu\zeta(L)^\vee,\\
\tau_\lambda^{(2)\vee}(E_{1,2}^{(2)})\zeta(L)^\vee &= -(\nu-\lambda_2+1)\zeta\Big(L+\begin{pmatrix}0 & 0\\ & 1\end{pmatrix}\Big)^\vee,\\
\tau_\lambda^{(2)\vee}(E_{2,1}^{(2)})\zeta(L)^\vee &= -(\lambda_1-\nu+1)\zeta\Big(L-\begin{pmatrix}0 & 0\\ & 1\end{pmatrix}\Big)^\vee
\end{aligned}$$

for $L=\begin{pmatrix}\lambda_1 & \lambda_2\\ & \nu\end{pmatrix}\in\mathbf{G}(\lambda)$.

The tensor product $V_{(2,1)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}$ has the following irreducible decomposition as a $U(2)$-module:

$$V_{(2,1)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}\simeq\begin{cases}V_{\lambda+(2,1)}^{(2)}\oplus V_{\lambda+(1,2)}^{(2)} & \text{if } \lambda_1>\lambda_2,\\ V_{\lambda+(2,1)}^{(2)} & \text{if } \lambda_1=\lambda_2.\end{cases}$$

We specify each component of this decomposition as follows.

**Lemma 5.1.** *Retain the notation.*
*(i) We define a linear map* $I_\lambda^{(2,1)}\colon V_{\lambda+(2,1)}^{(2)}\to V_{(2,1)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}$ *by*

$$\begin{aligned}
I_\lambda^{(2,1)}(\zeta(L)) =&(\lambda_1+2-\nu)\zeta\begin{pmatrix}2 & 1\\ & 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2\\ & \nu-1\end{pmatrix}\\
&+(\nu-\lambda_2-1)\zeta\begin{pmatrix}2 & 1\\ & 2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2\\ & \nu-2\end{pmatrix}\\
&\Big(L=\begin{pmatrix}\lambda_1+2 & \lambda_2+1\\ & \nu\end{pmatrix}\in\mathbf{G}(\lambda+(2,1))\Big).
\end{aligned}$$

*Then* $I_\lambda^{(2,1)}$ *is a* $U(2)$*-homomorphism.*
*(ii) When* $\lambda_1>\lambda_2$*, we define a linear map* $I_\lambda^{(1,2)}\colon V_{\lambda+(1,2)}^{(2)}\to V_{(2,1)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}$ *by*

$$\begin{aligned}
I_\lambda^{(1,2)}(\zeta(L)) =\zeta\begin{pmatrix}2 & 1\\ & 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2\\ & \nu-1\end{pmatrix}-\zeta\begin{pmatrix}2 & 1\\ & 2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2\\ & \nu-2\end{pmatrix}\\
\Big(L=\begin{pmatrix}\lambda_1+1 & \lambda_2+2\\ & \nu\end{pmatrix}\in\mathbf{G}(\lambda+(1,2))\Big).
\end{aligned}$$

*Then* $I_\lambda^{(1,2)}$ *is a* $U(2)$*-homomorphism.*

**Proof.** If suffices to confirm that

$$I_\lambda^{(d_1,d_2)}\left(\tau_{\lambda+(d_1,d_2)}^{(2)}(E_{i,j}^{(2)})\zeta(L)\right)=(\tau_{(2,1)}^{(2)}\otimes\tau_\lambda^{(2)})(E_{i,j}^{(2)})I_\lambda^{(d_1,d_2)}(\zeta(L))$$

for $(d_1,d_2)\in\{(2,1),(1,2)\}$, $L\in\mathbf{G}(\lambda+(d_1,d_2))$ and $1\le i,j\le 2$. We can confirm these equalities by direct computation. ■

The tensor product $V_{(-1,-2)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}$ has the following irreducible decomposition as a $U(2)$-module:

$$V_{(-1,-2)}^{(2)}\otimes_{\mathbb{C}}V_\lambda^{(2)}\simeq\begin{cases}V_{\lambda+(-1,-2)}^{(2)}\oplus V_{\lambda+(-2,-1)}^{(2)} & \text{if } \lambda_1>\lambda_2,\\ V_{\lambda+(-1,-2)}^{(2)} & \text{if } \lambda_1=\lambda_2.\end{cases}$$

For $(d_1, d_2) \in \{(-1,-2),\ (-2,-1)\}$, we set

$$I_\lambda^{(d_1,d_2)} = (T_{(2,1)} \otimes T_{\widehat{\lambda}}) \circ I_{\widehat{\lambda}}^{(-d_2,-d_1)} \circ T_{\lambda+(d_1,d_2)}.$$

Then $I_\lambda^{(d_1,d_2)} \colon V_{\lambda+(d_1,d_2)}^{(2)} \to V_{(-1,-2)}^{(2)} \otimes_{\mathbb{C}} V_\lambda^{(2)}$ is a $U(2)$-homomorphism. For $L = \left(\begin{smallmatrix} \lambda_1-2 & & \lambda_2-1 \\ & \nu & \end{smallmatrix}\right) \in \mathbf{G}(\lambda+(-2,-1))$, the image of $\zeta(L)$ under $I_{\widehat{\lambda}}^{(-2,-1)}$ is given by

$$I_\lambda^{(-2,-1)}(\zeta(L)) = \zeta\left(\begin{smallmatrix} -1 & & -2 \\ & -1 & \end{smallmatrix}\right) \otimes \zeta\left(\begin{smallmatrix} \lambda_1 & & \lambda_2 \\ & \nu+1 & \end{smallmatrix}\right) - \zeta\left(\begin{smallmatrix} -1 & & -2 \\ & -2 & \end{smallmatrix}\right) \otimes \zeta\left(\begin{smallmatrix} \lambda_1 & & \lambda_2 \\ & \nu+2 & \end{smallmatrix}\right). \tag{5.1}$$

### 5.2. Differential equations of the $\nu$-components for $n = 2$.

**Proposition 5.2.** *We use the notation in Theorem 3.2. Then, for $\mu_1 - 1 \geq \nu \geq \mu_2$, we have*

$$\begin{aligned}
&\left\{\frac{1}{2}\left(\frac{d}{dt} - (\mu_1+\mu_2-\nu-1)\frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right) + (\mu_1-\nu)\frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}\right\} \phi_\mu[\nu+1](\mathrm{a}[t]) \\
&- (\mu_1-\nu)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu](\mathrm{a}[t]) = 0, \\
&\left\{\frac{1}{2}\left(\frac{d}{dt} + (\mu_1+\mu_2-\nu)\frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right) + (\nu-\mu_2+1)\frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}\right\} \phi_\mu[\nu](\mathrm{a}[t]) \\
&- (\nu-\mu_2+1)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu+1](\mathrm{a}[t]) = 0.
\end{aligned}$$

**Proof.** We define a natural $K$-homomorphism $P_\Pi \colon \mathfrak{p}_{\mathbb{C}} \otimes_{\mathbb{C}} H_{\mu,K} \to H_{\mu,K}$ by $X \otimes f \mapsto \Pi_\mu(X)f$, and set

$$\tilde{\iota}_{\mu,(1,2)} = P_\Pi \circ (I_{\mathfrak{p}}^+ \otimes_{\mathbb{C}} \iota_\mu) \circ I_\mu^{(1,2)}, \qquad \tilde{\iota}_{\mu,(-2,-1)} = P_\Pi \circ (I_{\mathfrak{p}}^- \otimes_{\mathbb{C}} \iota_\mu) \circ I_\mu^{(-2,-1)}.$$

Because of the property (b) of $\Pi_\mu$ in §2.7, we note that the $K$-homomorphisms $\tilde{\iota}_{\mu,(1,2)} \colon V_{\mu+(1,2)}^{(n)} \to H_{\Pi,K}$ and $\tilde{\iota}_{\mu,(-2,-1)} \colon V_{\mu+(-2,-1)}^{(n)} \to H_{\Pi,K}$ are the zero maps. Hence, for $\mu_1 \geq \nu' \geq \mu_2$ and $\mu_1 - 1 \geq \nu \geq \mu_2$, we have

$$\begin{aligned}
&R(E_{1,3}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu+1 & \end{smallmatrix}\right)\Big) - R(E_{2,3}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu & \end{smallmatrix}\right)\Big) \\
&= \left(\Phi_{\Pi_\mu} \circ (\iota_\mu^\vee \boxtimes \tilde{\iota}_{\mu,(1,2)})\right)\left(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1+1 & & \mu_2+2 \\ & \nu+2 & \end{smallmatrix}\right)\right) = 0, \\
&R(E_{3,1}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu & \end{smallmatrix}\right)\Big) + R(E_{3,2}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu+1 & \end{smallmatrix}\right)\Big) \\
&= -\left(\Phi_{\Pi_\mu} \circ (\iota_\mu^\vee \boxtimes \tilde{\iota}_{\mu,(-2,-1)})\right)\left(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu' & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1-2 & & \mu_2-1 \\ & \nu-1 & \end{smallmatrix}\right)\right) = 0.
\end{aligned}$$

From these equalities, we have

$$\begin{aligned}
&R(E_{1,3}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu+1 & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu+1 & \end{smallmatrix}\right)\Big)(\mathrm{a}[t]) \\
&- R(E_{2,3}^{(3)})\phi_\mu\Big(\zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu+1 & \end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix} \mu_1 & & \mu_2 \\ & \nu & \end{smallmatrix}\right)\Big)(\mathrm{a}[t]) = 0,
\end{aligned}$$

$$R(E^{(3)}_{3,1})\phi_\mu\Big(\zeta\left(\begin{smallmatrix}\mu_1 & \mu_2\\ \nu\end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix}\mu_1 & \mu_2\\ \nu\end{smallmatrix}\right)\Big)(\mathrm{a}[t])$$
$$+ R(E^{(3)}_{3,2})\phi_\mu\Big(\zeta\left(\begin{smallmatrix}\mu_1 & \mu_2\\ \nu\end{smallmatrix}\right)^\vee \boxtimes \zeta\left(\begin{smallmatrix}\mu_1 & \mu_2\\ \nu+1\end{smallmatrix}\right)\Big)(\mathrm{a}[t]) = 0$$

for $\mu_1 - 1 \geq \nu \geq \mu_2$. Applying Proposition 2.2, Lemma 4.4 and the formulas of the action of $\mathfrak{k}_{\mathbf{C}}$ in §5.1 to these equalities, we obtain the assertion. ■

### 5.3. Proof of Theorem 3.2.

**Proof of Theorem 3.2.** For $\nu \in B(\mu)$, we set $\varphi_\nu(y) = \mathrm{ch}(t)^{\mu_1+\mu_2-\nu}\phi_\mu[\nu](\mathrm{a}[t])$ with $y = 1 - \mathrm{ch}(t)^2$. From (3.2), we know that

$$\varphi_\nu(0) = 1 \qquad (\nu \in B(\mu)). \tag{5.2}$$

The system of differential equations for $\{\phi_\mu[\nu](\mathrm{a}[t])\}_{\nu\in B(\mu)}$ in Proposition 5.2 is translated to the following system of differential equations for $\{\varphi_\nu(y)\}_{\nu\in B(\mu)}$:

$$\left\{y(1-y)\frac{d}{dy} + (\mu_2-1)y + \mu_1 - \nu\right\}\varphi_{\nu+1}(y) = (\mu_1-\nu)\varphi_\nu(y),$$
$$\left(y\frac{d}{dy} + \nu - \mu_2 + 1\right)\varphi_\nu(y) = (\nu-\mu_2+1)\varphi_{\nu+1}(y)$$

for $\mu_1 - 1 \geq \nu \geq \mu_2$. Combining these equations, we have

$$\Big\{(1-y)y\frac{d^2}{dy^2} + \{(\mu_1-\mu_2+2) - (\nu-2\mu_2+3)y\}\frac{d}{dy}$$
$$- (\nu-\mu_2+1)(-\mu_2+1)\Big\}\varphi_\nu(y) = 0 \tag{5.3}$$

for $\mu_1 \geq \nu \geq \mu_2$. This equation is the Gaussian hypergeometric differential equation (2.7) with $a = \nu - \mu_2 + 1$, $b = -\mu_2 + 1$ and $c = \mu_1 - \mu_2 + 2$, and has a unique (up to scalar multiple) solution which is regular at $y = 0$. By the normalization (5.2), we have

$$\varphi_\nu(y) = {}_2F_1\left(\begin{matrix}\nu-\mu_2+1,\ -\mu_2+1\\ \mu_1-\mu_2+2\end{matrix};y\right)$$
$$= (1-y)^{\mu_1+\mu_2-\nu}{}_2F_1\left(\begin{matrix}\mu_1-\nu+1,\ \mu_1+1\\ \mu_1-\mu_2+2\end{matrix};y\right)$$

for $\mu_1 \geq \nu \geq \mu_2$. Here the second equality is by (2.8). ■

## 6. Matrix coefficients of large discrete series representations of $SU(3,1)$

Throughout this section, we set $n = 3$.

**6.1. Decomposition of tensor products and injectors for $n = 3$.** Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda_3$. The set of G-patterns of type $\lambda$ is given by

$$\mathbf{G}(\lambda) = \left\{ L = \left(\begin{smallmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \nu_1 & \nu_2 \\ l\end{smallmatrix}\right) \;\middle|\; \begin{matrix}\nu_1,\nu_2,l\in\mathbb{Z},\ \nu_1\geq l\geq\nu_2,\\ \lambda_1\geq\nu_1\geq\lambda_2\geq\nu_2\geq\lambda_3\end{matrix} \right\}.$$

Let $B(\lambda)$ be the set defined in §2.5, that is,

$$B(\lambda) = \left\{ \nu = (\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \lambda \geq \nu_1 \geq \lambda_2 \geq \nu_2 \geq \lambda_3 \right\}.$$

For $\nu = (\nu_1, \nu_2) \in B(\lambda)$, we set

$$P_\lambda(\nu) = \begin{cases} \dfrac{(\lambda_2 - \nu_2)(\lambda_1 - \nu_2 + 1)}{(\nu_1 - \nu_2)(\nu_1 - \nu_2 + 1)} & \text{if } \nu_1 > \nu_2, \\ \lambda_1 - \lambda_2 + 1 & \text{if } \nu_1 = \nu_2, \end{cases}$$

$$\widehat{P}_\lambda(\nu) = P_{\widehat{\lambda}}(\widehat{\nu}) = \begin{cases} \dfrac{(\nu_1 - \lambda_2)(\nu_1 - \lambda_3 + 1)}{(\nu_1 - \nu_2)(\nu_1 - \nu_2 + 1)} & \text{if } \nu_1 > \nu_2, \\ \lambda_2 - \lambda_3 + 1 & \text{if } \nu_1 = \nu_2. \end{cases}$$

Let $\{\zeta(L)\}_{L \in \mathbf{G}(\lambda)}$ be the basis of $V_\lambda^{(3)}$ in Lemma 4.2, and take its dual basis $\{\zeta(L)^\vee\}_{L \in \mathbf{G}(\lambda)}$ of $V_\lambda^{(3)\vee}$. We set $\zeta(L) = 0$ and $\zeta(L)^\vee = 0$ if $L$ is not a G-pattern. Then the basis $E_{i,j}^{(3)}$ $(1 \leq i, j \leq 3)$ of $\mathfrak{u}(3)_\mathbb{C}$ act on these bases by the following formulas:

$$\begin{aligned}
&\tau_\lambda^{(3)}(E_{1,1}^{(3)})\zeta(L) = (\lambda_1 + \lambda_2 + \lambda_3 - \nu_1 - \nu_2)\zeta(L), \\
&\tau_\lambda^{(3)}(E_{2,2}^{(3)})\zeta(L) = (\nu_1 + \nu_2 - l)\zeta(L), \qquad \tau_\lambda^{(3)}(E_{3,3}^{(3)})\zeta(L) = l\zeta(L), \\
&\tau_\lambda^{(3)}(E_{1,2}^{(3)})\zeta(L) = (\nu_2 - \lambda_3)\zeta\left(L - \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;1 \\ 0 \end{smallmatrix}\right)\right) + (\nu_1 - l)\widehat{P}_\lambda(\nu)\zeta\left(L - \left(\begin{smallmatrix} 0\;0\;0 \\ 1\;0 \\ 0 \end{smallmatrix}\right)\right), \\
&\tau_\lambda^{(3)}(E_{2,3}^{(3)})\zeta(L) = (l - \nu_2)\zeta\left(L - \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;0 \\ 1 \end{smallmatrix}\right)\right), \\
&\tau_\lambda^{(3)}(E_{1,3}^{(3)})\zeta(L) = -(\nu_2 - \lambda_3)\zeta\left(L - \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;1 \\ 1 \end{smallmatrix}\right)\right) + (l - \nu_2)\widehat{P}_\lambda(\nu)\zeta\left(L - \left(\begin{smallmatrix} 0\;0\;0 \\ 1\;0 \\ 1 \end{smallmatrix}\right)\right), \\
&\tau_\lambda^{(3)}(E_{2,1}^{(3)})\zeta(L) = (\lambda_1 - \nu_1)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 1\;0 \\ 0 \end{smallmatrix}\right)\right) + (l - \nu_2)P_\lambda(\nu)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;1 \\ 0 \end{smallmatrix}\right)\right), \\
&\tau_\lambda^{(3)}(E_{3,2}^{(3)})\zeta(L) = (\nu_1 - l)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;0 \\ 1 \end{smallmatrix}\right)\right), \\
&\tau_\lambda^{(3)}(E_{3,1}^{(3)})\zeta(L) = (\lambda_1 - \nu_1)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 1\;0 \\ 1 \end{smallmatrix}\right)\right) - (\nu_1 - l)P_\lambda(\nu)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;1 \\ 1 \end{smallmatrix}\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\tau_\lambda^{(3)\vee}(E_{1,1}^{(3)})\zeta(L)^\vee = -\,(\lambda_1 + \lambda_2 + \lambda_3 - \nu_1 - \nu_2)\zeta(L)^\vee, \\
&\tau_\lambda^{(3)\vee}(E_{2,2}^{(3)})\zeta(L)^\vee = -\,(\nu_1 + \nu_2 - l)\zeta(L)^\vee, \qquad \tau_\lambda^{(3)\vee}(E_{3,3}^{(3)})\zeta(L)^\vee = -l\zeta(L)^\vee, \\
&\tau_\lambda^{(3)\vee}(E_{1,2}^{(3)})\zeta(L)^\vee = -\,(\nu_2 - \lambda_3 + 1)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;1 \\ 0 \end{smallmatrix}\right)\right)^\vee \\
&\qquad\qquad - (\nu_1 - l + 1)\widehat{P}_\lambda(\nu + (1,0))\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 1\;0 \\ 0 \end{smallmatrix}\right)\right)^\vee, \\
&\tau_\lambda^{(3)\vee}(E_{2,3}^{(3)})\zeta(L)^\vee = -\,(l - \nu_2 + 1)\zeta\left(L + \left(\begin{smallmatrix} 0\;0\;0 \\ 0\;0 \\ 1 \end{smallmatrix}\right)\right)^\vee,
\end{aligned}$$

$$\tau_\lambda^{(3)\vee}(E_{1,3}^{(3)})\zeta(L)^\vee = (\nu_2-\lambda_3+1)\zeta\left(L+\begin{pmatrix}0\ 0\ 0\\ 0\ 1\\ 1\end{pmatrix}\right)^\vee - (l-\nu_2+1)\widehat{P}_\lambda(\nu+(1,0))\zeta\left(L+\begin{pmatrix}0\ 0\ 0\\ 1\ 0\\ 1\end{pmatrix}\right)^\vee,$$

$$\tau_\lambda^{(3)\vee}(E_{2,1}^{(3)})\zeta(L)^\vee = -(\lambda_1-\nu_1+1)\zeta\left(L-\begin{pmatrix}0\ 0\ 0\\ 1\ 0\\ 0\end{pmatrix}\right)^\vee - (l-\nu_2+1)P_\lambda(\nu-(0,1))\zeta\left(L-\begin{pmatrix}0\ 0\ 0\\ 0\ 1\\ 0\end{pmatrix}\right)^\vee,$$

$$\tau_\lambda^{(3)\vee}(E_{3,2}^{(3)})\zeta(L)^\vee = -(\nu_1-l+1)\zeta\left(L-\begin{pmatrix}0\ 0\ 0\\ 0\ 0\\ 1\end{pmatrix}\right)^\vee,$$

$$\tau_\lambda^{(3)\vee}(E_{3,1}^{(3)})\zeta(L)^\vee = -(\lambda_1-\nu_1+1)\zeta\left(L-\begin{pmatrix}0\ 0\ 0\\ 1\ 0\\ 1\end{pmatrix}\right)^\vee + (\nu_1-l+1)P_\lambda(\nu-(0,1))\zeta\left(L-\begin{pmatrix}0\ 0\ 0\\ 0\ 1\\ 1\end{pmatrix}\right)^\vee,$$

for $L=\begin{pmatrix}\lambda\\ \nu\\ l\end{pmatrix}=\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1\ \nu_2\\ l\end{pmatrix}\in\mathbf{G}(\lambda)$.

The purpose of this subsection is to specify each irreducible component of $V_{e_1-e_4}^{(3)}\otimes_{\mathbb{C}}V_\lambda^{(3)}$ as a $U(3)$-module. First, we specify each irreducible component of $V_{e_1-e_4}^{(3)}\otimes_{\mathbb{C}}V_\lambda^{(3)}$ as a $U(2)$-module. Since

$$V_{e_1-e_4}^{(3)}=V_{e_1-e_4,(2,1)}^{(3)}\oplus V_{e_1-e_4,(1,1)}^{(3)},\qquad V_\lambda^{(3)}=\bigoplus_{\nu\in B(\lambda)}V_{\lambda,\nu}^{(3)},$$

we have

$$V_{e_1-e_4}^{(3)}\otimes_{\mathbb{C}}V_\lambda^{(3)}=\bigoplus_{\substack{\nu\in B(\lambda)\\ i\in\{1,2\}}}V_{e_1-e_4,(i,1)}^{(3)}\otimes_{\mathbb{C}}V_{\lambda,\nu}^{(3)}\simeq\bigoplus_{\substack{\nu\in B(\lambda)\\ i\in\{1,2\}}}V_{(i,1)}^{(2)}\otimes_{\mathbb{C}}V_\nu^{(2)}. \tag{6.1}$$

For $(d_1,d_2)\in\{(2,1),(1,2),(1,1)\}$ and $\nu=(\nu_1,\nu_2)\in\Lambda_2$ such that $\nu-(d_1,d_2)\in B(\lambda)$, we define a $\mathbb{C}$-linear map $J_{\lambda,\nu}^{(d_1,d_2)}\colon V_\nu^{(2)}\to V_{e_1-e_4}^{(3)}\otimes_{\mathbb{C}}V_\lambda^{(3)}$ by

$$J_{\lambda,\nu}^{(2,1)}\left(\zeta\begin{pmatrix}\nu_1\ \nu_2\\ l\end{pmatrix}\right)=(\nu_1-l)\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1-2\ \nu_2-1\\ l-1\end{pmatrix} + (l-\nu_2)\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1-2\ \nu_2-1\\ l-2\end{pmatrix},$$

$$J_{\lambda,\nu}^{(1,2)}\left(\zeta\begin{pmatrix}\nu_1\ \nu_2\\ l\end{pmatrix}\right)=\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1-1\ \nu_2-2\\ l-1\end{pmatrix}-\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1-1\ \nu_2-2\\ l-2\end{pmatrix},$$

$$J_{\lambda,\nu}^{(1,1)}\left(\zeta\begin{pmatrix}\nu_1\ \nu_2\\ l\end{pmatrix}\right)=\zeta\begin{pmatrix}2\ 1\ 1\\ 1\ 1\\ 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1-1\ \nu_2-1\\ l-1\end{pmatrix}\qquad(\nu_1\geq l\geq\nu_2).$$

Then, from Lemmas 4.3, 5.1 and the decomposition (6.1), we have

$$\mathrm{Hom}_{U(2)}(V_\nu^{(2)},V_{e_1-e_4}^{(3)}\otimes_{\mathbb{C}}V_\lambda^{(3)})=\mathbb{C}J_{\lambda,\nu}^{(1,1)}\oplus\mathbb{C}J_{\lambda,\nu}^{(2,1)}\oplus\mathbb{C}J_{\lambda,\nu}^{(1,2)}$$

for $\nu = (\nu_1, \nu_2) \in \Lambda_2$, where we put $J^{(d_1,d_2)}_{\lambda,\nu} = 0$ if $\nu - (d_1, d_2) \notin B(\lambda)$.

**Lemma 6.1.** *We use the above notation. When $J^{(d_1,d_2)}_{\lambda,\nu} \neq 0$, the elements $E^{(3)}_{1,2}$ and $E^{(3)}_{2,1}$ of $\mathfrak{u}(3)_{\mathbb{C}}$ act on $J^{(d_1,d_2)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big)$ $(\nu_1 \geq l \geq \nu_2)$ by the following formulas:*

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{1,2}) J^{(2,1)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\nu_2 - \lambda_3 - 1) J^{(2,1)}_{\lambda,\nu-(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2-1 \\ & l\end{pmatrix}\Big) \\ &+ (\nu_1 - l)\widehat{P}_{\lambda}(\nu - (2,1)) J^{(2,1)}_{\lambda,\nu-(1,0)}\Big(\zeta\begin{pmatrix}\nu_1-1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &+ (\nu_1 - l) J^{(1,1)}_{\lambda,\nu-(1,0)}\Big(\zeta\begin{pmatrix}\nu_1-1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &- (\nu_2 - \lambda_3 - 1)\zeta\begin{pmatrix}2 & 1 & 1 \\ & 2 & 1 \\ & 2 & \end{pmatrix} \otimes \zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ & \nu_1-2 & \nu_2-2 \\ & l-2 & \end{pmatrix},\end{aligned}$$

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{1,2}) J^{(1,2)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\nu_2 - \lambda_3 - 2) J^{(1,2)}_{\lambda,\nu-(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2-1 \\ & l\end{pmatrix}\Big) \\ &+ (\nu_1 - l)\widehat{P}_{\lambda}(\nu - (1,2)) J^{(1,2)}_{\lambda,\nu-(1,0)}\Big(\zeta\begin{pmatrix}\nu_1-1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &+ J^{(1,1)}_{\lambda,\nu-(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2-1 \\ & l\end{pmatrix}\Big) \\ &- \widehat{P}_{\lambda}(\nu - (1,2))\zeta\begin{pmatrix}2 & 1 & 1 \\ & 2 & 1 \\ & 2 & \end{pmatrix} \otimes \zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ & \nu_1-2 & \nu_2-2 \\ & l-2 & \end{pmatrix},\end{aligned}$$

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{1,2}) J^{(1,1)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\nu_2 - \lambda_3 - 1) J^{(1,1)}_{\lambda,\nu-(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2-1 \\ & l\end{pmatrix}\Big) \\ &+ (\nu_1 - l)\widehat{P}_{\lambda}(\nu - (1,1)) J^{(1,1)}_{\lambda,\nu-(1,0)}\Big(\zeta\begin{pmatrix}\nu_1-1 & \nu_2 \\ & l\end{pmatrix}\Big),\end{aligned}$$

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{2,1}) J^{(2,1)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\lambda_1 - \nu_1 + 2) J^{(2,1)}_{\lambda,\nu+(1,0)}\Big(\zeta\begin{pmatrix}\nu_1+1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &+ (l - \nu_2) P_{\lambda}(\nu - (2,1)) J^{(2,1)}_{\lambda,\nu+(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2+1 \\ & l\end{pmatrix}\Big) \\ &- (\lambda_1 - \nu_1 + 2)\zeta\begin{pmatrix}2 & 1 & 1 \\ & 2 & 1 \\ & 1 & \end{pmatrix} \otimes \zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ & \nu_1-1 & \nu_2-1 \\ & l-1 & \end{pmatrix},\end{aligned}$$

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{2,1}) J^{(1,2)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\lambda_1 - \nu_1 + 1) J^{(1,2)}_{\lambda,\nu+(1,0)}\Big(\zeta\begin{pmatrix}\nu_1+1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &+ (l - \nu_2) P_{\lambda}(\nu - (1,2)) J^{(1,2)}_{\lambda,\nu+(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2+1 \\ & l\end{pmatrix}\Big) \\ &+ P_{\lambda}(\nu - (1,2))\zeta\begin{pmatrix}2 & 1 & 1 \\ & 2 & 1 \\ & 1 & \end{pmatrix} \otimes \zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ & \nu_1-1 & \nu_2-1 \\ & l-1 & \end{pmatrix},\end{aligned}$$

$$\begin{aligned}(\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{2,1}) J^{(1,1)}_{\lambda,\nu}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2 \\ & l\end{pmatrix}\Big) =& (\lambda_1 - \nu_1 + 1) J^{(1,1)}_{\lambda,\nu+(1,0)}\Big(\zeta\begin{pmatrix}\nu_1+1 & \nu_2 \\ & l\end{pmatrix}\Big) \\ &+ (l - \nu_2) P_{\lambda}(\nu - (1,1)) J^{(1,1)}_{\lambda,\nu+(0,1)}\Big(\zeta\begin{pmatrix}\nu_1 & \nu_2+1 \\ & l\end{pmatrix}\Big) \\ &+ \zeta\begin{pmatrix}2 & 1 & 1 \\ & 2 & 1 \\ & 1 & \end{pmatrix} \otimes \zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ & \nu_1-1 & \nu_2-1 \\ & l-1 & \end{pmatrix}.\end{aligned}$$

**Proof.** Direct computation. ∎

The tensor product $V^{(3)}_{e_1-e_4} \otimes_{\mathbb{C}} V^{(3)}_{\lambda}$ has the following irreducible decomposition as a $U(3)$-module:

$$V^{(3)}_{e_1-e_4} \otimes_{\mathbb{C}} V^{(3)}_{\lambda} \simeq \begin{cases} V^{(3)}_{\lambda+e_1-e_4} \oplus V^{(3)}_{\lambda+e_2-e_4} \oplus V^{(3)}_{\lambda+e_3-e_4} & \text{if } \lambda_1 > \lambda_2 > \lambda_3, \\ V^{(3)}_{\lambda+e_1-e_4} \oplus V^{(3)}_{\lambda+e_2-e_4} & \text{if } \lambda_1 > \lambda_2 = \lambda_3, \\ V^{(3)}_{\lambda+e_1-e_4} \oplus V^{(3)}_{\lambda+e_3-e_4} & \text{if } \lambda_1 = \lambda_2 > \lambda_3, \\ V^{(3)}_{\lambda+e_1-e_4} & \text{if } \lambda_1 = \lambda_2 = \lambda_3. \end{cases}$$

We specify each component of this decomposition as follows.

**Proposition 6.2.** *We use the above notation. For $i \in \{1,2,3\}$, we define a linear map $I^{e_i-e_4}_{\lambda} \colon V^{(3)}_{\lambda+e_i-e_4} \to V^{(3)}_{e_1-e_4} \otimes_{\mathbb{C}} V^{(3)}_{\lambda}$ by*

$$I^{e_i-e_4}_{\lambda}(\zeta(L)) = \sum_{(d_1,d_2)\in\{(2,1),(1,2),(1,1)\}} C^{i}_{(d_1,d_2)}(\lambda,\nu) J^{(d_1,d_2)}_{\lambda,\nu}\left(\zeta\begin{pmatrix} \nu_1 & & \nu_2 \\ & l & \end{pmatrix}\right)$$

$$\left(L = \begin{pmatrix} \lambda+e_i-e_4 \\ \nu \\ l \end{pmatrix} = \begin{pmatrix} \lambda_1+1+\delta_{i,1} & & \lambda_2+1+\delta_{i,2} & & \lambda_3+1+\delta_{i,3} \\ & \nu_1 & & \nu_2 & \\ & & l & & \end{pmatrix} \in \mathbf{G}(\lambda+e_i-e_4)\right)$$

*with*

$$C^{1}_{(1,1)}(\lambda,\nu) = (\lambda_1-\nu_1+2)(\lambda_1-\nu_2+3),$$
$$C^{1}_{(1,2)}(\lambda,\nu) = (\lambda_1-\nu_1+2)(\nu_2-\lambda_3-1),$$
$$C^{1}_{(2,1)}(\lambda,\nu) = \begin{cases} \dfrac{(\lambda_1-\nu_2+3)(\nu_1-\lambda_2-1)(\nu_1-\lambda_3)}{(\nu_1-\nu_2)(\nu_1-\nu_2+1)} & \textit{if } \nu_1 > \nu_2, \\ (\lambda_1-\lambda_2+2)(\lambda_2-\lambda_3+1) & \textit{if } \nu_1 = \nu_2, \end{cases}$$

$$C^{2}_{(1,1)}(\lambda,\nu) = \lambda_2-\nu_2+2, \qquad C^{2}_{(1,2)}(\lambda,\nu) = \nu_2-\lambda_3-1,$$
$$C^{2}_{(2,1)}(\lambda,\nu) = \begin{cases} -\dfrac{(\lambda_2-\nu_2+2)(\nu_1-\lambda_3)}{(\nu_1-\nu_2)(\nu_1-\nu_2+1)} & \textit{if } \nu_1 > \nu_2, \\ -(\lambda_2-\lambda_3+2) & \textit{if } \nu_1 = \nu_2, \end{cases}$$

$$C^{3}_{(1,1)}(\lambda,\nu) = 1, \qquad C^{3}_{(1,2)}(\lambda,\nu) = -1,$$
$$C^{3}_{(2,1)}(\lambda,\nu) = \begin{cases} -\dfrac{\nu_1-\lambda_2-1}{(\nu_1-\nu_2)(\nu_1-\nu_2+1)} & \textit{if } \nu_1 > \nu_2, \\ -1 & \textit{if } \nu_1 = \nu_2. \end{cases}$$

*Then $I^{e_i-e_4}_{\lambda}$ is a $U(3)$-homomorphism for $i \in \{1,2,3\}$.*

**Proof.** It suffices to show that $I^{e_i-e_4}_{\lambda}$ is a $\mathfrak{u}(3)_{\mathbb{C}}$-homomorphism for $i \in \{1,2,3\}$, since $U(3)$ is connected. The Lie algebra $\mathfrak{u}(3)_{\mathbb{C}}$ is generated by $E^{(3)}_{1,2}$, $E^{(3)}_{2,1}$ and $\iota_{2,3}(\mathfrak{u}(2)_{\mathbb{C}})$. Moreover, $J^{(d_1,d_2)}_{\lambda,\nu}$ is a $\mathfrak{u}(2)_{\mathbb{C}}$-homomorphisms for $(d_1,d_2) \in \{(2,1),(1,2),(1,1)\}$. Hence, it suffices to confirm that

$$I^{e_i-e_4}_{\lambda}\left(\tau^{(3)}_{\lambda+e_i-e_4}(E^{(3)}_{1,2})\zeta(L)\right) = (\tau^{(3)}_{e_1-e_4} \otimes \tau^{(3)}_{\lambda})(E^{(3)}_{1,2}) I^{e_i-e_4}_{\lambda}(\zeta(L)),$$

$$I_\lambda^{e_i-e_4}\left(\tau^{(3)}_{\lambda+e_i-e_4}(E^{(3)}_{2,1})\zeta(L)\right)=(\tau^{(3)}_{e_1-e_4}\otimes\tau^{(3)}_\lambda)(E^{(3)}_{2,1})I_\lambda^{e_i-e_4}(\zeta(L))$$

for $i\in\{1,2,3\}$ and $L\in\mathbf{G}(\lambda+e_i-e_4)$. We can confirm these equalities using Lemma 6.1 and the following formulas:

$$\begin{aligned}
&\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\quad\lambda_2\quad\lambda_3\\ \nu_1-2\ \nu_2-2\\ l-2\end{pmatrix}\\
&=\frac{1}{\nu_1-\nu_2+1}\left\{J^{(2,1)}_{\lambda,\nu-(0,1)}\left(\zeta\begin{pmatrix}\nu_1\ \nu_2-1\\ l\end{pmatrix}\right)-(\nu_1-l)J^{(1,2)}_{\lambda,\nu-(1,0)}\left(\zeta\begin{pmatrix}\nu_1-1\ \nu_2\\ l\end{pmatrix}\right)\right\},\\
&\zeta\begin{pmatrix}2\ 1\ 1\\ 2\ 1\\ 1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\quad\lambda_2\quad\lambda_3\\ \nu_1-1\ \nu_2-1\\ l-1\end{pmatrix}\\
&=\frac{1}{\nu_1-\nu_2+1}\left\{J^{(2,1)}_{\lambda,\nu+(1,0)}\left(\zeta\begin{pmatrix}\nu_1+1\ \nu_2\\ l\end{pmatrix}\right)+(l-\nu_2)J^{(1,2)}_{\lambda,\nu+(0,1)}\left(\zeta\begin{pmatrix}\nu_1\ \nu_2+1\\ l\end{pmatrix}\right)\right\}.
\end{aligned}$$

■

The tensor product $V^{(3)}_{-e_3+e_4}\otimes_{\mathbb{C}}V^{(3)}_\lambda$ has the following irreducible decomposition as a $U(3)$-module:

$$V^{(3)}_{-e_3+e_4}\otimes_{\mathbb{C}}V^{(3)}_\lambda\simeq\begin{cases}V^{(3)}_{\lambda-e_3+e_4}\oplus V^{(3)}_{\lambda-e_2+e_4}\oplus V^{(3)}_{\lambda-e_1+e_4} & \text{if } \lambda_1>\lambda_2>\lambda_3,\\ V^{(3)}_{\lambda-e_3+e_4}\oplus V^{(3)}_{\lambda-e_1+e_4} & \text{if } \lambda_1>\lambda_2=\lambda_3,\\ V^{(3)}_{\lambda-e_3+e_4}\oplus V^{(3)}_{\lambda-e_2+e_4} & \text{if } \lambda_1=\lambda_2>\lambda_3,\\ V^{(3)}_{\lambda-e_3+e_4} & \text{if } \lambda_1=\lambda_2=\lambda_3.\end{cases}$$

For $i\in\{1,2,3\}$, we set

$$I_\lambda^{-e_i+e_4}=(T_{e_1-e_4}\otimes T_{\widehat{\lambda}})\circ I_{\widehat{\lambda}}^{e_{4-i}-e_4}\circ T_{\lambda-e_i+e_4}.$$

Then $I_\lambda^{-e_i+e_4}\colon V^{(3)}_{\lambda-e_i+e_4}\to V^{(3)}_{-e_3+e_4}\otimes_{\mathbb{C}}V^{(3)}_\lambda$ is a $U(3)$-homomorphism. For $i\in\{1,2\}$ and $L=\begin{pmatrix}\lambda\\ \nu\\ l\end{pmatrix}=\begin{pmatrix}\lambda_1\ \lambda_2\ \lambda_3\\ \nu_1\ \nu_2\\ l\end{pmatrix}\in\mathbf{G}(\lambda)$ such that $\lambda_1-1\geq\nu_1\geq\lambda_2$, $\lambda_2-\delta_{i,2}\geq\nu_2\geq\lambda_3$, we define the element $\zeta_i^{(-)}(L)$ of the $\tau^{(3)}_{\lambda-e_2+e_4}\oplus\tau^{(3)}_{\lambda-e_1+e_4}$-component of $V^{(3)}_{-e_3+e_4}\otimes_{\mathbb{C}}V^{(3)}_\lambda$ by

$$\begin{aligned}
\zeta_1^{(-)}(L)=&\frac{\lambda_2-\nu_2}{\lambda_1-\lambda_2+1}I_\lambda^{-e_2+e_4}\left(\zeta\begin{pmatrix}\lambda_1-1\ \lambda_2-2\ \lambda_3-1\\ \nu_1-1\ \nu_2-1\\ l-1\end{pmatrix}\right)\\
&-\frac{(\nu_1-\lambda_2+1)(\lambda_1-\nu_2+1)}{\lambda_1-\lambda_2+1}I_\lambda^{-e_1+e_4}\left(\zeta\begin{pmatrix}\lambda_1-2\ \lambda_2-1\ \lambda_3-1\\ \nu_1-1\ \nu_2-1\\ l-1\end{pmatrix}\right)\\
=&-(\nu_1-\nu_2+1)\zeta\begin{pmatrix}-1\ -1\ -2\\ -1\ -2\\ -2\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\quad\lambda_2\ \lambda_3\\ \nu_1+1\ \nu_2\\ l+1\end{pmatrix}\\
&+(\nu_1-\nu_2+1)\zeta\begin{pmatrix}-1\ -1\ -2\\ -1\ -2\\ -1\end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1\quad\lambda_2\ \lambda_3\\ \nu_1+1\ \nu_2\\ l\end{pmatrix}
\end{aligned}$$

$$-(\nu_1-\lambda_2+1)\zeta\begin{pmatrix}-1 & -1 & -2\\ & -1 \; -1 & \\ & -1 & \end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ & \nu_1 \; \nu_2 & \\ & l & \end{pmatrix},$$

$$\begin{aligned}\zeta_2^{(-)}(L)=&\frac{\nu_1-\nu_2}{\lambda_1-\lambda_2+1}I_\lambda^{-e_2+e_4}\left(\zeta\begin{pmatrix}\lambda_1-1 & \lambda_2-2 & \lambda_3-1\\ & \nu_1-1 \; \nu_2-1 & \\ & l-1 & \end{pmatrix}\right)\\&+\frac{(\nu_1-\nu_2)(\lambda_1-\nu_1)}{\lambda_1-\lambda_2+1}I_\lambda^{-e_1+e_4}\left(\zeta\begin{pmatrix}\lambda_1-2 & \lambda_2-1 & \lambda_3-1\\ & \nu_1-1 \; \nu_2-1 & \\ & l-1 & \end{pmatrix}\right)\\=&-(\nu_1-l)\zeta\begin{pmatrix}-1 & -1 & -2\\ & -1 \; -2 & \\ & -2 & \end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ & \nu_1 \; \nu_2+1 & \\ & l+1 & \end{pmatrix}\\&-(l-\nu_2)\zeta\begin{pmatrix}-1 & -1 & -2\\ & -1 \; -2 & \\ & -1 & \end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ & \nu_1 \; \nu_2+1 & \\ & l & \end{pmatrix}\\&+(\nu_1-\nu_2)\zeta\begin{pmatrix}-1 & -1 & -2\\ & -1 \; -1 & \\ & -1 & \end{pmatrix}\otimes\zeta\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ & \nu_1 \; \nu_2 & \\ & l & \end{pmatrix}.\end{aligned}$$

### 6.2. Differential equations of the $\nu$-components for $n=3$.

**Proposition 6.3.** *We use the notation in Theorem 3.3. Then, we have the following system of differential equations:*
*(i) For $\mu_1\ge\nu_1\ge\mu_2\ge\nu_2>\mu_3$, it holds that*

$$\begin{aligned}&(\nu_1-\nu_2+1)\left\{\frac{1}{2}\left(\frac{d}{dt}-(\mu_1+\mu_2+\mu_3-\nu_1-\nu_2)\frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right)\right.\\&\left.+(\mu_1+\mu_2-\nu_1-\nu_2+2)\frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}\right\}\phi_\mu[\nu](\mathrm{a}[t])\\&-(\nu_1-\mu_2)(\mu_1-\nu_1+1)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu-(1,0)](\mathrm{a}[t])\\&-(\mu_2-\nu_2+1)(\mu_1-\nu_2+2)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu-(0,1)](\mathrm{a}[t])=0.\end{aligned}$$

*(ii) For $\mu_1>\nu_1\ge\mu_2\ge\nu_2\ge\mu_3$, it holds that*

$$\begin{aligned}&\left\{\frac{1}{2}\left(\frac{d}{dt}+(\mu_1+\mu_2+\mu_3-\nu_1-\nu_2)\frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right)+(\nu_1-\mu_3+2)\frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}\right\}\phi_\mu[\nu](\mathrm{a}[t])\\&-(\nu_1-\mu_3+2)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu+(1,0)](\mathrm{a}[t])=0.\end{aligned}$$

*(iii) For $\mu_1>\nu_1\ge\mu_2>\nu_2\ge\mu_3$, it holds that*

$$\begin{aligned}&-\left\{\frac{1}{2}\left(\frac{d}{dt}+(\mu_1+\mu_2+\mu_3-\nu_1-\nu_2)\frac{\mathrm{sh}(t)}{\mathrm{ch}(t)}\right)+(\nu_2-\mu_3+1)\frac{\mathrm{ch}(t)}{\mathrm{sh}(t)}\right\}\phi_\mu[\nu](\mathrm{a}[t])\\&+(\nu_2-\mu_3+1)\frac{1}{\mathrm{sh}(t)}\phi_\mu[\nu+(0,1)](\mathrm{a}[t])=0.\end{aligned}$$

**Proof.** We define the natural $K$-homomorphism $P_\Pi\colon\mathfrak{p}_\mathbb{C}\otimes_\mathbb{C}H_{\mu,K}\to H_{\mu,K}$ by $X\otimes f\mapsto\Pi_\mu(X)f$, and set

$$\tilde{\iota}_{\mu,+}=P_\Pi\circ(I_\mathfrak{p}^+\otimes_\mathbb{C}\iota_\mu),\qquad\qquad\tilde{\iota}_{\mu,-}=P_\Pi\circ(I_\mathfrak{p}^-\otimes_\mathbb{C}\iota_\mu).$$

Because of the property (b) of $\Pi_\mu$ in §2.7, the image of $I_\mu^{e_3-e_4}(\zeta(L))$ under $\tilde{\iota}_{\mu,+}$ and the images of $\zeta_1^{(-)}(L)$ and $\zeta_2^{(-)}(L)$ under $\tilde{\iota}_{\mu,-}$ are 0. Hence, we have

$$\left(\Phi_{\Pi_\mu}\circ(\iota_\mu^\vee\boxtimes\tilde{\iota}_{\mu,+})\right)\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes I_\lambda^{e_3-e_4}\left(\zeta\left(\begin{smallmatrix}\mu_1+1\ \mu_2+1\ \mu_3+2\\ \nu_1+1\ \ \nu_2+1\\ l+1\end{smallmatrix}\right)\right)\right)=0,$$

$$\left(\Phi_{\Pi_\mu}\circ(\iota_\mu^\vee\boxtimes\tilde{\iota}_{\mu,-})\right)\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta_i^{(-)}\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)\right)=0\qquad(i\in\{1,2\}).$$

These equations are equivalent to the following equations:

$$\begin{aligned}
&-\frac{(\nu_1-\mu_2)}{(\nu_1-\nu_2)(\nu_1-\nu_2+1)}\left\{(\nu_1-l)R(E_{2,4}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \ \ \mu_2\ \mu_3\\ \nu_1-1\ \nu_2\\ l\end{smallmatrix}\right)\right)\right.\\
&\left.+(l-\nu_2)R(E_{3,4}^{(4)})\phi_\mu\left(\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \ \ \mu_2\ \mu_3\\ \nu_1-1\ \nu_2\\ l-1\end{smallmatrix}\right)\right)\right\}\\
&-R(E_{2,4}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \ \ \mu_3\\ \nu_1\ \nu_2-1\\ l\end{smallmatrix}\right)\right)\\
&+R(E_{3,4}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \ \ \mu_3\\ \nu_1\ \nu_2-1\\ l-1\end{smallmatrix}\right)\right)\\
&+R(E_{1,4}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)\right)=0,
\end{aligned}$$

$$\begin{aligned}
&(\nu_1-\nu_2+1)R(E_{4,3}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \ \ \mu_2\ \mu_3\\ \nu_1+1\ \nu_2\\ l+1\end{smallmatrix}\right)\right)\\
&+(\nu_1-\nu_2+1)R(E_{4,2}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \ \ \mu_2\ \mu_3\\ \nu_1+1\ \nu_2\\ l\end{smallmatrix}\right)\right)\\
&+(\nu_1-\mu_2+1)R(E_{4,1}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)\right)=0,
\end{aligned}$$

$$\begin{aligned}
&(\nu_1-l)R(E_{4,3}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \ \ \mu_3\\ \nu_1\ \nu_2+1\\ l+1\end{smallmatrix}\right)\right)\\
&-(l-\nu_2)R(E_{4,2}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \ \ \mu_3\\ \nu_1\ \nu_2+1\\ l\end{smallmatrix}\right)\right)\\
&-(\nu_1-\nu_2)R(E_{4,1}^{(4)})\phi_\mu\left(\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)^\vee\boxtimes\zeta\left(\begin{smallmatrix}\mu_1\ \mu_2\ \mu_3\\ \nu_1\ \nu_2\\ l\end{smallmatrix}\right)\right)=0.
\end{aligned}$$

If we restrict them to $A$ and apply Proposition 2.2, Lemma 4.4 and the formulas of the action of $\mathfrak{k}_{\mathbf{C}}$ in §6.1, these equations become differential equations in the statement. ■

### 6.3. Proof of Theorem 3.3.

**Proof of Theorem 3.3.** For $\nu=(\nu_1,\nu_2)\in B(\mu)$, we set

$$\varphi_\nu(y)=\mathrm{ch}(t)^{\mu_1+\mu_2+\mu_3-\nu_1-\nu_2}\phi_\mu[\nu](\mathrm{a}[t])$$

with $y = 1 - \mathrm{ch}(t)^2$. From (3.2), we know that

$$\varphi_\nu(0) = 1 \qquad (\nu \in B(\mu)). \tag{6.2}$$

The system of differential equations for $\{\phi_\mu[\nu](\mathrm{a}[t])\}_{\nu\in B(\mu)}$ in Proposition 6.3 is translated to the following system of differential equations for $\{\varphi_\nu(y)\}_{\nu\in B(\mu)}$:

$$\begin{aligned}&(\nu_1 - \nu_2 + 1)\left\{(1-y)y\frac{d}{dy} + (\mu_3 - 2)y + \mu_1 + \mu_2 - \nu_1 - \nu_2 + 2\right\}\varphi_\nu(y)\\ &= (\nu_1 - \mu_2)(\mu_1 - \nu_1 + 1)\varphi_{\nu-(1,0)}(y) + (\mu_2 - \nu_2 + 1)(\mu_1 - \nu_2 + 2)\varphi_{\nu-(0,1)}(y),\end{aligned} \tag{6.3}$$

$$\left(y\frac{d}{dy} + \nu_1 - \mu_3 + 2\right)\varphi_\nu(y) = (\nu_1 - \mu_3 + 2)\varphi_{\nu+(1,0)}(y), \tag{6.4}$$

$$\left(y\frac{d}{dy} + \nu_2 - \mu_3 + 1\right)\varphi_\nu(y) = (\nu_2 - \mu_3 + 1)\varphi_{\nu+(0,1)}(y). \tag{6.5}$$

As special cases of (6.3) and (6.5), we have

$$\left\{(1-y)y\frac{d}{dy} + (\mu_3 - 2)y + \mu_1 - \nu_2 + 1\right\}\varphi_{(\mu_2,\nu_2+1)}(y) = (\mu_1 - \nu_2 + 1)\varphi_{(\mu_2,\nu_2)}(y),$$

$$\left(y\frac{d}{dy} + \nu_2 - \mu_3 + 1\right)\varphi_{(\mu_2,\nu_2)}(y) = (\nu_2 - \mu_3 + 1)\varphi_{(\mu_2,\nu_2+1)}(y)$$

for $\mu_2 - 1 \geq \nu_2 \geq \mu_3$. Combining these equations, we have

$$\begin{aligned}&\left\{(1-y)y\frac{d^2}{dy^2} + \{(\mu_1 - \mu_3 + 3) - (\nu_2 - 2\mu_3 + 4)y\}\frac{d}{dy}\right.\\ &\left.- (\nu_2 - \mu_3 + 1)(-\mu_3 + 2)\right\}\varphi_{(\mu_2,\nu_2)}(y) = 0\end{aligned} \tag{6.6}$$

for $\mu_2 \geq \nu_2 \geq \mu_3$. This equation is the Gaussian hypergeometric differential equation (2.7) with $a = \nu_2 - \mu_3 + 1$, $b = -\mu_3 + 2$ and $c = \mu_1 - \mu_3 + 3$, and has a unique (up to scalar multiple) solution which is regular at $y = 0$. By the normalization (6.2), we have

$$\begin{aligned}\varphi_{(\mu_2,\nu_2)}(y) &= {}_2F_1\left(\begin{matrix}\nu_2 - \mu_3 + 1,\ -\mu_3 + 2\\ \mu_1 - \mu_3 + 3\end{matrix}\ ; y\right)\\ &= {}_3F_2\left(\begin{matrix}\mu_2 - \mu_3 + 2,\ \nu_2 - \mu_3 + 1,\ -\mu_3 + 2\\ \mu_1 - \mu_3 + 3,\ \mu_2 - \mu_3 + 2\end{matrix}\ ; y\right)\end{aligned}$$

for $\mu_2 \geq \nu_2 \geq \mu_3$. Here the second equality is by (2.10). To obtain the general formula for $\mu_1 \geq \nu_1 \geq \mu_2$, we make use of (2.11) and (6.4), recursively. This is straightforward. Hence, we obtain the assertion. ∎

**6.4. Proof of Proposition 2.4.** In this subsection, we give the proof of Proposition 2.4.

**Lemma 6.4.** *For $z, a, b, c, d \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$ such that $a, d \notin \mathbb{Z}_{\leq 0}$, $|z| < 1$, we have*

$$ {}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix};z\right) = \sum_{j=0}^{m} \binom{m}{j} \frac{(b)_j(c)_j}{(a)_j(d)_j} z^j {}_2F_1\left(\begin{matrix} b+j,\ c+j \\ d+j \end{matrix};z\right). $$

**Proof.** Using the equalities (2.9) and

$$ \binom{m}{j} = (-1)^j \frac{(-m)_j}{j!}, \tag{6.7} $$

we have

$$ \begin{aligned} \frac{(a+m)_i}{(a)_i} &= \frac{\Gamma(a)\Gamma(a+m+i)}{\Gamma(a+i)\Gamma(a+m)} = {}_2F_1\left(\begin{matrix} -i,\ -m \\ a \end{matrix};1\right) = \sum_{j=0}^{\min\{i,m\}} \frac{(-i)_j(-m)_j}{(a)_j j!} \\ &= \sum_{j=0}^{\min\{i,m\}} \binom{m}{j} \frac{i(i-1)(i-2)\cdots(i-j+1)}{(a)_j}. \end{aligned} $$

Using this equality and $(b)_i = (b)_j(b+j)_{i-j}$, we have

$$ \begin{aligned} {}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix};z\right) &= \sum_{i=0}^{\infty} \frac{(a+m)_i(b)_i(c)_i}{(d)_i(a)_i} \frac{z^i}{i!} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{\min\{i,m\}} \binom{m}{j} \frac{(b)_i(c)_i}{(a)_j(d)_i} \frac{z^i}{(i-j)!} \\ &= \sum_{j=0}^{m} \binom{m}{j} \frac{(b)_j(c)_j}{(a)_j(d)_j} z^j \sum_{i=j}^{\infty} \frac{(b+j)_{i-j}(c+j)_{i-j}}{(d+j)_{i-j}} \frac{z^{i-j}}{(i-j)!}. \end{aligned} $$

Replacing $i \to i+j$, we obtain the assertion. ■

**Lemma 6.5.** *For $z, a, b, c \in \mathbb{C}$ and $l, m \in \mathbb{Z}_{\geq 0}$ such that*

$$ a \notin \mathbb{Z}_{\leq 0}, \quad (a-c+m, a-c) \in (\mathbb{C}^2)_{\mathrm{P}}, \quad (b, 1-c-j) \in (\mathbb{C}^2)_{\mathrm{P}} \quad (0 \leq j \leq m), $$

*we have*

$$ \sum_{j=0}^{m} \frac{(-m)_j(c)_j}{j!(a)_j} {}_2F_1\left(\begin{matrix} -l,\ b \\ 1-c-j \end{matrix};z\right) = \frac{(a-c)_m}{(a)_m} {}_3F_2\left(\begin{matrix} -l,\ b,\ a-c+m \\ 1-c,\ a-c \end{matrix};z\right). $$

**Proof.** Using the equality

$$ \frac{(b)_i(c)_j}{(1-c-j)_i} = \frac{(b)_i(c-i)_j}{(1-c)_i}, $$

we have

$$ \sum_{j=0}^{m} \frac{(-m)_j(c)_j}{j!(a)_j} {}_2F_1\left(\begin{matrix} -l,\ b \\ 1-c-j \end{matrix};z\right) = \sum_{j=0}^{m} \frac{(-m)_j(c)_j}{j!(a)_j} \sum_{i=0}^{l} \frac{(-l)_i(b)_i}{(1-c-j)_i} \frac{z^i}{i!} $$

$$= \sum_{i=0}^{l} \frac{(-l)_i (b)_i}{(1-c)_i} \frac{z^i}{i!} \sum_{j=0}^{m} \frac{(-m)_j (c-i)_j}{j!(a)_j}.$$

Since

$$\sum_{j=0}^{m} \frac{(-m)_j (c-i)_j}{j!(a)_j} = {}_2F_1\left(\begin{matrix} -m,\ c-i \\ a \end{matrix}; 1\right) = \frac{\Gamma(a)\Gamma(a-c+m+i)}{\Gamma(a+m)\Gamma(a-c+i)}$$
$$= \frac{(a-c)_m (a-c+m)_i}{(a)_m (a-c)_i}$$

(the second equality is by (2.9)), we have

$$\sum_{j=0}^{m} \frac{(-m)_j (c)_j}{j!(a)_j} {}_2F_1\left(\begin{matrix} -l,\ b \\ 1-c-j \end{matrix}; z\right) = \frac{(a-c)_m}{(a)_m} \sum_{i=0}^{l} \frac{(-l)_i (b)_i (a-c+m)_i}{(c)_i (a-c)_i} \frac{z^i}{i!}$$
$$= \frac{(a-c)_m}{(a)_m} {}_3F_2\left(\begin{matrix} -l,\ b,\ a-c+m \\ 1-c,\ a-c \end{matrix}; z\right).$$

■

**Proof of Proposition 2.4.** From Lemma 6.4, we have

$${}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix}; z\right) = \sum_{j=0}^{m} \binom{m}{j} \frac{(b)_j (c)_j}{(d)_j (a)_j} z^j {}_2F_1\left(\begin{matrix} b+j,\ c+j \\ d+j \end{matrix}; z\right).$$

By (2.8), we have

$${}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix}; z\right) = \sum_{\substack{0\le j\le m \\ d-b-c\le j}} \binom{m}{j} \frac{(b)_j (c)_j}{(d)_j (a)_j} z^j {}_2F_1\left(\begin{matrix} b+j,\ c+j \\ d+j \end{matrix}; z\right)$$
$$+ \sum_{\substack{0\le j\le m \\ j<d-b-c}} \binom{m}{j} \frac{(b)_j (c)_j}{(d)_j (a)_j} z^j (1-z)^{d-b-c-j} {}_2F_1\left(\begin{matrix} d-b,\ d-c \\ d+j \end{matrix}; z\right).$$

Applying Theorem 2.3, we find that the right hand side is equal to

$$\sum_{\substack{0\le j\le m \\ d-b-c\le j}} \binom{m}{j} \frac{(b)_j (c)_j}{(d)_j (a)_j} z^j \Big( Q_{[b+c-d+j,b-c,d-b-1]}(z) \log(1-z)$$
$$+ R_{[b+c-d+j,b-c,d-b-1]}(1-z) \Big)$$
$$+ \sum_{\substack{0\le j\le m \\ j<d-b-c}} \binom{m}{j} \frac{(b)_j (c)_j}{(d)_j (a)_j} z^j (1-z)^{d-b-c-j} \Big( Q_{[d-b-c-j,b-c,c+j-1]}(z) \log(1-z)$$
$$+ R_{[d-b-c-j,b-c,c+j-1]}(1-z) \Big).$$

Applying the equality

$$(1-z)^{d-b-c-j}Q_{[d-b-c-j,b-c,c+j-1]}(z)=Q_{[b+c-d+j,b-c,d-b-1]}(z)$$

(obtained from (2.8)), we know that ${}_3F_2\left(\begin{matrix} a+m,\ b,\ c \\ d,\ a \end{matrix};z\right)$ is equal to

$$\sum_{j=0}^{m}\binom{m}{j}\frac{(b)_j(c)_j}{(d)_j(a)_j}z^jQ_{[b+c-d+j,b-c,d-b-1]}(z)\log(1-z)+\tilde{R}_{[a,m;b,c,d]}(1-z).$$

Moreover, using the equalities (6.7),

$$(l+j)!=(l+1)_j\,l! \qquad (l,j\in\mathbb{Z}_{\geq 0})$$

and Lemma 6.5, we have

$$\begin{aligned}
&\sum_{j=0}^{m}\binom{m}{j}\frac{(b)_j(c)_j}{(d)_j(a)_j}z^jQ_{[b+c-d+j,b-c,d-b-1]}(z)\\
&=\frac{(-1)^{b+c-d+1}(d-2)!(d-1)!}{(b-1)!(d-b-1)!(c-1)!(d-c-1)!}z^{-d+1}\\
&\quad\times\sum_{j=0}^{m}\frac{(-m)_j(d-1)_j}{(a)_j j!}{}_2F_1\left(\begin{matrix} 1-d+b,\ 1-d+c \\ -d+2-j \end{matrix};z\right)\\
&=\tilde{Q}_{[a,m;b,c,d]}(z).
\end{aligned}$$

■

## References


[1] Collingwood, D. H., *Representations of rank one Lie groups*, Research Notes in Mathematics **137**, Pitman Advanced Publishing Program, Boston, MA, 1985.

[2] Fefferman, Ch. *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math. **26** (1974), 1–65.

[3] Godement, R., *Fonctions holomorphes de carré sommable dans le demi-plan de Siegel*, Séminaire Henri Cartan, 10e année, 1957/1958. Fonctions Automorphes, 2 vols., Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958.

[4] Hayata, T., H. Koseki, and T. Oda, *Matrix coefficients of the middle discrete series of* $\mathrm{SU}(2,2)$*, II. The explicit asymptotic expansion*, J. Funct. Anal. **259** (2010), 301–307.

[5] Ibukiyama, T., *On relations of dimensions of automorphic forms of* $\mathrm{Sp}(2,\mathbf{R})$ *and its compact twist* $\mathrm{Sp}(2)$*, I*, In: Automorphic forms and number theory (Sendai, 1983), Adv. Stud. Pure Math. **7**, North-Holland, Amsterdam, 1985, 7–30.

[6] Kato, S., *A dimension formula for a certain space of automorphic forms of* $\mathrm{SU}(p,1)$, Math. Ann. **266** (1984), 457–477.

[7] —, *A dimension formula for a certain space of automorphic forms of* $\mathrm{SU}(p,1)$*, II: The case of* $\Gamma(N)$ *with* $N \geq 3$, Tohoku Math. J. (2) **37** (1985), 571–584, and *Correction*, Tohoku Math. J. (2), **38** (1986), 643.

[8] Knapp, A. W., *Representation theory of semisimple groups.* Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the original of 1986.

[9] Koseki, H., *On a comparison of trace formulas for* $\mathrm{GU}(1,2)$ *and* $\mathrm{GU}(3)$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 467–521.

[10] Molev, A. I., *Gelfand-Tsetlin bases for classical Lie algebras*, In: Handbook of Algebra **4**, Elsevier/North-Holland, Amsterdam, 2006, 109–170.

[11] Schmid, W., *On the realization of the discrete series of a semisimple Lie group*, Rice Univ. Studies **56** (1971), 99–108.

[12] Taniguchi, K., Discrete series Whittaker functions of $\mathrm{SU}(n,1)$ and $\mathrm{Spin}(2n,1)$, J. Math. Sci. Univ. Tokyo **3** (1996), 331–377.

[13] Tsuzuki, M., *Real Shintani functions and multiplicity free property for the symmetric pair* $(\mathrm{SU}(2,1),\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(1)))$, J. Math. Sci. Univ. Tokyo **4** (1997), 663–727.

[14] —, *Real Shintani functions on* $\mathrm{U}(n,1)$, J. Math. Sci. Univ. Tokyo **8** (2001), 609–688.

[15] Vidūnas, R., *Degenerate Gauss hypergeometric functions*, Kyushu J. Math. **61** (2007), 109–135.

[16] Vogan Jr., D. A., *Gel′fand-Kirillov dimension for Harish-Chandra modules*, Invent. Math. **48** (1978), 75–98.

[17] Wallach, N. R., and J. A. Wolf, *Completeness of Poincaré series for automorphic forms associated to the integrable discrete series*, In: Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math. **40**, Birkhäuser Boston, Boston, MA, 1983, 265–281.

[18] Whittaker, E. T., and G. N. Watson, "A course of modern analysis," Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1996. An introduction to the general theory of infinite processes and of analytic functions–with an account of the principal transcendental functions, Reprint of the fourth edition of 1927.

[19] Zhelobenko, D. P., *On Gel′fand-Zetlin bases for classical Lie algebras*, In: Representations of Lie groups and Lie algebras (Budapest, 1971), Akad. Kiadó, Budapest, 1985, 79–106.

Takahiro Hayata
Department of Applied
Mathematics and Physics
Graduate School of Science
and Engineering
Yamagata University
Yonezawa, 992-8510, Japan
hayata@yz.yamagata-u.ac.jp

Harutaka Koseki
Department of Mathematics
Faculty of Education, Mie University
1577, Kurimamachiya-cho
Tsushi, 514-8507, Japan
E-mail: h-koseki@edu.mie-u.ac.jp

Tadashi Miyazaki
Department of Mathematics
College of Liberal Arts and Sciences
Kitasato University
1-15-1, Kitasato
Minamiku Sagamihara
Kanagawa, 252-0373, Japan
E-mail: miyaza@kitasato-u.ac.jp

Takayuki Oda
Graduate School of
Mathematical Sciences
University of Tokyo
3-8-1, Komaba, Meguroku
Tokyo, 153-8914, Japan
E-mail: takayuki@ms.u-tokyo.ac.jp

Received March 4, 2014
and in final form July 4, 2014