On Kottwitz's Conjecture for Twisted Involutions

Meinolf Geck

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Abstract. Motivated by problems on nilpotent orbital integrals for real Lie groups, Kottwitz (2000) formulated a conjecture concerning the relationship between Kazhdan–Lusztig cells of a finite Coxeter group W and its conjugacy classes of $\diamond$-twisted involutions, where $\diamond$ is an involutory graph automorphism of W. In this paper, we study this relationship in type D_n and all cases where $\diamond$ is non-trivial. Combined with work of Kottwitz himself, Casselmann, Marberg, and joint work of Bonnafé, Halls and the author, this completes the proof of Kottwitz's Conjecture for all $W, \diamond$.
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1. Introduction

Let W be a finite Coxeter group with generating set S. We assume that we have a map $w \mapsto w^\diamond$ which is a group automorphism of W such that $S^\diamond = S$ and $(w^\diamond)^\diamond = w$ for all $w \in W$. An element $w \in W$ is called a $\diamond$-twisted involution if $w^\diamond = w^{-1}$. Given such an element $w \in W$, Kottwitz [12] defined a character $\Upsilon_w^\diamond$ of W which only depends on the $\diamond$-conjugacy class of w and which is remarkable for various reasons:

(1) The decomposition of $\Upsilon_w^\diamond$ into irreducible characters is related to Lusztig's non-abelian Fourier transforms [14, Chap. 4]. See also the introduction of [12] for further explanations about the relevance of $\Upsilon_w^\diamond$ for real Lie groups.

(2) By Lusztig and Vogan [22] there is a natural lift of $\sum_w \Upsilon_w^\diamond$ (sum over all w up to $\diamond$-conjugacy) to the generic one-parameter Iwahori-Hecke algebra associated with W, S. (By [20], there is even a version for arbitrary Coxeter groups.)

(3) Kottwitz [12] conjectures that, for any left cell Γ of W in the sense of Kazhdan–Lusztig [11], the number of elements in the intersection of Γ with

the ⋄-conjugacy class of w equals the scalar product of $\Upsilon_w^\diamond$ with the character afforded by Γ.

Following Kottwitz, we say that we are in the "split" case if $w^\diamond = w$ for all $w \in W$; otherwise, we are in the "quasi-split" case. If W is irreducible, then the quasi-split cases to consider are as follows:

- W of type A_n, E_6 and ⋄ conjugation with the longest element in W;

- W of type D_n and ⋄ the non-trivial graph automorphism of order 2;

- W of type F_4, $I_2(m)$ and ⋄ the non-trivial graph automorphism.

Previous work by Casselman [2], Kottwitz [12], Marberg [23], Bonnafé, Halls and the author [1], [4], [5] shows that the conjecture in (3) holds in the split case – modulo some results on type D_n which were used in [1] without proof. The purpose of this paper is to provide the missing proofs for type D_n and to settle the quasi–split case as well.

In Section 2, we introduce Kottwitz's involution module, both the split and the quasi-split version. In (2.6) we show that this coincides with the module constructed by Lusztig and Vogan [22]. (The identification in the split case already appeared in [7, §2].) We then also discuss various examples: first of all, the case where ⋄ is given by conjugation with the longest element in W; furthermore, the cases where W is of type F_4, $I_2(m)$ and ⋄ is the non-trivial graph automorphism.

In Section 3, which may be of independent interest, we clarify some notoriously troublesome issues concerning those irreducible characters of a Coxeter group of type D_n which are not invariant under the graph automorphism of order 2. The main result is Proposition 3.7 which establishes a strengthened version of "Pieri's Rule" for these characters. This was used without proof in [1] to remove some ambiguities in the determination of the character of the split version of Kottwitz's involution module for type D_n.

In Section 4, we consider Lusztig's leading coefficients of character values of Iwahori–Hecke algebras. In [14, Chap. 12], [17], Lusztig has used his classification of the unipotent characters of a finite reductive group to determine the leading coefficients in the split case. Here, we extend at least some of these results to the quasi-split case. The main difficulty consists in carefully choosing extensions of ⋄-invariant characters of W to the semidirect product of W with $\langle \diamond \rangle \subseteq \mathrm{Aut}(W)$. The applications to the quasi-split case in type D_n are contained in Proposition 4.11.

Finally, in Section 5, we complete the proof of Kottwitz's conjecture for type D_n. The main idea is to treat the split and the quasi-split case at the same time. For this purpose, we develop a modified version of Kottwitz's conjecture for type B_n, where we consider the left cells with respect to a suitable weight function in the sense of Lusztig [18]. The main result in this section is Theorem 5.3. This involves the construction of a modified version of Kottwitz's involution module in Lemma 5.2. At first sight, this new combined setting seems to make things more complicated (which is certainly true from a technical point of view); but, in fact, I do not see any way how to carry out the argument separately for the split and the quasi-split case.

We shall use standard results and notation concerning the (complex) characters of finite groups. If χ, ψ are two class functions on a finite group G, then $\langle \chi, \psi \rangle_G$ denotes the usual scalar product for which the irreducible characters of G form an orthonormal basis. If $H \subseteq G$ is a subgroup and ψ is a class function on H, then $\mathrm{Ind}_H^G(\psi)$ denotes the induced class function on G. We denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of G.

2. Kottwitz's twisted involution module

Let W be a finite Coxeter group with generating set S. For $w \in W$, we denote by $\ell(w)$ the length of w. We assume that we have a map $w \mapsto w^\diamond$ which is an automorphism of W such that $S^\diamond = S$ and $(w^\diamond)^\diamond = w$ for all $w \in W$. We say that two elements $w, w' \in W$ are $\diamond$-conjugate if there exists some $x \in W$ such that $w' = x^\diamond w x^{-1}$. This defines an equivalence relation on W, and the corresponding equivalence classes will be called the $\diamond$-conjugacy classes of W. The subgroup

$$C_W^\diamond(w) := \{ x \in W \mid x^\diamond w = wx \}$$

is called the $\diamond$-centraliser of w in W. Let Φ be the root system of W and $\Phi = \Phi^+ \amalg \Phi^-$ be the partition into positive and negative roots determined by S. Since $S^\diamond = S$, the automorphism $w \mapsto w^\diamond$ defines a permutation of the simple roots in Φ. We shall assume that this permutation induces a map $\alpha \mapsto \alpha^\diamond$ on all of Φ such that

$$w^\diamond(\alpha^\diamond) = w(\alpha)^\diamond \qquad \text{for all } w \in W \text{ and } \alpha \in \Phi.$$

Definition 2.1. An element $w \in W$ is called a "$\diamond$-twisted involution" if $w^\diamond = w^{-1}$. Given such an element $w \in W$, let Φ_w be the set of all $\alpha \in \Phi$ such that $w(\alpha) = -\alpha^\diamond$. Then, by Kottwitz [12, 2.1, 4.2], we can define a linear character $\varepsilon_w \colon C_W^\diamond(w) \to \{\pm 1\}$ as follows. For $x \in C_W^\diamond(w)$ we have $\varepsilon_w(x) = (-1)^k$ where k is the number of positive roots $\alpha \in \Phi_w$ such that $x(\alpha)$ is negative. Then set

$$\Upsilon_w^\diamond := \mathrm{Ind}_{C_W^\diamond(w)}^W \big(\varepsilon_w \big).$$

Remark 2.2. Let $C \subseteq W$ be any subset which is a union of $\diamond$-conjugacy classes of $\diamond$-twisted involutions in W. If C is a single $\diamond$-conjugacy class, then we certainly have $\Upsilon_w^\diamond = \Upsilon_{w'}^\diamond$ for all $w, w' \in C$. In general, we set

$$\Upsilon_C^\diamond = \sum_w \Upsilon_w^\diamond$$

where w runs over a set of representatives of the $\diamond$-conjugacy classes contained in C. In particular, this applies to the set of all $\diamond$-twisted involutions in W.

Remark 2.3. Assume that $w^\diamond = w$ for all $w \in W$; we just write this as $\diamond = 1$. Then a $\diamond$-twisted involution w is just an ordinary involution in W. Furthermore, $\Upsilon_w^\diamond$ is the character of the involution module in the "split case"; see [12, §2]. So, here, this character will be denoted by Υ_w^1.

If ◇ is non-trivial, then Kottwitz [12] formulated Definition 2.1 in a slightly different way, using the semidirect product of W with the automorphism given by ◇. The two versions are equivalent by the following remark.

Remark 2.4. Let $\widetilde{W}$ be the semidirect product of W with the subgroup $\langle ◇ \rangle \subseteq \mathrm{Aut}(W)$. Thus, $\widetilde{W}$ is generated by W and an additional element γ such that $\gamma w \gamma = w^◇$ for all $w \in W$. (If $◇ = 1$, then $\gamma = 1$ and $\widetilde{W} = W$; otherwise, $\gamma^2 = 1$.) The natural action of W on Φ can be extended to $\widetilde{W}$ such that $\gamma(\alpha) = \alpha^◇$ for all $\alpha \in \Phi$.

First of all, this shows that $w \in W$ is a ◇-twisted involution if and only if γw is an ordinary involution in $\widetilde{W}$. Furthermore, two elements $w, w' \in W$ are ◇-conjugate if and only if $\gamma w, \gamma w'$ are conjugate in $\widetilde{W}$. Consequently, the map $w \mapsto \gamma w$ defines a bijection between the ◇-conjugacy classes of W and the ordinary conjugacy classes of $\widetilde{W}$ which are contained in the coset $\gamma W \subseteq \widetilde{W}$. We have $C_{\widetilde{W}}(\gamma w) = C_W^◇(w)$ for all $w \in W$.

Remark 2.5. For any subset $I \subseteq S$ we denote by $W_I \subseteq W$ the corresponding parabolic subgroup and by w_I the longest element in W_I. Let C be a ◇-conjugacy class of ◇-twisted involutions. Then there exists a subset $I \subseteq S$ such that $w_I \in C$ and $s^◇ w_I = w_I s$ for all $s \in I$; furthermore, $I^◇ = I$ and w_I has minimum length in C. (See [8, Prop. 3.2.10] for the case where ◇ is the identity and He [9, Lemma 3.6] for the general case.) If we take $w = w_I$, then one easily sees that the set of roots Φ_{w_I} is just the parabolic subsystem $\Phi_I \subseteq \Phi$ corresponding to I. Let $\Pi_I \subseteq \Phi_I^+$ be the set of simple roots. Then we have:

$$W_I \text{ is a normal subgroup of } C_W^◇(w_I). \tag{a}$$

Indeed, since $s^◇ w_I = w_I s$ for all $s \in I$, we certainly have $W_I \subseteq C_W^◇(w_I)$. Now let $x \in C_W^◇(w_I)$ and $\alpha \in \Phi_I$. Then $w_I(x(\alpha)) = x^◇(w_I(\alpha)) = -x^◇(\alpha^◇) = -x(\alpha)^◇$ and so $x(\alpha) \in \Phi_{w_I} = \Phi_I$, as required. Thus, (a) is proved. Consequently, by Howlett [10, Cor. 3], we have a semidirect product decomposition

$$C_W^◇(w_I) = Y \ltimes W_I \qquad \text{where} \qquad Y := \{y \in C_W^◇(w_I) \mid y(\Pi_I) = \Pi_I\}. \tag{b}$$

Note that Y is contained in the set of distinguished left coset representatives of W_I in W; in particular, each element of Y sends all positive roots in Φ_I to positive roots. We conclude that

$$\varepsilon_{w_I}(yw) = (-1)^{\ell(w)} \qquad \text{for all } y \in Y \text{ and } w \in W_I. \tag{c}$$

This provides an explicit description of $\Upsilon_{w_I}^◇$ which will be useful in several places below.

2.6. Let C be a ◇-conjugacy class of ◇-twisted involutions in W. Let M be a $\mathbb{Q}$-vector space with a basis $\{a_w \mid w \in C\}$. By Lusztig and Vogan [22, 7.1] (see also [20]), it is known that M is a $\mathbb{Q}[W]$-module, where a generator $s \in S$ acts via the following formula:

$$s.a_w = \begin{cases} -a_w & \text{if } s^◇ w = ws \text{ and } \ell(ws) < \ell(w), \\ a_{s^◇ ws} & \text{otherwise.} \end{cases}$$

Let $w \in C$ be fixed. As discussed in [22, 6.3], we obtain a group homomorphism

$$\eta_w \colon C_W^\diamond(w) \to \{\pm 1\}$$

such that $x.a_w = \eta_w(x)a_w$ for all $x \in C_W^\diamond(w)$; furthermore, M is isomorphic to the $\mathbb{Q}[W]$-module obtained by inducing η_w from $C_W^\diamond(w)$ to W. We claim that

$\boxed{\Upsilon_w^\diamond \text{ is the character afforded by the } \mathbb{Q}[W]\text{-module } M.}$

If $\diamond$ is trivial, this is shown in [7, §2]. The argument in the general case is similar. Indeed, let $I \subseteq S$ and $w_I \in C$ be as in Remark 2.5. Then we have $C_W^\diamond(w_I) = Y \ltimes W_I$. Thus, it will be sufficient to show that

$$\eta_{w_I}(yw) = (-1)^{\ell(w)} \qquad \text{for all } y \in Y \text{ and } w \in W_I.$$

We argue as in [7, Lemma 2.1]. For $s \in I$, we have $s.a_{w_I} = -a_{w_I}$ and so $\eta_{w_I}(s) = -1$. Consequently, we have $\eta_{w_I}(w) = (-1)^{\ell(w)}$ for all $w \in W_I$. Thus, it remains to show that $y.a_{w_I} = a_{w_I}$ for all $y \in Y$. We shall in fact show that $x.a_{w_I} = a_{x^\diamond w_I x^{-1}}$ where x is any distinguished left coset representative of W_I in W. We proceed by induction on $\ell(x)$. If $x = 1$, the assertion is clear. Now assume that $x \neq 1$ and choose $s \in S$ such that $\ell(sx) < \ell(x)$. By Deodhar's Lemma [8, 2.1.2], we also have that $z := sx$ is a distinguished left coset representative. Hence, using induction, we have $z.a_{w_I} = a_{z^\diamond w_I z^{-1}}$ and so

$$x.a_{w_I} = s.a_{z^\diamond w_I z^{-1}}.$$

Let $u = z^\diamond w_I z^{-1}$. Given the formula for the action of a generator on the basis elements of M, it now suffices to show that either $s^\diamond u \neq us$ or that $\ell(us) > \ell(u)$. Assume, if possible, that none of these two conditions is satisfied, that is, we have $s^\diamond u = us$ and $\ell(us) < \ell(u)$; in particular, $\ell(szw_I(z^\diamond)^{-1}) = \ell(su^{-1}) < \ell(u^{-1}) = \ell(zw_I(z^\diamond)^{-1})$. But then the "Exchange Lemma" (see [8, Exc. 1.6]) and the fact that $\ell(szw_I) = \ell(z) + \ell(w_I) + 1$ imply that $szw_I(z^\diamond)^{-1} = zw_I z'$ where $\ell(z') < \ell(z)$. Since $s^\diamond u = us$, we have $zw_I(z^\diamond)^{-1}s^\diamond = szw_I(z^\diamond)^{-1} = zw_I z'$ and so $(z^\diamond)^{-1}s^\diamond = z'$. This would imply that $\ell(z') = \ell((z^\diamond)^{-1}s^\diamond) = \ell((sz)^\diamond) = \ell(sz) > \ell(z)$, a contradiction. Hence, the assumption was wrong and so $x.a_{w_I} = a_{x^\diamond w_I x^{-1}}$, as required. Thus, the above claim is proved.

Next, we briefly recall the definition of Kazhdan–Lusztig cell modules.

2.7. Let $\mathbf{H}$ be the generic one-parameter Iwahori–Hecke algebra associated with (W, S), over the ring of Laurent polynomials $A = \mathbb{Z}[v, v^{-1}]$ in an indeterminate v. Thus, $\mathbf{H}$ has a basis $\{T_w \mid w \in W\}$ and, for any $s \in S$ and $w \in W$, the multiplication is given by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + (v - v^{-1})T_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\{\mathbf{C}_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of $\mathbf{H}$. For any $w \in W$, we have

$$\mathbf{C}_w = T_w + \sum_{y \in W, y < w} (-1)^{\ell(w) + \ell(y)} v^{l(w) - l(y)} P_{y,w}(v^{-1}) T_y,$$

where $P_{y,w} \in \mathbb{Z}[v]$ are the polynomials defined in [11, Theorem 1.1] and $y < w$ denotes the Bruhat–Chevalley order. We write

$$\mathbf{C}_x \mathbf{C}_y = \sum_{z \in W} h_{x,y,z} \mathbf{C}_z \qquad \text{where } h_{x,y,z} \in A \text{ for all } x, y, z \in W.$$

Let $\leqslant_L$ be the pre-order relation on W defined in [11]; for any $w \in W$, we have

$$\mathbf{HC}_w \subseteq \sum_{y \in W, y \leqslant_L w} A\mathbf{C}_y.$$

For $y, w \in W$, we write $y \sim_L w$ if $y \leqslant_L w$ and $w \leqslant_L y$. This defines an equivalence relation on W; the equivalence classes are called the *left cells* of W. Let Γ be such a left cell. Let $[\Gamma]_A$ be a free A-module with a basis $\{e_x \mid x \in \Gamma\}$. By the definition of $\sim_L$, this is an **H**-module where the action is given by

$$\mathbf{C}_x.e_y = \sum_{z \in \Gamma} h_{x,y,z} e_z \qquad \text{for all } x \in W \text{ and } y \in \Gamma.$$

Then we obtain a $\mathbb{Q}[W]$-module $[\Gamma]_1$ by extension of scalars via the unique ring homomorphism $A \to \mathbb{Q}$ such that $v \mapsto 1$. We identify $[\Gamma]_1$ with its character.

Conjecture 2.8 (Kottwitz [12]).   *Let $w \in W$ be a $\diamond$-twisted involution and C be its $\diamond$-conjugacy class in W. Let Γ be a left cell in W. Then*

$$\boxed{\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = |C \cap \Gamma|.}$$

Note that, by Remark 2.4, the above formulation is indeed equivalent to the original formulation by Kottwitz. The above formulation covers both the case where $\diamond$ is the identity ("split" case) and the case where $\diamond$ is non-trivial ("quasi-split" case).

Remark 2.9.   The map $\diamond$ induces an A-algebra automorphism $h \mapsto h^\diamond$ on **H** such that $T_w^\diamond = T_{w^\diamond}$ for all $w \in W$. One easily checks that $\mathbf{C}_w^\diamond = \mathbf{C}_{w^\diamond}$ for all $w \in W$. Consequently, $\diamond$ permutes the left cells of W, the right cells of W and the two-sided cells of W.

Now let $\mathfrak{C}$ be a two-sided cell of W and $\Gamma \subseteq \mathfrak{C}$ be a left cell. Let C be a $\diamond$-conjugacy class of $\diamond$-twisted involutions in W. Then we claim that

$$C \cap \Gamma = \varnothing \qquad \text{unless} \qquad \mathfrak{C}^\diamond = \mathfrak{C}.$$

Indeed, assume that $\mathfrak{C}^\diamond \neq \mathfrak{C}$ and that there exists some $w \in C \cap \Gamma$. Then $w^{-1} = w^\diamond \in \Gamma^\diamond \subseteq \mathfrak{C}^\diamond$ and, hence, $w^{-1} \notin \mathfrak{C}$. But this contradicts [14, Lemma 5.2(iii)] which shows that, for any $w \in \mathfrak{C}$, we also have $w^{-1} \in \mathfrak{C}$. Thus, the above claim is proved.

This statement provides a first test case for Conjecture 2.8: we will have to show that then we also have $\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = 0$ for $w \in C$; see Remark 3.13 below for type D_n.

Example 2.10. Let $w_0 \in W$ be the longest element and assume that w_0 is not central in W. Then define $w^\diamond = w_0 w w_0$ for $w \in W$. One easily checks that the following hold.

(a) $w \in W$ is a $\diamond$-twisted involution if and and only if $w_0 w$ is an involution.

(b) Let $w \in W$ be a $\diamond$-twisted involution and C its $\diamond$-conjugacy class. Then $C_W^\diamond(w) = C_W(w_0 w)$ and $C = w_0 \mathcal{C}$ where $\mathcal{C}$ is the ordinary conjugacy class of $w_0 w$ in W.

Now let $w \in W$ be a $\diamond$-twisted involution and Γ be a left cell in W. Then Γw_0 and $w_0 \Gamma$ are also left cells and we have $[w_0 \Gamma]_1 = [\Gamma w_0]_1 = [\Gamma]_1 \otimes \varepsilon$; see [14, 5.14], [18, 11.7]. Consequently, we obtain

$$|C \cap \Gamma| = |w_0(\mathcal{C} \cap w_0\Gamma)| = |\mathcal{C} \cap w_0\Gamma|,$$

where C is the $\diamond$-conjugacy class of w and $\mathcal{C}$ is the ordinary conjugacy class of $w_0 w$ in W. On the other hand, by [12, 5.3.1], we have

$$\langle \Upsilon_{w_0 w}^1, \chi \otimes \varepsilon \rangle_W = \langle \Upsilon_w^\diamond, \chi \rangle_W \qquad \text{for all } \chi \in \mathrm{Irr}(W).$$

This yields that

$$\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = \langle \Upsilon_{w_0 w}^1, [\Gamma]_1 \otimes \varepsilon \rangle_W = \langle \Upsilon_{w_0 w}^1, [w_0\Gamma]_1 \rangle_W.$$

So, if the split version of Kottwitz's conjecture holds for W, then we have

$$\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_W = \langle \Upsilon_{w_0 w}^1, [w_0\Gamma]_1 \rangle_W = |\mathcal{C} \cap w_0\Gamma| = |C \cap \Gamma|,$$

that is, the quasi-split version (with respect to $\diamond$) also holds.

This discussion applies, in particular, to (W, S) of type A_n, D_{2n+1} E_6, $I_2(2m+1)$. The split version of Kottwitz's conjecture holds in type A_n, as already observed by Kottwitz himself [12]; see also [1, Exp. 5.10]. For type D_{2n+1}, see [1, Cor. 8.6] and Corollary 5.4 below. Finally, Casselman [2] has verified that the split version holds in type E_6; see Marberg [23] for the dihedral groups.

Example 2.11. Let $m \geqslant 2$ and (W, S) be of type $I_2(2m)$ where $S = \{s_1, s_2\}$ and $s_1 s_2$ has order $2m$. Assume that $s_1^\diamond = s_2$ and $s_2^\diamond = s_1$. By Remark 2.5, it is clear that, up to $\diamond$-conjugacy, 1 is the only $\diamond$-twisted involution; let C_1 denote its $\diamond$-conjugacy class. Let us consider the corresponding character $\Upsilon_1^\diamond$. We have

$$\mathrm{Irr}(W) = \{1, \varepsilon, \varepsilon_1, \varepsilon_2, \chi_1, \chi_2, \dots, \chi_{m-1}\}$$

where 1 is the trivial character, ε is the sign character, ε_1 and ε_2 are two further characters of degree 1 and each χ_j has degree 2; see [8, 5.3.4]. Here, the notation is such that $\varepsilon_1(s_1) = \varepsilon_2(s_2) = 1$ and $\varepsilon_1(s_2) = \varepsilon_2(s_1) = -1$; furthermore, χ_j is determined by the condition that $\chi_j(s_1 s_2) = 2\cos(\pi j/m)$. Now note that

$$C_W^\diamond(1) = \{x \in W \mid x^\diamond = x\} = \{1, w_0 (= \text{ longest element})\}.$$

Furthermore, $\Phi_1 = \varnothing$ and so ε_1 is the trivial character of $C_W^\diamond(1)$. Then we find that

$$\Upsilon_1^\diamond = \mathrm{Ind}_{C_W^\diamond(1)}^W(1) = \begin{cases} 1 + \varepsilon + \varepsilon_1 + \varepsilon_2 + \displaystyle\sum_{1 \leqslant j \leqslant (m-2)/2} 2\chi_{2j} & \text{if } m \text{ is even,} \\ 1 + \varepsilon + \displaystyle\sum_{1 \leqslant j \leqslant (m-1)/2} 2\chi_{2j} & \text{if } m \text{ is odd.} \end{cases}$$

Next, we consider the left cells of W. To simplify notation, write $1_k = s_1 s_2 s_1 \cdots$ (k factors) and $2_k = s_2 s_1 s_2 \cdots$ (k factors); in particular, $1_{2m} = 2_{2m}$ is the longest element in W. By [18, 8.8], the left cells are

$$\Gamma_0 := \{1_0\}, \qquad \Gamma_1 := \{1_1, 2_2, 1_3, \ldots, 1_{2m-1}\},$$
$$\Gamma_2 := \{2_1, 1_2, 2_3, \ldots, 2_{2m-1}\}, \qquad \Gamma_{2m} := \{1_{2m}\}.$$

Thus, we have

$$|C_1 \cap \Gamma_i| = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 2m, \\ m - 1 & \text{if } i = 1 \text{ or } i = 2. \end{cases}$$

The characters of the left cell modules are given by:

$$[\Gamma_0]_1 = 1, \quad [\Gamma_1]_1 = \varepsilon_1 + \sum_{1 \leqslant j \leqslant m-1} \chi_j, \quad [\Gamma_2]_1 = \varepsilon_2 + \sum_{1 \leqslant j \leqslant m-1} \chi_j, \quad [\Gamma_{2m}]_1 = \varepsilon;$$

see, for example, [6, Exp. 2.2.8]. Consequently, we have

$$\langle \Upsilon_1^\diamond, [\Gamma_i]_1 \rangle_W = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 2m, \\ m - 1 & \text{if } i = 1 \text{ or } i = 2. \end{cases}$$

Hence, we see that Kottwitz's conjecture holds in this case.

Example 2.12. Let (W, S) be of type F_4, where $S = \{s_0, s_1, s_2, s_3\}$ is such that $s_0 s_1$ and $s_2 s_3$ have order 3 and $s_1 s_2$ has order 4. Assume that $s_0^\diamond = s_3$, $s_1^\diamond = s_2$, $s_2^\diamond = s_1$ and $s_3^\diamond = s_0$. Let $C_1 = \{x^\diamond x^{-1} \mid x \in W\}$ be the $\diamond$-conjugacy class containing $w = 1$. Using Remark 2.5 we find that C_1 is the only $\diamond$-conjugacy class of $\diamond$-twisted involutions in W. We have $C_W^\diamond(1) = \{x \in W \mid x^\diamond = x\}$; this is a dihedral group of order 16, generated by $s_0 s_3$ and $(s_1 s_2)^2$. Since $\Phi_1 = \varnothing$, we obtain by a direct computation (which can be done by hand):

$$\Upsilon_1^\diamond = \mathrm{Ind}_{C_W^\diamond(1)}^W(1) = 1_1 + 1_4 + 2_1 + 2_2 + 2_3 + 2_4 + 2 \cdot 4_1 + 9_1 + 9_2 + 9_3 + 9_4 + 6_1 + 12_1,$$

where we use the notation for $\mathrm{Irr}(W)$ as in [8, Table C.3 (p. 413)]. Using a computer algebra system capable of computing Kazhdan–Lusztig cells, it is straightfoward to check that Conjecture 2.8 holds. For example, using PyCox [4], the left cells of W and the characters of the corresponding left cell modules are obtained by the following commands:

```
>>> W=coxeter("F",4); l=klcells(W,1,v)[0]
>>> ch=[leftcellleadingcoeffs(W,1,v,c)['char'] for c in l]
>>> chartable(W)['charnames']
```

The last command gives the labelling of $\mathrm{Irr}(W)$. The set C_1 is obtained by:

```
>>> p=[3,2,1,0]      # the permutation on S={0,1,2,3}
>>> C1=noduplicates([W.reducedword(w[::-1]+[p[s] for s in w],W)
                                        for w in allwords(W)]
```

Further explanations are available through the online help in PyCox.

Remark 2.13. The map $w \mapsto w^\diamond$ on W also induces an operation on $\mathrm{Irr}(W)$, which we denote by $\chi \mapsto \chi^\diamond$. We have $\chi^\diamond(w) = \chi(w^\diamond)$ for all $w \in W$. Let $\widetilde{W} = \langle W, \gamma \rangle$ be the semidirect product as in Remark 2.4. By standard results on Clifford theory, we have

$$\mathrm{Ind}_W^{\widetilde{W}}(\chi) \in \mathrm{Irr}(\widetilde{W}) \qquad \text{for all } \chi \in \mathrm{Irr}(W) \text{ such that } \chi^\diamond \neq \chi.$$

On the other hand, let us denote

$$\mathrm{Irr}^\diamond(W) := \{\chi \in \mathrm{Irr}(W) \mid \chi^\diamond = \chi\}.$$

Then each $\chi \in \mathrm{Irr}^\diamond(W)$ has exactly two extensions to $\widetilde{W}$, which differ only by a sign on elements in the coset γW.

Remark 2.14. Let $\mathfrak{C}$ be a two-sided cell of W. We set

$$\mathrm{Irr}(W \mid \mathfrak{C}) = \{\chi \in \mathrm{Irr}(W) \mid \langle [\Gamma]_1, \chi \rangle_W \neq 0 \text{ for some left cell } \Gamma \subseteq \mathfrak{C}\}.$$

Alternatively, we have $\mathrm{Irr}(W \mid \mathfrak{C}) := \{\chi \in \mathrm{Irr}(W) \mid \chi \sim_{LR} w \text{ for some } w \in \mathfrak{C}\}$, where the relation "$\chi \sim_{LR} w$" is defined in [14, 5.1 (p. 139)]. (This easily follows from the definitions; see, for example, [14, 2.2.18].) Thus, we obtain a partition

$$\mathrm{Irr}(W) = \coprod_{\mathfrak{C}} \mathrm{Irr}(W \mid \mathfrak{C}).$$

where $\mathfrak{C}$ runs over all two-sided cells in W. Now recall from Remark 2.9 that $\diamond$ permutes the left cells of W. One easily sees that, for a left cell Γ of W, we have

$$\mathrm{trace}(T_w, [\Gamma^\diamond]_A) = \mathrm{trace}(T_{w^\diamond}, [\Gamma]_A) \qquad \text{for all } w \in W.$$

This certainly implies that $\mathrm{Irr}(W \mid \mathfrak{C}^\diamond) = \{\chi^\diamond \mid \chi \in \mathrm{Irr}(W \mid \mathfrak{C})\}$.

Remark 2.15. Having dealt with type F_4 and the dihedral groups, we shall assume from now on that W is a Weyl group and that $\diamond$ is "ordinary" in the sense of [14, 3.1], that is, whenever $s, t \in S$ are in the same $\diamond$-orbit, then the product st has order 2 or 3. This has the following consequences.

(a) Each $\chi \in \mathrm{Irr}(W)$ can be realised over $\mathbb{Q}$. (This is a well-known fact; see, for example, [8, 6.3.8].)

(b) The two extensions of any $\chi \in \mathrm{Irr}^\diamond(W)$ to $\widetilde{W}$ can also be realised over $\mathbb{Q}$. (See [14, Prop. 3.2]).

(c) If $\mathfrak{C}$ is a two-sided cell of W such that $\mathfrak{C}^\diamond = \mathfrak{C}$, then $\mathrm{Irr}(W \mid \mathfrak{C}) \subseteq \mathrm{Irr}^\diamond(W)$. (See [14, 4.17].)

In any case, as far as the quasi-split version of Kottwitz's conjecture is concerned, it now remains to deal with type D_n and the non-trivial graph automorphism of order 2.

3. On the irreducible characters in type D_n

In this section we fix some notation concerning the irreducible characters of Coxeter groups of classical type. This is especially relevant in type D_n for n even, where there are characters which are not invariant under the graph automorphism of order 2; it will be important for us to know exactly how to distinguish these characters from each other. In Corollary 3.8, we establish a strengthened "branching rule" for type D_n. Then, in (3.9) and Proposition 3.12, we state explicit formulae for the decomposition of Υ_w^1 and $\Upsilon_w^\diamond$.

3.1. For $\chi \in \operatorname{Irr}(W)$, let $\mathbf{b}_\chi$ denote the smallest integer $i \geqslant 0$ such that χ occurs in the ith symmetric power of the standard reflection representation of W. For example, if ε is the sign character of W, then

$$\mathbf{b}_\varepsilon = |T| = \ell(w_0)$$

where $w_0 \in W$ is the longest element and $T = \{wsw^{-1} \mid s \in S, w \in W\}$ is the set of all reflections in W (see [8, 5.3.1(a)]). Let $W' \subseteq W$ be a subgroup generated by reflections and let $T' = W' \cap T$. Let ε' be the sign character of W'. By a result due to Macdonald (see [8, 5.2.11]), there is a unique $\chi \in \operatorname{Irr}(W)$ such that $\mathbf{b}_\chi = |T'|$ and

$$\operatorname{Ind}_{W'}^W(\varepsilon') = \chi + \text{combination of various } \psi \in \operatorname{Irr}(W) \text{ such that } \mathbf{b}_\psi > \mathbf{b}_\chi.$$

We shall denote this character by

$$\chi := \mathbf{j}_{W'}^W(\varepsilon').$$

This "**j**-induction" can be used to systematically construct all the irreducible characters of W of type A_{n-1}, B_n and D_n; see [8, Chap. 5].

Example 3.2. Let $n \geqslant 1$ and $W = \mathfrak{S}_n$ be the symmetric group, where the generators are given by the basic transpositions $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. (We also set $\mathfrak{S}_0 = \{1\}$.) It is well-known that the irreducible characters of $\mathfrak{S}_n$ are parametrized by the partitions of n; we write this as

$$\operatorname{Irr}(\mathfrak{S}_n) = \{\chi^\alpha \mid \alpha \vdash n\}.$$

This labelling is determined as follows; see, for example, [8, 5.4.7]. Given a partition $\alpha \vdash n$, we denote by α^* denote the transpose partition. Let $\mathfrak{S}_{\alpha^*} \subseteq \mathfrak{S}_n$ be the corresponding Young subgroup; we have $\mathfrak{S}_{\alpha^*} \cong \mathfrak{S}_{\alpha_1^*} \times \mathfrak{S}_{\alpha_2^*} \times \ldots \times \mathfrak{S}_{\alpha_k^*}$, where $\alpha_1^*, \alpha_2^*, \ldots, \alpha_k^*$ are the parts of α^*. Let ε_{α^*} be the sign character of $\mathfrak{S}_{\alpha^*}$. Then

$$\chi^\alpha = \mathbf{j}_{\mathfrak{S}_{\alpha^*}}^{\mathfrak{S}_n}(\varepsilon_{\alpha^*}) \qquad \text{and} \qquad \mathbf{b}_{\chi^\alpha} = n(\alpha) := \sum_{1 \leqslant i \leqslant l}(i-1)\alpha_i$$

where $\alpha = (\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_l \geqslant 0)$.

Example 3.3. Let $n \geqslant 1$ and $\widetilde{W}_n$ be a Coxeter group of type B_n, with generators $\{t, s_1, s_2, \ldots, s_{n-1}\}$ and diagram given as follows.
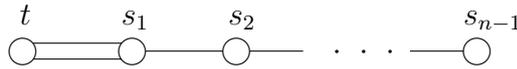
(We also set $\widetilde{W}_0 = \{1\}$.) The irreducible characters of $\widetilde{W}_n$ are parametrised by pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. We write this as

$$\mathrm{Irr}(\widetilde{W}_n) = \{\widetilde{\chi}^{(\alpha,\beta)} \mid (\alpha, \beta) \vdash n\}.$$

For $(\alpha, \beta) \vdash n$, there is a reflection subgroup $\widetilde{W}_{\alpha,\beta} \subseteq \widetilde{W}_n$ of type
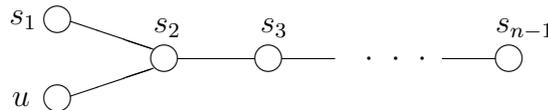
$$D_{\alpha_1} \times D_{\alpha_2} \times \ldots \times D_l \times B_{\beta_1} \times B_{\beta_2} \times \ldots \times B_{\beta_k}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_l$ are the parts of α and $\beta_1, \beta_2, \ldots, \beta_k$ are the parts of β. Let $\widetilde{\varepsilon}_{\alpha,\beta}$ be the sign character on $\widetilde{W}_{\alpha,\beta}$. Then, by [8, 5.5.1, 5.5.3], we have

$$\widetilde{\chi}^{(\alpha,\beta)} = \mathbf{j}_{\widetilde{W}_{\alpha,\beta}}^{\widetilde{W}_n}(\widetilde{\varepsilon}_{\alpha,\beta}) \qquad \text{and} \qquad \mathbf{b}_{\widetilde{\chi}^{(\alpha,\beta)}} = 2n(\alpha) + 2n(\beta) + |\beta|.$$

Note also that $\widetilde{W}_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n$ and there is a corresponding description of $\mathrm{Irr}(\widetilde{W}_n)$ in terms of Clifford theory; see [8, 5.5.6].

Example 3.4. Let $n \geqslant 2$ and W_n be a Coxeter group of type D_n, with generators $u, s_1, \ldots, s_{n-1}$ and diagram given as follows.

Let $w \mapsto w^\diamond$ be defined by $u^\diamond = s_1$, $s_1^\diamond = u$ and $s_i^\diamond = s_i$ for $2 \leqslant i \leqslant n - 1$. Then we can identify the semidirect product $W_n \rtimes \langle \diamond \rangle$ (see Remark 2.4) with a Coxeter group $\widetilde{W}_n$ of type B_n, with generators $t, s_1, \ldots, s_{n-1}$ and diagram as in Example 3.3. We have an embedding $W_n \hookrightarrow \widetilde{W}_n$ given by the map

$$u \mapsto ts_1t, \quad s_1 \mapsto s_1, \quad s_2 \mapsto s_2, \quad \ldots, \quad s_{n-1} \mapsto s_{n-1}.$$

Under this identification, we have $w^\diamond = twt$ for all $w \in W_n$. (Thus, the generator t is the "additional" element denoted by γ in Remark 2.4; by convention, we also set $W_0 = W_1 = \{1\}$, where $\widetilde{W}_0 = \{1\}$ and $\widetilde{W}_1 = \{1, t\}$.) This provides a convenient setting for classifying the irreducible characters of W_n. Given $(\alpha, \beta) \vdash n$, we denote by $\chi^{[\alpha,\beta]}$ the restriction of $\widetilde{\chi}^{(\alpha,\beta)} \in \mathrm{Irr}(\widetilde{W}_n)$ to W_n. Then we have (see [8, 5.6.1, 5.6.2]):

(a) If $\alpha \neq \beta$, then $\chi^{[\alpha,\beta]} = \chi^{[\beta,\alpha]} \in \mathrm{Irr}(W_n)$. We have

$$\mathbf{b}_{\chi^{[\alpha,\beta]}} = 2n(\alpha) + 2n(\beta) + \min\{|\alpha|, |\beta|\}.$$

(b) If $\alpha = \beta$, then $\chi^{[\alpha,\beta]} = \chi^{[\alpha,+]} + \chi^{[\alpha,-]}$ where $\chi^{[\alpha,+]}$, $\chi^{[\alpha,-]}$ are distinct irreducible characters of W_n. We have

$$\mathbf{b}_{\chi^{[\alpha,+]}} = \mathbf{b}_{\chi^{[\alpha,-]}} = 4n(\alpha) + n/2.$$

Furthermore, all irreducible characters of W_n arise in this way. Of course, the second case can only occur if n is even. In this case, the two characters $\chi^{[\alpha,\pm]}$ are explicitly given as follows; see [14, 4.6.2]. Let

$$H_n^+ = \langle s_1, s_2, \ldots, s_{n-1}\rangle \qquad \text{and} \qquad H_n^- = \langle u, s_2, \ldots, s_{n-1}\rangle.$$

Both H_n^+, H_n^- are isomorphic to $\mathfrak{S}_n$. Let $\alpha \vdash n/2$ and $\mathfrak{S}_{2\alpha^*}$ be the corresponding Young subgroup in $\mathfrak{S}_n$ where $2\alpha^*$ denotes the partition of n obtained by multiplying all parts of α by 2. We have corresponding subgroups $H_{2\alpha^*}^+ \subseteq H_n^+$ and $H_{2\alpha^*}^- \subseteq H_n^-$. Then

$$\chi^{[\alpha,+]} = \mathbf{j}_{H_{2\alpha^*}^+}^{W_n}(\varepsilon_{2\alpha^*}^+) \qquad \text{and} \qquad \chi^{[\alpha,-]} = \mathbf{j}_{H_{2\alpha^*}^-}^{W_n}(\varepsilon_{2\alpha^*}^-)$$

where $\varepsilon_{2\alpha^*}^+$ denotes the sign character of $H_{2\alpha^*}^+$ and $\varepsilon_{2\alpha^*}^-$ denotes the sign character of $H_{2\alpha^*}^-$. (This is also discussed in [8, §5.6] but [8, 5.6.3] has to be reformulated as above.)

We take this occasion to correct an error in [8]. (This will actually be essential for the proof of the strengthened "branching rule" in Corollary 3.8.) Let ε be the sign character of W_n. In [8, Rem. 5.6.5], it is stated that $\chi^{[\alpha,+]} \otimes \varepsilon = \chi^{[\alpha^*,+]}$, where α^* denotes the conjugate partition. This can be easily seen to be wrong already in small examples. The correct statement is as follows.

Lemma 3.5.   *Assume that $n \geqslant 2$ is even and let $\alpha \vdash n/2$.*

(a) *If $n/2$ is even, then $\chi^{[\alpha,+]} \otimes \varepsilon = \chi^{[\alpha^*,+]}$ and $\chi^{[\alpha,-]} \otimes \varepsilon = \chi^{[\alpha^*,-]}$.*

(b) *If $n/2$ is odd, then*

$$\chi^{[\alpha,+]} \otimes \varepsilon = \chi^{[\alpha^*,-]} \qquad \text{and} \qquad \chi^{[\alpha,-]} \otimes \varepsilon = \chi^{[\alpha^*,+]}.$$

(c) *Let $\sigma_{n/2} := s_1 s_3 \cdots s_{n-1} \in W_n$. (Note that $\sigma_{n/2}$ is a product of $n/2$ pairwise commuting generators s_i.) Then*

$$\chi^{[\alpha,+]}(\sigma_{n/2}) - \chi^{[\alpha,-]}(\sigma_{n/2}) = (-1)^{n/2}\, 2^{n/2}\, \chi^\alpha(1)$$

where χ^α denotes the irreducible character of $\mathfrak{S}_{n/2}$ labelled by α.

Proof.   In [8, 10.4.6] (see also [24, Thm. 5.1]), we find the definition of a collection of irreducible characters of W_n, which we denote here by
$$\{\psi^{[\alpha,\pm]} \mid \alpha \vdash n/2\}, \text{ such that}$$
$$\chi^{[\alpha,+]} + \chi^{[\alpha,-]} = \psi^{[\alpha,+]} + \psi^{[\alpha,-]} \qquad \text{for all } \alpha \vdash n/2;$$
furthermore, it is shown there that

$$\psi^{[\alpha,+]}(\sigma_{n/2}) - \psi^{[\alpha,-]}(\sigma_{n/2}) = 2^{n/2}\, \chi^\alpha(1).$$

Note that this identity allows us to distinguish $\psi^{[\alpha,+]}, \psi^{[\alpha,-]}$ one from another. Tensoring with ε, we obtain

$$(\psi^{[\alpha,+]} \otimes \varepsilon)(\sigma_{n/2}) - (\psi^{[\alpha,-]} \otimes \varepsilon)(\sigma_{n/2}) = \varepsilon(\sigma_{n/2})\, 2^{n/2}\, \chi^\alpha(1) = (-1)^{n/2}\, 2^{n/2}\, \chi^\alpha(1).$$

Now, by [8, 5.5.6], we have $\widetilde{\chi}^{(\alpha,\alpha)} \otimes \widetilde{\varepsilon} = \widetilde{\chi}^{(\alpha^*,\alpha^*)}$ where $\widetilde{\varepsilon}$ is the sign character of $\widetilde{W}_n$ (an extension of ε). This already implies that

$$\psi^{[\alpha,+]} \otimes \varepsilon = \psi^{[\alpha^*,\pm]} \qquad \text{and} \qquad \psi^{[\alpha,-]} \otimes \varepsilon = \psi^{[\alpha^*,\mp]}.$$

Comparing with the identity

$$\psi^{[\alpha^*,+]}(\sigma_{n/2}) - \psi^{[\alpha^*,-]}(\sigma_{n/2}) = 2^{n/2} \chi^{\alpha^*}(1) = 2^{n/2} \chi^{\alpha^*}(1),$$

we conclude that the desired description of the effect of tensoring with ε holds for the characters $\psi^{[\alpha,\pm]}$. Once this is shown, we can proceed as follows. By the computation in [8, 10.4.10], we have

$$\left\langle \mathrm{Ind}_{H_{2\alpha^*}^+}^{W_n}(1_{2\alpha^*}), \psi^{[\alpha^*,+]} - \psi^{[\alpha^*,-]} \right\rangle_{W_n} = 1,$$

where $1_{2\alpha^*}$ stands for the trivial character of $H_{2\alpha^*}^+$. Consequently, we also have

$$\left\langle \mathrm{Ind}_{H_{2\alpha^*}^+}^{W_n}(\varepsilon_{2\alpha^*}), \psi^{[\alpha^*,+]} \otimes \varepsilon - \psi^{[\alpha^*,-]} \otimes \varepsilon' \right\rangle_{W_n} = 1,$$

where $\varepsilon_{2\alpha^*}$ is the sign character of $H_{2\alpha^*}^+$. Comparing with the definition of $\chi^{[\alpha,\pm]}$ in Example 3.4, we conclude that we must have

$$\chi^{[\alpha,+]} = \psi^{[\alpha^*,+]} \otimes \varepsilon \qquad \text{and} \qquad \chi^{[\alpha,-]} = \psi^{[\alpha^*,-]} \otimes \varepsilon$$

for all $\alpha \vdash n/2$. This yields (a), (b), (c). ∎

Remark 3.6. Assume that $n \geqslant 2$ is even and let $\sigma_{n/2} = s_1 s_3 \cdots s_{n-1} \in W_n$ as above. Let C_0 be the conjugacy class of $\sigma_{n/2}$ in W_n. By [8, Prop. 3.4.12], we have $tC_0t \neq C_0$ and $\{C_0, tC_0t\}$ is the only pair of conjugacy classes of involutions with this property. Furthermore, a direct computation shows (see also the formula in [24, 4.3]):

$$|C_{\mathfrak{S}_n}(\sigma_{n/2})| = 2^{n/2}(n/2)! \qquad \text{and} \qquad |C_{W_n}(\sigma_{n/2})| = 2^n(n/2)!.$$

We can now state the following strengthening of the induction formula in [8, 6.4.9].

Proposition 3.7. *Assume that $n \geqslant 2$ is even. Let $r \in \{2, 4, \ldots, n\}$ and consider the parabolic subgroup $W' = W_{n-r} \times H_r$ where $W_{n-r} = \langle u, s_1, \ldots, s_{n-r-1} \rangle$ (type D_{n-r}) and $H_r = \langle s_{n-r+1}, \ldots, s_{n-1} \rangle \cong \mathfrak{S}_r$. Let $\alpha' \vdash (n-r)/2$ and denote by ε_r the sign character on the factor H_r. Then*

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',+]} \boxtimes \varepsilon_r\right) = \sum_\alpha \chi^{[\alpha,+]} \quad + \quad \text{``further terms''},$$

where the sum runs over all partitions $\alpha \vdash n/2$ whose Young diagram is obtained from that of α' by adding $r/2$ boxes, with no two boxes in the same row; the expression "further terms" stands for a sum of various $\chi \in \mathrm{Irr}(W_n)$ which can be extended to $\widetilde{W}_n$.

Proof.   By [8, Prop. 6.4.9] (and its proof), we already know that

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',+]}\boxtimes\varepsilon_r\right)=\sum_\alpha\chi^{[\alpha,\mu_\alpha]}\quad+\quad\text{``further terms''},$$

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',-]}\boxtimes\varepsilon_r\right)=\sum_\alpha\chi^{[\alpha,-\mu_\alpha]}\quad+\quad\text{``further terms''},$$

where the sums run over all partitions $\alpha\vdash n/2$ as above and where $\mu_\alpha\in\{\pm1\}$. So it remains to determine the signs μ_α. First note that the two "further terms" must be equal since the above induced characters are conjugate to each other under t. Hence, we have

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',+]}\otimes\varepsilon_r\right)-\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',-]}\otimes\varepsilon_r\right)=\sum_\alpha\left(\chi^{[\alpha,\mu_\alpha]}-\chi^{[\alpha,-\mu_\alpha]}\right)$$

where the sum runs over all partitions $\alpha\vdash n/2$ as above. To determine the signs, we evaluate both sides of this identity on the special element $\sigma_{n/2}$. Let C_0 denote the conjugacy class of $\sigma_{n/2}$. We have $\sigma_{n/2}\in W'$ and so $C_0\cap W'\neq\varnothing$; furthermore, $C_0\cap W'$ can only contain elements $w\in W'$ such that w,twt are not conjugate (see Remark 3.6). Consequently, $C_0\cap W'$ is just the conjugacy class of W' containing $\sigma_{n/2}$. Now note that

$$\sigma_{n/2}=\sigma_{(n-r)/2}\times s_{n-r+1}s_{n-r+3}\cdots s_{n-1}\in W'=W_{n-2}\times H_r.$$

This yields

$$\left(\chi^{[\alpha',\pm]}\boxtimes\varepsilon_r\right)(\sigma_{n/2})=\chi^{[\alpha',\pm]}(\sigma_{(n-r)/2})\varepsilon_r(s_{n-r+1}s_{n-r+3}\cdots s_{n-1})$$
$$=(-1)^{r/2}\chi^{[\alpha',\pm]}(\sigma_{(n-r)/2})$$

and so

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',\pm]}\boxtimes\varepsilon_r\right)(\sigma_{n/2})=(-1)^{r/2}\frac{|C_{W_n}(\sigma_{n/2})|}{|C_{W'}(\sigma_{n/2})|}\chi^{[\alpha',\pm]}(\sigma_{(n-r)/2}).$$

Furthermore, by the formulae in Remark 3.6, we have

$$\frac{|C_{W_n}(\sigma_{n/2})|}{|C_{W'}(\sigma_{(n-2)/2})|}=[\mathfrak{S}_{n/2}:\mathfrak{S}_{(n-r)/2}\times\mathfrak{S}_{r/2}]\,2^{r/2}.$$

Thus, using also Lemma 3.5 (applied to W_{n-r}), we conclude that

$$\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',+]}\otimes\varepsilon_r\right)(\sigma_{n/2})-\mathrm{Ind}_{W'}^{W_n}\left(\chi^{[\alpha',-]}\otimes\varepsilon_r\right)(\sigma_{n/2})$$
$$=(-1)^{n/2}\,2^{n/2}\,[\mathfrak{S}_{n/2}:\mathfrak{S}_{(n-r)/2}\times\mathfrak{S}_{r/2}]\,\chi^{\alpha'}(1).$$

On the other hand, Lemma 3.5 applied to W_n yields that

$$\sum_\alpha\left(\chi^{[\alpha,\mu_\alpha]}(\sigma_{n/2})-\chi^{[\alpha,-\mu_\alpha]}(\sigma_{n/2})\right)=(-1)^{n/2}\,2^{n/2}\sum_\alpha\mu_\alpha\,\chi^\alpha(1),$$

where the sum runs over all partitions $\alpha \vdash n/2$ whose Young diagram is obtained from that of α' by adding $r/2$ boxes, with no two boxes in the same row. Now, by "Pieri's Rule" for the characters of $\mathfrak{S}_{n/2}$ (see [8, 6.1.7]), we have

$$[\mathfrak{S}_{n/2} : \mathfrak{S}_{(n-r)/2} \times \mathfrak{S}_{r/2}] \, \chi^{\alpha'}(1) = \sum_{\alpha} \chi^{\alpha}(1)$$

where the sum runs over all α as above. Hence, we conclude that

$$(-1)^{n/2} \Big(\operatorname{Ind}_{W'}^{W_n} \big(\chi^{[\alpha',+]} \big)(\sigma_{n/2}) - \operatorname{Ind}_{W'}^{W_n} \big(\chi^{[\alpha',-]} \big)(\sigma_{n/2}) \Big) = 2^{n/2} \Big(\sum_{\alpha} \chi^{\alpha}(1) \Big)$$

$$\geqslant 2^{n/2} \Big(\sum_{\alpha} \mu_{\alpha} \chi^{\alpha}(1) \Big) = (-1)^{n/2} \Big(\sum_{\alpha} \big(\chi^{\alpha,\mu_{\alpha}}(\sigma_{n/2}) - \chi^{[\alpha,-\mu_{\alpha}]}(\sigma_{n/2}) \big) \Big).$$

Since the left hand side equals the right hand side, the inequality must be an equality which means that $\mu_{\alpha} = 1$ for all $\alpha \vdash n/2$, as desired. ∎

Taking the special case $r = 2$, we obtain the following strengthened version of the "branching rule" for type D_n.

Corollary 3.8. *Assume that $n \geqslant 2$ is even. Consider the parabolic subgroup $W' = W_{n-2} \times H_2$ where $W_{n-2} = \langle u, s_1, \ldots, s_{n-3} \rangle$ (type D_{n-2}) and $H_2 = \langle s_{n-1} \rangle$. Let $\alpha' \vdash (n-2)/2$ and denote by ε_1 the sign character on the factor H_2. Then*

$$\operatorname{Ind}_{W'}^{W_n} \big(\chi^{[\alpha',+]} \boxtimes \varepsilon_1 \big) = \sum_{\alpha} \chi^{[\alpha,+]} \quad + \quad \text{"further terms"},$$

where the sum runs over all partitions $\alpha \vdash n/2$ such that α is obtained by increasing one part of α' by 1; the expression "further terms" stands for a sum of various $\chi \in \operatorname{Irr}(W_n)$ which can be extended to W_n. In particular,

$$\Big\langle \operatorname{Ind}_{W'}^{W_n} \big(\chi^{[\alpha',+]} \boxtimes \varepsilon_1 \big), \chi^{[\alpha,-]} \Big\rangle_{W_n} = 0 \quad \text{for all } \alpha \vdash n/2.$$

Finally, we are able to describe the decomposition of Kottwitz's characters Υ_w^1 and $\Upsilon_w^\diamond$ for W_n into irreducible characters.

3.9. Assume that $n \geqslant 2$ is even. Consider the element $\sigma_{n/2} \in W_n$ in Remark 3.6. Let $\Upsilon_{\sigma_{n/2}}^1$ be the character of the split version of Kottwitz's involution module for W_n; see Remark 2.3. By [12, §3.3], we have

$$\Upsilon_{\sigma_{n/2}}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha,\nu_{\alpha}]} \quad \text{and} \quad \Upsilon_{t\sigma_{n/2}t}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha,-\nu_{\alpha}]}$$

where $\nu_{\alpha} \in \{\pm 1\}$ for all $\alpha \vdash n/2$. But note that these signs have not been determined in [12]. Using an inductive argument based on Corollary 3.8, it is shown in [1, Prop. 8.4] that $\nu_{\alpha} = 1$ for all $\alpha \vdash n/2$. Thus, we have

$$\Upsilon_{\sigma_{n/2}}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha,+]} \quad \text{and} \quad \Upsilon_{t\sigma_{n/2}t}^1 = \sum_{\alpha \vdash n/2} \chi^{[\alpha,-]}.$$

3.10.   A complete set of representatives of the conjugacy classes of involutions in $\widetilde{W}_n$ is given as follows. Let l, j be non-negative integers such that $l + 2j \leqslant n$. Then set

$$\sigma_{l,j} := t_1 \cdots t_l s_{l+1} s_{l+3} \cdots s_{l+2j-1} \in \widetilde{W}_n,$$

where $t_1 := t$ and $t_i := s_{i-1} t_{i-1} s_{i-1}$ for $2 \leqslant i \leqslant n$. Note that $\sigma_{l,j}$ is the longest element in a parabolic subgroup of $\widetilde{W}_n$ of type $B_l \times A_1 \times \ldots \times A_1$, where the A_1 factor is repeated j times. In particular, $\sigma_{l,j}$ is central in this parabolic subgroup and $\sigma_{l,j}$ has minimal length in its conjugacy class; see also [8, 3.2.10]. Every involution in $\widetilde{W}_n$ is conjugate to exactly one of the elements $\sigma_{l,j}$. Note that $\sigma_{l,j} \in W_n$ if and only if l is even. Furthermore, if n is odd, then every involution in W_n is conjugate (in W_n) to exactly one of the elements $\sigma_{l,j}$ where l is even. Assume now that n is even. Then

$$\sigma_{0,n/2} = s_1 s_3 \cdots s_{n-1} \in W_n$$

is the element already introduced in Remark 3.6. Let C_0 be the conjugacy class of $\sigma_{0,n/2}$ in W_n. Recall from Remark 3.6 that $C_0^\diamond \neq C_0$ and that $\{C_0, C_0^\diamond\}$ is the only pair of conjugacy classes of involutions in W_n with this property.

3.11.   There is an alternative labelling of $\mathrm{Irr}(\widetilde{W}_n)$ in terms of Lusztig's "symbols". To describe this in more detail, let $(\alpha, \beta) \vdash n$ and consider the corresponding irreducible character $\widetilde{\chi} = \widetilde{\chi}^{(\alpha,\beta)} \in \mathrm{Irr}(\widetilde{W}_n)$; see Example 3.3. Choose $m \geqslant 1$ such that we can write

$$\alpha = (0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_m) \qquad \text{and} \qquad \beta = (0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \ldots \leqslant \beta_m).$$

As in [13, §1], [14, §4.18], we have a corresponding "symbol" with two rows of equal length

$$\Lambda_m(\widetilde{\chi}) := \begin{pmatrix} \lambda_1, \lambda_2, \ldots, \lambda_m \\ \mu_1, \mu_2, \ldots, \mu_m \end{pmatrix}$$

where $\lambda_i := \alpha + i - 1$ and $\mu_i = \beta_i + i - 1$ for $1 \leqslant i \leqslant m$. We associate with $\widetilde{\chi}$ the following invariants. First, we set

$$c(\widetilde{\chi}) = c(\alpha, \beta) := \text{ number of } i \in \{1, \ldots, m\} \text{ such that } \mu_i \notin \{\lambda_1, \lambda_2, \ldots, \lambda_m\},$$

$$d_0(\widetilde{\chi}) = d_0(\alpha, \beta) := \beta_1 + \sum_{2 \leqslant i \leqslant m} \sup\{\alpha_{i-1}, \beta_i\}.$$

In particular, if $\alpha = \beta$, then $c(\alpha, \alpha) = 0$ and $d_0(\alpha, \alpha) = n/2$. Next, we set

$$\mathbf{a}_{\widetilde{\chi}}^\diamond = \mathbf{a}_{(\alpha,\beta)}^\diamond := \sum_{1 \leqslant i \leqslant j \leqslant m} \min\{\lambda_i, \lambda_j\} + \sum_{1 \leqslant i \leqslant j \leqslant m} \min\{\mu_i, \mu_j\}$$

$$+ \sum_{1 \leqslant i,j \leqslant m} \min\{\lambda_i, \mu_j\} - \frac{1}{6}m(m-1)(4m-5);$$

see [14, 4.6.3], [18, 22.14]. (Note that these definitions do not depend on the choice of m.)

(a) We say that $\widetilde{\chi}$ is "$\diamond$-special" if $\mu_1 \leqslant \lambda_1 \leqslant \mu_2 \leqslant \lambda_2 \leqslant \ldots \leqslant \mu_m \leqslant \lambda_m$.

In particular, if $\widetilde{\chi}$ is $\diamond$-special then either $\alpha = \beta$ or $|\beta| < |\alpha|$. Assume now that $\alpha \neq \beta$. Then $\chi = \chi^{[\alpha,\beta]} \in \mathrm{Irr}(W_n)$ and the two extensions of χ to $\widetilde{W}_n$ are $\widetilde{\chi}^{(\alpha,\beta)}$ and $\widetilde{\chi}^{(\beta,\alpha)}$. Following Lusztig [16, 17.2], we say that $\chi^{(\alpha,\beta)}$ is the "*preferred extension*" of χ if the symbol $\Lambda_m(\widetilde{\chi})$ has the following property: the smallest entry which appears in only one row appears in the lower row. Using also [14, 4.6.4], one easily verifies that:

(b) $\widetilde{\chi}$ is $\diamond$-special if and only if $\mathbf{a}_{\widetilde{\chi}}^{\diamond} = \mathbf{b}_\chi$ and $\widetilde{\chi}$ is the preferred extension of χ.

This property played a role in the proof of the main result of [7] for type D_n, and it will also play a role in the proof of Proposition 4.9 below.

Proposition 3.12 (Kottwitz [12, §3.3 and §5.4]). *Let l, j be non-negative integers such that $l + 2j \leqslant n$; consider the coresponding involution $\sigma_{l,j} \in \widetilde{W}_n$. Thus, if l is even, then $\sigma_{l,j} \in W_n$ in an "ordinary" involution; on the other hand, if l is odd, then $t\sigma_{l,j} \in W_n$ is a $\diamond$-twisted involution. Let*

$$\Upsilon'_{l,j} := \begin{cases} \Upsilon^1_{\sigma_{l,j}} & \text{if } l \text{ is even,} \\ \Upsilon^{\diamond}_{t\sigma_{l,j}} & \text{if } l \text{ is odd,} \end{cases}$$

Then the following hold, where (α, β) is any pair of partitions such that $|\alpha| + |\beta| = n$ and, as before, $\chi^{[\alpha,\beta]}$ denotes the restriction of $\widetilde{\chi}^{(\alpha,\beta)} \in \mathrm{Irr}(\widetilde{W}_n)$ to W_n.

(a) *We have $\langle \Upsilon'_{l,j}, \chi^{[\alpha,\beta]} \rangle_{W_n} = 0$ unless $\widetilde{\chi}^{(\alpha,\beta)}$ is $\diamond$-special and $|\beta| = j$.*

(b) *If $\widetilde{\chi}^{(\alpha,\beta)}$ is $\diamond$-special and $|\beta| = j$, then*

$$\langle \Upsilon'_{l,j}, \chi^{[\alpha,\beta]} \rangle_{W_n} = \binom{c(\alpha, \beta)}{j + l - d_0(\alpha, \beta)} \qquad \textit{(binomial coefficient)};$$

in particular, the multiplicity is zero unless $d_0(\alpha, \beta) \leqslant j + l \leqslant d_0(\alpha, \beta) + c(\alpha, \beta)$.

(c) *If $j < n/2$, then $\langle \Upsilon'_{l,j}, \chi^{[\alpha,\beta]} \rangle_{W_n} = 0$ unless $\alpha \neq \beta$; if $2j = n$, then $\langle \Upsilon'_{l,j}, \chi^{[\alpha,\beta]} \rangle_{W_n} = 0$ unless $\alpha = \beta$.*

Proof. First assume that $j < n/2$; then $\sigma_{l,j}$ is not the special element $\sigma_{n/2}$ in Remark 3.6. The desired multiplicities in (a) and (b) are explicitly determined in [12, §3.3 and §5.4]. A similar argument applies to the case $j = n/2$ and $l = 0$. Using Example 3.4, we obtain

$$\langle \Upsilon'_{l,j}, \chi^{[\alpha,\alpha]} \rangle_{W_n} = \langle \Upsilon^1_{\sigma_{n/2}}, \chi^{[\alpha,+]} + \chi^{[\alpha,-]} \rangle_{W_n} = 1$$

in this case, as already mentioned in (3.9). ∎

In order to prove Kottwitz's conjecture, our task now is to find similar formulae for the numbers of elements in the intersections of left cells with ordinary or $\diamond$-conjugacy classes of involutions in W_n.

Remark 3.13. Let $\mathfrak{C}$ be a two-sided cell of W_n such that $\mathfrak{C}^\diamond \neq \mathfrak{C}$. By [14, 4.6.10, 5.25], this can only happen if n is even; in this case, we have

$$\mathrm{Irr}(W_n \mid \mathfrak{C}) = \{\chi\} \qquad \text{where} \qquad \chi = \chi^{[\alpha,\pm]} \text{ for some } \alpha \vdash n/2.$$

Note that this implies that $[\Gamma]_1 = \chi$ for every left cell $\Gamma \subseteq \mathfrak{C}$. (More precisely, the above statement on $\mathrm{Irr}(W_n \mid \mathfrak{C})$ implies that $[\Gamma]_1$ is a multiple of χ; but then [14, 12.17] shows that $[\Gamma]_1$ is multiplicity-free.) Now let C be a $\diamond$-conjugacy class of $\diamond$-twisted involutions in W_n. In Remark 2.9, we have seen that $C \cap \Gamma = \varnothing$. We can now also show that

$$\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_{W_n} = 0 \qquad \text{for } w \in C.$$

Indeed, we are in the case where $l \neq 1$ is odd in Proposition 3.12. Then the formulae show that all irreducible constituents of $\Upsilon_w^\diamond$ are of the form $\chi^{[\alpha,\beta]}$ where $\alpha \neq \beta$. Hence, we must have $\langle \Upsilon_w^\diamond, [\Gamma]_1 \rangle_{W_n} = 0$ since $[\Gamma]_1 = \chi^{[\alpha,\pm]}$ for some $\alpha \vdash n/2$.

4. The extended Iwahori–Hecke algebra

The aim of this section is to establish certain positivity results for leading coefficients of character values of Iwahori–Hecke algebras in the quasi-split case. The analogous results in the split case were shown by Lusztig [14, 7.1], [17, 3.14]. The arguments are similar in the quasi-split case but some additional work is required in choosing the correct extensions of the characters in $\mathrm{Irr}^\diamond(W)$. Then Proposition 4.11 formulates the main applications to type D_n. We shall need a number of results from Lusztig's book [14] and [17] so, as in Remark 2.15, we assume that W is a Weyl group and that $\diamond$ is "ordinary" in the sense of [14, 3.1]. Let $\widetilde{W} = \langle W, \gamma \rangle$ be the semidirect product as in Remark 2.4.

4.1. We begin by recalling some results concerning the representation theory of the (split semisimple) algebra $\mathbf{H}_K = K \otimes_A \mathbf{H}$, where $K = \mathbb{Q}(v)$ is the field of fractions of A. (See (2.7) for the definition of $\mathbf{H}$.) Via the specialisation $v \mapsto 1$, we obtain a canonical bijection

$$\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathbf{H}_K), \qquad \chi \leftrightarrow \chi_v;$$

see [14, 3.3]. We have $\chi_v(T_w) \in A$ for all $w \in W$. For $\chi \in \mathrm{Irr}(W)$, we define

$$\mathbf{a}_\chi := \min\{i \geqslant 0 \mid v^i \chi_v(T_w) \in \mathbb{Z}[v] \text{ for all } w \in W\}.$$

Then there are well-defined integers $c_{w,\chi} \in \mathbb{Z}$ such that

$$v^{\mathbf{a}_\chi} \chi_v(T_w) \equiv (-1)^{\ell(w)} c_{w,\chi} \mod v\mathbb{Z}[v] \qquad \text{for all } w \in W.$$

These are Lusztig's "leading coefficients of character values"; see [14, Chap. 5], [17]. Note that the sum of all terms $c_{w,\chi}^2$ ($w \in W$) is a strictly positive number. Consequently, there is a well-defined positive rational number f_χ such that

$$\sum_{w \in W} c_{w,\chi}^2 = \chi(1) f_\chi.$$

In fact, it turns out that $f_\chi > 0$ is an integer; see [14, 4.1]. We have the following relation; see [14, Cor. 5.8]:

$$\sum_{w \in \Gamma} c_{w,\chi}^2 = f_\chi \langle [\Gamma]_1, \chi \rangle_W \qquad \text{for any left cell } \Gamma \text{ of } W.$$

In particular, if $\mathfrak{C}$ is a two-sided cell of W and $c_{w,\chi} \neq 0$ for some $w \in \mathfrak{C}$, then $\chi \in \mathrm{Irr}(W \mid \mathfrak{C})$. We will now have to consider "quasi-split" versions of these constructions.

4.2. As in [14, 3.3], we can define an extended algebra $\widetilde{\mathbf{H}}$ with an A-basis $\{T_\sigma \mid \sigma \in \widetilde{W}\}$. For this purpose, we define a function $L \colon \widetilde{W}_n \to \mathbb{Z}$ by $L(\gamma^i w) = \ell(w)$ for any $w \in W_n$ and $i = 0, 1$; in particular, L is an extension of the length function ℓ on W. Note that $L(\gamma) = 0$ and $L(\gamma w) = L(w^{-1}\gamma)$ for all $w \in W$. Then the multiplication in $\widetilde{\mathbf{H}}$ is given as follows:

$$T_\sigma T'_{\sigma'} = T_{\sigma\sigma'} \quad \text{if } \sigma, \sigma' \in \widetilde{W} \text{ are such that } L(\sigma\sigma') = L(\sigma) + L(\sigma'),$$
$$T_s^2 = T_1 + (v - v^{-1})T_s \quad \text{if } s \in S.$$

Thus, $\mathbf{H}$ can be identified with the A-submodule of $\widetilde{\mathbf{H}}$ spanned by all T_w ($w \in W$); note also that $\widetilde{\mathbf{H}} = \mathbf{H}$ if $\diamond = 1$. Let $\{\mathbf{C}_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of $\mathbf{H}$ as defined in (2.7). Following [17, 3.1(a)], we extend this to a basis of $\widetilde{\mathbf{H}}$ by setting

$$\mathbf{C}_{\gamma w} = T_{\gamma w} + \sum_{y \in W, y < w} (-1)^{\ell(w)-\ell(y)} v^{\ell(w)-\ell(y)} P_{y,w} T_{\gamma y} \qquad (w \in W).$$

Correspondingly, we also have notions of left, right and two-sided cells in $\widetilde{W}$; in order to avoid any confusion with the analogous notions for W itself, we shall call them the left, right and two-sided L-cells in $\widetilde{W}$. One easily sees that we have the following relations between the cells in W and the L-cells in $\widetilde{W}$.

(a) If Γ is a left cell in W, then $\Gamma^+ := \Gamma \cup \gamma\Gamma$ is a left L-cell of $\widetilde{W}$. All left L-cells of $\widetilde{W}$ arise in this way.

(b) If Γ, Γ^+ are as in (a), then the corresponding characters of $W, \widetilde{W}$ are related by

$$[\Gamma^+]_1 = \mathrm{Ind}_W^{\widetilde{W}}([\Gamma]_1).$$

(c) If $\mathfrak{C}$ is a two-sided cell of W, then $\mathfrak{C}^+ := \mathfrak{C} \cup \gamma\mathfrak{C} \cup \mathfrak{C}\gamma \cup \gamma\mathfrak{C}\gamma$ is a two-sided L-cell of $\widetilde{W}$. All two-sided L-cells of $\widetilde{W}$ arise in this way.

(See [15, §16], [17, 3.1]; a related setting is considered in [6, 2.4.9].)

4.3. We extend the constructions in (4.1) to $\widetilde{\mathbf{H}}$. Let $K = \mathbb{Q}(v)$ be the field of fractions of A. Then $\widetilde{\mathbf{H}}_K := K \otimes_A \widetilde{\mathbf{H}}$ is a split semisimple algebra. Via the specialisation $v \mapsto 1$, we obtain a canonical bijection

$$\mathrm{Irr}(\widetilde{W}) \leftrightarrow \mathrm{Irr}(\widetilde{\mathbf{H}}_K), \qquad \widetilde{\chi} \leftrightarrow \widetilde{\chi}_v$$

which is compatible with the restriction of characters from $\widetilde{W}$ to W on the one side, and with the restriction of characters from $\widetilde{\mathbf{H}}$ to $\mathbf{H}$ on the other side; see [14, 3.3]. We have $\widetilde{\chi}_v(T_\sigma) \in A$ for all $\sigma \in \widetilde{W}$. For $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$, we define

$$\mathbf{a}_{\widetilde{\chi}} := \min\{i \geqslant 0 \mid v^i \chi_v(T_\sigma) \in \mathbb{Z}[v] \text{ for all } \sigma \in \widetilde{W}\}.$$

Then there are well-defined integers $c_{\sigma,\widetilde{\chi}} \in \mathbb{Z}$ such that

$$v^{\mathbf{a}_{\widetilde{\chi}}} \widetilde{\chi}_v(T_\sigma) \equiv (-1)^{L(\sigma)} c_{\sigma,\widetilde{\chi}} \bmod v\mathbb{Z}[v] \qquad \text{for all } \sigma \in \widetilde{W}.$$

Again the sum of all terms $c_{\sigma,\widetilde{\chi}}^2$ $(\sigma \in \widetilde{W})$ is a strictly positive number. Consequently, there is a well-defined positive rational number $f_{\widetilde{\chi}}$ such that

$$\sum_{\sigma \in \widetilde{W}} c_{\sigma,\widetilde{\chi}}^2 = \widetilde{\chi}(1)\, f_{\widetilde{\chi}}.$$

The relation between $\mathbf{a}_{\widetilde{\chi}}$, $f_{\widetilde{\chi}}$ and the analogous invariants for the ireducible characters of W are given as follows; see [6, Prop. 2.4.14]. Let $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$ and $\chi \in \mathrm{Irr}(W)$ be such that χ occurs in the restriction of $\widetilde{\chi}$ to W. Then

$$\mathbf{a}_{\widetilde{\chi}} = \mathbf{a}_\chi \qquad \text{and} \qquad \widetilde{\chi}(1)\, f_{\widetilde{\chi}} = 2\,\chi(1)\, f_\chi. \tag{a}$$

Assume now that $\widetilde{\chi}$ is an extension of some $\chi = \chi^\diamond \in \mathrm{Irr}(W)$. Then we have the following relation with the left cells of W; see [14, Cor. 5.8]:

$$\sum_{w \in \Gamma} c_{\gamma w,\widetilde{\chi}}^2 = f_\chi \, \langle [\Gamma]_1, \chi \rangle_W \qquad \text{for any left cell } \Gamma \text{ of } W. \tag{b}$$

In particular, if $\mathfrak{C}$ is a two-sided cell of W and $c_{\gamma w,\widetilde{\chi}} \neq 0$ for some $w \in W$, then $\chi \in \mathrm{Irr}(W \mid \mathfrak{C})$.

Remark 4.4. Let $\chi \in \mathrm{Irr}(W)$ and assume that $\chi^\diamond \neq \chi$. Then $\widetilde{\chi} := \mathrm{Ind}_W^{\widetilde{W}}(\chi) \in \mathrm{Irr}(\widetilde{W})$. Correspondingly, if V is an $\mathbf{H}_K$-module affording χ_v, then

$$\widetilde{V} = \widetilde{\mathbf{H}}_K \otimes_{\mathbf{H}_K} V$$

is a $\widetilde{\mathbf{H}}_K$-module affording $\widetilde{\chi}_v$; note that $\widetilde{\mathbf{H}}_K$ is free as an $\mathbf{H}_K$-module, with basis $\{T_1, T_\gamma\}$. One easily sees that this implies that

$$\widetilde{\chi}_v(T_w) = \chi_v(T_w) + \chi_v^\diamond(T_w) \quad \text{and} \quad \widetilde{\chi}_v(T_{\gamma w}) = 0 \quad \text{for all } w \in W.$$

Hence, we obtain that

$$c_{w,\widetilde{\chi}} = c_{w,\chi} + c_{w,\chi^\diamond} \quad \text{and} \quad c_{\gamma w,\widetilde{\chi}} = 0 \quad \text{for all } w \in W.$$

4.5. Recall from [14, 4.1] that $\chi \in \mathrm{Irr}(W)$ is called "*special*" if $\mathbf{a}_\chi = \mathbf{b}_\chi$. By [14, 4.14.2, 5.25], the sets $\mathrm{Irr}(W \mid \mathfrak{C})$ (where $\mathfrak{C} \subseteq W$ is a two-sided cell) are explicitly known; in particular, it it is known that each set $\mathrm{Irr}(W \mid \mathfrak{C})$ contains a unique special character.

Now let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}$ and define

$$\mathrm{Irr}(\widetilde{W} \mid \widetilde{\mathfrak{C}}) := \big\{ \widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}) \mid \langle \mathrm{Ind}_W^{\widetilde{W}}(\chi), \widetilde{\chi} \rangle_{\widetilde{W}} \neq 0 \text{ for some } \chi \in \mathrm{Irr}(W \mid \mathfrak{C}) \big\}$$

where $\mathfrak{C}$ is a two-sided cell of W such that $\widetilde{\mathfrak{C}} = \mathfrak{C}^+$. (One easily sees that this does not depend on the choice of $\mathfrak{C}$; note that there only is a choice if $\mathfrak{C} \neq \mathfrak{C}^\diamond$.) Using (4.2), we see that we obtain a partition

$$\mathrm{Irr}(\widetilde{W}) = \coprod_{\widetilde{\mathfrak{C}}} \mathrm{Irr}(\widetilde{W} \mid \widetilde{\mathfrak{C}})$$

where $\widetilde{\mathfrak{C}}$ runs over all two-sided L-cells in $\widetilde{W}$. We now define what it means for an irreducible character $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$ to be "$\diamond$-*special*".

(a) Assume that the restriction of $\widetilde{\chi}$ to W is not irreducible. Then we say that $\widetilde{\chi}$ is $\diamond$-special if $\widetilde{\chi} = \mathrm{Ind}_W^{\widetilde{W}}(\chi)$ for some special $\chi \in \mathrm{Irr}(W)$. (Note that χ is special if and only if $\chi^\diamond$ is special.)

(b) Otherwise, $\widetilde{\chi}$ is the extension of some $\chi \in \mathrm{Irr}(W)$. In this case, we say that $\widetilde{\chi}$ is $\diamond$-special if χ is special and $\widetilde{\chi}$ is the "*preferred extension*" of χ in the sense of Lusztig [16, 17.2].

With the above definitions, each set $\mathrm{Irr}(\widetilde{W} \mid \widetilde{\mathfrak{C}})$ will also contain a unique $\diamond$-special character of $\widetilde{W}$. Note also that, if W is of type D_n and $\diamond$ is as in Example 3.4, then these definitions are consistent with those in (3.11).

Example 4.6. Let $W = W_n$ be of type D_n. Let $\mathfrak{C}$ be a two-sided cell of W_n and $\chi_0 \in \mathrm{Irr}(W_n \mid \mathfrak{C})$ be the unique special character. Let Γ be a left cell contained in $\mathfrak{C}$. Then the following hold:

(a) $[\Gamma]_1$ is multiplicity-free with exactly f_{χ_0} irreducible constituents (one of which is χ_0); furthermore, Γ contains exactly f_{χ_0} involutions of W_n. (See [14, 12.17].)

Now let $\diamond$ be the non-trivial graph automorphism as in Example 3.4. We identify $\widetilde{W}$ with a group $\widetilde{W}_n$ of type B_n. First we note:

(b) $L \colon \widetilde{W}_n \to \mathbb{Z}$ is a weight function in the sense of Lusztig [18]. Thus, $\widetilde{\mathbf{H}}$ is the generic Iwahori–Hecke algebra associated with $\widetilde{W}_n, L$ as in [18]; furthermore, the notions of left, right and two-sided L-cells of $\widetilde{W}_n$ in (4.2) correspond exactly to the analogous notions in [18].

(c) We have $\mathbf{a}_{\widetilde{\chi}} = \mathbf{a}_{\widetilde{\chi}}^\diamond$ for all $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}_n)$. (See [18, 22.14].)

Let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}_n$ and $\widetilde{\chi}_0 \in \mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}})$ be the unique $\diamond$-special character. Let $\widetilde{\Gamma}$ be a left L-cell contained in $\widetilde{\mathfrak{C}}$. Then we have, where $c(\widetilde{\chi}_0)$ is defined in (3.11):

(d) $[\widetilde{\Gamma}]_1$ is multiplicity-free with exactly $f_{\widetilde{\chi}_0} = 2^{c(\widetilde{\chi}_0)}$ irreducible constituents (one of which is $\widetilde{\chi}_0$); furthermore, $\widetilde{\Gamma}$ contains exactly $f_{\widetilde{\chi}_0}$ involutions of $\widetilde{W}_n$.

Indeed, there are two cases. Assume first that $\widetilde{\chi}_0$ is an extension of a special character $\chi_0 \in \mathrm{Irr}(W_n)$. Let $\mathfrak{C}$ be the two-sided cell of W_n such that $\chi_0 \in \mathrm{Irr}(W_n \mid \mathfrak{C})$. By (2.15)(c), we have $\mathrm{Irr}(W_n \mid \mathfrak{C}) \subseteq \mathrm{Irr}^{\diamond}(W_n)$. So (a) and (4.2)(b) imply that $[\widetilde{\Gamma}]_1$ is multiplicity-free with exactly $2f_{\chi_0}$ irreducible constituents (one of which is $\widetilde{\chi}_0$). By (4.3)(a), we have $f_{\widetilde{\chi}_0} = 2f_{\chi_0}$. Now assume that $\widetilde{\chi}_0$ is obtained by inducing a special $\chi_0 \in \mathrm{Irr}(W_n)$ to $\widetilde{W}_n$. Then n is even and $\chi_0 = \chi^{[\alpha,\pm]}$ for some $\alpha \vdash n/2$. Again, let $\mathfrak{C}$ be the left cell of W_n such that $\chi_0 \in \mathrm{Irr}(W_n \mid \mathfrak{C})$. Then $[\Gamma]_1 = \chi_0$ for any left cell $\Gamma \subseteq \mathfrak{C}$; see Remark 3.13. So (4.2)(b) implies that $[\widetilde{\Gamma}]_1 = \widetilde{\chi}_0 = \widetilde{\chi}^{(\alpha,\alpha)}$ is irreducible. By (4.3)(b), we have $f_{\widetilde{\chi}_0} = f_{\chi_0}$. In both cases, the equality $f_{\widetilde{\chi}_0} = 2^{c(\widetilde{\chi}_0)}$ follows from the explicit formula in [18, 22.14]; note that $c(\widetilde{\chi}_0) = 0$ in the second case. This completes the proof of the statement concerning the decomposition of $[\widetilde{\Gamma}]_1$. To prove the statement concerning the involutions in $\widetilde{\Gamma}$, we argue as follows. By (b), we can apply [3, Theorem 1.1] to $\widetilde{\mathbf{H}}$ which shows that the number of (ordinary) involutions in $\widetilde{\Gamma}$ equals the number of irreducible constituents of $[\widetilde{\Gamma}]_1$ (counting multiplicities). Since $[\widetilde{\Gamma}]_1$ is multiplicity-free, this yields the desired statement.

4.7. Assume that $(W, S, \diamond)$ arises from a connected reductive algebraic group $\mathbf{G}$ and a Frobenius map F, corresponding to some $\mathbb{F}_q$-rational structure on $\mathbf{G}$. Thus, W is the Weyl group of $\mathbf{G}$ with respect to an F-stable maximal torus which is contained in an F-stable Borel subgroup of $\mathbf{G}$; furthermore, $w \mapsto w^{\diamond}$ is the map induced by F on W. Let $G = \mathbf{G}^F$. For each $\chi \in \mathrm{Irr}^{\diamond}(W)$, let $R_{\widetilde{\chi}}$ be the corresponding "almost character" of G (see [14, 3.7]), where $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$ is an extension of χ. (If we choose another extension $\widetilde{\chi}'$ of χ, then $R_{\widetilde{\chi}'} = \pm R_{\widetilde{\chi}}$.) Let

$$\mathrm{Uch}(G) = \{\rho \in \mathrm{Irr}(G) \mid \langle R_{\widetilde{\chi}}, \rho \rangle_G \neq 0 \text{ for some } \chi \in \mathrm{Irr}^{\diamond}(W)\}$$

be the set of *unipotent characters* of G. For any two-sided cell $\mathfrak{C}$ of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$, we denote by $\mathrm{Uch}(G \mid \mathfrak{C})$ the set of all $\rho \in \mathrm{Uch}(G)$ such that $\langle R_{\widetilde{\chi}}, \rho \rangle_G \neq 0$ for some $\chi \in \mathrm{Irr}(W \mid \mathfrak{C})$. By the "Disjointness Theorem" [14, 6.16], we obtain a partition

$$\mathrm{Uch}(G) = \coprod_{\mathfrak{C}} \mathrm{Uch}(G \mid \mathfrak{C})$$

where $\mathfrak{C}$ runs over all two-sided cells of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$. All the multiplicities $\langle R_{\widetilde{\chi}}, \rho \rangle_G$ are explicitly described by [14, Main Theorem 4.23]; this involves a certain Fourier matrix and a function $\Delta \colon \mathrm{Uch}(G) \to \{\pm 1\}$.

To state the following result, we introduce the following notation. For $\chi \in \mathrm{Irr}^{\diamond}(W)$, we set

$$c^*_{\gamma w, \widetilde{\chi}} := (-1)^{\mathbf{a}_\chi + l(w)}\, c_{\gamma w, \widetilde{\chi}} \qquad \text{for all } w \in W,$$

where $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$ is an extension of χ to $\widetilde{W}$.

Proposition 4.8 (Lusztig [14, 7.1], [17]). *In the above setting, let $\mathfrak{C}$ be a two-sided cell of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$. Assume there exists some $\chi_0 \in \mathrm{Irr}^{\diamond}(W)$ and*

an extension $\widetilde{\chi}_0 \in \mathrm{Irr}(\widetilde{W})$ of χ_0 such that

$$\Delta(\rho)\langle R_{\widetilde{\chi}_0}, \rho\rangle_G > 0 \qquad \text{for all } \rho \in \mathrm{Uch}(G^F \mid \mathfrak{C}). \tag{$*$}$$

Then $c^*_{\gamma w, \widetilde{\chi}_0} \geqslant 0$ for all $w \in \mathfrak{C}$; furthermore, $c^*_{\gamma w, \widetilde{\chi}_0} > 0$ for all $w \in \mathfrak{C}$ such that $w^\diamond, w^{-1}$ belong to the same left cell of W.

Proof. First consider the inequality $c^*_{\gamma w, \widetilde{\chi}_0} \geqslant 0$ for all $w \in \mathfrak{C}$. In [14, 7.1], this is proved assuming that F acts trivially on W and $\Delta(\rho) = 1$ for all $\rho \in \mathrm{Uch}(G \mid \mathfrak{C})$. But the same proof gives the more general statement above. Let us briefly sketch the main ingredients. For each $\chi \in \mathrm{Irr}^\diamond(W)$ (other than χ_0), let us fix some extension $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W})$. Let $w \in \mathfrak{C}$ and consider the class function

$$R_{\gamma w} := \sum_{\chi \in \mathrm{Irr}^\diamond(W)} c_{\gamma w, \widetilde{\chi}} R_{\widetilde{\chi}}$$

on G. (Note that this is independent of any choices.) Since $w \in \mathfrak{C}$, we have that $R_{\gamma w}$ is a linear combination of the unipotent characters in $\mathrm{Uch}(G \mid \mathfrak{C})$; see [14, 5.2]. Since the functions $\{R_{\widetilde{\chi}} \mid \chi \in \mathrm{Irr}^\diamond(W)\}$ form an orthonormal system (see [14, 3.9]), we obtain

$$c_{\gamma w, \widetilde{\chi}_0} = \sum_{\rho \in \mathrm{Uch}(G|\mathfrak{C})} \langle R_{\widetilde{\chi}_0}, \rho\rangle_G \langle R_{\gamma w}, \rho\rangle_G.$$

Now let $\rho \in \mathrm{Uch}(G \mid \mathfrak{C})$ be such that the corresponding terms in the above sum are non-zero. Then [14, 6.19] shows that $(-1)^{\mathbf{a}_{\chi_0} + \ell(w)} = \Delta(\rho)$. Hence, we obtain

$$c^*_{\gamma w, \widetilde{\chi}_0} = (-1)^{\mathbf{a}_{\chi_0} + \ell(w)} c_{\gamma w, \widetilde{\chi}_0} = \sum_{\rho \in \mathrm{Uch}(G|\mathfrak{C})} \Delta(\rho)\langle R_{\widetilde{\chi}_0}, \rho\rangle_G \langle R_{\gamma w}, \rho\rangle_G.$$

Now, by the "Disjointness Theorem" [14, 6.17], $R_{\gamma w}$ is an actual character of G and so $\langle R_{\gamma w}, \rho\rangle_G \geqslant 0$ for all $\rho \in \mathrm{Uch}(G \mid \mathfrak{C})$. Since $(*)$ is assumed to hold, we conclude that $c_{\gamma w, \widetilde{\chi}_0} \geqslant 0$ for all $w \in \mathfrak{C}$.

Finally, assume that $w \in \mathfrak{C}$ is such that $w^\diamond, w^{-1}$ belong to the same left cell of W. Then we must prove that $c^*_{\gamma w, \widetilde{\chi}_0} \neq 0$. Now, the above expression for $c^*_{\gamma w, \widetilde{\chi}_0}$ (together with $(*)$) shows that it will be sufficient to prove that that there exists some $\rho \in \mathrm{Uch}(G \mid \mathfrak{C})$ such that $\langle R_{\gamma w}, \rho\rangle_G \neq 0$. For this purpose, it is enough to show that $R_{\gamma w} \neq 0$. Furthermore, since the class functions $\{R_{\widetilde{\chi}}\}$ are linearly independent, it will be sufficient to show that $c_{\gamma w, \widetilde{\chi}} \neq 0$ for some $\chi \in \mathrm{Irr}^\diamond(W)$. But this follows by an argument involving Lusztig's asymptotic algebra $\widetilde{\mathbf{J}}$; see [17, 3.1]. Indeed, this algebra has a basis $\{t_\sigma \mid \sigma \in \widetilde{W}\}$ where the structure constants are integers. It is known that $\widetilde{\mathbf{J}}$ is a "based ring" in the sense of [17, §1]; see [17, 3.1(j)]. This has several consequences. First of all, by [17, 3.1(k)], the elements σ, σ^{-1} belong to the same left L-cell in $\widetilde{W}$ if and only if $t_\sigma^2 \neq 0$. Furthermore, by [17, 1.2(b)], we have $t_\sigma^2 \neq 0$ if and only some irreducible character of $\widetilde{\mathbf{J}}$ has a non-zero value on t_σ. Finally, by [17, 3.4(a), (e)], the leading coefficients $c_{\sigma, \widetilde{\psi}}$ can be interpreted (up to signs) as the values of the irreducible characters of $\widetilde{\mathbf{J}}$ on t_σ. Thus, we have:

$$\sigma, \sigma^{-1} \text{ belong to the same left } L\text{-cell} \qquad \Leftrightarrow \qquad c_{\sigma, \widetilde{\psi}} \neq 0 \text{ for some } \widetilde{\psi} \in \mathrm{Irr}(\widetilde{W}).$$

Now return to our element $w \in \mathfrak{C}$ such that $w^{\diamond}, w^{-1}$ belong to the same left cell of W. Since $\diamond$ permutes the left cells of W, we also have that $w, (w^{\diamond})^{-1}$ belong to the same left cell of W. Consequently, since $(\gamma w)^{-1} = w^{-1} \gamma = \gamma (w^{\diamond})^{-1}$, the elements $\gamma w, (\gamma w)^{-1}$ belong to the same left L-cell of $\widetilde{W}$. So the above equivalence shows that there exists some $\widetilde{\psi} \in \text{Irr}(\widetilde{W})$ such that $c_{\gamma w, \widetilde{\psi}} \neq 0$. But then Remark 4.4 implies that $\widetilde{\psi}$ must be an extension of some $\psi \in \text{Irr}^{\diamond}(W)$, as required.   ∎

Proposition 4.9.   *Assume that we are in the setting of (4.7). Let $\mathfrak{C}$ be a two-sided cell of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$. Let $\chi_0 \in \text{Irr}(W)$ be the unique special character in $\text{Irr}(W \mid \mathfrak{C})$; we have $\chi_0^{\diamond} = \chi_0$. Then condition $(*)$ in Proposition 4.8 holds if $\widetilde{\chi}_0$ is the $\diamond$-special extension of χ_0 in the sense of (4.5).*

Proof.   By standard reduction arguments, it is enough to prove this in the case where W is irreducible. If F acts trivially on W, then the multiplicity formula in [14, Main Theorem 4.23] shows that $\langle R_{\widetilde{\chi}_0}, \rho \rangle_G = \Delta(\rho)$ for all $\rho \in \text{Uch}(G \mid \mathfrak{C})$. (The special character χ_0 corresponds to the pair $(1, 1)$ in the set $\mathcal{M}(\mathcal{G}_{\mathfrak{C}})$ where $\mathcal{G}_{\mathfrak{C}}$ is the finite group associated with $\mathfrak{C}$ and $\mathcal{M}(\mathcal{G}_{\mathfrak{C}})$ is defined in [14, 4.14].) Hence, the assertion is clear in this case. So let us now assume that F does not act trivially on W. Then we only have 2 cases to consider:

(a) W is of type E_6 or A_n and $\diamond$ is given by conjugation with the longest element.

(b) $W = W_n$ is of type D_n and $\diamond$ is given as in Example 3.4.

Assume that we are in case (a). Then $\chi^{\diamond} = \chi$ for all $\chi \in \text{Irr}(W)$. Given χ, the "preferred extension" $\widetilde{\chi}$ is determined by the condition that γw_0 acts as $(-1)^{\mathbf{a}_{\chi}}$ in a representation affording $\widetilde{\chi}$. The assertion then follows from the description of the Fourier matrix in [14, 4.19] and the Δ-function in [14, p. 124]. Now assume that we are in case (b). The "preferred extensions" are described in (3.11). We have $\Delta(\rho) = 1$ for all $\rho \in \text{Uch}(G)$; see [14, 6.18.5, 6.19]. The assertion now follows from the description of the Fourier matrix in [14, 4.18]; see also [13, Theorem 3.15]. (This has also been discussed in some detail in the proof of [7, Theorem 5.1].)   ∎

Lemma 4.10 ("The basic identity"; cf. [1, §4]).   *Let $\mathfrak{C}$ be a two-sided cell of W such that $\mathfrak{C}^{\diamond} = \mathfrak{C}$. Let Γ be a left cell of W such that $\Gamma \subseteq \mathfrak{C}$ and C be a $\diamond$-conjugacy class of $\diamond$-twisted involutions of W. Then*

$$\langle [\Gamma]_1, \chi \rangle_W \sum_{w \in C \cap \mathfrak{C}} c_{\gamma w, \widetilde{\chi}} = \chi(1) \sum_{w \in C \cap \Gamma} c_{\gamma w, \widetilde{\chi}} \qquad \text{for all } \chi \in \text{Irr}^{\diamond}(W),$$

where $\widetilde{\chi} \in \text{Irr}(\widetilde{W})$ denotes a fixed extension of $\chi \in \text{Irr}^{\diamond}(W)$ to $\widetilde{W}$.

Proof.   If $\diamond = 1$, then this is proved in [1, Lemma 4.1]. The general case is completely analogous. First, as in the proof of [1, Lemma 2.2], one verifies that $T_{s^{\diamond}} Z = Z T_s$ for all $s \in S$, where

$$Z := \sum_{w \in C} (-1)^{\ell(w)} T_w \in \mathbf{H}.$$

Consequently, the element $\widetilde{Z} := \sum_{w \in C} (-1)^{\ell(w)} T_{\gamma w} \in \widetilde{\mathbf{H}}$ lies in the centre of $\widetilde{\mathbf{H}}$. Once this is established, the proof proceeds exactly as in [1, Lemma 4.1]. All the required properties of the leading coefficients $c_{\gamma w, \widetilde{\chi}}$ and the structure constants of the Kazhdan–Lusztig basis of $\widetilde{\mathbf{H}}$ hold by [17, 3.1–3.4]. ∎

We now apply the above results to type D_n.

Proposition 4.11. *Let $W = W_n$ be of type D_n and $\diamond$ be as in Example 3.4. We identify $\widetilde{W}$ with a group $\widetilde{W}_n$ of type B_n. Let $\mathfrak{C}$ be a two-sided cell of W_n and $\chi_0 \in \mathrm{Irr}(W_n)$ be the unique special character in $\mathrm{Irr}(W_n \mid \mathfrak{C})$. Then the following hold.*

(a) *Assume that $\mathfrak{C}^\diamond = \mathfrak{C}$ and let $\widetilde{\chi}_0 \in \mathrm{Irr}(\widetilde{W}_n)$ be the preferred extension of $\chi_0 = \chi_0^\diamond$. Then*

$$c_{tw,\widetilde{\chi}_0}^* = 1 \qquad \text{for all } \diamond\text{-twisted involutions } w \in \mathfrak{C}.$$

furthermore, each left cell $\Gamma \subseteq \mathfrak{C}$ contains exactly f_{χ_0} $\diamond$-twisted involutions.

(b) *Let C be a $\diamond$-conjugacy class of $\diamond$-twisted involutions in W_n. Then*

$$|C \cap \mathfrak{C}| = \chi_0(1)|C \cap \Gamma| \qquad \text{for any left cell } \Gamma \subseteq \mathfrak{C}.$$

Proof. (a) Let $w \in \mathfrak{C}$ be a $\diamond$-twisted involution. Then $w^\diamond = w^{-1}$ belong to the same left cell. By Propositions 4.8 and 4.9, we conclude that $c_{tw,\widetilde{\chi}_0}^* > 0$. In order to show that $c_{tw,\widetilde{\chi}_0}^* = 1$, we use a counting argument. Let $\Gamma \subseteq \mathfrak{C}$ be any left cell. Since $c_{tw,\widetilde{\chi}_0}^* > 0$ for all $\diamond$-twisted involutions $w \in \Gamma$, we have

$$(\text{number of } \diamond\text{-twisted involutions in } \mathfrak{C}) \leqslant \sum_{w \in \mathfrak{C}} c_{tw,\widetilde{\chi}_0}^2,$$

with equality only if $c_{tw,\widetilde{\chi}_0}^* = 1$ for all $\diamond$-twisted involutions $w \in \Gamma$. By (4.3)(b), the right hand side equals $f_{\chi_0} \langle [\Gamma]_1, \chi_0 \rangle_{W_n}$. By Example 4.6(a), we have $\langle [\Gamma]_1, \chi_0 \rangle_{W_n} = 1$ and so
$$(\text{number of } \diamond\text{-twisted involutions in } \Gamma) \leqslant f_{\chi_0}.$$

Now consider the left L-cell $\Gamma^+ = \Gamma \cup t\Gamma$ of $\widetilde{W}_n$. Using Example 4.6(a) and Remark 2.4, the above inequality can be rephrased as:

$$(\text{number of ordinary involutions in } \Gamma^+) \leqslant 2f_{\chi_0},$$

where equality holds if and only if equality holds in all the previous inequalities. But then Example 4.6(d) shows that all the previous inequalities must be equalities; note that $f_{\widetilde{\chi}_0} = 2f_{\chi_0}$ in this case. In particular, $c_{tw,\widetilde{\chi}_0}^* = 1$ for all $\diamond$-twisted involutions $w \in \Gamma$. It also follows that the number of these $\diamond$-twisted involutions equals f_{χ_0}. Thus, (a) is proved.

(b) If $\mathfrak{C}^\diamond \neq \mathfrak{C}$, then both sides of the equality are zero by Remark 2.9. So let us now assume that $\mathfrak{C}^\diamond = \mathfrak{C}$ and let $\widetilde{\chi}_0$ be as in (a). By Example 4.6(a), we have $\langle [\Gamma]_1, \chi_0 \rangle_{W_n} = 1$. It remains to use the identity in Lemma 4.10. ∎

5. Twisted involutions in type D_n

Throughout this section we place ourselves in the setting of Example 3.4. Thus, $n \geqslant 2$ and $W = W_n$ is a Coxeter group of type D_n, with generators $u, s_1, \ldots, s_{n-1}$. Let $w \mapsto w^\diamond$ be defined by $u^\diamond = s_1$, $s_1^\diamond = u$ and $s_i^\diamond = s_i$ for $2 \leqslant i \leqslant n-1$. We identify the semidirect product $\widetilde{W} = W \rtimes \langle \diamond \rangle$ with the Coxeter group $\widetilde{W}_n$ of type B_n with generators $t, s_1, \ldots, s_{n-1}$ as in Example 3.3. Recall that, under this identification, we have

$$w^\diamond = twt \qquad \text{for all } w \in W_n.$$

Let $L \colon \widetilde{W}_n \to \mathbb{Z}$ be defined as in (4.2). We have already noted in Example 4.6(c) that L is a weight function in the sense of Lusztig [18]; explicitly, we have

$$L(t) = 0 \qquad \text{and} \qquad L(s_1) = L(s_2) = \ldots = L(s_{n-1}) = 1.$$

We shall consider the left, right and two-sided L-cells of $\widetilde{W}_n$. In Theorem 5.3, we state a modified version of Kottwitz's conjecture for all (ordinary) conjugacy classes of involutions in $\widetilde{W}_n$. This crucially relies on the construction of the modified involution module in Lemma 5.2. Then note that any involution in $\widetilde{W}_n$ is either an ordinary involution in W_n or corresponds to a $\diamond$-twisted involution in W_n. In Corollary 5.4, we will see that the modified version of Kottwitz's conjecture for $\widetilde{W}_n$ encapsulates both the split and the quasi-split version of Kottwitz's conjecture for W_n.

Throughout, it will be convenient to allow also the possibility that $n = 0, 1$, where $\widetilde{W}_0 = \{1\}$, $\widetilde{W}_1 = \{1, t\}$ and $W_0 = W_1 = \{1\}$.

Remark 5.1. We have the following relation between the length function ℓ on W_n and the length function $\widetilde{\ell}$ on $\widetilde{W}_n$. Let $w \in W_n$ and $1 \leqslant i \leqslant n-1$. Then we have

$$\ell(ws_i) < \ell(w) \quad \Longleftrightarrow \quad \widetilde{\ell}(ws_i) < \widetilde{\ell}(w) \quad \Longleftrightarrow \quad \widetilde{\ell}(tws_i) < \widetilde{\ell}(tw).$$

Indeed, first note that, by [8, 1.4.12], we have $\widetilde{\ell}(w) = \ell(w) + \ell_t(w)$, where $\ell_t(w)$ denotes the number of occurrences of t in a reduced expression of w in terms of the generators of $\widetilde{W}_n$. This implies the first equivalence. To prove the second equivalence, we distinguish two cases. Suppose first that $\widetilde{\ell}(tw) > \widetilde{\ell}(w)$. Now, if $\widetilde{\ell}(ws_i) < \widetilde{\ell}(w)$, then $\widetilde{\ell}(tws_i) \leqslant \widetilde{\ell}(w) < \widetilde{\ell}(tw)$, as required. Conversely, assume that $\widetilde{\ell}(tws_i) < \widetilde{\ell}(tw)$. Since $\widetilde{\ell}(tw) > \widetilde{\ell}(w)$, this implies $\widetilde{\ell}(tws_i) = \widetilde{\ell}(w)$. So we must have $\widetilde{\ell}(ws_i) < \widetilde{\ell}(w)$ by [8, Lemma 1.2.6]. (Otherwise, we would have $tws_i = w$ and so t, s_i would be conjugate, a contradiction.) The argument for the case $\widetilde{\ell}(tw) < \widetilde{\ell}(w)$ is similar.

Lemma 5.2. *Let $\mathcal{C}$ be an (ordinary) conjugacy class of involutions of $\widetilde{W}_n$. Let $\widetilde{M}$ be a $\mathbb{Q}$-vector space with a basis $\{\widetilde{a}_\sigma \mid \sigma \in \mathcal{C}\}$. Then $\widetilde{M}$ is a $\mathbb{Q}[\widetilde{W}_n]$-module, where the action is given by:*

$$t.\widetilde{a}_\sigma = \widetilde{a}_{t\sigma t},$$

$$s_i.\widetilde{a}_\sigma = \begin{cases} -\widetilde{a}_\sigma & \text{if } s_i\sigma = \sigma s_i \text{ and } \widetilde{\ell}(\sigma s_i) < \widetilde{\ell}(\sigma), \\ \widetilde{a}_{s_i\sigma s_i} & \text{otherwise,} \end{cases}$$

for $1 \leqslant i \leqslant n-1$. *Furthermore, let* $\widetilde{\Upsilon}_{\mathcal{C}}$ *denote the character of* $\widetilde{W}_n$ *afforded by* $\widetilde{M}$.

(a) *If* $\mathcal{C} \subseteq W_n$ *and* $\mathcal{C}$ *is a single conjugacy class in* W_n, *then the restriction of* $\widetilde{\Upsilon}_{\mathcal{C}}$ *to* W_n *is Kottwitz's character* $\Upsilon_{\mathcal{C}}^1$ *for* W_n *(split case, see Remark 2.3).*

(b) *If* $\mathcal{C} \subseteq W_n$ *and* $\mathcal{C}$ *consists of two conjugacy classes in* W_n, *then* n *is even and the restriction of* $\widetilde{\Upsilon}_{\mathcal{C}}$ *to* W_n *equals the sum of the two characters* $\Upsilon_{\sigma_{n/2}}^1$ *and* $\Upsilon_{t\sigma_{n/2}t}^1$ *in (3.9).*

(c) *If* $\mathcal{C} \subseteq W_n t$, *then* $C = \{t\sigma \mid \sigma \in \mathcal{C}\}$ *is a* $\diamond$-*conjugacy class of* $\diamond$-*twisted involutions in* W_n *and the restriction of* $\widetilde{\Upsilon}_{\mathcal{C}}$ *to* W_n *is Kottwitz's character* $\Upsilon_C^{\diamond}$ *for* W_n *(quasi-split case).*

Proof. If $\mathcal{C} \subseteq W_n$, then this result is contained in [7, Prop. 2.4], with the only difference that the length condition is expressed in terms of the length ℓ on W_n. But then the first equivalence in Remark 5.1 allows us to rewrite this condition as above. By [7, Rem. 2.2], the character of the restriction of $\widetilde{M}$ to W_n is $\Upsilon_{\mathcal{C}}^1$. This yields (a) if $\mathcal{C}$ is a single conjugacy class in W_n; otherwise, n must be even and $\mathcal{C} = C_0 \cup tC_0 t$ where C_0 is the conjugacy class of the element $\sigma_{n/2}$ in Remark 3.6; this yields (b).

Now assume that $\mathcal{C} \subseteq tW_n$. Then recall from Remark 2.4 that $C := \{t\sigma \mid \sigma \in \mathcal{C}\}$ is a $\diamond$-conjugacy class of $\diamond$-twisted involutions in W_n. Let M be a $\mathbb{Q}$-vector space with a basis $\{a_w \mid w \in C\}$. By [22, 7.1] (see also [20]), we already know that M is a $\mathbb{Q}[W_n]$-module, where the action is given by the following formulae for any $s \in \{u, s_1, \ldots, s_{n-1}\}$:

$$s.a_w = \begin{cases} -a_w & \text{if } s^{\diamond}w = ws \text{ and } \ell(ws) < \ell(w), \\ a_{s^{\diamond}ws} & \text{otherwise.} \end{cases}$$

By (2.6), the character of W_n afforded by M is $\Upsilon_C^{\diamond}$. Via the linear map $M \to \widetilde{M}$, $a_w \mapsto \widetilde{a}_{tw}$, we can transport this action to $\widetilde{M}$. The action of W_n on $\widetilde{M}$ is given by the following formulae for any $s \in \{u, s_1, \ldots, s_{n-1}\}$:

$$s.\widetilde{a}_{\sigma} = \begin{cases} -\widetilde{a}_{\sigma} & \text{if } s\sigma = \sigma s \text{ and } \ell(t\sigma s) < \ell(t\sigma), \\ \widetilde{a}_{s\sigma s} & \text{otherwise.} \end{cases}$$

Now, by [21, 0.4], this action can be extended to $\widetilde{W}_n$ via the formulae:

$$t.\widetilde{a}_{\sigma} = \widetilde{a}_{t\sigma t},$$
$$s_i.\widetilde{a}_{\sigma} = \begin{cases} -\widetilde{a}_{\sigma} & \text{if } s_i\sigma = \sigma s_i \text{ and } \ell(t\sigma s_i) < \ell(t\sigma), \\ \widetilde{a}_{s_i\sigma s_i} & \text{otherwise;} \end{cases}$$

Then Remark 5.1 shows that the conditions involving the length function ℓ on W_n can be rewritten in terms of the length function $\widetilde{\ell}$ on W_n. ∎

We can now state the main result of this section.

Theorem 5.3. *Let $\mathcal{C}$ be any (ordinary) conjugacy class of involutions in $\widetilde{W}_n$. Then*

$$\boxed{\langle \widetilde{\Upsilon}_{\mathcal{C}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n} = |\mathcal{C} \cap \widetilde{\Gamma}| \qquad \text{for any left } L\text{-cell } \widetilde{\Gamma} \subseteq \widetilde{W}_n;}$$

here, $\widetilde{\Upsilon}_{\mathcal{C}}$ is the character of the $\mathbb{Q}[\widetilde{W}_n]$-module $\widetilde{M}$ in Lemma 5.2.

The proof will be given in (5.10), at the end of this section.

Corollary 5.4. *Assuming the truth of Theorem 5.3, both the split and the quasi-split version of Kottwitz's Conjecture 2.8 hold for W_n.*

Proof. As far as the split version is concerned, the argument is given in [1, Cor. 8.6]; this also uses the formulae in (3.9) for $\Upsilon^1_{\sigma_{n/2}}$ and $\Upsilon^1_{t\sigma_{n/2}t}$ (where the signs ν_α have been fixed). Now consider the quasi-split case. Let C be a $\diamond$-conjugacy class of $\diamond$-twisted involutions of W_n. Then $\mathcal{C} := \{tw \mid w \in C\}$ is a conjugacy class of involutions of $\widetilde{W}_n$ which is contained in the coset tW_n; see Remark 2.4. Let Γ be a left cell of W_n and $\widetilde{\Gamma} = \Gamma \cup t\Gamma$ be the corresponding left L-cell of $\widetilde{W}_n$. Then, using (4.2)(b), Lemma 5.2(c), Frobenius reciprocity and Theorem 5.3, we obtain

$$\langle \Upsilon^{\diamond}_C, [\Gamma]_1 \rangle_{W_n} = \langle \widetilde{\Upsilon}_{\mathcal{C}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n} = |\mathcal{C} \cap \widetilde{\Gamma}|.$$

Since $\mathcal{C} \subseteq tW_n$, the right hand side equals $|\mathcal{C} \cap t\Gamma| = |t(C \cap \Gamma)| = |C \cap \Gamma|$, as required. ∎

We now turn to the proof of Theorem 5.3; this will require a number of preparations.

Proposition 5.5. *Let $\mathcal{C}$ be a conjugacy class of involutions in $\widetilde{W}_n$. Assume that $\sigma_{l,j} \in \mathcal{C}$ where $l, j \geqslant 0$ are such that $l + 2j \leqslant n$; see (3.10). Then, using the notation in (3.11), we have*

$$\widetilde{\Upsilon}_{\mathcal{C}} = \sum_{(\alpha,\beta)} \binom{c(\alpha,\beta)}{j + l - d_0(\alpha,\beta)} \widetilde{\chi}^{(\alpha,\beta)}$$

where the sum runs over all $(\alpha,\beta) \vdash n$ such that $\widetilde{\chi}^{(\alpha,\beta)}$ is $\diamond$-special, $|\beta| = j$ and $d_0(\alpha,\beta) \leqslant j + l \leqslant d_0(\alpha,\beta) + c(\alpha,\beta)$.

Proof. Assume first that n is even, $l = 0$ and $j = n/2$. Then, by (3.9) and Lemma 5.2(b), the restriction of $\widetilde{\Upsilon}_{\mathcal{C}}$ to W_n equals

$$\Upsilon^1_{\sigma_{n/2}} + \Upsilon^1_{t\sigma_{n/2}t} = \sum_{\alpha \vdash n/2} \left(\chi^{[\alpha,+]} + \chi^{[\alpha,-]} \right).$$

So Frobenius reciprocity immediately implies that $\widetilde{\Upsilon}_{\mathcal{C}} = \sum_{\alpha \vdash n/2} \chi^{(\alpha,\alpha)}$, in accordance with the formula stated above. Now assume that $j < n/2$ and let (α,β) be a pair of partitions such that $|\alpha| + |\beta| = n$. Then Frobenius reciprocity, Proposition 3.12 and Lemma 5.2 show that

$$\langle \widetilde{\Upsilon}_{\mathcal{C}}, \mathrm{Ind}_{W_n}^{\widetilde{W}_n}(\chi^{[\alpha,\beta]}) \rangle_{\widetilde{W}_n} = 0 \qquad \text{unless} \qquad \alpha \neq \beta, \ |\beta| = j \text{ and } \widetilde{\chi}^{(\alpha,\beta)} \text{ is } \diamond\text{-special};$$

furthermore, if $\alpha \neq \beta$, $|\beta| = j$ and $\widetilde{\chi}^{(\alpha,\beta)}$ is $\diamond$-special, then

$$\big\langle \widetilde{\Upsilon}_{\mathcal{C}}, \mathrm{Ind}_{W_n}^{\widetilde{W}_n}(\chi^{[\alpha,\beta]}) \big\rangle_{\widetilde{W}_n} = \text{ binomial coefficient as above.}$$

Now let $\alpha \neq \beta$. Since

$$\mathrm{Ind}_{W_n}^{\widetilde{W}_n}(\chi^{[\alpha,\beta]}) = \widetilde{\chi}^{(\alpha,\beta)} + \widetilde{\chi}^{(\beta,\alpha)},$$

it will be sufficient to show that

$$\big\langle \widetilde{\Upsilon}_{\mathcal{C}}, \widetilde{\chi}^{(\alpha,\beta)} \big\rangle_{\widetilde{W}_n} = 0 \qquad \text{unless} \qquad |\alpha| > |\beta|. \qquad\qquad (*)$$

Now, by Example 3.10, we can assume that $\sigma = \sigma_{l,j}$ is the longest element in a parabolic subgroup $\widetilde{W}_I \subseteq \widetilde{W}_n$ where $I \subseteq \{t, s_1, \ldots, s_{n-1}\}$; furthermore, σ is central in $\widetilde{W}_I$. Then $C_{\widetilde{W}_n}(\sigma) = \widetilde{Y} \ltimes \widetilde{W}_I$ where $\widetilde{Y}$ is a certain set of distinguished coset representatives of $\widetilde{W}_I$ in $\widetilde{W}_n$; see [7, §2]. By the argument in [7, Lemma 2.1] (see also [7, Rem. 3.3]; this is essentially the same argument as in (2.6)), we have

$$\widetilde{M} \cong \mathrm{Ind}_{C_{\widetilde{W}_n}(\sigma)}^{\widetilde{W}_n}(\widetilde{\varepsilon}_\sigma)$$

where the homomorphism $\widetilde{\varepsilon}_\sigma \colon C_{\widetilde{W}_n}(\sigma) \to \{\pm 1\}$ is given by

$$\widetilde{\varepsilon}_\sigma(yw') = (-1)^{\widetilde{\ell}(w') - \ell_t(w')} \qquad \text{for all } y \in \widetilde{Y} \text{ and } w' \in \widetilde{W}_I.$$

Then $(*)$ is shown in [7, Lemma 3.4]. (Note that, in [7], it is generally assumed that $\mathcal{C} \subseteq W_n$ but in the proof of [7, Lemma 3.4], this assumption is irrelevant.) ∎

Remark 5.6. Let $X \subseteq \widetilde{W}_n$ be any subset which is a union of (ordinary) conjugacy classes of involutions in $\widetilde{W}_n$. Then we set

$$\widetilde{\Upsilon}_X = \sum_{\mathcal{C}} \widetilde{\Upsilon}_{\mathcal{C}}$$

where $\mathcal{C}$ runs over the conjugacy classes contained in X. In particular, this applies to the set of all involutions in $\widetilde{W}_n$, which we denote by $\mathcal{I}_n$. With this notation, we have

$$\boxed{\widetilde{\Upsilon}_{\mathcal{I}_n} = \sum_{\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}_n)\ \diamond\text{-special}} 2^{c(\widetilde{\chi})}\, \widetilde{\chi}.}$$

This immediately follows from Proposition 5.5, by summing over all $l, j \geqslant 0$ such that $l + 2j \leqslant n$.

Lemma 5.7. Let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}_n$ and $\mathcal{C}$ be any ordinary conjugacy class of involutions in $\widetilde{W}_n$. Then

$$|\mathcal{C} \cap \widetilde{\mathfrak{C}}| = \widetilde{\chi}_0(1)|\mathcal{C} \cap \widetilde{\Gamma}| \qquad \text{for any left } L\text{-cell } \widetilde{\Gamma} \subseteq \widetilde{\mathfrak{C}},$$

where $\widetilde{\chi}_0 \in \mathrm{Irr}(\widetilde{W}_n)$ is the unique $\diamond$-special character in $\mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}})$.

Proof.   Let $\mathfrak{C}$ be a two-sided cell of W_n such that $\widetilde{\mathfrak{C}} = \mathfrak{C}^+$; see (4.2). Then we also have $\widetilde{\Gamma} = \Gamma \cup t\Gamma$ for some left cell $\Gamma \subseteq \mathfrak{C}$ of W_n. Let $\chi_0 \in \mathrm{Irr}(W_n \mid \mathfrak{C})$ be the unique special character. By [1, Exp. 5.5], we already know that

$$|\mathcal{C} \cap \mathfrak{C}| = \chi_0(1)|\mathcal{C} \cap \Gamma| \qquad \text{if } \mathcal{C} \subseteq W_n. \tag{$*$}$$

We now distinguish two cases.

Case 1. Assume that $\mathfrak{C}^\diamond = \mathfrak{C}$. Then $\widetilde{\mathfrak{C}} = \mathfrak{C}^+ = \mathfrak{C} \cup t\mathfrak{C}$. Furthermore, $\widetilde{\chi}_0$ is the preferred extension of χ_0; in particular, $\widetilde{\chi}_0(1) = \chi_0(1)$. Now, if $\mathcal{C} \subseteq W_n$, then $\mathcal{C} \cap \widetilde{\mathfrak{C}} = \mathcal{C} \cap \mathfrak{C}$ and $\mathcal{C} \cap \widetilde{\Gamma} = \mathcal{C} \cap \Gamma$. So the assertion holds by $(*)$.

On the other hand, if $\mathcal{C} \subseteq tW_n$, then $C := \{t\sigma \mid \sigma \in \mathcal{C}\}$ is a $\diamond$-conjugacy class of $\diamond$-twisted involutions in W_n. In this case, we have $\mathcal{C} \cap \widetilde{\mathfrak{C}} = t(C \cap \mathfrak{C})$ and $\mathcal{C} \cap \widetilde{\Gamma} = t(C \cap \Gamma)$. So the assertion holds by Proposition 4.11(b).

Case 2. Assume that $\mathfrak{C}^\diamond \neq \mathfrak{C}$. Then $\widetilde{\mathfrak{C}} = \mathfrak{C}^+ = \mathfrak{C} \cup t\mathfrak{C} \cup \mathfrak{C}t \cup t\mathfrak{C}t$. If $\mathcal{C} \subseteq tW_n$, then both sides of the desired identity are zero. (Indeed, we have $\mathfrak{C}^{-1} = \mathfrak{C}$ by [14, 5.2(iii)] and so, if $1 = (tw)^2 = twtw = w^\diamond w$, then $w^\diamond = w^{-1} \in \mathfrak{C} \cap \mathfrak{C}^\diamond$, a contradiction.) Now assume that $\mathcal{C} \subseteq W_n$. Then

$$|\mathcal{C} \cap \widetilde{\mathfrak{C}}| = |\mathcal{C} \cap \mathfrak{C}| + |\mathcal{C} \cap t\mathfrak{C}t| = 2\,|\mathcal{C} \cap \mathfrak{C}|.$$

By $(*)$, we have $|\mathcal{C} \cap \mathfrak{C}| = \chi_0(1)|\mathcal{C} \cap \Gamma|$ and so $|\mathcal{C} \cap \widetilde{\mathfrak{C}}| = \chi_0(1)|\mathcal{C} \cap \widetilde{\Gamma}|$. It remains to note that, since $\mathfrak{C}^\diamond \neq \mathfrak{C}$, we have $\chi_0^\diamond \neq \chi_0$ and so $\widetilde{\chi}_0$ is obtained by inducing χ_0 from W_n to $\widetilde{W}_n$; in particular, $\widetilde{\chi}_0(1) = 2\chi_0(1)$. ∎

Let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}_n$. We say that "*Kottwitz's Modified Conjecture holds for $\widetilde{\mathfrak{C}}$*" if, for any conjugacy class of involutions $\mathcal{C}$ in $\widetilde{W}_n$, we have

$$\langle \widetilde{\Upsilon}_{\mathcal{C}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n} = |\mathcal{C} \cap \widetilde{\Gamma}| \qquad \text{for all left } L\text{-cells } \widetilde{\Gamma} \subseteq \widetilde{\mathfrak{C}}.$$

The following remark, together with (5.9), will provide the basis for an inductive proof of Theorem 5.3 where we proceed one two-sided L-cell at a time.

5.8.   Let $\widetilde{w}_0 \in \widetilde{W}_n$ be the longest element. Let $\mathcal{C}$ be a conjugacy class of involutions of $\widetilde{W}_n$. Let $\widetilde{M}$ be the corresponding $\mathbb{Q}[\widetilde{W}_n]$-module as in Lemma 5.2. Since $\widetilde{w}_0$ is central in $\widetilde{W}_n$, the set $\mathcal{C}\widetilde{w}_0$ also is a conjugacy class of involutions in $\widetilde{W}_n$. Let $\widetilde{M}_0$ be the corresponding $\mathbb{Q}[\widetilde{W}_n]$-module. Then we have

$$\widetilde{M}_0 \cong \widetilde{M} \otimes \widetilde{\varepsilon}' \qquad \text{and} \qquad \widetilde{\Upsilon}_{\mathcal{C}\widetilde{w}_0} = \widetilde{\Upsilon}_{\mathcal{C}} \otimes \varepsilon', \tag{a}$$

where $\widetilde{\varepsilon}' \colon W_n \to \{\pm 1\}$ is the homomorphism given by $\widetilde{\varepsilon}'(t) = 1$ and $\widetilde{\varepsilon}'(s_i) = -1$ for $1 \leqslant i \leqslant n-1$. This follows by an argument entirely analogous to that in [1, Lemma 6.2]. Now let $\widetilde{\Gamma}$ be a left L-cell of $\widetilde{W}_n$. Then $\widetilde{\Gamma}\widetilde{w}_0$ also is a left L-cell of $\widetilde{W}_n$; see [18, 11.7]. We show that

$$[\widetilde{\Gamma}\widetilde{w}_0]_1 = [\widetilde{\Gamma}_1] \otimes \varepsilon'. \tag{b}$$

Indeed, by (4.2)(b), $[\widetilde{\Gamma}]_1$ is obtained by inducing $[\Gamma]_1$ from W_n to $\widetilde{W}_n$ where Γ is a left cell of W_n such that $\widetilde{\Gamma} = \Gamma \cup t\Gamma$. If n is even, then $\widetilde{w}_0 \in W_n$ and this is the

longest element in W_n. So $\widetilde{\Gamma}\widetilde{w}_0 = (\Gamma\widetilde{w}_0) \cup t(\Gamma\widetilde{w}_0)$. By [14, Lemma 5.14], we have $[\Gamma\widetilde{w}_0]_1 = [\Gamma]_1 \otimes \varepsilon$ where ε is the sign character of W_n. Since ε is the restriction of ε' to W_n, this implies (b) in this case. Now assume that n is odd. Then $w_0 := t\widetilde{w}_0 \in W_n$ and this is the longest element in W_n; furthermore, $\Gamma = \Gamma^\diamond = t\Gamma t$. So we obtain $\widetilde{\Gamma}\widetilde{w}_0 = \Gamma w_0 \cup t(\Gamma w_0)$. Using once more [14, Lemma 5.14], this implies (b) in this case as well. We conclude that

$$\langle \widetilde{\Upsilon}_{\mathcal{C}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n} = \langle \widetilde{\Upsilon}_{\mathcal{C}\widetilde{w}_0}, [\widetilde{\Gamma}\widetilde{w}_0]_1 \rangle_{\widetilde{W}_n} \qquad \text{for any left } L\text{-cell } \widetilde{\Gamma} \text{ of } \widetilde{W}_n. \qquad \text{(c)}$$

In particular, Kottwitz's Modified Conjecture holds for a two-sided L-cell $\widetilde{\mathfrak{C}}$ if and only if it holds for the two-sided L-cell $\widetilde{\mathfrak{C}}\widetilde{w}_0$.

Next, there is a standard combinatorial procedure by which certain arguments about two-sided cells can be reduced to so-called "cuspidal" two-sided cells. This appears and is used at various places in Lusztig's work; see, for example, [14, 8.1] and [16, 17.13]. We have also used it in [1, §7] to deal with Kottwitz's conjecture in type B_n. Let us explicitly describe this procedure in our present context.

5.9. For any $r \in \{0, 1, \ldots, n\}$, we have a parabolic subgroup

$$\widetilde{W}_{n-r} = \langle t, s_1, \ldots, s_{n-r-1} \rangle \subseteq \widetilde{W}_n$$

of type B_{n-r} (where $\widetilde{W}_0 = \{1\}$ and $\widetilde{W}_1 = \{1, t\}$). Following Lusztig [13, 1.8], [14, 4.1], we define an additive map

$$\mathbf{J}_r \colon \mathbb{Z}[\mathrm{Irr}(\widetilde{W}_{n-r})] \to \mathbb{Z}[\mathrm{Irr}(\widetilde{W}_n)]$$

as follows. Consider the parabolic subgroup $\widetilde{W}' = \widetilde{W}_{n-r} \times H_r \subseteq \widetilde{W}_n$ where $H_r = \langle s_{n-r+1}, \ldots, s_{n-1} \rangle \cong \mathfrak{S}_r$. Let $\widetilde{\chi}' \in \mathrm{Irr}(\widetilde{W}_{n-r})$ and ε_r be the sign character on H_r. Since $\mathbf{a}_{\varepsilon_r} = r(r-1)/2$, we have the implication

$$\left\langle \mathrm{Ind}_{\widetilde{W}'}^{\widetilde{W}_n}(\widetilde{\chi}' \boxtimes \varepsilon_r), \widetilde{\chi} \right\rangle_{\widetilde{W}_n} \neq 0 \qquad \Rightarrow \qquad \mathbf{a}_{\widetilde{\chi}} \geqslant \mathbf{a}_{\widetilde{\chi}'} + r(r-1)/2$$

for all $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}_n)$. We set

$$\mathbf{J}_r(\widetilde{\chi}') := \sum_{\widetilde{\chi}} \left\langle \mathrm{Ind}_{\widetilde{W}'}^{\widetilde{W}_n}(\widetilde{\chi}' \boxtimes \varepsilon_r), \widetilde{\chi} \right\rangle_{\widetilde{W}_n} \widetilde{\chi}$$

where the sum runs over all $\widetilde{\chi} \in \mathrm{Irr}(\widetilde{W}_n)$ such that $\mathbf{a}_{\widetilde{\chi}} = \mathbf{a}_{\widetilde{\chi}'} + r(r-1)/2$. Now let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}_n$. Following [14, 8.1], [16, 17.13], we say that $\widetilde{\mathfrak{C}}$ is "*smoothly induced*" if there exists some $r \in \{1, \ldots, n\}$ and a two-sided cell L-cell $\widetilde{\mathfrak{C}}'$ of $\widetilde{W}_{n-r}$ such that $\mathbf{J}_r$ establishes a bijection

$$\mathrm{Irr}(\widetilde{W}_{n-r} \mid \widetilde{\mathfrak{C}}') \to \mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}}), \qquad \widetilde{\chi}' \mapsto \mathbf{J}_r(\widetilde{\chi}'). \qquad \text{(a)}$$

As in [1, Remark 7.2], one easily sees that then the following holds:

$$c(\widetilde{\chi}_0) = c(\widetilde{\chi}_0'), \qquad d_0(\widetilde{\chi}_0) = d_0(\widetilde{\chi}_0') + \lfloor r/2 \rfloor \qquad \text{and} \qquad |\beta| = |\beta'| + \lfloor r/2 \rfloor, \quad \text{(b)}$$

where $\widetilde{\chi}_0 = \widetilde{\chi}^{(\alpha,\beta)} \in \mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}})$ and $\widetilde{\chi}'_0 = \widetilde{\chi}^{(\alpha',\beta')} \in \mathrm{Irr}(\widetilde{W}_{n-r} \mid \widetilde{\mathfrak{C}}')$ are the unique $\diamond$-special characters. Now $\{w_{0,r}\}$ is a two-sided cell in H_r where $w_{0,r} \in H_r$ is the longest element; furthermore, $\mathrm{Irr}(H_r \mid \{w_{0,r}\}) = \{\varepsilon_r\}$. Consequently, $\widetilde{\mathfrak{C}}'w_{0,r} \subseteq \widetilde{W}'$ is a two-sided L-cell (with respect to the restriction of L to $\widetilde{W}'$); thus, using also [19, 43.11(b)], we have

$$\widetilde{\mathfrak{C}}'w_{0,r} \subseteq \widetilde{\mathfrak{C}} \qquad \text{and} \qquad \mathrm{Irr}(\widetilde{W}' \mid \widetilde{\mathfrak{C}}'w_{0,r}) = \{\widetilde{\chi}' \boxtimes \varepsilon_r \mid \widetilde{\chi}' \in \mathrm{Irr}(\widetilde{W}_{n-r} \mid \widetilde{\mathfrak{C}}')\}. \qquad \text{(c)}$$

Finally, the point of these definitions is that every two-sided L-cell $\widetilde{\mathfrak{C}}$ of $\widetilde{W}_n$ is either itself smoothly induced, or the two-sided L-cell $\widetilde{\mathfrak{C}}\widetilde{w}_0$ is smoothly induced (where $\widetilde{w}_0 \in \widetilde{W}_n$ is the longest element), or $n = d^2$ for some $d \geqslant 2$ in which case $\widetilde{\mathfrak{C}}$ is uniquely determined; in the last case, $\widetilde{\mathfrak{C}}$ is called "cuspidal" and determined by the condition that the unique $\diamond$-special character in $\mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}})$ is $\widetilde{\chi}^{(\alpha,\beta)}$ where $\alpha = (1,2,\ldots,d-1)$ and $\beta = (0,1,2,\ldots,d-2)$; see [13, 3.17], [14, 8.1].

5.10.   Proof of Theorem 5.3. We proceed by induction on n. If $n = 0$, then $\widetilde{W}_0 = \{1\}$ and the assertion is clear. Now assume that $n \geqslant 1$. Let $\widetilde{\mathfrak{C}}$ be a two-sided L-cell of $\widetilde{W}_n$. Assume first that $\widetilde{\mathfrak{C}}$ is smoothly induced; let $r \in \{1,\ldots,n\}$ and $\widetilde{\mathfrak{C}}' \subseteq \widetilde{W}_{n-r}$ be as in (5.9). Let $\mathcal{C}$ be any conjugacy class of involutions in $\widetilde{W}_n$. Assume that $\sigma_{l,j} \in \mathcal{C}$ where $l,j \geqslant 0$ are such that $l + 2j \leqslant n$; we write $\mathcal{C} = \mathcal{C}_{l,j}$. Let $\widetilde{\chi}_0 = \widetilde{\chi}^{(\alpha,\beta)} \in \mathrm{Irr}(\widetilde{W}_n \mid \widetilde{\mathfrak{C}})$ be the unique $\diamond$-special character. Let $\widetilde{\Gamma}$ be a left L-cell of $\widetilde{W}_n$ such that $\widetilde{\Gamma} \subseteq \widetilde{\mathfrak{C}}$. By Example 4.6(d), we have $\langle [\widetilde{\Gamma}]_1, \widetilde{\chi}_0 \rangle_{\widetilde{W}_n} = 1$. Hence, using Proposition 5.5, we already know that

$$\langle \widetilde{\Upsilon}_{\mathcal{C}_{l,j}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n} = \begin{cases} \dbinom{c(\alpha,\beta)}{j + l - d_0(\alpha,\beta)} & \text{if } |\beta| = j, \\ 0 & \text{otherwise.} \end{cases}$$

We will now show that

$$|\mathcal{C}_{l,j} \cap \widetilde{\Gamma}| \leqslant \langle \widetilde{\Upsilon}_{\mathcal{C}_{l,j}}, [\widetilde{\Gamma}]_1 \rangle_{\widetilde{W}_n}. \qquad (\dagger)$$

This is seen as follows. If $\mathcal{C}_{l,j} \cap \widetilde{\Gamma} = \varnothing$, this is clear. Now assume that $\mathcal{C}_{l,j} \cap \widetilde{\Gamma} \neq \varnothing$. By Lemma 5.7, the cardinality $|\mathcal{C}_{l,j} \cap \widetilde{\Gamma}|$ does not depend on the left L-cell $\widetilde{\Gamma} \subseteq \widetilde{\mathfrak{C}}$. Thus, it will be enough to prove $(\dagger)$ for one particular left L-cell in $\widetilde{\mathfrak{C}}$. We will choose such a left L-cell as follows. Consider the two-sided L-cell $\widetilde{\mathfrak{C}}'$ of $\widetilde{W}_{n-r}$. As in (5.9), let $w_{0,r} \in H_r$ be the longest element. Then $\widetilde{\mathfrak{C}}'w_{0,r}$ is a two-sided L-cell in $\widetilde{W}' = \widetilde{W}_{n-r} \times H_r$ and we have $\widetilde{\mathfrak{C}}'w_{0,r} \subseteq \widetilde{\mathfrak{C}}$; see (5.9)(c). Now let $\widetilde{\Gamma}'$ be a left L-cell of $\widetilde{W}_{n-r}$ which is contained in $\widetilde{\mathfrak{C}}'$. Then $\widetilde{\Gamma}'w_{0,r}$ is a left L-cell of $\widetilde{W}'$. Let $\widetilde{\Gamma} \subseteq \widetilde{\mathfrak{C}}$ be the left L-cell of $\widetilde{W}_n$ such that

$$\widetilde{\Gamma}'w_{0,r} \subseteq \widetilde{\Gamma}.$$

By Example 4.6(d) and (5.9)(b), the left L-cells $\widetilde{\Gamma}'$, $\widetilde{\Gamma}'w_{0,r}$ and $\widetilde{\Gamma}$ all contain the same number of involutions. Hence, all the involutions in $\widetilde{\Gamma}$ are already contained in $\widetilde{\Gamma}'w_{0,r} \subseteq \widetilde{W}'$. Consequently, we have

$$\mathcal{C}_{l,j} \cap \widetilde{\Gamma} = (\mathcal{C}_{l,j} \cap \widetilde{W}') \cap \widetilde{\Gamma}'w_{0,r}.$$

By an argument entirely analogous to that in [1, Theorem 7.3] (see the paragraph following ($\triangle$) in the proof theoreof), the assumption that $\mathcal{C}_{l,j} \cap \widetilde{\Gamma} \neq \varnothing$ now implies that $\mathcal{C}_{l,j} \cap \widetilde{W}'$ is the conjugacy class containing the element $\sigma_{l,j'} w_{0,r}$ where $j = j' + k$ and $k = \lfloor r/2 \rfloor$. We conclude that

$$|\mathcal{C}_{l,j} \cap \widetilde{\Gamma}| = |\mathcal{C}' \cap \widetilde{\Gamma}'| \text{ where } \mathcal{C}' \subseteq \widetilde{W}_{n-r} \text{ is the conjugacy class containing } \sigma_{l,j'}.$$

Hence, using the equality $j' = j - \lfloor r/2 \rfloor$ and (5.9)(b), we see that (†) holds by induction. Once this is established, it actually follows that we must have equality in (†). Indeed, by Example 4.6(d), we have

$$\sum_{l,j} |\mathcal{C}_{l,j} \cap \widetilde{\Gamma}| = \text{ (number of involutions in } \widetilde{\Gamma}) = 2^{c(\widetilde{\chi}_0)}$$

where the sum runs over all $l, j \geqslant 0$ such that $l + 2j \leqslant n$. But we obtain the same result when we sum the binomial coefficients giving the right hand side of (†) over all l, j as above. Hence, all the inequalities in (†) must be equalities. Thus, Kottwitz's Modified Conjecture holds for $\widetilde{\mathfrak{C}}$. Then (5.8) shows that Kottwitz's Modified Conjecture also holds for $\widetilde{\mathfrak{C}} \widetilde{w}_0$. By the remarks at the end of (5.9), these arguments cover all non-cuspidal two-sided L-cells of $\widetilde{W}_n$. So it remains to show that Kottwitz's Modified Conjecture holds for the unique cuspidal two-sided L-cell of $\widetilde{W}_n$ where $n = d^2$ for some $d \geqslant 2$. But this follows from a formal argument based on Lemma 5.7, exactly as in the proof of [1, Theorem 7.3]. ∎

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Meinolf Geck
IAZ - Lehrstuhl für Algebra
Universität Stuttgart
Pfaffenwaldring 57
D–70569 Stuttgart, Germany
meinolf.geck@mathematik.uni-stuttgart.de