

Conjugacy Classes of n -Tuples in Semi-Simple Jordan Algebras

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Abstract. Let J be a finite-dimensional semi-simple Jordan algebra over an algebraically closed field of characteristic 0. In this article, the diagonal action of the automorphism group of J on the n -fold product $J \times \cdots \times J$ is studied. In particular, it is shown that the orbit through an n -tuple $x = (x_1, \dots, x_n)$ is closed if and only if the Jordan subalgebra generated by the elements x_1, \dots, x_n is semi-simple.

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1. Introduction

Let J be a finite-dimensional semi-simple Jordan algebra over an algebraically closed field \mathbb{K} of characteristic $\text{char}(\mathbb{K}) = 0$. Denote by $\text{Aut}(J)$ the automorphism group of J , which naturally acts on J . In this article, geometric properties of the diagonal action of $\text{Aut}(J)$ on the n -fold product of $J \times \cdots \times J$ are studied. In particular, an algebraic characterization of the closed orbits is given.

In the case of a reductive linear algebraic group G with Lie algebra \mathfrak{g} , R. W. Richardson studied geometric properties of the diagonal action of G on the n -fold product of the Lie algebra \mathfrak{g} , see [Ric88]. In particular, he showed that the orbit through an n -tuple $x = (x_1, \dots, x_n)$ is closed if and only if the Lie algebra generated by x_1, \dots, x_n is reductive in \mathfrak{g} . Moreover, the closure of $G \cdot x$ contains 0 precisely if x_1, \dots, x_n are contained in a nilpotent subalgebra of \mathfrak{g} which consists of nilpotent elements. These results obtained by R. W. Richardson generalize the results in the case $n = 1$, i.e. the adjoint action of a reductive linear algebraic group on its Lie algebra. There, an orbit through an element x is closed if and only if the element x is semi-simple. Furthermore, if $x = x_s + x_n$ is the decomposition of x into its semi-simple and nilpotent part, then the orbit through x_s is the unique closed orbit in the closure of the orbit through x .

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Similar results can be obtained in the case of Jordan algebras:

Theorem 1.1 (see Section 6). *Let J be a semi-simple Jordan algebra over an algebraically closed field of characteristic 0, with automorphism group $\text{Aut}(J)$. Consider the diagonal action of $\text{Aut}(J)$ on the n -fold product $J \times \dots \times J$, and let $x = (x_1, \dots, x_n) \in J \times \dots \times J$. Then:*

1. *The $\text{Aut}(J)$ -orbit through the n -tuple x is closed if and only if the subalgebra $A(x)$ generated by x_1, \dots, x_n is semi-simple.*
2. *The closure of the orbit $\text{Aut}(J) \cdot x$ contains 0 if and only if $A(x)$ is solvable.*

Moreover, the closed orbit in the closure of an orbit $\text{Aut}(J) \cdot x$ can be described in the following way:

Theorem 1.2 (see Section 7). *Let $x = (x_1, \dots, x_n) \in J \times \dots \times J$, and let $A(x)$ be the subalgebra generated by x_1, \dots, x_n . Moreover, let $R(x)$ be the radical of $A(x)$, $S(x) \subseteq A(x)$ a semi-simple subalgebra with $A(x) = S(x) + R(x)$, and $s_j \in S(x)$, $r_j \in R(x)$ such that $x_j = s_j + r_j$. Then $\text{Aut}(J) \cdot s = \text{Aut}(J) \cdot (s_1, \dots, s_n)$ is the unique closed orbit in the closure of $\text{Aut}(J) \cdot x$.*

The methods to prove the above statements are similar to those used by R.W. Richardson in [Ric88] in the case of Lie groups and Lie algebras.

The key ingredient for the proof is the existence of appropriate one-parameter subgroups. Firstly, it is needed that if a reductive linear algebraic group G acts on an affine variety X , then for any $x \in X$ there exists a one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists and is contained in the unique closed orbit in the closure of the orbit $G \cdot x$ (see [Bir71]).

Secondly, the following proposition on the existence of one-parameter subgroups in the automorphism group of a semi-simple Jordan algebra J satisfying suitable properties with respect to subalgebras of J is needed:

Proposition 1.3 (see Section 6). *Let J be a semi-simple Jordan algebra. Let A be a subalgebra of J , R the radical of A , and let S be a semi-simple subalgebra such that $A = S + R$. Then there exists a one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$ such that*

$$\begin{aligned} A &\subseteq J_{\geq 0}(\lambda), \\ S &\subseteq J_0(\lambda), \text{ and} \\ R &\subseteq J_{> 0}(\lambda), \end{aligned}$$

where $J_k(\lambda) = \{z \in J \mid \lambda(t) \cdot z = t^k z \text{ for all } t \in \mathbb{K}^\times\}$ for $k \in \mathbb{Z}$, $J_{\geq 0}(\lambda) = \bigoplus_{k \geq 0} J_k(\lambda)$, and $J_{> 0}(\lambda) = \bigoplus_{k > 0} J_k(\lambda)$.

To prove this proposition, the structure of the Lie subalgebra

$$[A, A] \oplus A = \left\{ \sum_{i=1}^l [L(a_i), L(b_i)] + L(a) \mid a_i, b_i, a \in A \right\} \subseteq \mathfrak{gl}(J)$$

is studied, where $L(z)$ denotes the left-multiplication $J \rightarrow J$ with z for any $z \in J$. Then, a result on the existence of one-parameter subgroups in Lie algebra invariant under an involution is applied (see [Ric88], Proposition 12.4).

I would like to thank E. B. Vinberg for drawing my attention to the results of R. W. Richardson ([Ric88]) and raising the question whether similar statements hold true in the case of Jordan algebras. Moreover, I am grateful for his interest in this work and the invitation to Lomonosov Moscow State University in April 2014, where part of this paper was written.

2. Basic definitions

In the following, a few basic definitions related to Jordan algebras are recalled, see also e.g. [BK66] and [FK94].

Definition 2.1 (Jordan algebra). A Jordan algebra is a vector space J (over a field \mathbb{K}) together with a bilinear map $J \times J \rightarrow J$, $(x, y) \mapsto x \circ y$, satisfying

- (i) $x \circ y = y \circ x$ and
- (ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$

for all $x, y \in J$.

For any $x \in V$ let $L(x) : J \rightarrow J$ denote the linear map defined by $L(x)(y) = x \circ y$.

The second property in the definition of a Jordan algebra is equivalent to requiring that $L(x)$ and $L(x^2)$ commute for any $x \in J$, i.e. $[L(x), L(x^2)] = 0$.

In the following, we will assume the field \mathbb{K} to have characteristic $\text{char}(\mathbb{K}) = 0$. Moreover, the dimension of the vector space J of a Jordan algebra will be assumed to be finite.

Example 2.2. If A is any associative algebra (over \mathbb{K}), then a Jordan algebra structure on A can be defined by setting

$$x \circ y = \frac{1}{2}(xy + yx).$$

A Jordan algebra J is called special if there exists a finite-dimensional associative algebra A such that J is isomorphic to a subalgebra of the Jordan algebra (A, \circ) .

Definition 2.3. A derivation of a Jordan algebra J is a linear transformation $D : J \rightarrow J$ such that $D(x \circ y) = Dx \circ y + x \circ Dy$ for all $x, y \in J$, which is equivalent to $[D, L(x)] = L(Dx)$ for any $x \in J$.

Let $\text{Der}(J)$ denote the set of derivations of J . This is a Lie algebra with respect to the usual bracket $[D_1, D_2] = D_1D_2 - D_2D_1$.

If $x, y \in J$, then the commutator $[L(x), L(y)] : J \rightarrow J$ is a derivation of J (see e.g. [FK94], Proposition II.4.1).

Definition 2.4. A derivation D of a Jordan algebra J is called inner if there are elements x_i, y_i , $i = 1, \dots, l$, such that $D = \sum_{i=1}^l [L(x_i), L(y_i)]$.

Definition 2.5 (quadratic representation). Let J be Jordan algebra. For any $x \in J$ define a linear map $P(x) : J \rightarrow J$, $P(x) = 2L(x)^2 - L(x^2)$. The assignment $P : J \rightarrow \text{End}(J)$, $x \mapsto P(x)$, is called the quadratic representation of J .

3. Structure Theory of Jordan Algebras

In this section, facts about the structure theory of Jordan algebras, which will be needed later, are collected. For more details see [Alb47] and [Jac51].

Using the results of [Alb47] on the structure theory of Jordan algebras, the radical of a Jordan algebra can be defined as follows:

Definition 3.1. The radical of a Jordan algebra J is the greatest ideal of J consisting of nilpotent elements.

Definition 3.2. A Jordan algebra whose radical vanishes is called semi-simple. It is called solvable if it equals its own radical.

Definition 3.3. Let J be a Jordan algebra. Define the trace form $\tau = \tau_J$ on J by setting $\tau(x, y) = \text{Tr}(L(x \circ y))$ for any $x, y \in J$.

Definition 3.4. An inner product (\cdot, \cdot) on a Jordan algebra J is called associative if $(x \circ y, z) = (x, y \circ z)$ for all $x, y, z \in J$. This is equivalent to requiring that each left multiplication $L(x)$, $x \in J$, is symmetric with respect to the inner product.

Remark 3.5 (c.f. Proposition II.4.3 in [FK94]). The trace form τ on a Jordan algebra J is an associative symmetric bilinear form, since

$$\tau(x \circ z, y) - \tau(x, z \circ y) = \text{Tr}(L((x \circ z) \circ y - x \circ (z \circ y))) = \text{Tr}([L(y), L(x)], L(z)) = 0.$$

Theorem 3.6 (see [Alb47], § 8). A Jordan algebra J is semi-simple if and only if its trace form τ_J is non-degenerate.

Theorem 3.7 (see e.g. [Koe99], Chapter III, Theorem 10). A Jordan algebra J containing an identity element e is semi-simple if there exists a non-degenerate associative symmetric bilinear form on J .

Proposition 3.8 (see [Jac51], Theorem 9.2). Let J be a Jordan algebra, and S a semi-simple subalgebra of J . Then any derivation $D : S \rightarrow J$ can be extended to an inner derivation of J , i.e. there exist elements $a_i, b_i \in J$, $i = 1, \dots, l$, such that $D = \sum_{i=1}^l [L(a_i), L(b_i)]$.

Definition 3.9. Let J be a Jordan algebra. Two subalgebras A_1 and A_2 of J are called strictly conjugate if there exists nilpotent derivations D_1, \dots, D_k of V

such that

$$(\exp(D_1) \circ \dots \circ \exp(D_k))(A_1) = A_2.$$

For Jordan algebras, there is an analogue of the Levi-Malcev theorem for Lie algebras:

Theorem 3.10 (c.f. [Pen51], and [Jac51], Theorem 9.3). *Let J be a Jordan algebra (over an algebraically closed field) with radical R . Then there exists a semi-simple subalgebra S of J such that $J = S \oplus R$ as vector spaces, $S \cap R = 0$, and $J/R \cong S$, i.e. J is the semi-direct product of S with R .*

If S' is any semi-simple subalgebra of J , then S' is strictly conjugate to a subalgebra of S . Moreover, the automorphism mapping S' onto a subalgebra of S can be chosen to be of the form $\exp(D)$ for an inner nilpotent derivation D of J with

$$D = \sum_{i=1}^l [L(z_i), L(r_i)], \quad z_i \in J, r_i \in R.$$

Remark 3.11. If $J = S + R$ is a Jordan algebra with radical R and $S \subseteq J$ semi-simple, then any derivation of the form $D = \sum_{i=1}^l [L(z_i), L(r_i)]$, $z_i \in J$, $r_i \in R$ is nilpotent.

For any ideal $A \subseteq J$ define $A^{(1)} = A$, and inductively $A^{(k)} = A^{(k-1)} A^{(k-1)} = \{\sum_i a_i b_i \mid a_i, b_i \in A^{(k-1)}\}$. Since R is the radical of J , there is $n \in \mathbb{N}$ with $R^{(n)} = \{0\}$. The chain $A = A^{(1)} \supseteq A^{(2)} \supseteq \dots$ is a chain of subalgebras of J , but $A^{(k)}$ is in general not an ideal in J due to the non-associativity of J . Therefore, one cannot expect to have $D(R^{(k)}) \subseteq R^{(k+1)}$ in general, and the nilpotence of D does not directly follow from the existence of $n \in \mathbb{N}$ with $R^{(n)} = \{0\}$.

For a proof of the nilpotence of D see e.g. Corollary 8.4 in [Jac51].

4. Automorphism and structure group

Definition 4.1. Define the automorphism group of a Jordan algebra J to be $\text{Aut}(J) = \{g \in \text{GL}(J) \mid g(x \circ y) = (gx) \circ (gy) \text{ for all } x, y \in J\}$.

The automorphism group of any Jordan algebra J is an algebraic subgroup of $\text{GL}(J)$.

Remark 4.2. The Lie algebra of $\text{Aut}(J)$ is the Lie algebra $\text{Der}(J)$ of derivations.

Moreover, in the case of a semi-simple Jordan algebra all derivations are inner.

Definition 4.3. Let J be a semi-simple Jordan algebra with trace form τ . For any invertible linear transformation $g \in \text{GL}(J)$, let ${}^t g$ denote the transpose of g with respect to τ . Define the structure group $\text{Str}(J)$ of J to be $\text{Str}(J) = \{g \in \text{GL}(J) \mid P(gx) = gP(x) {}^t g \text{ for any } x \in J\}$, and let $\mathfrak{str}(J)$ denote its Lie algebra.

Remark 4.4. If J is semi-simple, then $\text{Aut}(J)$ and $\text{Str}(J)$ are reductive subgroups of $\text{GL}(J)$, see e.g. [Jac68], Chapter VIII, Theorem 3.

Remark 4.5. For the Lie algebra $\mathfrak{str}(J)$ of the structure group J we have

$$\mathfrak{str}(J) = \text{Der}(J) \oplus J$$

since every element $X \in \mathfrak{str}(J)$ can be written uniquely as $X = D + L(a)$, where $D \in \text{Der}(J)$ and $a \in J$ (see e.g. [FK94], Proposition VIII.2.6). The Lie algebra structure on $\text{Der}(J) \oplus J$ is thus given by

$$[D_1 + a_1, D_2 + a_2] = ([D_1, D_2] + [L(a_1), L(a_2)]) + (D_1(a_2) - D_2(a_1))$$

for $D_1, D_2 \in \text{Der}(J)$ and $a_1, a_2 \in J$.

Note that $\text{Der}(A) \oplus A$ is a Lie algebra with this definition of the bracket for any Jordan algebra A ; c.f. Exercise 2, Chapter I in [Sat80]. The homomorphism of Lie algebras $\text{Der}(A) \oplus A \rightarrow \mathfrak{gl}(A)$, $D + a \mapsto D + L(a)$, is injective for any Jordan algebra A with an identity element.

Remark 4.6. If J is a complex semi-simple Jordan algebra, then it is the complexification of a real Euclidean Jordan algebra $J_{\mathbb{R}}$ (c.f. Theorem VIII.5.2 in [FK94]), where a real Jordan algebra is called Euclidean (or compact) if its trace form τ is positive-definite. The automorphism group of the Euclidean Jordan algebra $J_{\mathbb{R}}$ is a closed subgroup of the orthogonal group $O(J_{\mathbb{R}}, \tau)$ and thus compact. The connected component of $\text{Aut}(J)$ is the complexification of the connected component of $\text{Aut}(J_{\mathbb{R}})$, which shows again that $\text{Aut}(J)$ is a complex reductive Lie group.

Proposition 4.7 ([FK94], Proposition VIII.2.3). *The automorphism group $\text{Aut}(J)$ is a subgroup of the structure group $\text{Str}(J)$. Furthermore, if e denotes the identity element of J , then $\text{Aut}(J)$ is the isotropy of $\text{Str}(J)$ in e .*

5. One-parameter subgroups

In the following, let \mathbb{K} be algebraically closed, and let \mathbb{K}^{\times} denote the set of non-zero elements in \mathbb{K} .

The following result on the existence of one-parameter subgroups will be frequently used in the next sections when studying closed orbits and orbits closures of the automorphism group of a Jordan algebra in the n -fold product of the Jordan algebra.

Theorem 5.1 (see [Bir71], Theorem 4.2). *Let G be a reductive algebraic group and $\rho : G \rightarrow \text{GL}(V)$ a representation of G . Let $x \in V$ and Y be the unique closed orbit in the closure of $G \cdot x$. Then there exists a one-parameter subgroup $\lambda : \mathbb{K}^{\times} \rightarrow G$ such that $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists and $\lim_{t \rightarrow 0} \lambda(t) \cdot x = y \in Y$.*

Let J be a semi-simple Jordan algebra and let $\lambda : \mathbb{K}^{\times} \rightarrow \text{Aut}(J)$ be a one-parameter subgroup. As the irreducible representations of \mathbb{K}^{\times} are given by

$t \mapsto t^k$, $k \in \mathbb{Z}$, we have the decomposition

$$J = \bigoplus_{k \in \mathbb{Z}} J_k(\lambda),$$

where $J_k(\lambda) = \{z \in J \mid \lambda(t) \cdot z = t^k z \text{ for all } t \in \mathbb{K}^\times\}$. This defines a grading of the Jordan algebra J in the sense that $J_k(\lambda) \circ J_l(\lambda) \subseteq J_{k+l}(\lambda)$. We set

$$\begin{aligned} J_{\geq 0}(\lambda) &= \bigoplus_{k \geq 0} J_k(\lambda) = \{z \in J \mid \lim_{t \rightarrow 0} \lambda(t) \cdot z \text{ exists}\}, \\ J_0(\lambda) &= \{z \in J \mid \lambda(t) \cdot z = z \text{ for all } t \in \mathbb{K}^\times\}, \text{ and} \\ J_{> 0}(\lambda) &= \bigoplus_{k > 0} J_k(\lambda) = \{z \in J_{\geq 0}(\lambda) \mid \lim_{t \rightarrow 0} \lambda(t) \cdot z = 0\}. \end{aligned}$$

Remark 5.2. Note that the subspaces $J_{\geq 0}(\lambda)$, $J_{> 0}(\lambda)$, and $J_0(\lambda)$ are subalgebras of J .

The subalgebra $J_0(\lambda)$ is semi-simple since the trace form τ_J of J restricted to $J_0(\lambda)$ is a non-degenerate associative symmetric bilinear form and the identity e is contained in $J_0(\lambda)$.

Furthermore, $J_{> 0}(\lambda)$ is the radical of $J_{\geq 0}(\lambda)$ and $J_{\geq 0}(\lambda)$ is the semi-direct product of $J_0(\lambda)$ and $J_{> 0}(\lambda)$.

Definition 5.3. For any one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$, define moreover the following subspaces of $\mathfrak{h} = \mathfrak{str}(J) = \text{Der}(J) \oplus J$:

$$\begin{aligned} \mathfrak{p}(\lambda) &= \mathfrak{h}_{\geq 0}(\lambda) = \{X \in \mathfrak{h} \mid \lim_{t \rightarrow 0} \lambda(t) \cdot X \text{ exists}\} \\ \mathfrak{h}^\lambda &= \mathfrak{h}_0(\lambda) = \{X \in \mathfrak{h} \mid \lambda(t) \cdot X = X \text{ for all } t \in \mathbb{K}^\times\} \\ \mathfrak{u}(\lambda) &= \mathfrak{h}_{> 0}(\lambda) = \{X \in \mathfrak{p}(\lambda) \mid \lim_{t \rightarrow 0} \lambda(t) \cdot X = 0\} \end{aligned}$$

6. Closed orbits of the automorphism group

In the following, let J denote a semi-simple Jordan algebra over \mathbb{K} , where \mathbb{K} is algebraically closed.

The automorphism group $\text{Aut}(J)$ acts diagonally on the n -fold product $J \times \dots \times J$ of J :

$$\text{Aut}(J) \times (J \times \dots \times J) \rightarrow J, g(x_1, \dots, x_n) = (gx_1, \dots, gx_n).$$

The goal is now to characterize the closed orbits in $J \times \dots \times J$ under this action. An $\text{Aut}(J)$ -orbit through an n -tuple (x_1, \dots, x_n) is closed if and only if the orbit of the connected group $G = \text{Aut}(J)^\circ$ through (x_1, \dots, x_n) is closed since $\text{Aut}(J)$ is an algebraic subgroup of $\text{GL}(J)$ and hence has finitely many connected components.

Let H denote the connected component of the structure group $\text{Str}(J)$ of J , and $\mathfrak{h} = \mathfrak{str}(J)$ its Lie algebra.

Definition 6.1. For any n -tuple $x = (x_1, \dots, x_n)$ in $J \times \dots \times J$, define $A(x)$ to be the subalgebra of J generated by x_1, \dots, x_n . Moreover, let $\mathfrak{l}(x)$ be the Lie subalgebra

$$[A(x), A(x)] \oplus A(x) = \left\{ \sum_{i=1}^k [L(y_i), L(y'_i)] + z \mid y_i, y'_i, z \in A(x) \right\} \subseteq \text{Der}(J) \oplus J = \mathfrak{h}.$$

Example 6.2. Let $J = \text{Sym}_2(\mathbb{C})$. We have $G = \text{Aut}(J)^\circ = \text{SO}_2(\mathbb{C})$ and $H = \text{GL}_2(\mathbb{C})$ where G and H act via $g \cdot X = gX^t g$ for $X \in \text{Sym}_2(\mathbb{C})$.

- (i) Let $n = 2$, and $x = \left(\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \right)$. Then $A(x) = \text{Sym}_2(\mathbb{C})$, and $\mathfrak{l}(x) = \mathfrak{gl}_2(\mathbb{C})$. Remark that $A(x)$ is a simple Jordan algebra even though both $\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$ are nilpotent elements of $\text{Sym}_2(\mathbb{C})$.
- (ii) Let $n = 2$, and $x = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \right)$. Then $A(x) = \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and since $[L \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, L \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}] = 0$ we also get $\mathfrak{l}(x) = \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \mathbb{C} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$.

Definition 6.3. Let G be a linear algebraic group with Lie algebra \mathfrak{g} . An algebraic Lie subalgebra \mathfrak{c} of \mathfrak{g} is a Lie subalgebra such that there exists an algebraic subgroup C of G with Lie algebra \mathfrak{c} .

Definition 6.4. Let G be a reductive linear algebraic group, and \mathfrak{g} its Lie algebra. A Lie subalgebra \mathfrak{c} of \mathfrak{g} is called reductive in \mathfrak{g} if the adjoint representation of \mathfrak{g} restricted to \mathfrak{c} is completely reducible.

Remark 6.5. A Lie subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is reductive in \mathfrak{g} if and only if its algebraic hull, the smallest algebraic Lie subalgebra of \mathfrak{g} containing \mathfrak{c} , is reductive in \mathfrak{g} ; c.f. [Hoc81], § VIII.3.

Define an involution Θ on H by setting

$$\Theta : H \rightarrow H, \Theta(h) = {}^t h^{-1},$$

where the transpose ${}^t h$ of the element h is taken with respect to the trace form $\tau = \tau_J$ of J . This map Θ is a group automorphism of H . The differential of $\Theta : H \rightarrow H$ is given by

$$\theta : \mathfrak{h} \rightarrow \mathfrak{h}, \theta(X) = -{}^t X.$$

Its fixed point set is exactly $\text{Der}(J)$, which is the Lie algebra \mathfrak{g} of the automorphism group $\text{Aut}(J)$, and moreover we have $\theta|_J = -\text{id}_J$. Therefore, the connected component of the fixed point group H^Θ is precisely the connected component G of the automorphism group $\text{Aut}(J)$.

Let A be any subalgebra of J and $A = S + R$ be a Levi decomposition of A , where S is a semi-simple subalgebra of A and R is the radical (c.f. Theorem 3.10). Next, the structure of $[A, A] \oplus A = \left\{ \sum_{i=1}^k [L(y_i), L(y'_i)] + L(z) \mid y_i, y'_i, z \in A \right\}$ as a subset of $\mathfrak{h} = \mathfrak{str}(J) \subset \mathfrak{gl}(J)$ is examined. Note that $[A, A] \oplus A$ is θ -invariant.

The centre Z_S of S is defined to be the subset of S consisting of the elements which operator-commute with all other elements of S in S , i.e. $s \in Z_S$ if and only if the the derivation $[L(s), L(t)]$ vanishes (on S) for every $t \in S$.

Remark 6.6. Let $S = \bigoplus_i S_i$ a decomposition of S into simple subalgebras S_i (c.f. Theorem 11 in [Alb47]). By Satz 5.1, Kapitel I, of [BK66] the centre Z_{S_i} of each simple subalgebra S_i is $\mathbb{K}e_i$, where e_i is the identity element of S_i . Therefore, the centre of S is $Z_S = \bigoplus_i Z_{S_i} = \bigoplus_i \mathbb{K}e_i$.

The structure Lie algebra $\mathfrak{str}(S) \cong \text{Der}(S) \oplus S$ of S is reductive in $\mathfrak{gl}(S)$, all derivations $X \in \text{Der}(S)$ are inner, and its centre is Z_S . Let S' be the subspace of S defined by

$$S' = \text{Der}(S)(S) = \left\{ \sum_i [L(s_i), L(t_i)](s) \mid s_i, t_i, s \in S \right\}.$$

Since $\text{Der}(S) \oplus S$ is a reductive Lie algebra, its derived algebra

$$[\text{Der}(S) \oplus S, \text{Der}(S) \oplus S] = \text{Der}(S) \oplus (\text{Der}(S)(S)) = \text{Der}(S) \oplus S'$$

is semi-simple and $\text{Der}(S) \oplus S = (\text{Der}(S) \oplus S') \oplus Z_S$, which implies $S = S' \oplus Z_S$, and $\text{Der}(S) = [S' \oplus Z_S, S' \oplus Z_S] = [S', S'] = \{ \sum_i [L(r_i), L(s_i)] \mid r_i, s_i \in S' \}$. Furthermore, we have $S' = (Z_S)^\perp = \{ z \in J \mid \tau_J(z, w) = 0 \text{ for all } w \in Z_S \}$.

Example 6.7. Let $S = \text{Sym}_2(\mathbb{C})$. Then $[\text{Sym}_2(\mathbb{C}), \text{Sym}_2(\mathbb{C})] \oplus \text{Sym}_2(\mathbb{C}) \cong \mathfrak{gl}_2(\mathbb{C})$ and $[S, S] \cong \mathfrak{so}_2(\mathbb{C})$. The centre of $S = \text{Sym}_2(\mathbb{C})$ is $Z_S = \mathbb{C} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$$S' = \{ x \in \text{Sym}_2(\mathbb{C}) \mid \text{Tr}(x) = 0 \} = \text{Sym}_2(\mathbb{C}) \cap \mathfrak{sl}_2(\mathbb{C}),$$

and $[S', S'] \oplus S' \cong \mathfrak{sl}_2(\mathbb{C})$. In particular, $S' \subset S$ is not a subalgebra of the Jordan algebra S .

Lemma 6.8. *The Lie subalgebra $[S, S] \oplus S \subset \mathfrak{gl}(J)$ is reductive in $\mathfrak{gl}(J)$.*

Proof. The structure Lie algebra $\mathfrak{str}(S) \cong \text{Der}(S) \oplus S$ of $S \subseteq J$ is reductive. The inclusion map $S \hookrightarrow J$ extends to an injective homomorphism of Lie algebras $\varphi : \mathfrak{str}(S) \hookrightarrow \mathfrak{str}(J) \subset \mathfrak{gl}(J)$, where an element $X + s \in \text{Der}(S) \oplus S$, $X = \sum_i [L(r_i), L(s_i)]$, $r_i, s_i \in S$, is mapped to $\sum_i [L(r_i), L(s_i)] + L(s)$, now considered as a transformation of the Jordan algebra J (c.f. [Jac51], Lemma 8.3). The image of φ is $[S, S] \oplus S$.

Using the preceding remark, it follows that $[S', S'] \oplus S'$ is a semi-simple Lie algebra. It thus remains to show that the centre Z of $[S, S] \oplus S$ consists of semi-simple elements. If $S = \bigoplus_i S_i$ is again a decomposition of S into simple subalgebras S_i , and e_i denotes the identity element of S_i , the centre Z is spanned by the left multiplications $L(e_i) \in \mathfrak{gl}(J)$. Since the element e_i is the identity of S_i , it is in particular idempotent, i.e. $e_i^2 = e_i$, as an element of J . This implies that $L(e_i) : J \rightarrow J$ is diagonalisable and the only possible eigenvalues of $L(e_i)$ are $1, \frac{1}{2}, 0$ (see e.g. [Alb47], § 5, Lemma 5). In particular, the transformation $L(e_i) : J \rightarrow J$ is semi-simple. ■

Remark 6.9. The proof of the lemma shows moreover that $[S, S] \oplus S$ is an algebraic Lie subalgebra of $\mathfrak{gl}(J)$.

Remark 6.10. Using the notion of a representation of a Jordan algebra (see e.g. [Jac51]), the lemma could also be proven in the following way:

Since S is a semi-simple Jordan algebra, the representation $S \rightarrow \text{End}(J)$, $x \mapsto L(x)$, is completely reducible (c.f. [Jac51], § 8). Consequently, the representation of $[S, S] \oplus S \subset \mathfrak{gl}(J)$ on J is also completely reducible and thus $[S, S] \oplus S$ is reductive in $\mathfrak{gl}(J)$.

Lemma 6.11. *The subspace $[A, R] \oplus R$ is a nilpotent ideal in $[A, A] \oplus A \subseteq \mathfrak{str}(J)$ and is nilpotent on J , i.e. there is $m \in \mathbb{N}$ such that $\{f_1 \circ \dots \circ f_m \mid f_i \in [A, R] \oplus R\} = 0$*

Proof. We have $[A, R] \subset [A, R] \oplus R$ and for $a, b \in A$, $r \in R$ we have $[a, [L(b), L(r)]] = -(b(ra) - r(ba)) \in R$ since R is an ideal in A . Therefore, $[A, [A, R] \oplus R] \subset [A, R] \oplus R$ and consequently $[[A, A], [A, R] \oplus R] \subseteq [A, R] \oplus R$. Hence, $[A, R] \oplus R$ is an ideal in $[A, A] \oplus A$.

To show that $[A, R] \oplus R \subseteq \mathfrak{h} = \mathfrak{str}(J)$ is a nilpotent Lie subalgebra consisting of nilpotent endomorphisms, similar arguments can be used as are needed to show that a derivation of A of the form $\sum_i [L(a_i), L(r_i)]$, $a_i \in A$, $r_i \in R$, is nilpotent (c.f. Remark 3.11). A proof is given for example in [Jac51], Corollary 8.2. ■

As a corollary of the preceding lemmata on the structure of $[A, A] \oplus A$ we get the following:

Corollary 6.12. *The Lie subalgebra $[A, A] \oplus A \subseteq \mathfrak{str}(J) \subseteq \mathfrak{gl}(J)$ might be written as*

$$[A, A] \oplus A = ([S, S] \oplus S) \oplus ([A, R] \oplus R),$$

for $A = S + R$, where again R is the radical of A and S is a Levi factor. The Lie subalgebra $[S, S] \oplus S$ is reductive in $\mathfrak{gl}(J)$ and $[A, R] \oplus R$ is an ideal which is nilpotent on J .

Proposition 6.13. *The Lie subalgebra $[A, A] \oplus A \subseteq \mathfrak{str}(J) \subseteq \mathfrak{gl}(J)$ is algebraic.*

Proof. As remarked before, the Lie subalgebra $[S, S] \oplus S \subset \mathfrak{gl}(J)$ is algebraic. Moreover, the Lie subalgebra $[A, R] \oplus R \subseteq \mathfrak{gl}(J)$ is an algebraic Lie subalgebra since it is nilpotent on J .

As $[A, A] \oplus A = ([A, R] \oplus R) \oplus ([S, S] \oplus S)$, and both $[A, R] \oplus R$ and $[S, S] \oplus S$ are algebraic, it follows that $[A, A] \oplus A$ is an algebraic Lie subalgebra of $\mathfrak{str}(J) \subseteq \mathfrak{gl}(J)$ (c.f. [Hoc81], Chapter VIII, Theorem 3.4). ■

Proposition 6.14. *For any subalgebra $A \subseteq J$, $A = S + R$, where R is the radical of A and $S \subseteq A$ is a semi-simple subalgebra, there exists a one-parameter*

subgroup $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$ such that

$$\begin{aligned} A &\subseteq J_{\geq 0}(\lambda), \\ S &\subseteq J_0(\lambda), \text{ and} \\ R &\subseteq J_{> 0}(\lambda). \end{aligned}$$

Proof. Consider the Lie algebra $\mathfrak{l} = [A, A] \oplus A \subseteq \mathfrak{gl}(J)$. By Corollary 6.12, it might be written as $\mathfrak{l} = ([S, S] \oplus S) \oplus ([A, R] \oplus R)$, $[S, S] \oplus S$ is reductive in $\mathfrak{gl}(J)$ and $[A, R] \oplus R$ is a nilpotent ideal whose action on J is nilpotent.

Let H denote again the connected component of the structure group $\text{Str}(J)$, and $\mathfrak{h} = \mathfrak{str}(J)$ its Lie algebra. The Lie algebra $\mathfrak{l} \subseteq \mathfrak{h} = \mathfrak{str}(J)$ is θ -invariant and by Proposition 12.4 in [Ric88] there exists a one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow G = H^\theta$ such that $\mathfrak{l} \subseteq \mathfrak{p}(\lambda) = \mathfrak{h}_{\geq 0}(\lambda)$, $[S, S] \oplus S \subseteq \mathfrak{h}^\lambda = \mathfrak{h}_0(\lambda)$, and $[A, R] \oplus R \subseteq \mathfrak{u}(\lambda) = \mathfrak{h}_{> 0}(\lambda)$, and the statement of the proposition follows. ■

Theorem 6.15. *Let J be a complex semi-simple Jordan algebra and consider the diagonal action of the automorphism group $\text{Aut}(J)$ on the n -fold product $J \times \dots \times J$. Let $x = (x_1, \dots, x_n) \in J \times \dots \times J$. Then the orbit $\text{Aut}(J) \cdot x$ is closed if and only if $A(x)$ is a semi-simple Jordan algebra.*

Proof. First, let the orbit $\text{Aut}(J) \cdot x$, or equivalently the orbit $G \cdot x$ of the connected component $G = \text{Aut}(J)^\circ$, be closed. Assume now that $A(x)$ is not semi-simple, and let $A(x) = S(x) + R(x)$ be a decomposition of $A(x)$ such that $S(x)$ is semi-simple and $R(x) \neq 0$ is the radical of $A(x)$. By Proposition 6.14, there is a one-parameter subgroup $\mathbb{K}^\times \rightarrow G$ such that $A \subseteq J_{\geq 0}(\lambda)$, $S \subseteq J_0(\lambda)$, and $R \subseteq J_{> 0}(\lambda)$. Since $A(x) \subseteq J_{\geq 0}(\lambda)$, we have $x_1, \dots, x_n \in J_{\geq 0}(\lambda)$ and the limit $\lim_{t \rightarrow 0} \lambda(t) \cdot x$ exists. As we assumed that the orbit through x is closed, we get $y = (y_1, \dots, y_n) = \lim_{t \rightarrow 0} \lambda(t) \cdot (x_1, \dots, x_n) \in G \cdot x$. Consider the map

$$\psi_\lambda : J_{\geq 0}(\lambda) \rightarrow J_0(\lambda), \psi_\lambda(z) = \lim_{t \rightarrow 0} \lambda(t) \cdot z.$$

Remark that $\psi_\lambda|_{J_0(\lambda)} = \text{id}_{J_0(\lambda)}$ and $\psi_\lambda(r) = 0$ for all $r \in J_{> 0}(\lambda)$. By definition we have $\psi_\lambda(x_j) = y_j$, $j = 1, \dots, n$, and $\psi_\lambda(A(x)) = S(x)$ is semi-simple. Moreover, $y_j \in S(x)$ for all j and thus $A(y) \subseteq S(x)$. Since $A(x) = S(x) + R(x)$ is not semi-simple, $R(x) \neq 0$, and the dimension of $A(y) \subseteq S(x)$ is strictly smaller than the dimension of $A(x)$. At the same time, $A(x)$ and $A(y)$ are conjugated by an element in G because x and y lie in the same G -orbit, which yields a contradiction.

Now, let $A(x)$ be semi-simple and suppose now that the G -orbit through $x = (x_1, \dots, x_n)$ is not closed. By Theorem 5.1 there is a one-parameter group $\lambda : \mathbb{K}^\times \rightarrow G$ such that

$$y = (y_1, \dots, y_n) = \lim_{t \rightarrow 0} \lambda(t) \cdot x = \lim_{t \rightarrow 0} \lambda(t) \cdot (x_1, \dots, x_n)$$

exists and $G \cdot y$ is the unique closed G -orbit in the closure of $G \cdot x$. Since $y_j = \lim_{t \rightarrow 0} \lambda \cdot x_j$, $j = 1, \dots, n$, the subalgebra $A(x)$ is contained in $J_{\geq 0}(\lambda)$. The subalgebra $A(x)$ is semi-simple by assumption and by Theorem 3.10 $A(x)$

is strictly conjugate to a subalgebra of $J_0(\lambda)$ and there exists an inner nilpotent derivation $D = \sum_{i=1}^m [L(z_i), L(u_i)]$, $z_i \in J_{\geq 0}(\lambda)$, $u_i \in J_{> 0}(\lambda)$, of $J_{\geq 0}(\lambda)$ such that $\exp(D)(A(x)) \subseteq J_0(\lambda)$. Note that $D(J_k(\lambda)) \subseteq \bigoplus_{l>k} J_l(\lambda)$ for any k , which yields $\lim_{t \rightarrow 0} \lambda(t) \cdot (D(z)) = 0$ for arbitrary $z \in J_{\geq 0}(\lambda)$.

Consider again the map $\psi_\lambda : J_{\geq 0}(\lambda) \rightarrow J_0(\lambda)$, $z \mapsto \lim_{t \rightarrow 0} \lambda(t) \cdot z$. The sum $\exp(D)$ is a finite sum since D is nilpotent and we have

$$\psi_\lambda(\exp(D)(z)) = \lim_{t \rightarrow 0} \lambda(t) \cdot \left(\sum_{k=0}^N \frac{1}{k!} D^k(z) \right) = \lim_{t \rightarrow 0} \lambda(t) \cdot z = \psi_\lambda(z).$$

Using that $\exp(D)(z) \in J_0(\lambda)$ for any $z \in A(x)$, and thus $\psi_\lambda(\exp(D)(z)) = \exp(D)(z)$ for all $z \in A(x)$, we obtain $\psi_\lambda(z) = \exp(D)(z)$ for arbitrary $z \in A(x)$. The derivation D of $J_{\geq 0}(\lambda)$ can be extended to a derivation D of J because it is an inner derivation. Hence $\exp(D) \in G$. We have $x_i \in A(x)$ by definition and now get

$$y_i = \lim_{t \rightarrow 0} \lambda(t) \cdot x_i = \psi_\lambda(x_i) = \exp(D)x_i.$$

Therefore,

$$y = (y_1, \dots, y_n) = \lim_{t \rightarrow 0} \lambda(t) \cdot (x_1, \dots, x_n) = \exp(D)(x_1, \dots, x_n) \in G \cdot x,$$

which is a contradiction. ■

Definition 6.16. An n -tuple $x = (x_1, \dots, x_n) \in J \times \dots \times J$ is called unstable if $0 = (0, \dots, 0) \in J \times \dots \times J$ is contained in the closure $\overline{\text{Aut}(J) \cdot x}$ of the orbit $\text{Aut}(J) \cdot x$.

Proposition 6.17. An n -tuple $x = (x_1, \dots, x_n) \in J \times \dots \times J$ is unstable if and only if $A(x)$ is a solvable subalgebra of J .

Proof. Let x be unstable. Then $0 \in \overline{\text{Aut}(J) \cdot x}$ and 0 is the unique closed orbit in the closure of $\text{Aut}(J) \cdot x$. By Theorem 5.1 there exists a one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$ with $\lim_{t \rightarrow 0} \lambda(t) \cdot x = 0$. Therefore, x_1, \dots, x_n and thus also $A(x)$ are contained in $J_{> 0}(\lambda)$. Since $J_{> 0}(\lambda)$ is the radical of $J_{\geq 0}(\lambda) \subset J$, it is a solvable subalgebra of J . Hence, $A(x)$ is also solvable.

Now, let $A(x)$ be solvable. Then $A(x)$ is equal to its radical $R(x)$, and by Proposition 6.14 there is a one-parameter subgroup $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$ such that $A(x) \subseteq J_{> 0}(\lambda)$. Consequently, $\lim_{t \rightarrow 0} \lambda(t) \cdot y = 0$ for any $y \in A(x)$ and in particular $\lim_{t \rightarrow 0} \lambda(t) \cdot (x_1, \dots, x_n) = (0, \dots, 0) = 0 \in \overline{\text{Aut}(J) \cdot x}$. ■

Remark 6.18. Consider the case $n = 1$. Every element $x \in J$ can be written as $x = x_s + x_n$, $x_s, x_n \in \mathbb{C}[x]$, where x_s is semi-simple and x_n is nilpotent, and this decomposition is unique (see e.g. [BK66], Kapitel 1, § 3). It can be shown that the Jordan subalgebra $A(x)$ generated by x is semi-simple (solvable) if and only if x is semi-simple (nilpotent). Thus, the orbit $\text{Aut}(J) \cdot x$ is closed (unstable) if and only if x is semi-simple (nilpotent).

Remark 6.19. Let again $n = 1$ and consider the isomorphism $\text{Der}(J) \oplus J \rightarrow \mathfrak{str}(J)$, $D + z \mapsto D + L(z)$. It already follows from [KR69] (Proposition 3 and Theorem 4) that the $\text{Aut}(J)$ -orbit through an element $L(z) \in L(J) \subset \mathfrak{str}(J)$ is closed (unstable) in $L(J) \cong J$ if and only if $L(z) \in \mathfrak{str}(J) \subseteq \mathfrak{gl}(J)$ is a semi-simple (nilpotent) endomorphism. Using Peirce decomposition in J it follows that the element $z \in J$ is semi-simple as an element of the Jordan algebra if and only if $L(z) \in \mathfrak{str}(J)$ is semi-simple, and by Theorem 5, Chapter III, [Koe99], $z \in J$ is nilpotent precisely if the endomorphism $L(z)$ is nilpotent.

Remark 6.20. The statement in the case $n = 1$ could also be shown as follows: Consider again the isomorphism of Lie algebras $\text{Der}(J) \oplus J \rightarrow \mathfrak{str}(J) \subseteq \mathfrak{gl}(J)$, $D + z \mapsto D + L(z)$. As mentioned before, the group \mathbb{Z}_2 acts by automorphisms on the structure group $\text{Str}(J)$ via the involution $\Theta : \text{Str}(J) \rightarrow \text{Str}(J)$, $\Theta(h) = {}^t h^{-1}$, where ${}^t h$ denotes again the transpose of h with respect to the trace form τ_J of J . Let $\theta : \mathfrak{str}(J) \rightarrow \mathfrak{str}(J)$ again be the differential of Θ . The Lie subalgebra fixed by θ is $\text{Der}(J)$. Note that the Lie algebra spanned by $L(z)$ is stable under the involution θ . It is reductive in $\mathfrak{str}(J)$ if and only if $L(z)$ (or equivalently z) is semi-simple, and it is a Lie algebra acting nilpotently on J if and only if $L(z)$ (or equivalently z) is nilpotent. Since the orbit of the automorphism group $\text{Aut}(J)$ through $L(z)$ is contained in the subspace $L(J) \cong J$, the orbit $\text{Aut}(J) \cdot L(z) \subseteq \text{str}(J)$ is closed (unstable) if and only if the orbit $\text{Aut}(J) \cdot z \subseteq J$ is closed (unstable). Thus, by Theorem 13.2/13.3 in [Ric88], the orbit $\text{Aut}(J) \cdot z$ is closed (unstable) precisely if z is semi-simple (nilpotent).

Remark 6.21. Let again $n = 1$, let J be a complex semi-simple Jordan algebra, and let $J_{\mathbb{R}}$ be a Euclidean real form of J , $J = J_{\mathbb{R}} \oplus iJ_{\mathbb{R}}$. Denote the complex conjugation of an element $x \in J$ with respect to this decomposition by \bar{x} . If τ denotes the trace form on J , which is the \mathbb{C} -linear extension of the trace form of $J_{\mathbb{R}}$, then $\langle x, y \rangle = \tau(x, \bar{y})$ defines a positive-definite Hermitian product on J . This Hermitian product is invariant under the automorphism group $\text{Aut}(J_{\mathbb{R}})$ of $J_{\mathbb{R}}$. Let \mathfrak{k} be the Lie algebra of $\text{Aut}(J_{\mathbb{R}})$ and denote again by K the connected component of $\text{Aut}(J_{\mathbb{R}})$ and by $G = K^{\mathbb{C}}$ the connected component of $\text{Aut}(J)$. Then $\mu : J \rightarrow \mathfrak{k}^*$, $\mu^{\xi}(x) = \mu(x)(\xi) = \langle \xi(x), x \rangle$ for $x \in J$, $\xi \in \mathfrak{k}$, defines a K -equivariant moment map.

Assume now that $J_{\mathbb{R}}$ is simple. Using Lemma VI.1.1 of [FK94], there is a positive constant κ such that

$$\tau(Dx, y) = \kappa \text{Tr}(L(Dx) \circ L(y))$$

for any derivation D and $x, y \in J$. Therefore,

$$\begin{aligned} \mu^{\xi}(x) &= \tau(\xi(x), \bar{x}) = \kappa \text{Tr}(L(\xi(x)) \circ L(\bar{x})) \\ &= \kappa \text{Tr}([\xi, L(x)] \circ L(\bar{x})) = \kappa \text{Tr}(\xi \circ [L(x), L(\bar{x})]). \end{aligned}$$

The bilinear form on $\text{Der}(J)$ defined by $(D_1, D_2) \mapsto \text{Tr}(D_1 \circ D_2)$ is non-degenerate (c.f. [BK66], Kapitel IX, Lemma 3.2), and identifying \mathfrak{k}^* with \mathfrak{k} via a multiple of this bilinear form we get

$$\mu(x) = [L(x), L(\bar{x})].$$

The zero level set of μ is

$$\mu^{-1}(0) = \{x \in J \mid [L(x), L(\bar{x})] = 0\}.$$

Writing x as $x = x_1 + ix_2$ with $x_j \in J_{\mathbb{R}}$, $[L(x), L(\bar{x})] = 0$ is equivalent to $[L(x_1), L(x_2)] = 0$. If two elements $x_1, x_2 \in J_{\mathbb{R}}$ operator-commute, i.e. $[L(x_1), L(x_2)] = 0$, then x_1 and x_2 are simultaneously diagonalisable, meaning that there exists a Jordan frame c_1, \dots, c_k of $J_{\mathbb{R}}$, and $\alpha_j, \beta_j \in \mathbb{R}$ such that

$$x_1 = \sum_{j=1}^k \alpha_j c_j \quad \text{and} \quad x_2 = \sum_{j=1}^k \beta_j c_j,$$

see e.g. [FK94], Lemma X.2.2.

Let c_1, \dots, c_k now be a fixed Jordan frame of $J_{\mathbb{R}}$, and define the following subspace of J :

$$R_c = \left\{ \sum_{j=1}^k \alpha_j c_j \mid \alpha_j \in \mathbb{C} \right\}$$

Then,

$$\mu^{-1}(0) = K \cdot R_c,$$

since K acts transitively on the set of Jordan frames of $J_{\mathbb{R}}$.

Using that an orbit $G \cdot x$ is closed if and only if it meets the zero level set $\mu^{-1}(0)$ (see e.g. [KN79]), this also shows that the orbit $G \cdot x$ is closed if and only if $x \in J$ is a semi-simple element.

7. Orbit closures

For an n -tuple $x = (x_1, \dots, x_n) \in J \times \dots \times J$, let $A(x)$ denote again the subalgebra generated by x_1, \dots, x_n . Let $R(x)$ denote the radical of $A(x)$ and let $S(x)$ be a semi-simple subalgebra of $A(x)$ satisfying $A(x) = S(x) + R(x)$. Then there are unique elements $s_j \in S$, $r_j \in R$ such that $x_j = s_j + r_j$, and writing $s = (s_1, \dots, s_n)$, $r = (r_1, \dots, r_n)$ we get the decomposition $x = s + r$. This can be regarded as a ‘‘Levi-decomposition’’ of the n -tuple x .

Remark 7.1. Note that the decomposition of the n -tuple x into $x = s + r$ is in general not unique, but depends on the choice of the semi-simple subalgebra $S(x)$.

However, in the case $n = 1$ the subalgebra $A(x)$ is generated by one element and is thus associative. Consequently, decomposition $x = s + r$ is unique and is the usual decomposition of x into its semi-simple and nilpotent part.

More generally, the decomposition $x = s + r$ is unique whenever the elements of the radical $R(x)$ operator-commute with all elements in $A(x)$, i.e. $[A(x), R(x)] = 0$.

Proposition 7.2. *Let $x = s + r$ be a decomposition of an n -tuple $x \in J \times \dots \times J$ as above. Then $\text{Aut}(J) \cdot s$ is the unique closed orbit in the closure $\overline{\text{Aut}(J) \cdot x}$ of the $\text{Aut}(J)$ -orbit through x .*

Proof. Let $\lambda : \mathbb{K}^\times \rightarrow \text{Aut}(J)$ be a one-parameter subgroup such that $s \in J_0(\lambda)$ and $r \in J_{>0}(\lambda)$ (c.f. Proposition 6.14). Then $s = \lim_{t \rightarrow 0} \lambda(t) \cdot x$ is contained in $\overline{\text{Aut}(J) \cdot x}$.

The subalgebra $A(s)$ generated by s_1, \dots, s_n is contained in $S(x)$ since $s_j \in S(x)$ for each j . Moreover, $x_j = s_j + r_j \in A(s) + R(x)$. As $S_1 \oplus R(x)$ is a subalgebra of J for any subalgebra S_1 of $S(x)$, we get $A(s) = S(x)$, and $A(s)$ is semi-simple. It follows that the orbit $\text{Aut}(J) \cdot s$ is closed. ■

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