

The Differential of the Exponential Map for Reductive Homogeneous Spaces

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Abstract. The Helgason’s formula for a Taylor series related to the differential of the exponential map for symmetric spaces is generalized to reductive homogeneous spaces and to manifolds with torsion-free affine connections.

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1. Introduction

Let (\mathcal{M}, ∇) be a smooth manifold with an affine connection. For a point $p \in \mathcal{M}$, consider the exponential map

$$\text{Exp}_p : T_p\mathcal{M} \rightarrow \mathcal{M}.$$

Let $X \in T_p\mathcal{M}$ be a vector and let γ_X be the corresponding geodesic $\gamma_X(t) := \text{Exp}_p(tX)$. The domain $T_p\mathcal{M}$ is a linear space, hence the differential of the exponential map may be considered a linear map of the form

$$d_X \text{Exp}_p : T_p\mathcal{M} \rightarrow T_{\text{Exp}_p X}\mathcal{M}.$$

Denote by

$$\mathcal{I}_p(X) : T_p\mathcal{M} \rightarrow T_{\text{Exp}_p X}\mathcal{M}$$

the operator of parallel transport along the geodesic γ_X . Let $\mathcal{E}_p : T_p\mathcal{M} \rightarrow \text{End}_{\mathbb{R}}(T_p\mathcal{M})$ be the map defined by

$$\mathcal{E}_p(X) := \mathcal{I}_p(X)^{-1} \circ d_X \text{Exp}_p.$$

This is a smooth map between two linear spaces, hence its Taylor series is well defined. Our goal is to find this series explicitly in terms of the curvature tensor and the torsion tensor.

Essentially, the map was introduced by S. Helgason, who also proposed an interesting formula for it [5, Theorem 1]. This formula, however, does not give a

Taylor series and it is not very convenient to deal with, except for some special cases. Apparently, the series is only known for Lie groups and symmetric spaces. For a Lie group G with the usual $(-)$ connection we have

$$\mathcal{E}_e(X) = \frac{1 - e^{-\text{ad } X}}{\text{ad } X}, \quad X \in \mathfrak{g} \cong T_e G,$$

where e is the unit element and \mathfrak{g} is the Lie algebra of G . This is a classic, which is rather difficult to attribute. For a symmetric space G/H the series was found by Helgason [4], [6, Chap. 4, Theorem 4.1]:

$$\mathcal{E}_o(X) = \frac{\text{sh ad } X}{\text{ad } X}, \quad X \in \mathfrak{m} \cong T_o(G/H), \quad (1)$$

where \mathfrak{m} is the antisymmetric part in the canonical decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $o = eH$ is the origin.

In both cases this series or, to be more precise, the Taylor series of the determinant

$$J(X) = \det \mathcal{E}(X)$$

plays a fundamental role in the analysis on the given space. It is sufficient to mention the Duflo (Harish-Chandra) isomorphism [1, 2, 3]. A natural question is whether these results may be extended to more general homogeneous spaces and if an explicit formula for the Taylor series of \mathcal{E} may then be of some value. Admittedly, it is not clear at present in which generality the problem remains interesting.

In this paper we consider two cases. The first case is a general manifold with a torsion-free affine connection (Theorem 3.1). Apart from Lie theory, this result may be used in Riemannian geometry. The second case is a reductive homogeneous space (Theorem 4.1). This is the closest possible generalization of a symmetric space. From the geometric viewpoint, a reductive homogeneous space is a manifold with an affine connection satisfying $\nabla T = 0$ and $\nabla R = 0$, where T and R are the torsion tensor and the curvature tensor [8, Chapter X]. From the algebraic viewpoint, it is a homogeneous space G/H admitting a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The corresponding Taylor series can be found by either geometric or algebraic methods. However, for the best of author's knowledge, it has never been done. A related but different series was considered by I. Kantor [7].

In both cases we use a transparent geometric method, which is based on a formal solution of the Jacobi equation. Our method is general: it can be applied to any smooth manifold with an affine connection. However, we do not consider the general case for the sake of simplicity.

2. The Geometry

2.1. Parallel transport of the curvature tensor.

We recall here some properties of the covariant derivation along a geodesic. For our purposes it is convenient to consider a connection on a vector bundle

instead of an affine connection (which is a connection on the tangent bundle). Let $E \rightarrow \mathcal{M}$ be a smooth vector bundle on a smooth manifold. The space of smooth sections of E is denoted by $C^\infty(\mathcal{M}, E)$. We assume that the bundle is provided with a connection ∇ , which is a linear map

$$\nabla : C^\infty(\mathcal{M}, E) \rightarrow C^\infty(\mathcal{M}, T^*\mathcal{M} \otimes E),$$

satisfying the Leibniz rule

$$\nabla fu = df \otimes u + f\nabla u, \quad f \in C^\infty(\mathcal{M}), u \in C^\infty(\mathcal{M}, E).$$

Let $\gamma : I \rightarrow \mathcal{M}$ be a smooth curve, where $I \subset \mathbb{R}$ is an interval. The induced connection on the restricted bundle $\gamma^*E \rightarrow I$ is usually denoted by the symbol D . It is convenient to use this connection in the form of a differential operator $\frac{D}{dt}$, where $t \in I$ is the parameter on the curve. This operator is called a *covariant derivation along the curve* γ . In other words, for any section $u \in C^\infty(\mathcal{M}, E)$ we have the equality

$$\frac{D}{dt}\gamma^*u = \dot{\gamma} \cdot \gamma^*\nabla u, \tag{2}$$

where

$$\dot{\gamma}(t) = \frac{d}{dt}\gamma(t) \in T_{\gamma(t)}\mathcal{M}, \quad t \in I.$$

To be less pedantic, one may write (2) as an operator identity $\frac{D}{dt} = \nabla_{\dot{\gamma}}$.

We shall need the following simple, but useful lemma.

Lemma 2.1. *Let (\mathcal{M}, ∇) be a smooth manifold with an affine connection and let $E \rightarrow \mathcal{M}$ be a smooth vector bundle with a connection. Let $\gamma : I \rightarrow \mathcal{M}$ be a geodesic. Then for any $n \geq 1$ and any smooth section u of the bundle E*

$$\frac{D^n}{dt^n}\gamma^*u = \dot{\gamma}^{\otimes n} \cdot \gamma^*\nabla^n u, \tag{3}$$

where $\frac{D}{dt}$ is the covariant derivation along γ .

The equality (3) may also be written in the operator form $\frac{D^n}{dt^n} = \dot{\gamma}^{\otimes n} \cdot \nabla^n$. Apparently, this equality is well known, but the author does not know an appropriate reference. For this reason, we present here a proof. Note that for $n > 1$ the section $\nabla^n u$ on the right hand side of (3) depends on the affine connection on the manifold while the left hand side depends on the curve γ only.

Proof. It is an almost trivial fact that the derivation $\frac{D}{dt}$ can be canonically extended to the products of E with tensor bundles on \mathcal{M} and inherits all the common properties of the covariant derivation. In particular, it is compatible with contraction. For $n = 1$ the formula (3) is the definition of $\frac{D}{dt}$. If it is valid for some n , then we have

$$\frac{D^{n+1}}{dt^{n+1}}\gamma^*u = \frac{D}{dt} \frac{D^n}{dt^n}\gamma^*u = \frac{D}{dt} \dot{\gamma}^{\otimes n} \cdot \gamma^*\nabla^n u = \left(\frac{D}{dt} \dot{\gamma}^{\otimes n} \right) \cdot \gamma^*\nabla^n u + \dot{\gamma}^{\otimes n} \cdot \frac{D}{dt} \gamma^*\nabla^n u.$$

By assumption, γ is a geodesic, hence $\frac{D}{dt}\dot{\gamma} = 0$ and the first term on the right hand side vanishes. Applying (2) to the section $\nabla^n u \in C^\infty(\mathcal{M}, T^*\mathcal{M}^{\otimes n} \otimes E)$, we have $\frac{D}{dt}\gamma^*\nabla^n u = \dot{\gamma} \cdot \gamma^*\nabla^{n+1}u$. Thus,

$$\frac{D^{n+1}}{dt^{n+1}}\gamma^*u = \dot{\gamma}^{\otimes n+1} \cdot \gamma^*\nabla^{n+1}u,$$

and the equality (3) follows by induction. ■

The covariant derivation is closely related to parallel transport. As above, denote by $\mathcal{I}_p = \mathcal{I}_{p,E}$ the operator of parallel transport of the form

$$\mathcal{I}_p(X) : E_p \rightarrow E_{\text{Exp}_p X}$$

along the geodesic $\gamma_X(t) = \text{Exp}_p(tX)$. By the definition of parallel transport,

$$\frac{D}{dt}\mathcal{I}_p(tX)z = 0,$$

where $\frac{D}{dt}$ is the covariant derivation along γ_X and $z \in E_p$ is a constant. In a more general case, where $z = z(t)$ depends on the parameter t , we have the operator equality

$$\frac{D}{dt} \circ \mathcal{I}_p(tX) = \mathcal{I}_p(tX) \circ \frac{d}{dt}, \tag{4}$$

which will be used below.

After the above preliminaries we have come to the title of this section. Consider the operator $\mathcal{R}_p(X) \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$, defined by

$$\mathcal{R}_p(X)Y = \mathcal{I}_p(X)^{-1}R_{\text{Exp}_p X}(\mathcal{I}_p(X)X, \mathcal{I}_p(X)Y)\mathcal{I}_p(X)X, \quad Y \in T_p\mathcal{M},$$

where R_x denotes the curvature tensor at a point $x \in \mathcal{M}$. As it is well known, parallel transport is compatible with tensor operations. Thus, this operator can also be written in the form

$$\mathcal{R}_p(X)Y = (\mathcal{I}_p(X)^{-1}R_{\text{Exp}_p X})(X, Y)X.$$

In the latter equality the parallel transport operator $\mathcal{I}_p(X)$ is applied to the curvature tensor, i.e. it acts on the bundle $T_3^1\mathcal{M}$.

The Taylor series of \mathcal{R}_p may be described as follows. For $n \geq 0$ and $X \in T_p\mathcal{M}$ denote by $R_{p,X}^{(n)}$ the n -th order covariant derivation of the curvature tensor at the point p in the direction X . That is,

$$R_{p,X}^{(n)} = X^{\otimes n} \cdot (\nabla^n R)_p.$$

The symbol ‘ \cdot ’ on the right hand side denotes the contraction of the polyvector $X^{\otimes n}$ with the tensor $\nabla^n R$.

For X, n as above denote by $r_{p,n}(X) \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$ the linear operator defined by

$$r_{p,n}(X) : Y \mapsto R_{p,X}^{(n)}(X, Y)X, \quad Y \in T_p\mathcal{M}.$$

For the sake of convenience we usually omit the vector and the point in our writing. Hence: $r_n = r_{p,n}(X)$. For example,

$$r_0 : Y \mapsto R_p(X, Y)X, \quad r_1 : Y \mapsto (\nabla_X R)_p(X, Y)X,$$

etc. Obviously, r_n , as a function of X , is homogeneous of degree $n + 2$:

$$r_n(tX) = t^{n+2}r_n(X), \quad t \in \mathbb{R}.$$

Lemma 2.2. For $X \in T_p\mathcal{M}$ and $n \geq 0$,

$$\left. \frac{d^n}{dt^n} \mathcal{R}_p(tX) \right|_{t=0} = n(n-1)r_{n-2}(X). \tag{5}$$

If the curvature tensor R is parallel along the geodesic γ_X then

$$\mathcal{R}_p(tX) = t^2 r_0(X).$$

(For $n = 0$ and $n = 1$ the right hand side of (5) is zero by convention.)

Proof. Consider the geodesic $\gamma = \gamma_X$. For $Y \in T_p\mathcal{M}$ we have

$$\mathcal{R}_p(tX)Y = t^2(\mathcal{I}_p(tX)^{-1}R_{\gamma(t)})(X, Y)X,$$

hence

$$\left. \frac{d^n}{dt^n} \mathcal{R}_p(tX) \right|_{t=0} Y = n(n-1) \left(\left. \frac{d^{n-2}}{dt^{n-2}} \mathcal{I}_p(tX)^{-1}R_{\gamma(t)} \right) \right|_{t=0} (X, Y)X.$$

By (4),

$$\frac{d^{n-2}}{dt^{n-2}} \mathcal{I}_p(tX)^{-1}R_{\gamma(t)} = \mathcal{I}_p(tX)^{-1} \frac{D^{n-2}}{dt^{n-2}} R_{\gamma(t)}.$$

By Lemma 2.1,

$$\left. \frac{D^{n-2}}{dt^{n-2}} R_{\gamma(t)} \right|_{t=0} = X^{\otimes n-2} \cdot (\nabla^{n-2} R)_p = R_{p,X}^{(n-2)}.$$

Taking into account the equality $\mathcal{I}_p(0) = \mathbb{1}$ and the definition of r_{n-2} , we have (5). ■

Besides r_n and \mathcal{R} , we need similar operators for the torsion tensor, which will be denoted by $\mathbf{t}_p(X), \tau_p(X) \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$. By definition

$$\begin{aligned} \mathbf{t}_p(X) &: Y \mapsto T_p(X, Y), \\ \tau_p(X) &: Y \mapsto \mathcal{I}_p(X)^{-1}T_{\text{Exp}_p X}(\mathcal{I}_p(X)X, \mathcal{I}_p(X)Y) \end{aligned}$$

for $X, Y \in T_p\mathcal{M}$. We compute the operator τ in a special case only.

Lemma 2.3. If the torsion tensor T is parallel along the geodesic γ_X , then

$$\tau_p(X) = \mathbf{t}_p(X).$$

Proof. The proof is obvious, because $\tau_p(tX)Y = (\mathcal{I}_p(tX)^{-1}T_{\text{Exp}_p, tX})(tX, Y)$. ■

2.2. The Jacobi field.

For given vectors $X, Y \in T_p\mathcal{M}$ consider a family of geodesics

$$\gamma_s(t) = \text{Exp}_p(tX + tsY),$$

parametrized by a real number s taken from a neighbourhood of zero. Denote by $\gamma = \gamma_0 = \gamma_X$ the geodesic corresponding to $s = 0$. Let $J \in C^\infty(I, \gamma^*T\mathcal{M})$ be the vector field defined by

$$J_t = \left. \frac{d}{ds}\gamma_s(t) \right|_{s=0} \in T_{\gamma(t)}\mathcal{M}.$$

Equivalently,

$$J_t = (d_{tX} \text{Exp}_p) tY.$$

It is well known that a field of this kind is a Jacobi field [8, Chap. 8]. This means that it satisfies the equation

$$\frac{D^2}{dt^2} J_t = R_{\gamma(t)}(\dot{\gamma}(t), J_t)\dot{\gamma}(t) + \frac{D}{dt}T_{\gamma(t)}(\dot{\gamma}(t), J_t). \tag{6}$$

This equation may be reinterpreted as follows.

Proposition 2.4. *For any $X \in T_p\mathcal{M}$ the operator $\mathcal{E}_p(tX)$ satisfies the differential equation*

$$\left(t^2 \frac{d^2}{dt^2} + 2t \frac{d}{dt} \right) \mathcal{E}_p(tX) = \mathcal{R}_p(tX)\mathcal{E}_p(tX) + t \frac{d}{dt}\tau_p(tX)\mathcal{E}_p(tX), \tag{7}$$

with the initial conditions

$$\mathcal{E}_p(0) = \mathbb{1}, \quad \left. \frac{d}{dt}\mathcal{E}_p(tX) \right|_{t=0} = \mathbf{t}_p(X)/2.$$

Proof. By the definition of the operator \mathcal{E}_p , we have

$$\mathcal{E}_p(tX)tY = \mathcal{I}_p(tX)^{-1} \circ (d_{tX} \text{Exp}_p) tY = \mathcal{I}_p(tX)^{-1} J_t.$$

Thus, by (4) and (6),

$$\begin{aligned} \left(t \frac{d^2}{dt^2} + 2 \frac{d}{dt} \right) \mathcal{E}_p(tX)Y &= \frac{d^2}{dt^2} \mathcal{E}_p(tX)tY = \mathcal{I}_p(tX)^{-1} \frac{D^2}{dt^2} J_t \\ &= \mathcal{I}_p(tX)^{-1} R_{\gamma(t)}(\dot{\gamma}(t), J_t)\dot{\gamma}(t) + \frac{d}{dt} \mathcal{I}_p(tX)^{-1} T_{\gamma(t)}(\dot{\gamma}(t), J_t). \end{aligned}$$

On the other hand, by the definitions of the operators \mathcal{R} and τ , we have the equalities

$$\begin{aligned} \mathcal{R}_p(tX)\mathcal{E}_p(tX)Y &= t\mathcal{I}_p(tX)^{-1} R_{\gamma(t)}(\dot{\gamma}(t), J_t)\dot{\gamma}(t), \\ \tau_p(tX)\mathcal{E}_p(tX)Y &= \mathcal{I}_p(tX)^{-1} T_{\gamma(t)}(\dot{\gamma}(t), J_t). \end{aligned}$$

Comparing these equalities and taking into account that they are valid for any Y , we have (7). The equality $\mathcal{E}_p(0) = \mathbb{1}$ is obvious. The other initial condition is equivalent to

$$\frac{D}{dt}d_{tX} \text{Exp}_p \Big|_{t=0} = \frac{1}{2}t_p(X),$$

which may be verified directly. ■

3. Manifolds with torsion-free connections

3.1. The Taylor series.

In this section we compute the Taylor series of \mathcal{E}_p in the torsion-free case. We have to introduce some notation. Call a finite sequence of non-negative integers a *list*. The set of all lists (including the empty one) is denoted by Λ . The empty list is denoted by the symbol \emptyset ; to write down a nonempty list we use square brackets; for example,

$$\lambda = [2, 0, 1] \in \Lambda.$$

For every list $\lambda \in \Lambda$ there is a corresponding operator $r_\lambda \in \text{End}_{\mathbb{R}}(T_p\mathcal{M})$ which is a composition of simple operators r_n . Namely, if $\lambda = [n_1, n_2, \dots, n_k]$ then

$$r_\lambda = r_{n_1}r_{n_2} \cdots r_{n_k}.$$

By definition, $r_\emptyset = \mathbb{1}$.

We shall need three number functions on lists: the factorial $\lambda!$, the degree $|\lambda|$ and the denominator c_λ . By definition, $\emptyset! = 1, |\emptyset| = 0$. For $\lambda = [n_1, n_2, \dots, n_k]$ we define the factorial and the degree as follows:

$$\lambda! = \prod_{j=1}^k (n_j!), \quad |\lambda| = 2k + \sum_{j=1}^k n_j.$$

Obviously,

$$r_\lambda(tX) = t^{|\lambda|}r_\lambda(X), \quad t \in \mathbb{R},$$

which is where the term “degree” comes from.

The denominator is defined by $c_\emptyset = 1$ and by the recurrence relation

$$c_\lambda = |\lambda|(|\lambda| + 1)c_{\lambda'}, \quad \lambda \in \Lambda \setminus \{\emptyset\},$$

where λ' is obtained from λ by omitting the first element from the list. For example, $|[2, 0, 1]| = 2 \cdot 3 + 2 + 0 + 1 = 9$, hence

$$c_{[201]} = 9 \cdot 10c_{[01]} = 90 \cdot 30c_{[1]} = 90 \cdot 30 \cdot 12c_\emptyset = 32400.$$

In this notation, we have the following Taylor series (Theorem 3.1 below)

$$\mathcal{E}_p(X) = \sum_{\lambda \in \Lambda} \frac{1}{\lambda!c_\lambda} r_\lambda(X). \tag{8}$$

It may be shown that there are no nontrivial algebraic relations between the operators r_n on a general manifold. So, the series (8) is unique as a formal series in the non-commuting variables r_n . Note that for a list $\lambda = [0, 0, \dots, 0]$ which consists of k zeros we have $|\lambda| = 2k$. Hence

$$c_\lambda = 2k(2k+1)c_{\lambda'} = (2k+1)!.$$

Thus, the coefficient of the term $r_\lambda = r_0^k$ in the Taylor series is equal to $1/(2k+1)!$. This agrees with (1), because for a symmetric space we have $r_0(X) = (\text{ad } X)^2$, $X \in \mathfrak{m}$.

3.2. Proof for the Taylor series.

The result may be formulated as follows.

Theorem 3.1. *Let (\mathcal{M}, ∇) be a smooth manifold with a torsion-free affine connection. Let $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$. Then for any $n \geq 0$*

$$\left. \frac{1}{n!} \frac{d^n}{dt^n} \mathcal{E}_p(tX) \right|_{t=0} = \sum_{|\lambda|=n} \frac{1}{\lambda! c_\lambda} r_\lambda(X), \quad (9)$$

where the sum on the right hand side is taken over all the lists $\lambda \in \Lambda$ of degree n .

Proof. For $n \geq 0$ denote the left hand side of (9) by \mathcal{E}_n . Taking the n -th order derivative of the both sides of (7) at $t = 0$, we have by Lemma 2.2 the equality

$$n(n+1)\mathcal{E}_n = \sum_{k=0}^n \frac{k(k-1)}{k!} r_{k-2} \mathcal{E}_{n-k}.$$

For $n \geq 2$ the latter equality takes the form

$$\mathcal{E}_n = \frac{1}{n(n+1)} \sum_{m=0}^{n-2} \frac{1}{m!} r_m \mathcal{E}_{n-m-2}. \quad (10)$$

For $n = 0$ and $n = 1$ the equality

$$\mathcal{E}_n = \sum_{|\lambda|=n} \frac{1}{\lambda! c_\lambda} r_\lambda$$

follows from the initial conditions: $\mathcal{E}_0 = \mathbb{1}$, $\mathcal{E}_1 = 0$.

By induction, we may assume that (9) is valid for the operators \mathcal{E}_{n-m-2} on the right hand side of (10). We have then the equality

$$\mathcal{E}_n = \sum_{m=0}^{n-2} \sum_{|\mu|=n-m-2} \frac{1}{m! \mu! n(n+1) c_\mu} r_m r_\mu.$$

Denote $\lambda = [m, \mu]$ (thus $\lambda' = \mu$). One can see that the double sum on the right hand side can be replaced by a single sum taken over the lists λ of degree n . Taking into account the obvious equalities

$$\lambda! = m! \mu!, \quad c_\lambda = n(n+1) c_\mu, \quad r_\lambda = r_m r_\mu,$$

we obtain (9). ■

One can see that there are 13 lists of degree not greater than 6, namely $\emptyset, [0], [1], [2], [0, 0], [3], [1, 0], [0, 1], [4], [2, 0], [1, 1], [0, 2], [0, 0, 0]$. Computing the corresponding coefficients, we have the following corollary.

Corollary 3.2. *The function \mathcal{E}_p can be written in the form*

$$\begin{aligned} \mathcal{E}_p(X) = \mathbb{1} + \frac{1}{6}r_0 + \frac{1}{12}r_1 + \frac{1}{40}r_2 + \frac{1}{120}r_0^2 + \frac{1}{180}r_3 + \frac{1}{180}r_1r_0 + \frac{1}{360}r_0r_1 + \frac{1}{1008}r_4 + \\ + \frac{1}{504}r_2r_0 + \frac{1}{504}r_1^2 + \frac{1}{1680}r_0r_2 + \frac{1}{5040}r_0^3 + \rho_7(X), \end{aligned}$$

where $\rho_7(X) = O(|X|^7)$ as $X \rightarrow 0$.

In fact, the theorem may be extended to the general case as well. Let (\mathcal{M}, ∇) be a smooth manifold with an affine connection, which is not supposed to be torsion-free. The problem of computing the operator \mathcal{E}_p for a point $p \in \mathcal{M}$ may be reduced to the torsion-free case, because the exponential map does not depend on the torsion part of a connection. One can define a new connection $\tilde{\nabla}$ by

$$\tilde{\nabla}_X Y = \nabla_X Y - T(X, Y)/2, \tag{11}$$

where X, Y are vector fields on \mathcal{M} and T is the torsion tensor of ∇ . (Usually $\tilde{\nabla}$ is called the torsion-free part of ∇). Both connections have the same geodesics, hence the same exponential map. Thus,

$$\mathcal{E}_p(X) = S_p(X) \circ \tilde{\mathcal{E}}_p(X),$$

where $S_p(X) = \mathcal{I}_p(X)^{-1} \circ \tilde{\mathcal{I}}_p(X)$.

The functions $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{I}}$ are related to $\tilde{\nabla}$ while \mathcal{E} and \mathcal{I} are related to the original connection ∇ . Obviously, $\tilde{T} = 0$, hence $\tilde{\mathcal{E}}$ can be computed by (8); we have only to find the operators $\tilde{r}_n, n \geq 0$ and S_p . One can see that the curvature tensor \tilde{R} of $\tilde{\nabla}$ is

$$\tilde{R}(X, Y)Z = R(X, Y)Z + \frac{1}{4}T(X, T(Y, Z)) + \frac{1}{4}T(Y, T(Z, X)) + \frac{1}{2}T(Z, T(X, Y)),$$

hence

$$\tilde{r}_0 = r_0 + \mathbf{t}^2/4, \tag{12}$$

The operators \tilde{r}_n for $n > 0$ may be found by taking derivatives of \tilde{R} , though in the general case the computation is somewhat tedious. The operator S_p depends on the torsion tensor T .

Proposition 3.3. *For $X \in T_p\mathcal{M}$ the operator S_p is a solution of the equation*

$$\frac{d}{dt}S_p(tX) = \frac{1}{2t}\tau_p(tX)S_p(tX), \quad t > 0$$

with the initial condition $S_p(0) = \mathbb{1}$. If the torsion tensor T is parallel along the geodesic γ_X then

$$S_p(X) = \exp(\mathbf{t}_p(X)/2).$$

Proof. For $Y \in T_p\mathcal{M}$ and $t \neq 0$ we have

$$\begin{aligned} \frac{d}{dt}S_p(tX)Y &= \frac{d}{dt}\mathcal{I}_p(tX)^{-1}\tilde{\mathcal{I}}_p(tX)Y = \mathcal{I}_p(tX)^{-1}\frac{D}{dt}\tilde{\mathcal{I}}_p(tX)Y \\ &= \frac{1}{2}\mathcal{I}_p(tX)^{-1}T_{\gamma_X(t)}(\dot{\gamma}_X(t), \tilde{\mathcal{I}}_p(tX)Y) = \frac{1}{2t}\tau_p(tX)S_p(tX)Y. \end{aligned}$$

If T is parallel along γ_X then by Lemma 2.3

$$\frac{d}{dt}S_p(tX) = \frac{1}{2}\mathbf{t}_p(X)S_p(tX).$$

Obviously, $S_p(0) = \mathbb{1}$. ■

4. Reductive homogeneous spaces

4.1. The Taylor series for a reductive homogeneous space.

Let $\mathcal{M} = G/H$ be a reductive homogeneous space and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a decomposition of the Lie algebra \mathfrak{g} of the group G such that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. As usual, the subspace \mathfrak{m} can be identified with the tangent space at the origin $o = eH \in \mathcal{M}$, that is, $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h} = T_o\mathcal{M}$. For the sake of convenience we may assume \mathcal{M} to be connected and simply connected.

A reductive homogeneous space possesses a canonical connection, which only depends on the decomposition. This connection is G -invariant and its curvature tensor R and torsion tensor T are covariant constant. At the origin this tensors may be expressed in terms of the operations in the Lie algebra \mathfrak{g} as follows [8, Sec. X, Theorem 2.6]:

$$T_o(X, Y) = -[X, Y]_{\mathfrak{m}}, \quad R_o(X, Y)Z = -[[X, Y]_{\mathfrak{h}}, Z], \quad X, Y, Z \in \mathfrak{m} \cong T_o\mathcal{M}.$$

Our goal is to find the Taylor series of $\mathcal{E}_o(X)$, as a function of $X \in \mathfrak{m}$, in terms of the curvature tensor and the torsion tensor. It is not difficult to see that any operator of this kind can be constructed from two operators introduced above:

$$\mathbf{t} = \mathbf{t}(X) : Y \mapsto T_o(X, Y), \quad r = r(X) : Y \mapsto R_o(X, Y)X.$$

(We can write r instead of r_0 , because $\nabla R = 0$, hence $r_n = 0$ for $n > 0$.) Thus, $\mathcal{E}_o(X)$ may be presented as a series in two non-commuting variables r, \mathbf{t} . It should be mentioned that, by a well known argument, this function is real analytic, hence the series converges in a neighbourhood of the origin. Moreover, there are no algebraic relations between r and \mathbf{t} in general, hence this series must be unique as a formal series.

In this situation we cannot use Theorem 3.1 directly, because the canonical connection is not torsion-free unless \mathcal{M} is a symmetric space. However, we can apply the method sketched at the end of the previous section. In this case $S_o(X) = \exp(\mathbf{t}/2)$ and for $n > 0$ the operators \tilde{r}_n are defined by

$$\tilde{r}_n = \text{ad}(-\mathbf{t}/2)^n r. \tag{13}$$

Here $\text{ad}(\mathbf{t})$ denotes the commutator $\text{ad}(\mathbf{t})u = \mathbf{t}u - u\mathbf{t}$; for example,

$$\tilde{r}_1 = (r\mathbf{t} - \mathbf{t}r)/2, \tilde{r}_2 = (r\mathbf{t}^2 - 2\mathbf{t}r\mathbf{t} + \mathbf{t}^2r)/4 \text{ etc.}$$

Substituting (12) and (13) into (8), we can compute the series

$$\begin{aligned} \mathcal{E}_o(X) = & \mathbb{1} + \frac{1}{2}\mathbf{t} + \frac{1}{6}r + \frac{1}{6}\mathbf{t}^2 + \frac{1}{24}r\mathbf{t} + \frac{1}{24}\mathbf{t}r + \frac{1}{24}\mathbf{t}^3 + \frac{1}{120}r\mathbf{t}^2 + \frac{1}{120}\mathbf{t}r\mathbf{t} + \frac{1}{120}\mathbf{t}^2r + \\ & + \frac{1}{120}r^2 + \frac{1}{120}\mathbf{t}^4 + \rho(X); \quad \rho(X) = O(|X|^5), X \rightarrow 0. \end{aligned}$$

In the case $r\mathbf{t} = \mathbf{t}r$ the computation is much simpler, because $\tilde{r}_n = 0$ for $n > 0$ and $\tilde{\mathcal{E}}_o$ is a series in a single variable $\tilde{r}_0 = r + \mathbf{t}^2/4$. We have then

$$\mathcal{E}_o(X) = \exp(\mathbf{t}/2) \frac{\text{sh} \sqrt{r + \mathbf{t}^2/4}}{\sqrt{r + \mathbf{t}^2/4}}. \tag{14}$$

This method, however, is not very convenient for the general case. (Note that if $r\mathbf{t} = \mathbf{t}r$ then $\tilde{\nabla}\tilde{R} = 0$, which means that our space is actually a symmetric space in disguise. For a genuine reductive homogeneous space $r\mathbf{t} \neq \mathbf{t}r$).

4.2. The series for a reductive homogeneous space, final form.

Let $\mathbb{R}[r, \mathbf{t}]$ be the (commutative) algebra of polynomials in r, \mathbf{t} as formal variables and let $\mathbb{R}\langle r, \mathbf{t} \rangle$ be the free associative algebra in the same variables. There is a well known linear map

$$\sigma : \mathbb{R}[r, \mathbf{t}] \rightarrow \mathbb{R}\langle r, \mathbf{t} \rangle,$$

called the symmetrization map, which maps $\mathbb{R}[r, \mathbf{t}]$ onto the space of totally symmetric polynomials in $\mathbb{R}\langle r, \mathbf{t} \rangle$. It is defined uniquely by the condition

$$\sigma : (\alpha r + \beta \mathbf{t})^n \mapsto (\alpha r + \beta \mathbf{t})^n, \quad \alpha, \beta \in \mathbb{R}, n \geq 0.$$

For example, $\sigma(r) = r, \sigma(r^2\mathbf{t}) = (r^2\mathbf{t} + r\mathbf{t}r + \mathbf{t}r^2)/3$ etc. The symmetrization map is obviously homogeneous (assuming both variables to have the same degree), hence it may be extended to a map of formal series $\sigma : \mathbb{R}[[r, \mathbf{t}]] \rightarrow \mathbb{R}\langle\langle r, \mathbf{t} \rangle\rangle$.

Theorem 4.1. *Let (\mathcal{M}, ∇) be a smooth manifold with an affine connection. If the curvature tensor R and the torsion tensor T are parallel along any geodesic which passes through a given point $o \in \mathcal{M}$, then the Taylor series of the function \mathcal{E}_o is*

$$\mathcal{E}_o(X) = \sigma \left[\exp(\mathbf{t}/2) \frac{\text{sh} \sqrt{r + \mathbf{t}^2/4}}{\sqrt{r + \mathbf{t}^2/4}} \right], \tag{15}$$

where $X \in T_o\mathcal{M}$ and $r = r(X), \mathbf{t} = \mathbf{t}(X)$ are defined as above.

(The argument of σ on the right hand side of (15) is a formal series in *commuting* variables r and \mathbf{t}). A reductive homogeneous space with a canonical connection obviously satisfies the conditions of this theorem. The proof of (15) requires some calculation.

Lemma 4.2. For $k, l \geq 1$

$$kr\sigma(r^{k-1}\mathbf{t}^l) + l\mathbf{t}\sigma(r^k\mathbf{t}^{l-1}) = (k+l)\sigma(r^k\mathbf{t}^l).$$

This is a well known equality, which is an easy consequence of the definition of the symmetrization map.

Lemma 4.3. Let

$$\mathcal{E}_o(tX) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} t^n Q_n,$$

where the operators Q_n depend on X but do not depend on t . Then for $n \geq 0$

$$Q_n = \sum_{2k+l=n} \binom{k+l}{l} \sigma(r^k\mathbf{t}^l).$$

Proof. From Lemma 2.2 and Lemma 2.3 we have

$$\mathcal{R}_o(tX) = t^2 r, \quad \tau_o(tX) = \mathbf{t} \mathbf{t}.$$

Thus, (7) becomes

$$\left(t^2 \frac{d^2}{dt^2} + 2t \frac{d}{dt} \right) \mathcal{E}_o(tX) = t^2 r \mathcal{E}_o(tX) + t^2 \mathbf{t} \frac{d}{dt} \mathcal{E}_o(tX) + \mathbf{t} \mathcal{E}_o(tX).$$

From this equation we obtain the following recursion relation for the coefficients of the series $\mathcal{E}_o(tX)$:

$$Q_n = rQ_{n-2} + \mathbf{t}Q_{n-1}, \quad n \geq 2.$$

Also, we have $Q_0 = \mathbb{1}$, $Q_1 = \mathbf{t}$ from the initial conditions. Using Lemma 4.2, we can obtain the result by induction. ■

Lemma 4.4. For $k, l \geq 0$

$$\frac{(k+l)!}{l!(2k+l+1)!} = 2^{-l} \sum_{2i+j=l} \frac{(k+i)!}{i!j!(2k+2i+1)!}.$$

Proof. The equality is equivalent to

$$\sum_{2i+j=l} \binom{2k+l+1}{j} \binom{k+i}{i} = 2^l \binom{k+l}{l}.$$

This binomial identity can be proved by comparing the coefficients of z^l on the two sides of the following series equality

$$\sum_{i=0}^{\infty} \binom{k+i}{i} z^{2i} \cdot (1-z)^{-2k-2i-2} = (1-2z)^{-k-1}.$$

■

Proof (of Theorem 4.1). By Lemma 4.3,

$$\mathcal{E}_o(X) = \sigma \left[\sum_{k,l \geq 0} \frac{1}{(2k+l+1)!} \binom{k+l}{l} r^k \mathbf{t}^l \right].$$

On the other hand, by Lemma 4.4 we have the following equalities in the commutative algebra $\mathbb{R}[[r, \mathbf{t}]]$

$$\begin{aligned} \sum_{k,l \geq 0} \frac{1}{(2k+l+1)!} \binom{k+l}{l} r^k \mathbf{t}^l &= \sum_{k=0}^{\infty} \frac{r^k}{k!} \sum_{l=0}^{\infty} \frac{(k+l)!}{l!(2k+l+1)!} \mathbf{t}^l \\ &= \sum_{k=0}^{\infty} \frac{r^k}{k!} \sum_{i,j \geq 0} \frac{(k+i)!}{i!j!(2k+2i+1)!} \left(\frac{\mathbf{t}}{2}\right)^{2i+j} \\ &= \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\mathbf{t}}{2}\right)^j \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \left(\sum_{i+k=n} \binom{n}{i} r^k \left(\frac{\mathbf{t}}{2}\right)^{2i} \right) \\ &= \exp(\mathbf{t}/2) \frac{\text{sh} \sqrt{r + \mathbf{t}^2/4}}{\sqrt{r + \mathbf{t}^2/4}}. \quad \blacksquare \end{aligned}$$

5. Some remarks

Theorem 2 gives us a Taylor series for a reductive homogeneous space. The series for a Lie group or a symmetric space (1) are special cases of (15) (in fact, of (14)). Indeed, for a Lie group we have $r = 0$ and $\mathbf{t}(X) = -\text{ad } X$ and for a symmetric space $\mathbf{t} = 0$ and $r(X) = (\text{ad } X)^2$. One more example is a Lie group with the (0) connection; in this case $\mathbf{t} = 0, r(X) = (\text{ad } X)^2/4$, hence

$$\mathcal{E}_e(X) = \frac{\text{sh}(\text{ad } X/2)}{(\text{ad } X/2)}, \quad X \in \mathfrak{g}.$$

The symmetrization map is a right inverse of the natural homomorphism of algebras $\mathbb{R}\langle r, \mathbf{t} \rangle \rightarrow \mathbb{R}[[r, \mathbf{t}]]$, whose kernel is the ideal generated by $r\mathbf{t} - \mathbf{t}r$. By Lemma (4.3) the series belongs to the image of σ , thus (15) follows from (14). For this reason, the combinatorial part of the proof (Lemma 4.4) is not really necessary, but it is certainly instructive.

The symmetry in the variables r and \mathbf{t} was actually discovered by I. Kantor [7], though the series he considered was not \mathcal{E}_o but a related one. Note that, as a function of X , r is quadratic while \mathbf{t} is linear. So, this symmetry is somewhat unexpected, because it does not respect the natural degree.

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