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First extension groups of Verma modules and R-polynomials

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Abstract. We study the first extension groups between Verma modules. There was a conjecture which claims that the dimensions of the higher extension groups between Verma modules are the coefficients of R-polynomials defined by Kazhdan-Lusztig. This conjecture was known as the Gabber-Joseph conjecture (although Gabber and Joseph did not state.) However, Boe gives a counterexample to this conjecture. In this paper, we study how far are the dimensions of extension groups from the coefficients of R-polynomials.
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1. Introduction

The category $\mathcal{O}$ is introduced by Bernstein-Gelfand-Gelfand and plays an important role in the representation theory. One of the most important objects in $\mathcal{O}$ are the Verma modules and they are deeply investigated.

In this paper, we consider the Ext^i-groups between Verma modules. If $i = 0$, Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71] determine the dimension of this group. There are many studies about higher extension groups. One of these studies is a work of Gabber-Joseph. They proved some inequality between the dimensions of extension groups. After that, it is conjectured that this inequality is, in fact, an equality. Although not actually stated in [GJ81], this conjecture is known as the Gabber-Joseph conjecture. If this conjecture is true, then the dimension of extension groups are the coefficients of R-polynomial. However, Boe gives a counterexample to this conjecture [Boe92].

This conjecture is false even in the case of $i = 1$. In this paper, we consider how far the dimensions of extension groups from the coefficients of R-polynomials. Mazorchuk gives a formula of the dimension of the first extension group between Verma modules in a special case [Maz07, Theorem 32]. Our formula is a generalization of his formula.

Now we state our main theorem. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraic closed field K of characteristic zero. Fix its Borel subalgebra $\mathfrak{b}$ and

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a Cartan subalgebra $\mathfrak{h}$. Let Δ be the root system and ρ the half sum of positive roots. Fix a dominant integral element $\lambda \in \mathfrak{h}^*$. (It is sufficient to consider the integral case by a result of Soergel [Soe90, Theorem 11].) Let $M(x\lambda)$ be the Verma module with highest weight $x\lambda - \rho$ for $x \in W$. Then by a result of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71], if $x \geq y$, then there exists the unique (up to nonzero constant multiple) injective homomorphism $M(x\lambda) \to M(y\lambda)$. Hence we can regard $M(x\lambda)$ as a submodule of $M(\lambda)$. Then $M(w_0\lambda)$ is a submodule of $M(x\lambda)$ for all $x \in W$ where w_0 is the longest Weyl element. Hence we have the homomorphism $\operatorname{Ext}^1(M(x\lambda), M(y\lambda)) \to \operatorname{Ext}^1(M(w_0\lambda), M(\lambda))$. Let $V_\lambda(x, y)$ be the image of this homomorphism. We denote the unit element of W by e. Put $S_\lambda = \{s \in S \mid s(\lambda) = \lambda\}$. For integral $\lambda, \mu \in \mathfrak{h}^*$, let T_λ^μ be the translation functor from λ to μ.

Theorem 1.1 (Proposition 3.14, Theorem 4.3, Theorem 5.1)**.** *Let λ be a dominant integral element.*

(1) If λ is regular, then $V_\lambda(w_0, e)$ has a structure of W-module and it is isomorphic to $\mathfrak{h}^*$. For $s \in S$, we denote the element in $V_\lambda(w_0, e)$ corresponding to the simple root whose reflection is s by $v_s \in V_\lambda(w_0, e)$.

(2) Assume that λ is regular. For $x, y \in W$ and $s \in S$ such that $xs > x$, we have the following formula.

(a) If $ys < y$, then $V_\lambda(xs, y) = s(V_\lambda(x, ys))$.

(b) If $ys > y$, then $V_\lambda(xs, y) = Kv_s + s(V_\lambda(x, y))$.

(3) For general λ, the translation functor T_ρ^λ induces a linear map $V_\rho(w_0, e) \to V_\lambda(w_0, e)$. The kernel of this linear map is $\sum_{s \in S_\lambda} Kv_s$ and $T_\rho^\lambda(V_\rho(x, y)) = V_\lambda(x, y)$.

Notice that if λ is regular and $x \not\geq y$, then $\operatorname{Ext}^1(M(x\lambda), M(y\lambda)) = 0$. Hence $V_\lambda(x, y) = 0$. Since $V_\lambda(x, x) = 0$ for all $x \in W$, we can determine the space $V_\lambda(x, y)$ inductively via (2) if λ is regular. Combining (3), we can determine the space $V_\lambda(x, y)$. In particular, we can calculate the dimension of $V_\lambda(x, y)$. Namely, we can get the following theorem. For $x, y \in W$, let $R_{x,y}$ be the polynomial defined in [KL79, §2.]. Let $\ell(x)$ be the length of $x \in W$.

Theorem 1.2 (Theorem 4.4)**.** *Assume that λ is regular. Then $\dim V_\lambda(x, y)$ is the coefficient of q in $(-1)^{\ell(y)-\ell(x)-1} R_{y,x}(q)$.*

In other words, $V_\lambda(x, y)$ satisfies the Gabber-Joseph conjecture. Notice that the homomorphism $\operatorname{Ext}^1(M(x\lambda), M(y\lambda)) \to \operatorname{Ext}^1(M(w_0\lambda), M(\lambda))$ is not injective. It is easy to see that the kernel is isomorphic to $\operatorname{Hom}(M(x\lambda), M(\lambda)/M(y\lambda))$ if $x \geq y$ (Lemma 2.1).

We summarize the contents of this paper. In Section 2, we gather preliminaries and prove some easy facts. In Section 3, we define a W-module structure on $\operatorname{Ext}^1(M(w_0\lambda), M(\lambda))$ for a dominant integral regular λ. We also prove this module is isomorphic to $\mathfrak{h}^*$. The proof of the main theorem in the regular case is done in Section 4. We finish the proof of the main theorem in Section 5.

2. Preliminaries

Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraic closed field K of characteristic zero. Fix a Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$. These determine the BGG category $\mathcal{O}$ [BGG76]. We denote $\operatorname{Hom}_{\mathcal{O}}$ (resp. $\operatorname{Ext}^i_{\mathcal{O}}$) by Hom (resp. Ext^i). Denote the Weyl group of $\mathfrak{g}$ by W and let S be the set of its simple reflections. For $w \in W$, let $\ell(w)$ be its length. For $\lambda \in \mathfrak{h}^*$, let $\mathcal{O}_\lambda$ be the full-subcategory of $\mathcal{O}$ consisting of objects which have a generalized infinitesimal character λ.

In the rest of this paper, we only consider the objects which has a integral generalized infinitesimal character. By [Soe90, Theorem 11], it is sufficient to consider only the integral case.

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module with highest weight $\lambda - \rho$ where ρ is the half sum of positive roots. Assume that λ is dominant integral. Set $S_\lambda = S \cap \operatorname{Stab}_W \lambda$. Then $W_{S_\lambda} = \operatorname{Stab}_W \lambda$ is generated by S_λ. Put $W(S_\lambda) = \{w \in W \mid ws > w \text{ for all } s \in S_\lambda\}$. This gives a complete representative set of W/W_{S_λ}. For $x, y \in W(S_\lambda)$, $\operatorname{Hom}(M(x\lambda), M(y\lambda)) \neq 0$ if and only if $x \geq y$. Moreover, if $x \geq y$, then $\operatorname{Hom}(M(x\lambda), M(y\lambda))$ is one-dimensional and the nonzero homomorphisms from $M(x\lambda)$ to $M(y\lambda)$ are injective. Hence we can regard $M(x\lambda)$ as a submodule of $M(y\lambda)$. In the rest of this paper, we regard $M(x\lambda)$ as a submodule of $M(\lambda)$. Then $M(w_0\lambda)$ is a submodule of $M(x\lambda)$ for all $x \in W$.

Let $\lambda, \mu \in \mathfrak{h}^*$ be integral dominant elements. Then the translation functor $T_\lambda^\mu \colon \mathcal{O}_\lambda \to \mathcal{O}_\mu$ is defined [Jan79].

Assume that λ is a regular integral dominant element. Then for $s \in S$, the wall-crossing functor θ_s^λ is defined by $\theta_s^\lambda = T_\mu^\lambda T_\lambda^\mu$ where $\operatorname{Stab}_W \mu = \{e, s\}$. It is known that this functor is independent of μ. By the definition of θ_s^λ, there are natural transformations $\mathrm{Id} \to \theta_s^\lambda$ and $\theta_s^\lambda \to \mathrm{Id}$. Put $C_s^\lambda = \operatorname{Cok}(\mathrm{Id} \to \theta_s^\lambda)$ and $K_s^\lambda = \operatorname{Ker}(\theta_s^\lambda \to \mathrm{Id})$. Then C_s^λ (resp. K_s^λ) gives a right (resp. left) exact functor from $\mathcal{O}_\lambda$ to $\mathcal{O}_\lambda$. By the self-adjointness of θ_s^λ, $(C_s^\lambda, K_s^\lambda)$ is an adjoint pair. Moreover, the derived functor LC_s gives an auto-equivalence of the derived category $D^b(\mathcal{O}_\lambda)$ and its quasi-inverse is RK_s. For the action of C_s, K_s on Verma modules, the following formulas hold: Let $x \in W$ and $s \in S$ such that $xs > x$. Then $C_s(M(x)) = M(xs)$, $K_s(M(xs)) = M(x)$ and there exists an exact sequence $0 \to M(x)/M(xs) \to C_s(M(xs)) \to M(xs) \to 0$. Moreover, we have $L^kC_s(M(x)) = L^kC_s(M(xs)) = R^kK_s(M(xs)) = 0$ for all $k \geq 1$.

Let λ be the dominant integral element of $\mathfrak{h}^*$. Set

$$V_\lambda(x, y) = \operatorname{Im}(\operatorname{Ext}^1(M(x\lambda), M(y\lambda)) \to \operatorname{Ext}^1(M(w_0\lambda), M(\lambda)))$$

for $x, y \in W$. Put

$$V_\lambda = V_\lambda(w_0, e).$$

(1) of the following lemma is a part of the argument of the proof of [Maz07, Lemma 33]. (2) and (3) follows from (1).

Lemma 2.1. *Let λ be a dominant integral element.*

(1) Let P be the projective cover of $M(w_0\lambda)$. Let $M_i \subset P$ be the filtration such that $M_i/M_{i-1} \simeq \bigoplus_{\ell(w)=i, w\in W(S_\lambda)} M(w\lambda)$. Then we have

$$\begin{aligned}\mathrm{Ext}^1(M(x\lambda), M(\lambda)) &\xleftarrow{\sim} \mathrm{Hom}(M(x\lambda), P/M_0)\\ &\xleftarrow{\sim} \mathrm{Hom}(M(x\lambda), M_1/M_0) \simeq \bigoplus_{s\in S\setminus S_\lambda} \mathrm{Hom}(M(x\lambda), M(s\lambda)).\end{aligned}$$

(2) We have $\dim V_\lambda = \#(S \setminus S_\lambda)$.

(3) For $x, y \in W$ such that $x \geq y$, the kernel of $\mathrm{Ext}^1(M(x\lambda), M(y\lambda)) \to \mathrm{Ext}^1(M(w_0\lambda), M(\lambda))$ is isomorphic to $\mathrm{Hom}(M(x\lambda), M(\lambda)/M(y\lambda))$.

Proof. (1) follows from the proof of [Maz07, Lemma 33]. (2) follows from (1). We prove (3). We have the long exact sequence

$$\begin{aligned}&0 \to \mathrm{Hom}(M(x\lambda), M(y\lambda)) \to \mathrm{Hom}(M(x\lambda), M(\lambda))\\ &\to \mathrm{Hom}(M(x\lambda), M(\lambda)/M(y\lambda)) \to \mathrm{Ext}^1(M(x\lambda), M(y\lambda)) \to \mathrm{Ext}^1(M(x\lambda), M(\lambda))\end{aligned}$$

The morphism $\mathrm{Hom}(M(x\lambda), M(y\lambda)) \to \mathrm{Hom}(M(x\lambda), M(\lambda))$ is isomorphic by the classification of homomorphism between Verma modules.

Hence it suffices to prove that $\mathrm{Ext}^1(M(x\lambda), M(\lambda)) \to \mathrm{Ext}^1(M(w_0\lambda), M(\lambda))$ is injective. This follows from (1). ■

3. Weyl group action

In this section, fix a dominant regular integral element λ. Put $V = V_\lambda$, $V(x, y) = V_\lambda(x, y)$, $\theta_s = \theta_s^\lambda$, $C_s = C_s^\lambda$ and $M(x) = M(x\lambda)$. Since $M(x) \to \theta_s(M(x)) \to M(x)$ is zero, we have $C_s(M(x)) \to M(x)$.

We begin with the following lemma.

Lemma 3.1. *For M_1, M_2, N_1, N_2, take $a_i \in \mathrm{Ext}^1(M_i, N_i)$ for $i = 1, 2$. Take an exact sequence $0 \to N_i \xrightarrow{k_i} X_i \xrightarrow{p_i} M_i \to 0$ corresponding to a_i. Then for $f\colon N_1 \to N_2$ and $g\colon M_1 \to M_2$, there exists φ such that the following diagram commutes if and only if $f_* a_1 = g^* a_2$:*

$$\begin{array}{ccccccccc}0 & \longrightarrow & N_1 & \longrightarrow & X_1 & \longrightarrow & M_1 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & X_2 & \longrightarrow & M_2 & \longrightarrow & 0.\end{array}$$

Proof. First assume that there exists φ. Consider the following digram:

$$\begin{array}{ccccccccc}0 & \longrightarrow & N_1 & \longrightarrow & X_1 & \longrightarrow & M_1 & \longrightarrow & 0\\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & X' & \longrightarrow & M_1 & \longrightarrow & 0,\end{array}$$

where the left square is a push-forward. Then the exact sequence $0 \to N_2 \to X' \to M_1 \to 0$ corresponds to $f_* a_1$ and the existence of φ implies the existence of φ' such that the following digram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & X' & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & X_2 & \longrightarrow & M_2 & \longrightarrow & 0. \end{array}$$

The same argument implies the existence of the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N_2 & \longrightarrow & X' & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \varphi''} & & \| & & \\ 0 & \longrightarrow & N_2 & \longrightarrow & X'' & \longrightarrow & M_1 & \longrightarrow & 0, \end{array}$$

where $0 \to N_2 \to X'' \to M_1 \to 0$ corresponds to $g^* a_2$. The morphism φ'' should be isomorphism by 5-Lemma. Hence we have $f_* a_1 = g^* a_2$.

Conversely, we assume that $f_* a_1 = g^* a_2$. Consider the following diagram:

$$\begin{array}{ccccccc} \operatorname{Hom}(M_1, N_2) & \xrightarrow{k_{2*}} & \operatorname{Hom}(M_1, X_2) & \xrightarrow{p_{2*}} & \operatorname{Hom}(M_1, M_2) & \xrightarrow{\delta_2} & \operatorname{Ext}^1(M_1, N_2) \\ \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_1^*} & & \downarrow{\scriptstyle p_1^*} \\ \operatorname{Hom}(X_1, N_2) & \xrightarrow{k_{2*}} & \operatorname{Hom}(X_1, X_2) & \xrightarrow{p_{2*}} & \operatorname{Hom}(X_1, M_2) & \xrightarrow{\delta_1} & \operatorname{Ext}^1(X_1, N_2) \\ \downarrow{\scriptstyle k_2^*} & & \downarrow{\scriptstyle k_2^*} & & \downarrow{\scriptstyle k_2^*} & & \\ \operatorname{Hom}(N_1, N_2) & \xrightarrow{k_{2*}} & \operatorname{Hom}(N_1, X_2) & \xrightarrow{p_{2*}} & \operatorname{Hom}(N_1, M_2) & & \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_1} & & & & \\ \operatorname{Ext}^1(M_1, N_2) & \xrightarrow{k_{2*}} & \operatorname{Ext}^1(M_1, X_2) & & & & . \end{array}$$

Recall that $f \in \operatorname{Hom}(N_1, N_2)$ and $g \in \operatorname{Hom}(M_1, M_2)$. The assumption implies $\delta_1(f) = \delta_2(g)$. Hence $\delta_1 k_{2*}(f) = k_{2*}\delta_1(f) = k_{2*}\delta_2(g) = 0$. Hence there exists an element $\varphi' \in \operatorname{Hom}(X_1, X_2)$ such that $k_2^*(\varphi') = k_{2*}(f)$. Since $k_2^* p_{2*}(\varphi') = p_{2*}k_2^*(\varphi') = p_{2*}k_{2*}(f) = 0$, there exists $g' \in \operatorname{Hom}(M_1, M_2)$ such that $p_1^*(g') = p_{2*}(\varphi')$. Then $p_{2*}(\varphi') = p_1^*(g')$ and $p_1^*(\varphi') = k_{2*}(f)$. From the argument in the first part of this proof, we get $f_* a_1 = (g')^* a_2$, namely, we get $\delta_2(g') = \delta_1(f) = \delta_2(g)$. Hence there exists $r \in \operatorname{Hom}(M_1, X_2)$ such that $p_{2*}(r) = g - g'$. Set $\varphi = \varphi' + p_1^*(r)$. Then $p_{2*}(\varphi) = p_1^*(g)$ and $p_1^*(\varphi) = k_{2*}(f)$. This proves the lemma. ■

The following lemma is well-known. We give a proof for the sake of completeness.

Lemma 3.2. *Let $x \in W$ and $s \in S$ such that $x < xs$.*

(1) We have $\dim \operatorname{Ext}^1(M(xs), M(x)) = 1$. The basis is given by the exact sequence $0 \to M(x) \to \theta_s(M(x)) \to M(xs) \to 0$.

(2) The homomorphism $\operatorname{Ext}^1(M(xs), M(x)) \to \operatorname{Ext}^1(M(w_0), M(e))$ is injective and its image is independent of x.

Proof. Let μ be the integral dominant element such that $\mathrm{Stab}_W(\mu) = \{e, s\}$. Then $\theta_s = T_\mu^\lambda T_\lambda^\mu$.

(1) We have

$$\mathrm{Ext}^i(M(xs), \theta_s(M(x))) = \mathrm{Ext}^i(T_\lambda^\mu(M(xs)), T_\lambda^\mu(M(x))) = \mathrm{Ext}^i(M(x\mu), M(x\mu))$$

It is one-dimensional if $i = 0$ and zero if $i > 0$. Hence from the exact sequence

$$0 \to M(x) \to \theta_s(M(x)) \to M(xs) \to 0,$$

we have an exact sequence

$$0 \to \mathrm{Hom}(M(xs), M(x)) \to \mathrm{Hom}(M(xs), \theta_s(M(x))) \\ \to \mathrm{Hom}(M(xs), M(xs)) \to \mathrm{Ext}^1(M(xs), M(x)) \to 0.$$

By the dimension counting, $\mathrm{Hom}(M(xs), M(x)) \to \mathrm{Hom}(M(xs), \theta_s(M(x)))$ is isomorphic. Hence $\mathrm{Hom}(M(xs), M(xs)) \simeq \mathrm{Ext}^1(M(xs), M(x))$. The left hand side is one-dimensional.

(2) First we prove that $\mathrm{Ext}^1(M(xs), M(x)) \to \mathrm{Ext}^1(M(w_0), M(e))$ is injective. The kernel of the homomorphism is $\mathrm{Hom}(M(xs), M(e)/M(x))$ by Lemma 2.1 (3). We have an exact sequence $0 \to M(e) \to \theta_s(M(e)) \to M(s) \to 0$ and $0 \to M(x) \to \theta_s(M(x)) \to M(xs) \to 0$. Since $M(xs) \to M(s)$ is injective, we have that $M(e)/M(x) \to \theta_s(M(e)/M(x))$ is injective by the snake lemma. Hence it is sufficient to prove that $\mathrm{Hom}(M(xs), \theta_s(M(e)/M(x))) = 0$. We prove $\mathrm{Hom}(M(x\mu), M(\mu)/M(x\mu)) = 0$. This follows from the following exact sequence

$$0 \to \mathrm{Hom}(M(x\mu), M(x\mu)) \to \mathrm{Hom}(M(x\mu), M(\mu)) \\ \to \mathrm{Hom}(M(x\mu), M(\mu)/M(x\mu)) \to \mathrm{Ext}^1(M(x\mu), M(x\mu)) = 0.$$

We prove the independence of x. We have the following diagram

$$\begin{array}{ccc} M(xs) & \longrightarrow & M(x) \\ \downarrow & & \downarrow \\ M(s) & \longrightarrow & M(e). \end{array}$$

Hence we get the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(x) & \longrightarrow & \theta_s(M(x)) & \longrightarrow & M(xs) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M(e) & \longrightarrow & \theta_s(M(e)) & \longrightarrow & M(s) & \longrightarrow & 0. \end{array}$$

Therefore the image of

$$\mathrm{Ext}^1(M(xs), M(x)) \to \mathrm{Ext}^1(M(xs), M(e))$$

and

$$\mathrm{Ext}^1(M(s), M(e)) \to \mathrm{Ext}^1(M(xs), M(e))$$

coincide with each other by Lemma 3.1. Therefore, we get (2) ■

Take a basis v_s of $V(xs, x)$.

Lemma 3.3. *The set $\{v_s \mid s \in S\}$ is a basis of V.*

Proof. By Lemma 2.1 (1), we have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(M(s), M(e)) & \xrightarrow{\sim} & \bigoplus_{s' \in S} \mathrm{Hom}(M(s), M(s')) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(M(w_0), M(e)) & \xrightarrow{\sim} & \bigoplus_{s' \in S} \mathrm{Hom}(M(w_0), M(s')). \end{array}$$

We get the lemma. ■

We need the following lemma to define the action of $s \in S$ on V.

Lemma 3.4. *The linear map $\mathrm{Ext}^1(M(w_0), M(e)) \to \mathrm{Ext}^1(C_s(M(w_0)), M(e))$ is an isomorphism.*

Proof. By an exact sequence $0 \to M(w_0 s)/M(w_0) \to C_s(M(w_0)) \to M(w_0) \to 0$, we get

$$\begin{aligned} &\mathrm{Hom}(M(w_0 s)/M(w_0), M(e)) \to \mathrm{Ext}^1(M(w_0), M(e)) \\ &\qquad \to \mathrm{Ext}^1(C_s(M(w_0)), M(e)) \to \mathrm{Ext}^1(M(w_0 s)/M(w_0), M(e)) \end{aligned}$$

It is sufficient to prove that $\mathrm{Ext}^i(M(w_0 s)/M(w_0), M(e)) = 0$ for $i = 0, 1$. We have the following exact sequence

$$\begin{aligned} 0 &\to \mathrm{Hom}(M(w_0 s)/M(w_0), M(e)) \to \mathrm{Hom}(M(w_0 s), M(e)) \\ &\to \mathrm{Hom}(M(w_0), M(e)) \to \mathrm{Ext}^1(M(w_0 s)/M(w_0), M(e)) \\ &\to \mathrm{Ext}^1(M(w_0 s), M(e)) \to \mathrm{Ext}^1(M(w_0), M(e)). \end{aligned}$$

The homomorphism $\mathrm{Hom}(M(w_0 s), M(e)) \to \mathrm{Hom}(M(w_0), M(e))$ is an isomorphism. The kernel of $\mathrm{Ext}^1(M(w_0 s), M(e)) \to \mathrm{Ext}^1(M(w_0), M(e))$ is isomorphic to $\mathrm{Hom}(M(w_0 s), M(e)/M(e)) = 0$ by Lemma 2.1 (3). We get the lemma. ■

Using this lemma, we consider the following homomorphism:

$$\begin{aligned} V = \mathrm{Ext}^1(M(w_0), M(e)) &\to \mathrm{Ext}^1(LC_s(M(w_0)), LC_s(M(e))) \\ &\simeq \mathrm{Ext}^1(C_s(M(w_0)), M(s)) \\ &\to \mathrm{Ext}^1(C_s(M(w_0)), M(e)) \\ &\simeq \mathrm{Ext}^1(M(w_0), M(e)) = V. \end{aligned}$$

Define an action of s on V by the above homomorphism. In other words, for $v_i \in V$ and the corresponding exact sequences $0 \to M(e) \to X_i \to M(w_0) \to 0$, we have $s(v_1) = v_2$ if and only if there exists the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & C_s(M(e)) & \to & C_s(X_1) & \to & C_s(M(w_0)) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M(e) & \to & X_2 & \to & M(w_0) & \to & 0. \end{array}$$

Here, $C_s(M(e)) \to M(e)$ and $C_s(M(w_0)) \to M(w_0)$ are canonical homomorphisms.

The aim of this section is to show that this gives a structure of a W-module. Since $\{C_s\}$ satisfies the braid relations [MS05, Lemma 5.10], this action satisfies the braid relations.

Take an exact sequence $0 \to M(e) \to X \to M(w_0) \to 0$ and consider the following digram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M(e) & \longrightarrow & X & \longrightarrow & M(w_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \theta_s M(e) & \longrightarrow & \theta_s X & \longrightarrow & \theta_s M(w_0) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M(e) & \longrightarrow & X & \longrightarrow & M(w_0) & \longrightarrow & 0, \end{array}$$

here the vertical maps are natural transformations. The compositions of left and right vertical morphisms are zero. Hence composition of the middle vertical morphisms factors through $X \to M(w_0)$ and $M(e) \to X$. By this way, we get an element of $\mathrm{Hom}(M(w_0), M(e))$. This gives a morphism

$$\alpha_s\colon V = \mathrm{Ext}^1(M(w_0), M(e)) \to \mathrm{Hom}(M(w_0), M(e)).$$

Since we fix an inclusion $M(w_0) \hookrightarrow M(e)$, we regard $\alpha_s(v) \in K$ for $v \in V$. So we get $\alpha_s \in V^*$.

Lemma 3.5. *If $v \in \mathrm{Ext}^1(M(w_0), M(e))$ satisfies $\alpha_s(v) = 0$, then $s(v) = v$.*

Proof. Let $0 \to M(e) \to X \to M(w_0) \to 0$ be the corresponding exact sequence. The assumption means that $X \to \theta_s(X) \to X$ is zero. Hence $\theta_s(X) \to X$ factors through $\theta_s(X) \to C_s(X)$. Namely, we get the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_s(M(e)) & \longrightarrow & C_s(X) & \longrightarrow & C_s(M(w_0)) & \longrightarrow & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M(e) & \longrightarrow & X & \longrightarrow & M(w_0) & \longrightarrow & 0. \end{array}$$

This means that $s(v) = v$. ∎

Lemma 3.6. *We have $\alpha_s(v_s) = 2$ and $s(v_s) = -v_s$.*

Proof. We consider v_s as an element of $V(w_0, w_0 s)$. So we consider all things in $\{M \in \mathcal{O}_\lambda \mid [M : L(x)] = 0 \text{ for } x \neq w_0, w_0 s\}$. This category is equivalent to the regular integral block of the BGG category of $\mathfrak{g} = \mathfrak{sl}_2(K)$. So we may assume that $\mathfrak{g} = \mathfrak{sl}_2(K)$.

Set $P_0 = M(e)$, $P_1 = \theta_s(M(e))$. Then $P_0 \oplus P_1$ is a projective generator of $\mathcal{O}_\lambda$. Set $A = \mathrm{End}(P_0 \oplus P_1)$. Then $\mathcal{O}_\lambda$ is equivalent to the category of finitely generated right A-modules.

We have that $\dim \operatorname{Hom}(P_0, P_0) = \dim \operatorname{Hom}(P_0, P_1) = \dim \operatorname{Hom}(P_1, P_0) = 1$, $\dim \operatorname{Hom}(P_1, P_1) = 2$. Let $f\colon P_0 \to P_1$ and $g\colon P_1 \to P_0$ be the natural transformations. Then $\operatorname{Hom}(P_0, P_1) = Kf$, $\operatorname{Hom}(P_1, P_0) = Kg$ and $\operatorname{Hom}(P_1, P_1) = K\mathrm{id} + Kfg$. Moreover, $gf = 0$. Set $e_i\colon P \to P_i$ be the projection. Then $A = Ke_0 + Ke_1 + Kf + Kg + Kfg$, here f stands for $P_0 \oplus P_1 \to P_0 \xrightarrow{f} P_1 \to P_0 \oplus P_1$. ($g$ and fg are similar.)

First, we calculate $\alpha_s(v_s)$. To calculate it, we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_1/P_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & P_1 & \xrightarrow{j} & P_1^{\oplus 2} & \xrightarrow{q} & P_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle u} & & \\
0 & \longrightarrow & P_0 & \xrightarrow{i} & P_1 & \xrightarrow{p} & P_1/P_0 & \longrightarrow & 0,
\end{array}$$

here j is an inclusion $x \mapsto (x, 0)$ and q is second projection. Notice that we used $\theta_s(P_0) = \theta_s(P_1/P_0) = P_1$ and the second exact sequence splits.

We regard P_0 and P_1 as right A-modules. Then $P_0 = Ke_0 + Kg$, $P_1 = Ke_1 + Kf + Kfg$. We regard P_0 as a submodule of P_1, namely, $P_0 = Kf + Kfg$. We fix a inclusion $M(w_0) = P_1/P_0 \to M(e) = P_0$ by $[e_1] \mapsto fg$, here $[e_1]$ is an image of e_1. Set $h = fg$. Since $\operatorname{End}(P_0) = K\mathrm{id} + Kh$, b is given by $b = (\alpha_1 + \beta_1 h, \alpha_2 + \beta_2 h)$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in K$. Since a and i are both natural transformation. So $a = i = f$. By $bi = ja$, we have $\alpha_1 = 1$. Since c is given by $[e_1] \mapsto fg$, we get $\alpha_2 = 0$ and $\beta_2 = 1$ by $cp = qb$. So we have $b = (1 + \beta_1 h, h)$.

Next, we consider t. Define $\gamma_1, \gamma_2, \delta_1, \delta_2 \in K$ by $t = (\gamma_1 + \delta_1 h, \gamma_2 + \delta_2 h)$. The morphism s is given by $e_1 \mapsto fg$, $f \mapsto 0$ and $fg \mapsto 0$. Hence $tj = is$ implies that $\gamma_1 = 0$ and $\delta_1 = 1$. Since $uq = pt$, we get $\gamma_2 = 1$. Therefore, $t = (h, 1 + \delta_2 h)$. Hence the composition tb is given by $2h$. The image of e_1 under $P_1 \to \theta_s(P_1) \to P_1$ is $2h$. Since P_1/P_0 is given by $[e_1] \mapsto h$, this means $\alpha_s(v_s) = 2$.

Set $t' = (h, -1)\colon \theta_s(P_1) \simeq P_1^{\oplus 2} \to P_1$. Then we have $t'b = 0$ and $(-u)q = pt'$. This means that there exists a following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_s(P_0) & \longrightarrow & C_s(P_1) & \longrightarrow & C_s(P_1/P_0) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{s}} & & \downarrow{\scriptstyle \widetilde{t'}} & & \downarrow{\scriptstyle \widetilde{-u}} & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_1 & \longrightarrow & P_0/P_1 & \longrightarrow & 0.
\end{array}$$

Here $\widetilde{s}$ (resp. $\widetilde{t'}$, $\widetilde{-u}$) is the morphism induced by s (resp. t', $-u$). Hence we have $s(v_s) = -v_s$. ∎

Lemma 3.7. *For $v \in V$, we have $s(v) = v - \alpha_s(v) v_s$.*

Proof. Since $\alpha_s(v_s) \neq 0$, we have $V = Kv_s + \operatorname{Ker} \alpha_s$. So we may assume that $v \in \operatorname{Ker} \alpha_s$ or $v = v_s$. The first one is Lemma 3.5 and the second one is Lemma 3.6. ∎

Proposition 3.8. *The action of $s \in S$ on V defines a representation of W.*

Proof. We should prove $s^2 = 1$. This follows from Lemma 3.7 and Lemma 3.6. ■

In fact, V is isomorphic to $\mathfrak{h}^*$ as a W-module. We prove it in the rest of this section.

Lemma 3.9. *If $xs > x$ and $ys > y$, then $V(xs, ys) = s(V(x, y))$.*

Proof. This follows from $C_s(M(x)) \simeq M(xs)$ and $C_s(M(y)) \simeq M(ys)$. ■

Lemma 3.10. *Let $m'_{s,s'}$ be an order of $ss' \in \mathrm{GL}(V)$. Then there exists $v'_s \in K^\times v_s$ such that $s(v'_{s'}) = v'_{s'} - 2\cos(\pi/m'_{s,s'})v_s$ for all $s, s' \in S$.*

Proof. Since W is a finite group, V is defined over $\overline{\mathbb{Q}}$. Hence we may assume $K = \overline{\mathbb{Q}}$. Moreover, by the base change, we may assume that $K = \mathbb{C}$. Since W is a finite group, V has a W-invariant inner product $\langle \cdot, \cdot \rangle$. Take $v''_s \in \mathbb{C}v_s$ such that $\langle v''_s, v''_s \rangle = 2$. By Lemma 3.7, $s(v^*) = v^* - v^*(v_s)\alpha_s$ for $v^* \in V$. Hence the kernel of $s+1 \colon V^* \to V^*$ is $K\alpha_s$. The linear map $v \mapsto \langle v, v''_s \rangle$ belongs to $\mathrm{Ker}(s+1) = K\alpha_s$. Since $\langle v''_s, v''_s \rangle = \alpha_s(v_s)$, we have $s(v) = v - \langle v, v''_s \rangle v''_s$ by Lemma 3.7.

Put $a_{s,s'} = \langle v''_s, v''_{s'} \rangle$. Then we have $a_{s,s'} = \overline{a_{s',s}}$. By Lemma 3.7, $Kv'_s + Kv'_{s'}$ is stable under the action of ss'. Its characteristic polynomial is

$$t^2 + (2 - |a_{s,s'}|^2)t + 1.$$

Let α, α^{-1} be its eigenvalue. Since (W, S) is a Weyl group, $m'_{s,s'} \le 6$. Hence $\alpha = e^{2\pi\sqrt{-1}/m'_{s,s'}}$ or $\alpha = e^{-2\pi\sqrt{-1}/m'_{s,s'}}$. We get

$$|a_{s,s'}|^2 = 2 + \alpha + \alpha^{-1} = 2 + 2\cos 2\pi/m'_{s,s'} = (2\cos(\pi/m'_{s,s'}))^2.$$

Hence we have $a_{s,s'} = 2e^{\sqrt{-1}\theta_{s,s'}}\cos(\pi/m'_{s,s'})$ for some $\theta_{s,s'} \in \mathbb{R}$. Recall that the Coxeter graph of (W, S) is a tree. Hence we can choose $v'_s \in e^{\sqrt{-1}\mathbb{R}}v''_s$ such that $a_{s,s'} = 2\cos(\pi/m'_{s,s'})$. So we get the lemma. ■

Lemma 3.11. *For $x, y \in W$ such that $xs > x$, $x \ge y$ and $y < ys$, let A (resp. B) be the image of $\mathrm{Ext}^1(M(xs), M(ys)) \to \mathrm{Ext}^1(M(xs), M(y))$ (resp. $\mathrm{Ext}^1(M(xs), M(x)) \to \mathrm{Ext}^1(M(xs), M(y))$). Then $A \cap B = 0$ if and only if $x \not\ge ys$.*

Proof. By Lemma 3.2, B is one-dimensional. If $x \ge ys$, then we have a homomorphism $\mathrm{Ext}^1(xs, x) \to \mathrm{Ext}^1(xs, ys)$. So $B \subset A$.

On the other hand, assume that $A \cap B \ne 0$. Let $\mu \in \mathfrak{h}^*$ be a dominant integral element such that $\mathrm{Stab}_W(\mu) = \{e, s\}$. Then T_λ^μ induces an homomorphism $\mathrm{Ext}^1(M(xs), M(y)) \to \mathrm{Ext}^1(M(xs\mu), M(y\mu))$. Since $T_\lambda^\mu(M(xs)) = T_\lambda^\mu(M(x)) = M(x\mu)$, the image of B under T_λ^μ is zero. Hence the homomorphism

$$\mathrm{Ext}^1(M(xs), M(ys)) \to \mathrm{Ext}^1(M(xs\mu), M(y\mu))$$

has a kernel. This homomorphism is equal to

$$\mathrm{Ext}^1(M(xs), M(ys)) \to \mathrm{Ext}^1(M(xs), \theta_s(M(y))).$$

From an exact sequence

$$0 \to M(ys) \to \theta_s(M(y)) \to C_s(M(ys)) \to 0,$$

we have $\mathrm{Hom}(M(xs), C_s(M(ys)) \neq 0$. By the adjointness,

$$\mathrm{Hom}(M(x), C_s(M(ys))) = \mathrm{Hom}(K_s(M(xs)), M(ys)) = \mathrm{Hom}(M(x), M(ys)).$$

Hence we have $x \geq ys$. ■

Lemma 3.12. *For $s, s' \in S$, let $\langle s, s'\rangle$ be the group generated by $\{s, s'\}$. Then for all $x, y \in w_0\langle s, s'\rangle$, $\mathrm{Ext}^1(M(x), M(y)) \to \mathrm{Ext}^1(M(w_0), M(e))$ is injective.*

Proof. Denote the irreducible quotient of $M(x)$ by $L(x)$. Let $w \in \langle s, s'\rangle$ be the longest element. Put

$$\mathcal{O}' = \{M \in \mathcal{O}_\lambda \mid [M : L(z)] = 0 \text{ for } z \notin w_0\langle s, s'\rangle\}.$$

Then $\mathcal{O}'$ is equivalent to the regular integral block of the BGG category of semisimple Lie algebra of rank 2. Applying Lemma 2.1 (3) to this category, the kernel of

$$\mathrm{Ext}^1(M(x), M(y)) \to \mathrm{Ext}^1(M(w_0), M(w_0w))$$

is isomorphic to $\mathrm{Hom}(M(x), M(w_0w)/M(y))$. However, we have that

$$[M(w_0w) : L(x)] = [M(y) : L(x)] = 1.$$

Hence this space is zero. By Lemma 2.1 (3), the kernel of

$$\mathrm{Ext}^1(M(w_0), M(w_0w)) \to \mathrm{Ext}^1(M(w_0), M(e))$$

is isomorphic to $\mathrm{Hom}(M(w_0), M(e)/M(w_0w))$. It is zero since $[M(e) : L(w_0)] = [M(w_0w) : L(w_0)] = 1$. ■

Remark 3.13. By Lemma 3.11 and Lemma 3.12, we have the following. For $x, y \in w_0\langle s, s'\rangle$ such that $xs > x \geq y < ys$, $v_s \in V(xs, ys)$ if and only if $x \geq ys$. Since $s(v_s) = -v_s$ and $V(xs, ys) = s(V(x, y))$ (Lemma 3.9), under the same conditions, we have $v_s \in V(x, y)$ if and only if $x \geq ys$.

Proposition 3.14. *The W-representation V is isomorphic to the geometric representation defined in* [Bou02].

Proof. Let $w' \in \langle s, s'\rangle$ be the longest element and put $w = w_0w'$. Let $m_{s,s'}$ be an order of $ss' \in W$. It is sufficient to prove that $m_{s,s'}$ is an order of $ss' \in \mathrm{GL}(V)$.

Fix $s, s' \in S$ and set $m = m_{s,s'}$. Put $V' = Kv_s + Kv_{s'} = V(w_0, w)$. Then V' is $\langle s, s'\rangle$-stable. Let n be an order of $ss' \in \mathrm{GL}(V')$. Then $n|m$. We prove

$n = m$. Since (W, S) is a Weyl group, $m = 2, 3, 4, 6$. If $m = 2, 3$, then there is nothing to prove.

Assume that $n = 2$ and $m \neq 2$. Then by Lemma 3.10, $s'v'_s = v'_s$. Therefore, $v_s \in s'V(ws, w) = V(wss', ws')$ by Lemma 3.9. By Remark 3.13, $ss' \geq s's$. This is a contradiction since $m \neq 2$.

Assume that $(n, m) = (3, 6)$. Then we have $s's(v'_{s'}) = -v'_s$. Hence, $v_s \in s'sV(ws', w) = V(ws'ss', wss')$. By Remark 3.13, $s'ss' \geq ss's$. This is a contradiction since $m = 6$. ■

This proposition says that $V \simeq \mathfrak{h}^*$ as W-module. Let α be a simple root. By the proof, v_{s_α} corresponds to α up to constant multiple.

4. Regular case

As in the previous section, fix a regular integral dominant element λ. We continue to use the notation in the previous section. In this section, we determine $V(x, y) \subset V$.

We need the graded BGG category. By a result of Beilinson-Ginzburg-Soergel [BGS96], there exists a graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that $\mathcal{O}_\lambda$ is equivalent to the category of A-modules. Let $\widetilde{\mathcal{O}_\lambda}$ be the category of graded A-modules. Then we have the forgetful functor $\widetilde{\mathcal{O}_\lambda} \to \mathcal{O}_\lambda$. For $M = \bigoplus_i M_i \in \widetilde{\mathcal{O}_\lambda}$, define $M\langle n \rangle \in \widetilde{\mathcal{O}_\lambda}$ by $(M\langle n \rangle)_i = M_{i-n}$. There exists a module $\widetilde{M}(x) \in \widetilde{\mathcal{O}_\lambda}$ such that its image in $\mathcal{O}_\lambda$ is isomorphic to $M(x)$ and it is unique up to grading shift [BGS96, 3.11]. Take a grading of $\widetilde{M}(x)$ such that its top is in degree 0. Set $\widetilde{M}^0(x) = \widetilde{M}(x)\langle \ell(x) \rangle$. Then by the proof of [Maz07, Corollary 23], if $x \geq y$, then there exists a degree zero injective homomorphism $\widetilde{M}^0(x) \to \widetilde{M}^0(y)$.

For $M, N \in \widetilde{\mathcal{O}_\lambda}$, let $\operatorname{Hom}(M, N)_0$ be the space of homomorphisms of degree zero and $\operatorname{Ext}^i(M, N)_0$ its derived functors. The following lemma is proved in [Maz07, Theorem 32]. However, the author thinks, although his proof is correct, the shift in the statement of the theorem is wrong. So we give a proof (it is the same as in the proof in [Maz07]).

Lemma 4.1. *We have* $\operatorname{Ext}^1(\widetilde{M}^0(x), \widetilde{M}^0(e)\langle i \rangle)_0 = 0$ *if* $i \neq 2$. *Moreover, if* $i = 2$, *its dimension is equal to* $\#\{s \in S \mid s \leq x\}$.

Proof. We use the argument in the proof of Lemma 2.1 with a grading. Let $\widetilde{P}$ be the projective cover of $\widetilde{M}(w_0)$. Then it has a Verma flag M_i such that M_i/M_{i-1} is isomorphic to $\bigoplus_{\ell(x)=i} M(x)$ (here, we ignore the grading). The by Lemma 2.1 (1), we have $\operatorname{Hom}(\widetilde{M}^0(x\lambda), M_1/M_0)_0 \simeq \operatorname{Ext}^1(\widetilde{M}^0(x\lambda), M_0)$. We consider the grading of M_0 and M_1/M_0. Take $k \in \mathbb{Z}$ such that $M_0 \simeq \widetilde{M}(e)\langle k \rangle$. Then the multiplicity of $\widetilde{M}(w_0)\langle k \rangle$ in $\widetilde{M}(e)$ is 1 by [BGS96, Theorem 3.11.4]. Since $\widetilde{M}(w_0)\langle \ell(w_0) \rangle \subset \widetilde{M}(e)$, we have $k = \ell(w_0)$. By the same argument, we have

$M_1/M_0 \simeq \bigoplus_s \widetilde{M}(s)\langle \ell(w_0)-1\rangle$. So we have

$$\operatorname{Ext}^1(\widetilde{M}^0(x), \widetilde{M}^0(e)\langle i\rangle) \simeq \operatorname{Hom}\left(\widetilde{M}^0(x), \bigoplus_{s\in S} \widetilde{M}^0(s)\langle i-2\rangle\right).$$

We get the lemma. ■

We use the following abbreviations.

- $E^i(x\langle k\rangle, y\langle l\rangle) = \operatorname{Ext}^i(\widetilde{M}^0(x)\langle k\rangle, \widetilde{M}^0(y)\langle l\rangle)_0$.
- $E^i(x\langle k\rangle, y/z\langle l\rangle) = \operatorname{Ext}^i(\widetilde{M}^0(x)\langle k\rangle, \widetilde{M}^0(y)/\widetilde{M}^0(z)\langle l\rangle)_0$.

By the above lemma, $V(x,y)$ is the image of $E^1(x, y\langle 2\rangle) \to E^1(w_0, e\langle 2\rangle)$.

Lemma 4.2. *If $i \le 1$, then* $\operatorname{Ext}(\widetilde{M}^0(x), \widetilde{M}^0(y)\langle i\rangle)_0 = 0$.

Proof. We have $E^1(x, y\langle i\rangle) = E^0(x, e/y\langle i\rangle)$ by Lemma 2.1 (3) and the above lemma. Let L be the unique irreducible quotient of $\widetilde{M}^0(x)$. By [BGS96, Theorem 3.11.4], we have $[\widetilde{M}^0(e)/\widetilde{M}(y)\langle i\rangle : L] = 0$. Hence we have $E^0(x, e/y\langle i\rangle) = 0$. ■

There is a graded lift of θ_s [Str03]. We take a lift $\widetilde{\theta}_s$ of θ_s such that $\widetilde{\theta}_s$ is self-dual and there exist degree zero natural transformations $\operatorname{Id}\langle 1\rangle \to \widetilde{\theta}_s \to \operatorname{Id}\langle -1\rangle$. Let $\widetilde{C}_s$ (resp. $\widetilde{K}_s$) be the cokernel (resp. kernel) of $\operatorname{Id}\langle 1\rangle \to \widetilde{\theta}_s$ (resp. $\widetilde{\theta}_s \to \operatorname{Id}\langle -1\rangle$). Then $(\widetilde{C}_s, \widetilde{K}_s)$ is an adjoint pair. By [Str03, Theorem 3.6], we have the following formulas for $x \in W$ and $s \in S$ such that $xs > x$:

- $\widetilde{C}_s(\widetilde{M}^0(x)) = \widetilde{M}^0(xs)\langle -1\rangle$ and $L^k\widetilde{C}_s(\widetilde{M}^0(x)) = 0$ for $k > 0$.
- We have an exact sequence $0 \to (\widetilde{M}^0(x)/\widetilde{M}^0(xs))\langle 1\rangle \to C_s(\widetilde{M}^0(xs)) \to \widetilde{M}^0(xs)\langle -1\rangle \to 0$.
- $\widetilde{K}_s(\widetilde{M}^0(xs)) = \widetilde{M}^0(x)\langle 1\rangle$ and $R^k\widetilde{K}_s(\widetilde{M}^0(xs)) = 0$ for $k > 0$.

We also have that $L\widetilde{C}_s$ gives an auto-equivalence of $D^b(\widetilde{\mathcal{O}}_\lambda)$ and its quasi-inverse functor is $R\widetilde{K}_s$.

Now we prove the main theorem in the regular case. We have already proved (1) (Lemma 3.9).

Theorem 4.3. *Let $x, y \in W$ and $s \in S$ such that $xs > x \ge y$.*

(1) If $ys < y$, then $V(xs, y) = s(V(x, ys))$.

(2) If $ys > y$, then $V(xs, y) = Kv_s + s(V(x, y))$.

Proof. From (1), we have $s(V(x,y)) = V(xs,ys)$. Since $y < ys$, we have a homomorphism $\mathrm{Ext}^1(M(xs),M(ys)) \to \mathrm{Ext}^1(M(xs),M(y))$. Hence $V(xs,ys) \subset V(xs,y)$. We also have $\mathrm{Ext}^1(M(xs),M(x)) \to \mathrm{Ext}^1(M(xs),M(y))$. Since Kv_s is the image of $\mathrm{Ext}^1(M(xs),M(x))$, the right hand side is contained in the left hand side.

From an exact sequence

$$0 \to (\widetilde{M}^0(y)/\widetilde{M}^0(ys))\langle 2\rangle \to \widetilde{C}_s(\widetilde{M}^0(ys))\langle 1\rangle \to \widetilde{M}^0(ys) \to 0,$$

we have an exact sequence

$$\begin{aligned}\mathrm{Ext}^0(\widetilde{M}^0(xs), \widetilde{C}_s(\widetilde{M}^0(ys))\langle 1\rangle)_0 &\to E^0(xs,ys)\\ &\to E^1(xs, y/ys\langle 2\rangle) \to \mathrm{Ext}^1(\widetilde{M}^0(xs), \widetilde{C}_s(\widetilde{M}^0(ys))\langle 1\rangle)_0.\end{aligned}$$

Since $R\widetilde{K}_s$ is the quasi-inverse functor of $L\widetilde{C}_s$, for $i \geq 0$, we have

$$\begin{aligned}\mathrm{Ext}^i(\widetilde{M}^0(xs), C_s(\widetilde{M}^0(ys))\langle 1\rangle)_0 &= \mathrm{Hom}_{D^b(\widetilde{\mathcal{O}}_\lambda)}(\widetilde{M}^0(xs), L\widetilde{C}_s(\widetilde{M}^0(ys))\langle 1\rangle[i])_0\\ &= \mathrm{Hom}_{D^b(\widetilde{\mathcal{O}}_\lambda)}(R\widetilde{K}_s(\widetilde{M}^0(xs)), \widetilde{M}^0(ys)\langle 1\rangle[i])_0\\ &= \mathrm{Hom}_{D^b(\widetilde{\mathcal{O}}_\lambda)}(\widetilde{M}^0(x)\langle 1\rangle, \widetilde{M}^0(ys)\langle 1\rangle[i])_0\\ &= \mathrm{Ext}^i(\widetilde{M}^0(x), \widetilde{M}^0(ys))_0.\end{aligned}$$

If $i = 1$, then this is zero by Lemma 4.2. Hence we get an exact sequence

$$E^0(x,ys) \to E^0(xs,ys) \to E^1(xs, y/ys\langle 2\rangle) \to 0.$$

Assume that $x \not\geq ys$. Then $E^0(x,ys) = 0$. Therefore, we have that $\dim E^1(xs, y/ys\langle 2\rangle) = \dim E^0(xs,ys) = 1$. From an exact sequence

$$E^1(xs, ys\langle 2\rangle) \to E^1(xs, y\langle 2\rangle) \to E^1(xs, y/ys\langle 2\rangle),$$

the codimension of the image A of $E^1(xs, ys\langle 2\rangle) \to E^1(xs, y\langle 2\rangle)$ is less than or equal to 1. We also have that the image B of $E^1(xs, x\langle 2\rangle) \to E^1(xs, ys\langle 2\rangle)$ is one-dimensional by Lemma 3.2. By Lemma 3.11, $A \cap B = 0$. Hence $A + B = E^1(xs, y\langle 2\rangle)$. This implies the theorem in this case.

Next assume that $x \geq ys$. Then $E^0(x,ys) \to E^0(xs,ys)$ is isomorphic. Hence $E^1(xs, y/ys\langle 2\rangle) = 0$. Therefore, $E^1(xs, ys\langle 2\rangle) \to E^1(xs, y\langle 2\rangle)$ is surjective. This implies $V(xs,ys) \supset V(xs,y)$. ∎

As a corollary, we can determine the dimension of $V(x,y)$. Let $R_{y,x}(q)$ be the polynomial defined in [KL79].

Theorem 4.4. *Assume that λ is regular. Then $\dim V_\lambda(x,y)$ is the coefficient of q in $(-1)^{\ell(y)-\ell(x)-1}R_{y,x}(q)$.*

Proof. Put $n_{y,x} = \dim V(x,y)$. Then by the above theorem and its proof, for $x, y \in W$, $s \in S$ such that $xs > x \geq y$, we have

- If $ys < y$, then $n_{xs,y} = n_{x,ys}$.
- If $ys > y$ and $x \geq ys$, then $n_{xs,y} = n_{x,y}$.
- If $ys > y$ and $x \not\geq ys$, then $n_{xs,y} \leq n_{x,y} + 1$.

Let $r_{y,x}$ be the coefficient of q in $(-1)^{\ell(y)-\ell(x)-1}R_{y,x}(q)$. By [KL79, (2.0.b), (2.0.c)], the constant term of $(-1)^{\ell(y)-\ell(x)-1}R_{y,x}(q)$ is 0 or 1 and it is 1 if and only if $y \leq x$. Hence using [KL79, (2.0.b), (2.0.c)], we have

- If $ys < y$, then $r_{xs,y} = r_{x,ys}$.
- If $ys > y$ and $x \geq ys$, then $r_{xs,y} = r_{x,y}$.
- If $ys > y$ and $x \not\geq ys$, then $r_{xs,y} = r_{x,y} + 1$.

Hence we get $n_{y,x} \leq r_{y,x}$. To prove $n_{y,x} = r_{y,x}$, it is sufficient to prove $n_{w_0,x} = r_{w_0,x}$. We prove this by backward induction on $\ell(x)$. By Lemma 4.1, we have $n_{w_0,x} = \#\{s' \in S \mid w_0 s' \geq x\}$.

Take $s \in S$ such that $xs > x$. If $w_0 s \geq xs$, then $r_{w_0,x} = r_{w_0 s,x} = r_{w_0,xs} = n_{w_0,xs}$. If $w_0 s \not\geq xs$, then $r_{w_0,x} = r_{w_0 s,x} + 1 = r_{w_0,xs} + 1 = n_{w_0,xs} + 1$. We compare $X = \{s' \in S \mid w_0 s' \geq x\}$ and $Y = \{s' \in S \mid w_0 s' \geq xs\}$. Since $xs > x$, we have $Y \subset X$.

Assume $s' \in Y$ and $s' \neq s$. Then $s' \leq w_0 xs$. Hence s' appears in a reduced expression of $w_0 xs$. Since $s \neq s'$, s' appears in a reduced expression of $w_0 x$. Hence $s' \in X$. This implies $X \cap (S \setminus \{s\}) = Y \cap (S \setminus \{s\})$.

Since $xs > x$, we have $w_0 s \geq x$. Therefore, $s \in X$. Hence if $w_0 s \geq xs$, we have $X = Y$, this implies $n_{w_0,x} = n_{w_0,xs}$. If $w_0 s \not\geq xs$, we have $X = Y \amalg \{s\}$, this implies $n_{w_0,x} = n_{w_0,xs} + 1$. Therefore, $r_{w_0,x} = n_{w_0,x}$. ■

5. Singular case

In this section, we fix a dominant integral (may be singular) element $\lambda \in \mathfrak{h}^*$. We also fix a regular integral dominant element $\lambda_0 \in \mathfrak{h}^*$. Then the translation functor $T_{\lambda_0}^{\lambda}$ is defined and it gives $V_{\lambda_0}(x,y) \to V_\lambda(x,y)$. Recall the notation $S_\lambda = \{s \in S \mid s(\lambda) = \lambda\}$.

In this section, we prove the following theorem.

Theorem 5.1. *(1) The homomorphism $V_{\lambda_0}(x,y) \to V_\lambda(x,y)$ induced by the translation functor is surjective.*

(2) The kernel of $V_{\lambda_0} \to V_\lambda$ is $\sum_{s \in S_\lambda} K v_s$.

We use the notation $\widetilde{\mathcal{O}_\lambda}$, $M\langle n\rangle$ and $\mathrm{Hom}(M,N)_0$ which we use in the previous section. Then using the same argument in [Str03], $T_{\lambda_0}^{\lambda}$ and $T_{\lambda}^{\lambda_0}$ have graded lifts $\widetilde{T_{\lambda_0}^{\lambda}} \colon \widetilde{\mathcal{O}_{\lambda_0}} \to \widetilde{\mathcal{O}_\lambda}$ and $\widetilde{T_{\lambda}^{\lambda_0}} \colon \widetilde{\mathcal{O}_\lambda} \to \widetilde{\mathcal{O}_{\lambda_0}}$, respectively.

Using the argument in [Str03], we can prove the following properties. Put $\widetilde{\theta} = \widetilde{T_\lambda^{\lambda_0}}\widetilde{T_{\lambda_0}^\lambda}$. Set $W_{S_\lambda} = \mathrm{Stab}_W(\lambda)$ and let $w_\lambda \in W_{S_\lambda}$ be the longest element. Then we can take $\widetilde{T_{\lambda_0}^\lambda}$ and $\widetilde{T_\lambda^{\lambda_0}}$ such that $\widetilde{\theta}$ is self-dual and there exists a natural transformation $\mathrm{Id}\langle\ell(w_\lambda)\rangle \to \widetilde{\theta}$ and $\widetilde{\theta} \to \mathrm{Id}\langle-\ell(w_\lambda)\rangle$. Define a subset $W(S_\lambda)$ of W by $W(S_\lambda) = \{x \in W \mid xs > x \text{ for all } s \in S_\lambda\}$. Then for $x \in W(S_\lambda)$, $\widetilde{\theta}(\widetilde{M}^0(x\lambda_0))$ has a filtration M_i such that M_i/M_{i-1} is isomorphic to $\bigoplus_{\ell(w)=i, w\in W_{S_\lambda}} \widetilde{M}^0(xw\lambda_0)\langle\ell(w_\lambda) - 2\ell(w)\rangle$.

Proof of Theorem 5.1. We prove (1). By Theorem 4.3, for $y \in W(S_\lambda)$ and $w \in W_{S_\lambda}$, we have $V(x, yw) \subset V(x, y) \subset V(x, yw) + \sum_{s\in S_\lambda} Kv_s$. It is easy to see that v_s is in the kernel of $V_{\lambda_0} \to V_\lambda$. Hence we may assume that $y \in W(S_\lambda)$.

It is sufficient to prove that

$$\mathrm{Ext}^1(\widetilde{M}^0(x\lambda_0), \widetilde{M}^0(y\lambda_0)\langle 2\rangle)_0 \to \mathrm{Ext}^1(\widetilde{M}^0(x\lambda_0), \widetilde{\theta}(\widetilde{M}^0(y\lambda_0))\langle 2 - \ell(w_\lambda)\rangle)_0$$

is surjective. Let M be the cokernel of

$$\widetilde{M}^0(y\lambda_0)\langle 2\rangle \to \widetilde{\theta}(\widetilde{M}^0(y\lambda_0))\langle 2 - \ell(w_\lambda)\rangle.$$

Then it is sufficient to prove that $\mathrm{Ext}^1(\widetilde{M}^0(x\lambda_0), M)_0 = 0$. As we mentioned above, M has a filtration $\{M_i'\}_{i\geq 1}$ such that $M_i'/M_{i-1}' \simeq \bigoplus_{\ell(w)=i, w\in W_\lambda} \widetilde{M}^0(yw\lambda_0)\langle 2 - 2\ell(w)\rangle$. By Lemma 4.2, we have $\mathrm{Ext}^1(\widetilde{M}^0(x\lambda_0), \widetilde{M}^0(yw\lambda_0)\langle 2 - \ell(w)\rangle)_0 = 0$ if $\ell(w) > 0$. Hence we get (1).

We have (2) from (1) and (2) of Lemma 2.1. ∎

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