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The Splitting Problem for Complex Homogeneous Supermanifolds

Elizaveta Vishnyakova*

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Abstract. It is a classical result that any complex analytic Lie supergroup $\mathcal{G}$ is split [5], that is its structure sheaf is isomorphic to the structure sheaf of a certain vector bundle. However, there do exist non-split complex analytic homogeneous supermanifolds.

We study the question how to find out whether a complex analytic homogeneous supermanifold is split or non-split. Our main result is a description of left invariant gradings on a complex analytic homogeneous supermanifold $\mathcal{G}/\mathcal{H}$ in the terms of $\mathcal{H}$-invariants. As a corollary to our investigations we get some simple sufficient conditions for a complex analytic homogeneous supermanifold to be split in terms of Lie algebras.
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1. Introduction

A supermanifold is called split if its structure sheaf is isomorphic to the exterior power of a certain vector bundle. By Batchelor's Theorem any real supermanifold is non-canonically split. However, this is false in the complex analytic case. The property of a supermanifold to be split is very important for several reasons. For instance, in [2] it was shown that the moduli space of super Riemann surfaces is not projected (and in particular is not split) for genus $g \geqslant 5$. The physical meaning of this result is that [2]: "certain approaches to superstring perturbation theory that are very powerful in low orders have no close analog in higher orders". Another problem, when the property of a supermanifold to be split is very important, is the calculation of the cohomology group with values in a vector bundle over a supermanifold. In the split case we may use the well understood tools of complex analytic geometry. In the general case, several methods were suggested by Onishchik's school: spectral sequences, see e.g. [12]. All these methods connect the cohomology group with values in a vector bundle with the cohomology group with values in the corresponding split vector bundle.

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How do we determine whether a complex analytic supermanifold is split or non-split? Let me describe here some results in this direction that were obtained by Green, Koszul, Onishchik and Serov. In [3] Green described a moduli space with a marked point such that any non-marked point corresponds to a non-split supermanifold while the marked point corresponds to a split one. His idea was used for instance in [2]. The calculation of the Green moduli space is a difficult problem itself, and in many cases the method is difficult to apply. Furthermore, Onishchik and Serov [9, 10, 11] considered grading derivations, which correspond to $\mathbb{Z}$-gradings of the structure sheaf of a supermanifold. For example, it was shown that almost all super-grassmannians do not possess such derivations, i.e. their structure sheaves do not possess any $\mathbb{Z}$-gradings. Hence, in particular, they are non-split. The idea of grading derivations was independently used by Koszul. In [4] the following statement was proved: if the tangent bundle of a supermanifold $\mathcal{M}$ possesses a (holomorphic) connection then $\mathcal{M}$ is split. (Koszul's proof works in real and complex analytic cases.) In fact, it was shown that we can assign a grading derivation to any supermanifold with a connection and that this grading derivation is induced by a $\mathbb{Z}$-grading of a vector bundle.

Assume that a complex analytic supermanifold $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ is split. By definition this means that its structure sheaf $\mathcal{O}_\mathcal{M}$ is isomorphic to $\bigwedge \mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on the complex analytic manifold $\mathcal{M}_0$. The sheaf $\bigwedge \mathcal{E}$ is naturally $\mathbb{Z}$-graded and the isomorphism $\mathcal{O}_\mathcal{M} \simeq \bigwedge \mathcal{E}$ induces the $\mathbb{Z}$-grading in $\mathcal{O}_\mathcal{M}$. We call such gradings *split.* The main result of our paper is a description of those left invariant split gradings on a homogeneous superspace $\mathcal{G}/\mathcal{H}$ which are compatible with split gradings on $\mathcal{G}$. We also give sufficient conditions for pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g} = \operatorname{Lie}\mathcal{G}$ and $\mathfrak{h} = \operatorname{Lie}\mathcal{H}$, such that $\mathcal{G}/\mathcal{H}$ is split.

2. Complex analytic supermanifolds. Main definitions.

We will use the word "supermanifold" in the sense of Berezin and Leites, see [1], [7] and [8] for details. Throughout, we will be interested in the complex analytic version of the theory. Recall that a *complex analytic superdomain of dimension* $n|m$ is a $\mathbb{Z}_2$-graded ringed space

$$\mathcal{U} = \Big(U, \mathcal{F}_U \otimes \bigwedge(m)\Big),$$

where $\mathcal{F}_U$ is the sheaf of holomorphic functions on an open set $U \subset \mathbb{C}^n$ and $\bigwedge(m)$ is the exterior (or Grassmann) algebra with m generators. A *complex analytic supermanifold* of dimension $n|m$ is a $\mathbb{Z}_2$-graded ringed space that is locally isomorphic to a complex superdomain of dimension $n|m$.

Let $\mathcal{M} = (\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ be a complex analytic supermanifold and

$$\mathcal{J}_\mathcal{M} = (\mathcal{O}_\mathcal{M})_{\bar 1} + (\mathcal{O}_\mathcal{M})_{\bar 1}^2$$

be the subsheaf of ideals generated by odd elements in $\mathcal{O}_\mathcal{M}$. We put $\mathcal{F}_\mathcal{M} := \mathcal{O}_\mathcal{M}/\mathcal{J}_\mathcal{M}$. Then $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ is a usual complex analytic manifold. It is called the *reduction* or *underlying space* of $\mathcal{M}$. We will write $\mathcal{M}_0$ instead of $(\mathcal{M}_0, \mathcal{F}_\mathcal{M})$ for simplicity of notation. Morphisms of supermanifolds are just morphisms of the corresponding $\mathbb{Z}_2$-graded ringed spaces. If $f : \mathcal{M} \to \mathcal{N}$ is a morphism of supermanifolds, then we denote by f_0 the morphism of the underlying spaces $\mathcal{M}_0 \to \mathcal{N}_0$ and by f^* the morphism of the structure sheaves $\mathcal{O}_\mathcal{N} \to (f_0)_*(\mathcal{O}_\mathcal{M})$. If $x \in \mathcal{M}_0$ and $\mathfrak{m}_x$ is the maximal ideal of the local superalgebra $(\mathcal{O}_\mathcal{M})_x$, then the vector superspace $T_x(\mathcal{M}) := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}_0$.

Denote by $\mathcal{T}_\mathcal{M}$ the *tangent sheaf* or the *sheaf of vector fields* of $\mathcal{M}$. In other words, $\mathcal{T}_\mathcal{M}$ is the sheaf of derivations of the structure sheaf $\mathcal{O}_\mathcal{M}$. Since the sheaf $\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}_2$-graded, the tangent sheaf $\mathcal{T}_\mathcal{M}$ is also $\mathbb{Z}_2$-graded, i.e. there is the natural decomposition $\mathcal{T}_\mathcal{M} = (\mathcal{T}_\mathcal{M})_{\bar 0} \oplus (\mathcal{T}_\mathcal{M})_{\bar 1}$, where

$$(\mathcal{T}_\mathcal{M})_{\bar i} := \{v \in \mathcal{T}_\mathcal{M} \mid v\big((\mathcal{O}_\mathcal{M})_{\bar j}\big) \subset (\mathcal{O}_\mathcal{M})_{\bar j + \bar i}\}.$$

Let $\mathcal{M}_0$ be a complex analytic manifold and let $\mathcal{E}$ be the sheaf of holomorphic sections of a vector bundle over $\mathcal{M}_0$. Then the ringed space $(\mathcal{M}_0, \bigwedge \mathcal{E})$ is a supermanifold. In this case dim $\mathcal{M} = n|m$, where $n = \dim \mathcal{M}_0$ and m is the rank of the locally free sheaf $\mathcal{E}$.

Definition 2.1. A supermanifold $(\mathcal{M}_0, \mathcal{O}_\mathcal{M})$ is called *split* if $\mathcal{O}_\mathcal{M} \simeq \bigwedge \mathcal{E}$ for a locally free sheaf $\mathcal{E}$ on $\mathcal{M}_0$. The grading of $\mathcal{O}_\mathcal{M}$ induces by an isomorphism $\mathcal{O}_\mathcal{M} \simeq \bigwedge \mathcal{E}$ and the natural $\mathbb{Z}$-grading of $\bigwedge \mathcal{E} = \bigoplus_p \bigwedge^p \mathcal{E}$ is called *split grading.*

For example, all smooth supermanifolds are split by Batchelor's Theorem. In [4] it was shown that all complex analytic Lie supergroups are split too. In this paper we study the splitting problem for complex analytic homogeneous supermanifolds.

3. Lie supergroups and their homogeneous spaces

Lie supergroups and super Harish-Chandra pairs. A *Lie supergroup* is a group object in the category of supermanifolds, i.e. it is a supermanifold $\mathcal{G}$ with three morphisms: the multiplication morphism, the inversion morphism and the identity morphism, which satisfy the usual conditions, modeling the group axioms. In this case the underlying space $\mathcal{G}_0$ is a Lie group. The structure sheaf of a (complex analytic) Lie supergroup can be explicitly described in terms of the corresponding Lie superalgebra and underlying Lie group using super Harish-Chandra pairs (see [5] and [14] for more details). Let us describe this construction briefly.

Definition 3.1. A *super Harish-Chandra pair* is a pair $(\mathcal{G}_0, \mathfrak{g})$ that consists of a Lie group $\mathcal{G}_0$ and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar 0} \oplus \mathfrak{g}_{\bar 1}$ such that $\mathfrak{g}_{\bar 0} = \mathrm{Lie}\, \mathcal{G}_0$ provided with a representation $\mathrm{Ad} : \mathcal{G}_0 \to \mathrm{Aut}\, \mathfrak{g}$ of $\mathcal{G}_0$ in $\mathfrak{g}$ such that:

- Ad preserves the parity and induces the adjoint representation of $\mathcal{G}_0$ on $\mathfrak{g}_{\bar 0}$;
- the differential $(\mathrm{d}\,\mathrm{Ad})_e$ at the identity $e \in \mathcal{G}_0$ coincides with the adjoint representation ad of $\mathfrak{g}_{\bar 0}$ on $\mathfrak{g}$.

If a super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g})$ is given, it determines the Lie supergroup $\mathcal{G}$ in the following way, see [5]. Let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. It is clear that $\mathfrak{U}(\mathfrak{g})$ is a $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module, where $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$ is the universal enveloping algebra of $\mathfrak{g}_{\bar{0}}$. Recall that we denote by $\mathcal{F}_{\mathcal{G}_0}$ the structure sheaf of the manifold $\mathcal{G}_0$. The natural action of $\mathfrak{g}_{\bar{0}}$ on the sheaf $\mathcal{F}_{\mathcal{G}_0}$ gives rise to a structure of $\mathfrak{U}(\mathfrak{g}_{\bar{0}})$-module on $\mathcal{F}_{\mathcal{G}_0}(U)$ for any open set $U \subset \mathcal{G}_0$. Putting

$$\mathcal{O}_{\mathcal{G}}(U) = \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}_{\bar{0}})}\big(\mathfrak{U}(\mathfrak{g}), \mathcal{F}_{\mathcal{G}_0}(U)\big)$$

for every open $U \subset \mathcal{G}_0$, we get a sheaf $\mathcal{O}_{\mathcal{G}}$ of $\mathbb{Z}_2$-graded vector spaces. (Here we assume that the functions in $\mathcal{F}_{\mathcal{G}_0}(U)$ are even.) The enveloping superalgebra $\mathfrak{U}(\mathfrak{g})$ has a Hopf superalgebra structure. Using this structure we can define the product of elements from $\mathcal{O}_{\mathcal{G}}$ such that $\mathcal{O}_{\mathcal{G}}$ becomes a sheaf of superalgebras, see [5] and [14] for details. A supermanifold structure on $\mathcal{O}_{\mathcal{G}}$ is determined by the isomorphism $\Phi_{\mathfrak{g}} : \mathcal{O}_{\mathcal{G}} \to \mathrm{Hom}\,\big(\bigwedge(\mathfrak{g}_{\bar{1}}), \mathcal{F}_{\mathcal{G}_0}\big)$, $f \mapsto f \circ \gamma_{\mathfrak{g}}$, where

$$\gamma_{\mathfrak{g}} : \bigwedge(\mathfrak{g}_{\bar{1}}) \to \mathfrak{U}(\mathfrak{g}), \quad X_1 \wedge \cdots \wedge X_r \mapsto \frac{1}{r!} \sum_{\sigma \in S_r} (-1)^{|\sigma|} X_{\sigma(1)} \cdots X_{\sigma(r)}. \tag{1}$$

The following formulas define the multiplication morphism, the inversion morphism and the identity morphism respectively:

$$\begin{aligned} \mu^*(f)\,(X \otimes Y)\,(g,h) &= f\big(\,\mathrm{Ad}(h^{-1})(X) \cdot Y\big)\,(gh)\,; \\ \iota^*(f)(X)(g) &= f\big(\mathrm{Ad}(g)(S(X))\big)\,\big(g^{-1}\big)\,; \\ \varepsilon^*(f) &= f(1)(e). \end{aligned} \tag{2}$$

Here $X, Y \in \mathfrak{U}(\mathfrak{g})$, $f \in \mathcal{O}_{\mathcal{G}}$, $g, h \in \mathcal{G}_0$ and S is the antipode map of the Hopf superalgebra $\mathfrak{U}(\mathfrak{g})$. Here we identify the enveloping superalgebra $\mathfrak{U}(\mathfrak{g} \oplus \mathfrak{g})$ with the tensor product $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$.

Sometimes we will identify the Lie superalgebra $\mathfrak{g}$ of a Lie supergroup $\mathcal{G}$ with the tangent space $T_e(\mathcal{G})$ at $e \in \mathcal{G}_0$. The corresponding to $T \in T_e(\mathcal{G})$ left invariant vector field on $\mathcal{G}$ is given by

$$(\mathrm{id} \otimes T) \circ \mu^*, \tag{3}$$

where μ is the multiplication morphism of $\mathcal{G}$. (Recall that a vector field Y on $\mathcal{G}$ is called *left invariant* if $(\mathrm{id} \otimes Y) \circ \mu^* = \mu^* \circ Y$.) Denote by l_g and by r_g the left and right translations with respect to $g \in \mathcal{G}_0$, respectively. The morphisms l_g and r_g are given by the following formulas:

$$l_g^*(f)(X)(h) = f(X)(gh); \quad r_g^*(f)(X)(h) = f\big(\mathrm{Ad}(g^{-1})X\big)(hg), \tag{4}$$

where $f \in \mathcal{O}_{\mathcal{G}}$, $X \in \mathfrak{U}(\mathfrak{g})$ and $g, h \in \mathcal{G}_0$.

Homogeneous supermanifolds. An *action of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$* is a morphism $\nu : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ such that the usual conditions modeling group action axioms hold. Any vector $X \in T_e(\mathcal{G})$ defines the vector field on $\mathcal{M}$ by the following formula:

$$X \mapsto (X \otimes \mathrm{id}) \circ \nu^*. \tag{5}$$

Definition 3.2. An action ν is called *transitive* if ν_0 is a transitive action of the Lie group $\mathcal{G}_0$ on $\mathcal{M}_0$ and the vector fields (5) generates the tangent space $T_x(\mathcal{M})$ at any point $x \in \mathcal{M}_0$. In this case the supermanifold $\mathcal{M}$ is called $\mathcal{G}$*-homogeneous*. A supermanifold $\mathcal{M}$ is called *homogeneous*, if it possesses a transitive action of a Lie supergroup.

If a supermanifold $\mathcal{M}$ is $\mathcal{G}$-homogeneous and $\nu : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ is the corresponding transitive action, then $\mathcal{M}$ is isomorphic to the supermanifold $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is the isotropy subsupergroup of a certain point (see [16] for details). Recall that the underlying space of $\mathcal{G}/\mathcal{H}$ is the complex analytic manifold $\mathcal{G}_0/\mathcal{H}_0$ and the structure sheaf $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ of $\mathcal{G}/\mathcal{H}$ is given by

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} = \big\{ f \in (\pi_0)_*(\mathcal{O}_{\mathcal{G}}) \,\big|\, \mu^*_{\mathcal{G}\times\mathcal{H}}(f) = \mathrm{pr}^*(f) \big\}, \tag{6}$$

where $\pi_0 : \mathcal{G}_0 \to \mathcal{G}_0/\mathcal{H}_0$ is the natural map, $\mu_{\mathcal{G}\times\mathcal{H}}$ is the restriction of the multiplication map on $\mathcal{G}\times\mathcal{H}$ and $\mathrm{pr} : \mathcal{G}\times\mathcal{H} \to \mathcal{G}$ is the natural projection. Using (2) we can rewrite the condition $\mu^*_{\mathcal{G}\times\mathcal{H}}(f) = \mathrm{pr}^*(f)$ in the following way:

$$f\Big(\mathrm{Ad}\,(h^{-1})(X)Y\Big)(gh) = \begin{cases} f(X)(g), & Y \in \mathbb{C}; \\ 0, & Y \notin \mathbb{C}; \end{cases} \tag{7}$$

where $X \in \mathfrak{U}(\mathfrak{g})$, $Y \in \mathfrak{U}(\mathfrak{h})$, $\mathfrak{h} = \mathrm{Lie}\,\mathcal{H}$, $g \in \mathcal{G}_0$ and $h \in \mathcal{H}_0$.

Let $Y \in \mathfrak{g}$ and $f \in \mathcal{O}_{\mathcal{G}}$. Then the operator defined by the formula

$$Y(f)(X) = (-1)^{p(Y)} f(XY), \tag{8}$$

where $p(Y)$ is the parity of Y, is a left invariant vector field on $\mathcal{G}$. From (4), (7) and (8) it follows that

$f \in \mathcal{O}_{\mathcal{G}/\mathcal{H}}$ if and only if f is $\mathcal{H}_0$-right invariant, i.e. $r_h^*(f) = f$ for any $h \in \mathcal{H}_0$, and $Y(f) = 0$ for all $Y \in \mathfrak{h}_{\bar{1}}$, where $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$.

Sometimes we will consider also the left action $\mathcal{H}\times\mathcal{G} \to \mathcal{G}$ of a subsupergroup $\mathcal{H}$ on a Lie supergroup $\mathcal{G}$. The corresponding quotient supermanifold we will denote by $\mathcal{H}\backslash\mathcal{G}$.

More about split supermanifolds. Recall that a supermanifold $\mathcal{M}$ is called split if its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge\mathcal{E}$, where $\mathcal{E}$ is a locally free sheaf on $\mathcal{M}_0$. In this case, $\mathcal{O}_{\mathcal{M}}$ possesses the $\mathbb{Z}$-grading induced by the natural $\mathbb{Z}$-grading of $\bigwedge\mathcal{E} = \bigoplus\limits_p \bigwedge\limits^p \mathcal{E}$ and by isomorphism $\mathcal{O}_{\mathcal{M}} \simeq \bigwedge\mathcal{E}$. Such gradings of $\mathcal{O}_{\mathcal{M}}$ we call split.

Proposition 3.3. *Any Lie supergroup $\mathcal{G}$ is split.*

This statement follows from the fact that any Lie supergroup is determined by its super Harish-Chandra pair. A different proof of this result (probably the first one) was given in [4]. For completeness we give here another proof.

Proof. The underlying space $\mathcal{G}_0$ is a closed Lie subsupergroup of $\mathcal{G}$. Hence, there exists the homogeneous space $\mathcal{G}/\mathcal{G}_0$, which is isomorphic to the supermanifold $\mathcal{N}$ such that $\mathcal{N}_0$ is a point $\mathrm{pt} = \mathcal{G}_0/\mathcal{G}_0$ and $\mathcal{O}_{\mathcal{N}} \simeq \bigwedge(m)$, where $m = \dim \mathfrak{g}_{\bar{1}}$. By definition, the structure sheaf $\mathcal{O}_{\mathcal{N}}$ consists of all r_g-invariant functions, $g \in \mathcal{G}_0$. We have the natural map $\varphi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$, where $\varphi_0 : \mathcal{G}_0 \to \mathrm{pt}$ and $\varphi^* : \mathcal{O}_{\mathcal{N}} \to (\varphi_0)_*(\mathcal{O}_{\mathcal{G}})$ is the inclusion. It is known that $\varphi : \mathcal{G} \to \mathcal{G}/\mathcal{G}_0$ is a principal bundle (see [16]). Using the fact that the underlying space of $\mathcal{G}/\mathcal{G}_0$ is a point we get $\mathcal{G} \simeq \mathcal{N} \times \mathcal{G}_0$. Note that this is an isomorphism of supermanifolds but not of Lie supergroups. ∎

Example 3.4. As an example of a homogeneous non-split supermanifold we can cite the super-grassmannian $\mathbf{Gr}_{m|n,r|s}$ for $0 < r < m$ and $0 < s < n$. Super-grassmannians of other types are split (see Example 6.2).

Denote by `SSM` the category of split supermanifolds. Objects Ob `SSM` in this category are all split supermanifolds $\mathcal{M}$ with fixed split gradings. Further if $X, Y \in$ Ob `SSM`, we put

$$\mathrm{Hom}(X,Y) = \left\{ \begin{array}{l} \text{morphisms from } X \text{ to } Y \\ \text{preserving the split gradings} \end{array} \right\}$$

As in the category of supermanifolds, we can define in `SSM` a group object (*split Lie supergroup*), an action of a split Lie supergroup on a split supermanifold (*split action*) and a *split homogeneous supermanifold*.

There is a functor gr from the category of supermanifolds to the category of split supermanifolds. Let us briefly describe this construction. Let $\mathcal{M}$ be a supermanifold. Denote by $\mathcal{J}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{M}}$ the subsheaf of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$. Then by definition $\mathrm{gr}\,\mathcal{M} = (\mathcal{M}_0, \mathrm{gr}\,\mathcal{O}_{\mathcal{M}})$ is the split supermanifold with the structure sheaf

$$\mathrm{gr}\,\mathcal{O}_{\mathcal{M}} = \bigoplus_{p\geq 0} (\mathrm{gr}\,\mathcal{O}_{\mathcal{M}})_p, \quad \mathcal{J}^0_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}}, \quad (\mathrm{gr}\,\mathcal{O}_{\mathcal{M}})_p := \mathcal{J}^p_{\mathcal{M}}/\mathcal{J}^{p+1}_{\mathcal{M}}.$$

In this case $(\mathrm{gr}\,\mathcal{O}_{\mathcal{M}})_1$ is a locally free sheaf and there is a natural isomorphism of $\mathrm{gr}\,\mathcal{O}_{\mathcal{M}}$ onto $\bigwedge(\mathrm{gr}\,\mathcal{O}_{\mathcal{M}})_1$. If $\psi = (\psi_0, \psi^*) : \mathcal{M} \to \mathcal{N}$ is a morphism, then $\mathrm{gr}(\psi) = (\psi_0, \mathrm{gr}(\psi^*)) : \mathrm{gr}\,\mathcal{M} \to \mathrm{gr}\,\mathcal{N}$ is defined by

$$\mathrm{gr}(\psi^*)(f + \mathcal{J}^p_{\mathcal{N}}) := \psi^*(f) + \mathcal{J}^p_{\mathcal{M}} \text{ for } f \in (\mathcal{J}_{\mathcal{N}})^{p-1}.$$

Recall that by definition every morphism of supermanifolds is even and as a consequence sends $\mathcal{J}^p_{\mathcal{N}}$ into $\mathcal{J}^p_{\mathcal{M}}$.

Split Lie supergroups. Let $\mathcal{G}$ be a Lie supergroup with the supergroup morphisms μ, ι and ε: the multiplication, the inversion and the identity morphism, respectively. In this section we assign three split Lie supergroups $\mathcal{G}^1$, $\mathcal{G}^2$ and $\mathcal{G}^3$ to $\mathcal{G}$ and we show that these split Lie supergroups are pairwise isomorphic.

(1) The construction of $\mathcal{G}^1$ is very simple: we just apply functor gr to $\mathcal{G}$. Clearly, $\mathcal{G}^1 := \mathrm{gr}\,\mathcal{G}$ is a split Lie supergroup with the supergroup morphisms $\mathrm{gr}(\mu)$, $\mathrm{gr}(\iota)$ and $\mathrm{gr}(\varepsilon)$.

(2) Consider the super Harish-Chandra pair $(\mathcal{G}_0, \mathfrak{g}^2)$, where $\mathfrak{g}^2$ is the following Lie superalgebra: $\mathfrak{g}^2$ and $\mathfrak{g}$ are isomorphic as vector superspaces and the Lie bracket in $\mathfrak{g}^2$ is defined by the following formula:

$$[X,Y]_{\mathfrak{g}^2} = \begin{cases} [X,Y]_{\mathfrak{g}}, & \text{if } X,Y \in \mathfrak{g}_{\bar{0}} \text{ or } X \in \mathfrak{g}_{\bar{0}} \text{ and } Y \in \mathfrak{g}_{\bar{1}}; \\ 0, & \text{if } X,Y \in \mathfrak{g}_{\bar{1}}. \end{cases} \tag{9}$$

Denote by $\mathcal{G}^2$ the Lie supergroup corresponding to $(\mathcal{G}_0, \mathfrak{g}^2)$.

(3) Consider the sheaf $\mathcal{O}_{\mathcal{G}^3} := \mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$. For the ringed space $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ we can repeat the construction from Section 3. Indeed, this ringed space is clearly a supermanifold. Futhermore, the exterior algebra $\bigwedge \mathfrak{g}_{\bar{1}}$ is also a Hopf algebra. Therefore, we can define on $\mathcal{G}^3$ the multiplication, the inversion and the identity morphisms respectively by the following formulas:

$$\begin{aligned} (\mu^3)^*(f)\big(X \wedge Y\big)(g,h) &= f\big(\mathrm{Ad}(h^{-1})(X) \wedge Y\big)(gh); \\ (\iota^3)^*\big(f\big)(X)(g) &= f\big(\mathrm{Ad}(g)(S'(X))\big)\big(g^{-1}\big); \\ (\varepsilon^3)^*\big(f\big) &= f\big(1\big)(e). \end{aligned} \tag{10}$$

Here $X, Y \in \bigwedge \mathfrak{g}_{\bar{1}}$, $f \in \mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$, g, $h \in \mathcal{G}_0$ and S' is the antipode map of the Hopf superalgebra $\bigwedge \mathfrak{g}_{\bar{1}}$. Hence, $\mathcal{G}^3 := (\mathcal{G}_0, \mathcal{O}_{\mathcal{G}^3})$ is a Lie supergroup. Since

$$\mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}) = \bigoplus_{p \geq 0} \mathrm{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$$

is $\mathbb{Z}$-graded and the morphisms (10) preserve this $\mathbb{Z}$-grading, we see that $\mathcal{G}^3$ is a split Lie supergroup.

Later on we will need the explicit expression of left and right translations l'_g and r'_g in $\mathcal{G}^3$:

$$(l'_g)^*(f)(X)(h) = f(X)(gh); \quad (r'_g)^*(f)(X)(h) = f\big(\mathrm{Ad}(g^{-1})X\big)(hg), \tag{11}$$

where $f \in \mathrm{Hom}_{\mathbb{C}}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0})$, $X \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $g, h \in \mathcal{G}_0$.

In fact, all these split Lie supergroups are isomorphic. To show this we need the following lemma:

Lemma 3.5. *Let $\mathfrak{k}$ be a Lie superalgebra and $X_i, Y_j \in \mathfrak{k}_{\bar{1}}$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ be any elements. Assume that $[X_i, Y_j] = 0$ for any i, j. Then we have*

$$\gamma_{\mathfrak{k}}(X_1 \wedge \cdots \wedge X_r \wedge Y_1 \wedge \cdots \wedge Y_s) = \gamma_{\mathfrak{k}}(X_1 \wedge \cdots \wedge X_r) \cdot \gamma_{\mathfrak{k}}(Y_1 \wedge \cdots \wedge Y_s),$$

where $\gamma_{\mathfrak{k}}$ is given by (1).

Proof. A direct calculation. ∎

Proposition 3.6. *We have $\mathcal{G}^1 \simeq \mathcal{G}^2 \simeq \mathcal{G}^3$ in the category of Lie supergroups.*

Proof. **(a)** The statement $\mathcal{G}^1 \simeq \mathcal{G}^2$ was proven in [15], Theorem 3.

(b) Let us show that $\mathcal{G}^2 \simeq \mathcal{G}^3$. Applying Lemma 3.5 to $\mathfrak{g}^2$ and to any elements X_i, $Y_j \in \mathfrak{g}^2_{\bar{1}}$, we see that in this case $\gamma_{\mathfrak{g}^2}$ is not only isomorphism of super coalgebras but of Hopf superalgebras. In other words, the isomorphism

$$\Phi_{\mathfrak{g}^2} : \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g}^2_{\bar{0}})} \big(\mathfrak{U}(\mathfrak{g}^2), \mathcal{F}_{\mathcal{G}_0}\big) \to \mathrm{Hom}_{\mathbb{C}} \big(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}\big)$$

is an isomorphism of Lie supergroups. ∎

4. Split grading operators

Let again $\mathcal{M}$ be a supermanifold, $\mathrm{gr}\,\mathcal{M}$ be the corresponding split supermanifold and $\mathcal{J}$ be the sheaf of ideals generated by odd elements of $\mathcal{O}_\mathcal{M}$. We denote by $\mathcal{T} = \mathcal{D}er\mathcal{O}_\mathcal{M}$ and by $\mathrm{gr}\,\mathcal{T} = \mathcal{D}er(\mathcal{O}_{\mathrm{gr}\,\mathcal{M}})$ the tangent sheaf of $\mathcal{M}$ and of $\mathrm{gr}\,\mathcal{M}$, respectively. The sheaf $\mathcal{T}$ is naturally $\mathbb{Z}_2$-graded and the sheaf $\mathrm{gr}\,\mathcal{T}$ is naturally $\mathbb{Z}$-graded: the gradings are induced by the $\mathbb{Z}_2$ and $\mathbb{Z}$-grading of $\mathcal{O}_\mathcal{M}$ and $\mathrm{gr}\,\mathcal{O}_\mathcal{M}$, respectively. In other words, we have the decomposition:

$$\mathcal{T} = \mathcal{T}_{\bar{0}} \oplus \mathcal{T}_{\bar{1}}, \quad \mathrm{gr}\,\mathcal{T} = \bigoplus_{p\geq -1} (\mathrm{gr}\,\mathcal{T})_p.$$

The sheaves $\mathcal{T}$ and $\mathrm{gr}\,\mathcal{T}$ are related: this relation can be expressed by the following exact sequence:

$$0 \longrightarrow \mathcal{T}_{(2)\bar{0}} \longrightarrow \mathcal{T}_{\bar{0}} \xrightarrow{\alpha} (\mathrm{gr}\,\mathcal{T})_0 \longrightarrow 0, \tag{12}$$

where

$$\mathcal{T}_{(2)\bar{0}} = \{v \in \mathcal{T}_{\bar{0}} \mid v(\mathcal{O}_\mathcal{M}) \subset \mathcal{J}^2\}.$$

The morphism α in (12) is the composition of the natural morphism $\mathcal{T}_{\bar{0}} \to \mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and the isomorphism $\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}} \to (\mathrm{gr}\,\mathcal{T})_0$ that is given by

$$[w] \longmapsto \tilde{w}, \quad \tilde{w}\big(f + \mathcal{J}^{p+1}\big) := w(f) + \mathcal{J}^{p+1},$$

where $w \in \mathcal{T}_{\bar{0}}$, $[w]$ is the image of w in $\mathcal{T}_{\bar{0}}/\mathcal{T}_{(2)\bar{0}}$ and $f \in \mathcal{J}^p$.

Assume that the sheaf $\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}$-graded, i.e. $\mathcal{O}_\mathcal{M} = \bigoplus\limits_p (\mathcal{O}_\mathcal{M})_p$. Then we have the map $w : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ defined by $w(f) = pf$, where $f \in (\mathcal{O}_\mathcal{M})_p$. Such maps are called *grading operators* on $\mathcal{M}$.

Definition 4.1. We call a grading operator w on $\mathcal{M}$ a *split grading operator* if it corresponds to a split grading of $\mathcal{O}_\mathcal{M}$, see Definition 2.1.

In fact any split grading operator w on $\mathcal{M}$ is an even vector field on $\mathcal{M}$. Indeed, w is linear, it preserves the parity in $\mathcal{O}_\mathcal{M}$ and for $f \in (\mathcal{O}_\mathcal{M})_p$ and $g \in (\mathcal{O}_\mathcal{M})_q$ we have:

$$w(fg) = (p+q)fg = (pf)g + f(qg) = w(f)g + fw(g).$$

Note that $fg \in (\mathcal{O}_\mathcal{M})_{p+q}$.

By definition the sheaf $\mathrm{gr}\,\mathcal{O}_\mathcal{M}$ is $\mathbb{Z}$-graded. Denote by a the corresponding split grading operator.

Lemma 4.2. *1. A supermanifold $\mathcal{M}$ is split if and only if the vector field a is contained in* $\operatorname{Im} H^0(\alpha)$*, where*

$$H^0(\alpha) : H^0(\mathcal{M}_0, \mathcal{T}_{\bar 0}) \to H^0(\mathcal{M}_0, (\operatorname{gr}\mathcal{T})_0).$$

(We applied the functor $H^0(\mathcal{M}_0, -)$ to the sequence (12). We write $H^0(\alpha)$ instead of $H^0(\mathcal{M}_0, \alpha)$ for notational simplicity.)

2. If w is a split grading operator on $\mathcal{M}$, then any other split grading operator on $\mathcal{M}$ has the form $w + \chi$, where $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar 0})$.

Proof. 1. The statement of the lemma can be deduced from the following observation made by Koszul in [4, Lemma 1.1 and Section 3]. Let A be a commutative superalgebra over $\mathbb{C}$ and $\mathfrak{m}$ be a nilpotent ideal in A. An even derivation w of A is called *adapted to the filtration*

$$A \supset \mathfrak{m} \supset \mathfrak{m}^2 \ldots$$

if

$$(w - r\operatorname{id})(\mathfrak{m}^r) \subset \mathfrak{m}^{r+1} \text{ for any } r \geq 0.$$

Denote by $D^{ad}_{\mathfrak{m}}$ the set of all derivations adapted to $\mathfrak{m}$. In [4, Lemma 1.1] it was shown that $D^{ad}_{\mathfrak{m}}$ is not empty if and only if the filtration of A is splittable. Moreover, if $w \in D^{ad}_{\mathfrak{m}}$, then the corresponding splitting of A is given by eigenspaces of the derivation w: $A = \bigoplus_i A_i$, where A_i is the eigenspace of w with the eigenvalue i, and $\mathfrak{m}^r = A_r \oplus \mathfrak{m}^{r+1}$ for all $r \geq 0$.

We apply Koszul's observation to the sheaf of superalgebras $\mathcal{O}_{\mathcal{M}}$ and its subsheaf of ideals $\mathcal{J}$. The set $D^{ad}_{\mathcal{J}}$ is in this case the set of global derivations of $\mathcal{O}_{\mathcal{M}}$ adapted to the filtration

$$\mathcal{O}_{\mathcal{M}} \supset \mathcal{J} \supset \mathcal{J}^2 \supset \ldots. \tag{13}$$

Clearly, $D^{ad}_{\mathcal{J}}$ is not empty if and only if a is contained in $\operatorname{Im} H^0(\alpha)$. (Actually, $H^0(\alpha)(D^{ad}_{\mathcal{J}}) = a$.) Furthermore, if the supermanifold $\mathcal{M}$ is split, i.e. we have a split grading $\mathcal{O}_{\mathcal{M}} = \bigoplus_{p\geq 0}(\mathcal{O}_{\mathcal{M}})_p$, then $\mathcal{J}^q = \bigoplus_{p\geq q}(\mathcal{O}_{\mathcal{M}})_p$ and $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the split grading determine the splitting of the filtration (13) and the corresponding split grading operator belongs to $D^{ad}_{\mathcal{J}}$.

Conversely, if there exists $w \in D^{ad}_{\mathcal{J}}$, then we can decompose the sheaf $\mathcal{O}_{\mathcal{M}}$ into eigenspaces

$$(\mathcal{O}_{\mathcal{M}})_q := \{f \in \mathcal{O}_{\mathcal{M}} | w(f) = qf\}.$$

In this case the sheaves $\bigoplus\limits_p (\mathcal{O}_{\mathcal{M}})_p$ and $\operatorname{gr}\mathcal{O}_{\mathcal{M}}$ are isomorphic as $\mathbb{Z}$-graded sheaves of superalgebras since $\mathcal{J}^q = (\mathcal{O}_{\mathcal{M}})_q \oplus \mathcal{J}^{q+1}$. Hence, the supermanifold is split.

2. Applying the left-exact functor $H^0(\mathcal{M}_0, -)$ to (12), we get the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar 0}) \longrightarrow H^0(\mathcal{M}_0, \mathcal{T}_{\bar 0}) \overset{H^0(\alpha)}{\longrightarrow} H^0(\mathcal{M}_0, (\operatorname{gr}\mathcal{T})_0).$$

If w_1, w_2 are two split grading operators on $\mathcal{M}$, then

$$H^0(\alpha)(w_1) = H^0(\alpha)(w_2) = a,$$

according to the part 1. Therefore, $w_1 - w_2 \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar 0})$. The result follows. ∎

Example 4.3. Consider the supermanifold $\mathcal{G}_0\backslash\mathcal{G}$. Its structure sheaf is isomorphic to $\bigwedge(\mathfrak{g}_{\bar{1}})$ (compare with Example 3.3). Denote by (ε_i) the system of odd (global) coordinates on $\mathcal{G}_0\backslash\mathcal{G}$. An example of a split grading operator on the Lie supergroup $\mathcal{G}$ is $\sum \varepsilon^i X_i$. Here (X_i) is a basis of odd left invariant vector fields on $\mathcal{G}$ such that $X_i(\varepsilon^j)(e) = \delta_i^j$. We may produce other examples if we use right invariant vector fields or odd (global) coordinates on $\mathcal{G}/\mathcal{G}_0$.

By Lemma 4.2, any split grading operator on a Lie supergroup $\mathcal{G}$ is given by $\sum \varepsilon^i X_i + \chi$, where $\chi \in H^0(\mathcal{G}_0, \mathcal{T}_{(2)\bar{0}})$ is any vector field on $\mathcal{G}$.

5. Compatible split gradings on $\mathcal{G}/\mathcal{H}$

Compatible gradings on $\mathcal{G}/\mathcal{H}$. Let $\mathcal{G}$ be a Lie supergroup and $\mathcal{M} = \mathcal{G}/\mathcal{H}$ be a homogeneous supermanifold. As above we denote by $\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ the natural projection.

Definition 5.1. A split grading of the sheaf $\mathcal{O}_{\mathcal{G}} = \bigoplus\limits_p (\mathcal{O}_{\mathcal{G}})_p$ is called *compatible* with the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$ if the following holds:

$$f \in \mathcal{O}_{\mathcal{M}} \ \Rightarrow f_p \in \mathcal{O}_{\mathcal{M}} \text{ for all } p,$$

where $f = \sum f_p$ and $f_p \in (\pi_0)_*((\mathcal{O}_{\mathcal{G}})_p)$.

Let us take any split grading operator w on $\mathcal{G}$. Clearly, the corresponding split grading of $\mathcal{O}_{\mathcal{G}}$ is compatible with $\mathcal{O}_{\mathcal{M}}$ if and only if $w(\mathcal{O}_{\mathcal{M}}) \subset \mathcal{O}_{\mathcal{M}}$. It is not clear from Definition 5.1 that the compatible grading

$$(\mathcal{O}_{\mathcal{M}})_p = \mathcal{O}_{\mathcal{M}} \cap (\pi_0)_*((\mathcal{O}_{\mathcal{G}})_p) \tag{14}$$

of $\mathcal{O}_{\mathcal{M}}$, if it exists, is a split grading of $\mathcal{O}_{\mathcal{M}}$. However, the following proposition holds:

Proposition 5.2. *Assume that we have the $\mathbb{Z}$-grading:*

$$\mathcal{O}_{\mathcal{M}} = \bigoplus_{p \geq 0} (\mathcal{O}_{\mathcal{M}})_p,$$

where $(\mathcal{O}_{\mathcal{M}})_p$ are as in (14). Then this grading is a split grading.

Proof. The idea of the proof is to apply Lemma 4.2 to the grading operator $w' := w|_{\mathcal{O}_{\mathcal{M}}}$ on $\mathcal{M}$. Denote by $\mathcal{J}_{\mathcal{M}}$ and by $\mathcal{J}_{\mathcal{G}}$ the sheaves of ideals generated by odd elements of $\mathcal{O}_{\mathcal{M}}$ and $\mathcal{O}_{\mathcal{G}}$, respectively. Our aim is to show that

$$w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1} = pf + \mathcal{J}_{\mathcal{M}}^{p+1},$$

where $f \in \mathcal{J}_{\mathcal{M}}^p$. In other words, we want to show that $H^0(\alpha)(w')$ is a split grading operator for the grading of $\operatorname{gr} \mathcal{O}_{\mathcal{M}}$. (We use notations of Lemma 4.2.) We have:

$$(\operatorname{gr} \pi)^*(w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1}) = w(f) + \mathcal{J}_{\mathcal{G}}^{p+1} = pf + \mathcal{J}_{\mathcal{G}}^{p+1};$$
$$(\operatorname{gr} \pi)^*(pf + \mathcal{J}_{\mathcal{M}}^{p+1}) = pf + \mathcal{J}_{\mathcal{G}}^{p+1}.$$

Since the map $(\operatorname{gr} \pi)^*$ is injective, we get, $w'(f) + \mathcal{J}_{\mathcal{M}}^{p+1} = pf + \mathcal{J}_{\mathcal{M}}^{p+1}$. ∎

$\mathcal{H}$-invariant split grading operators. First of all let us consider the situation when a split grading operator w on $\mathcal{G}$ is invariant with respect to a Lie subsupergroup $\mathcal{H}$. In terms of super Harish-Chandra pairs this means:

$$\begin{array}{ll} r_h^* \circ w = w \circ r_h^*, & \text{for all} \quad h \in \mathcal{H}_0; \\ [Y, w] = 0, & \text{for all } Y \in \mathfrak{h}_{\bar{1}}. \end{array} \tag{15}$$

Here $(\mathcal{H}_0, \mathfrak{h})$ is the super Harish-Chandra pair of $\mathcal{H}$, r_h is the right translation and Y is an odd left invariant vector field.

Proposition 5.3. *Assume that w is an $\mathcal{H}$-invariant split grading operator on $\mathcal{G}$, i.e. equations (15) hold. Then $\mathcal{H}$ is an ordinary Lie group.*

Proof. The idea of the proof is to show that the Lie superalgebra $\mathfrak{h}$ of $\mathcal{H}$ has the trivial odd part: $\mathfrak{h}_{\bar{1}} = \{0\}$.

In Example 4.3 we saw that any split grading operator on $\mathcal{G}$ is given by $w = \sum \varepsilon^i X_i + \chi$. If Z is a vector field on $\mathcal{G}$, denote by $Z_e \in T_e(\mathcal{G})$ the corresponding tangent vector at the identity $e \in \mathcal{G}_0$. Consider the second equation in (15). At the point e, we have

$$[Y, w]_e = \big(\sum_i Y(\varepsilon^i) X_i - \sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y + [Y, \chi]\big)_e = 0$$

for any $Y \in \mathfrak{h}_{\bar{1}}$. Furthermore,

$$(\sum_i \varepsilon^i Y \circ X_i - \sum_i \varepsilon^i X_i \circ Y)_e = 0 \quad \text{and} \quad [Y, \chi]_e = 0,$$

because $\varepsilon^i(e) = 0$ and because $\chi \in H^0(\mathcal{M}_0, \mathcal{T}_{(2)\bar{0}})$. Therefore,

$$[Y, w]_e = \sum_i Y(\varepsilon^i)(e)(X_i)_e = 0$$

The tangent vectors $(X_i)_e$ form a basis in $T_e(\mathcal{G})_{\bar{1}}$, hence $Y(\varepsilon^i)(e) = 0$ for all i. The last statement is equivalent to $Y_e = 0$. Since Y is a left invariant vector field, we get $Y = 0$. The proof is complete. ∎

Remark 5.4. It is well known that the supermanifold $\mathcal{G}/\mathcal{H}$, where $\mathcal{H}$ is an ordinary Lie group, is split (see [5] or [14]). Therefore, the case of $\mathcal{H}$-invariant split grading operators does not lead to new examples of homogeneous split supermanifolds.

$\mathcal{G}_0$-left invariant split grading operators. Consider now a more general situation, when a split grading operator w leaves $\mathcal{O}_{\mathcal{M}}$ invariant. Let $f \in \mathcal{O}_{\mathcal{M}}$. Then $w(f) \in \mathcal{O}_{\mathcal{M}}$ if and only if

$$r_h^*(w(f)) = w(f) \quad \text{and} \quad Y(w(f)) = 0$$

for $h \in \mathcal{H}_0$ and $Y \in \mathfrak{h}_{\bar{1}}$. These conditions are equivalent to the following ones:

$$(r_h^* \circ w \circ (r_h^{-1})^* - w)|_{\mathcal{O}_{\mathcal{M}}} = 0; \quad [Y, w]|_{\mathcal{O}_{\mathcal{M}}} = 0. \tag{16}$$

Recall that $r_h^{-1} = r_{h^{-1}}$.

It seems to us that the system (16) is hard to solve in general. Consider now a special type of split grading operators, called $\mathcal{G}_0$-left invariant grading operators.

Definition 5.5. A split grading of $\mathcal{O}_\mathcal{G}$ is called $\mathcal{G}_0$*-left invariant* if it is invariant with respect to left translations. In other words, from $f \in (\mathcal{O}_\mathcal{G})_p$ it follows that $l_g^*(f) \in (\mathcal{O}_\mathcal{G})_p$ for all $g \in \mathcal{G}_0$.

It is easy to see that a split grading of $\mathcal{O}_\mathcal{G}$ is $\mathcal{G}_0$-left invariant if and only if the corresponding split grading operator w is invariant with respect to left translations: $l_g^* \circ w = w \circ l_g^*$, $g \in \mathcal{G}_0$. For example, the split grading operator $\sum \varepsilon^i X_i$ constructed in Example 4.3 is a $\mathcal{G}_0$-left invariant split grading operator, because ε^i are $\mathcal{G}_0$-left invariant functions and X_i are left invariant vector fields. In this section we will describe all such operators.

In Section 3 we have seen that the supermanifold $(\mathcal{G}_0, \mathrm{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}))$ is a Lie supergroup isomorphic to $\mathrm{gr}\, \mathcal{G}$. We need the following lemma:

Lemma 5.6. *The map*

$$\begin{aligned} \Phi_\mathfrak{g} : \mathcal{O}_\mathcal{G} &\to \mathrm{Hom}_\mathbb{C}(\textstyle\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}), \\ f &\mapsto f \circ \gamma_\mathfrak{g} \end{aligned}$$

from Section 3 is invariant with respect to left and right translations.

Proof. For any $h \in \mathcal{G}_0$, denote by r'_h and l'_h the right and the left translation in the Lie supergroup $\mathcal{G}^3 = (\mathcal{G}_0, \mathrm{Hom}_\mathbb{C}(\bigwedge \mathfrak{g}_{\bar{1}}, \mathcal{F}_{\mathcal{G}_0}))$, respectively. (See, (11)) Let us show that

$$(r'_h)^* \circ \Phi_\mathfrak{g} = \Phi_\mathfrak{g} \circ r_h^*. \tag{17}$$

Let us take $Z \in \bigwedge \mathfrak{g}_{\bar{1}}$ and $g, h \in \mathcal{G}_0$. Using (4) we have

$$\begin{gathered} \big[(r'_h)^* \circ \Phi_\mathfrak{g}\big](f)(Z)(g) = \Phi_\mathfrak{g}(f)\big(\mathrm{Ad}(h^{-1})(Z)\big)(gh) = \\ f\big(\gamma_\mathfrak{g}(\mathrm{Ad}(h^{-1})(Z))\big)(gh) = f\big(\mathrm{Ad}(h^{-1})(\gamma_\mathfrak{g}(Z))\big)(gh) = \\ r_h^*(f)\big(\gamma_\mathfrak{g}(Z)\big)(g) = [\Phi_\mathfrak{g} \circ r_h^*](f)(Z)(g). \end{gathered}$$

Similarly, we get

$$(l'_h)^* \circ \Phi_\mathfrak{g} = \Phi_\mathfrak{g} \circ l_h^*. \qquad \blacksquare$$

The following observation is known to experts, but we cannot find it in the literature:

Lemma 5.7. *The space of $\mathcal{G}_0$-left invariant vector fields $H^0(\mathcal{G}_0, \mathcal{T})^{\mathcal{G}_0}$ on a Lie supergroup $\mathcal{G}$ is isomorphic to $H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}$. The isomorphism is given by:*

$$f \otimes Z \stackrel{F}{\longmapsto} fZ,$$

where $f \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ and $Z \in \mathfrak{g}$.

Proof. Clearly, the map F is injective and its image is contained in the vector space $H^0(\mathcal{G}_0,\mathcal{T})^{\mathcal{G}_0}$. Let us show that any vector field v in $H^0(\mathcal{G}_0,\mathcal{T})^{\mathcal{G}_0}$ is contained in $\mathrm{Im}(F)$.

Let (X_i) and (Z_j) be a basis of odd and even left invariant (with respect to the supergroup $\mathcal{G}$) vector fields on $\mathcal{G}$, respectively. Assume that

$$v=\sum f^iX_i+\sum g^jZ_j,$$

where $f^i,g^j\in H^0(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}})$, be the decomposition of v with respect to this basis. We have:

$$\begin{aligned} l_g^*\circ v = & \sum l_g^*(f^i)l_g^*\circ X_i+\sum l_g^*(g^j)l_g^*\circ Z_j = \\ & \sum l_g^*(f^i)X_i\circ l_g^*+\sum l_g^*(g^j)Z_j\circ l_g^* = v\circ l_g^*. \end{aligned}$$

Therefore, $l_g^*(f^i)=f^i$ and $l_g^*(g^j)=g^j$ for all $g\in\mathcal{G}_0$. In other words, $f^i,g^j\in H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$. The proof is complete. ∎

The Lie supergroup $\mathcal{G}$ acts on the vector superspace $H^0(\mathcal{G}_0,\mathcal{T})^{\mathcal{G}_0}$. This action we can describe in terms of the corresponding super Harish-Chandra pair $(\mathcal{G}_0,\mathfrak{g})$ in the following way:

$$g\mapsto (X\mapsto r_g^*\circ X\circ(r_g^{-1})^*),\quad Y\mapsto (X\mapsto [Y,X]), \tag{18}$$

where $g\in\mathcal{G}_0$, $X\in H^0(\mathcal{G}_0,\mathcal{T})^{\mathcal{G}_0}$ and $Y\in\mathfrak{g}$. Note that this action is well-defined because $\mathcal{G}$-left and right actions on $H^0(\mathcal{G}_0,\mathcal{T})$ commute. The Lie supergroup $\mathcal{G}$ acts also on the vector superspace $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}$. This action is given by right translations r_g^* on $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ and by the formulas (18) on $\mathfrak{g}$ if we assume that $X\in\mathfrak{g}$. Clearly, the isomorphism F from Lemma 5.7 is equivariant. From now on we will identify $H^0(\mathcal{G}_0,\mathcal{T})^{\mathcal{G}_0}$ and $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}$ via isomorphism F from Lemma 5.7. If $\mathcal{H}$ is a Lie subsupergroup of $\mathcal{G}$ and $\mathfrak{h}=\mathrm{Lie}\,\mathcal{H}$ then $\mathfrak{g}/\mathfrak{h}$ is an $\mathcal{H}$-module.

Lemma 5.8. *Let us take a $\mathcal{G}_0$-left invariant split grading operator w. The vector field w satisfies (16) if and only if*

$$\overline{w}\in\big(H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}/\mathfrak{h}\big)^{\mathcal{H}}, \tag{19}$$

where $\overline{w}$ is the image of w by the natural mapping

$$H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{g}\to H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{g}/\mathfrak{h}.$$

Proof. Let $\overline{w}\in(H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}/\mathfrak{h})^{\mathcal{H}}$. It follows that

$$r_h^*\circ w\circ(r_h^{-1})^*-w\in H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{h},\ h\in\mathcal{H}_0,$$

and

$$[Y,w]\in H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{h},\ Y\in\mathfrak{h}.$$

Hence, the conditions (16) are satisfied.

On the other hand, if the conditions (16) are satisfied, then the vector fields $r_h^*\circ w\circ(r_h^{-1})^*-w$ and $[Y,w]$ are vertical with respect to the projection $\pi:\mathcal{G}\to\mathcal{G}/\mathcal{H}$. Therefore, $r_h^*\circ w\circ(r_h^{-1})^*-w$ and $[Y,w]$ belong to the superspace $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{h}$. It is equivalent to conditions (19). ∎

Now our aim is to describe the space $\big(H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}/\mathfrak{h}\big)^{\mathcal{H}_0}$. We have seen in Proposition 3.3 that the superspace $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ is isomorphic to $\bigwedge\mathfrak{g}_{\bar 1}^*$. Actually this isomorphism can be chosen in $\mathcal{G}_0$-equivariant way. More precisely, we need the following lemma.

Proposition 5.9. **a.** *We have*

$$\begin{aligned} H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{g} &\simeq \textstyle\bigwedge(\mathfrak{g}_{\bar 1}^*)\otimes\mathfrak{g} \qquad \text{as } \mathcal{G}_0\text{-modules},\\ H^0\big(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}\big)\otimes\mathfrak{g}/\mathfrak{h} &\simeq \textstyle\bigwedge(\mathfrak{g}_{\bar 1}^*)\otimes\mathfrak{g}/\mathfrak{h} \quad \text{as } \mathcal{H}_0\text{-modules},\end{aligned}$$

where the action of $\mathcal{G}_0$ on $\bigwedge(\mathfrak{g}_{\bar 1}^*)$ is standard.

b. *There exists a $\mathcal{G}_0$-left and right invariant split grading operator on $\mathcal{G}$.*

Proof. **a.** We have to show that there exists an $\mathcal{G}_0$-equivariant isomorphism

$$H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\xrightarrow{\beta}\textstyle\bigwedge\mathfrak{g}_{\bar 1}^*.$$

Then the map $\beta\otimes\mathrm{id}$ will provide the required isomorphism of $\mathcal{G}_0$-modules. Consider the Lie supergroup

$$\mathcal{G}^3=(\mathcal{G}_0,\mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge\mathfrak{g}_{\bar 1},\mathcal{F}_{\mathcal{G}_0}))$$

from Section 3. It follows from (4) that

$$H^0(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}^3})=\mathrm{Hom}_{\mathbb{C}}(\textstyle\bigwedge\mathfrak{g}_{\bar 1},\mathbb{C})=(\bigwedge\mathfrak{g}_{\bar 1})^*.$$

Note that the action of $\mathcal{G}_0$ on $(\bigwedge\mathfrak{g}_{\bar 1})^*$ by right translations in $\mathcal{G}^3$, denoted by $(r'_g)^*$, coincides with the standard action of $\mathcal{G}_0$ on $(\bigwedge\mathfrak{g}_{\bar 1})^*$. Indeed, let us take

$$f\in H^0(\mathcal{G}_0,\mathcal{O}_{\mathcal{G}^3})^{\mathcal{G}_0}=(\textstyle\bigwedge\mathfrak{g}_{\bar 1})^*.$$

By (11), we have:

$$(r'_g)^*(f)(X)(e)=(r'_g)^*(f)(X)(h)=f\big(\mathrm{Ad}(g^{-1})X\big)(hg)=f\big(\mathrm{Ad}(g^{-1})X\big)(e).$$

Here $g,h\in\mathcal{G}_0$, $X\in\bigwedge\mathfrak{g}_{\bar 1}$ and $e\in\mathcal{G}_0$ is the identity. It remains to note that by Lemma 5.6, the map $\Phi_{\mathfrak{g}}$ induces the equivariant isomorphism between the superspaces of left invariants $H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$ and $(\bigwedge\mathfrak{g}_{\bar 1})^*$.

b. We need to show that in the vector space

$$\big(\textstyle\bigwedge(\mathfrak{g}_{\bar 1}^*)\otimes\mathfrak{g}\big)^{\mathcal{G}_0}=\big(\bigwedge(\mathfrak{g}_{\bar 1}^*)\otimes\mathfrak{g}_{\bar 0}\big)^{\mathcal{G}_0}\oplus\big(\bigwedge(\mathfrak{g}_{\bar 1}^*)\otimes\mathfrak{g}_{\bar 1}\big)^{\mathcal{G}_0}$$

there exists points corresponding to split grading operators. This space always possesses a $\mathcal{G}_0$-invariant, precisely, the identity operator $\mathrm{id}\in\mathfrak{g}_{\bar 1}^*\otimes\mathfrak{g}_{\bar 1}$. The preimage of $\beta^{-1}(\mathrm{id})\in H^0(\mathrm{pt},\mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})\otimes\mathfrak{g}$ has the form $\sum\varepsilon^iX_i$ for some choice of local coordinates such that $X_i(\varepsilon^j)(e)=\delta_i^j$, see Example 4.3. We have seen that such vector fields correspond to $\mathcal{G}_0$-left invariant split grading operators on $\mathcal{G}$. ■

Denote by $\mathcal{T}_{\mathcal{G}}$ the tangent sheaf of a Lie group $\mathcal{G}$ and by $\overline{v}$ is the image of v by the natural mapping

$$H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g} \to H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{g}/\mathfrak{h}.$$

The result of our study is:

Theorem 5.10. *The following conditions are equivalent:*

a. *A homogeneous supermanifold $\mathcal{M} = \mathcal{G}/\mathcal{H}$ admits a $\mathcal{G}_0$-left invariant split grading that is induced by a grading of $\mathcal{O}_{\mathcal{G}}$ and the inclusion $\mathcal{O}_{\mathcal{M}} \subset (\pi_0)_*(\mathcal{O}_{\mathcal{G}})$.*

b. *There exists a $\mathcal{G}_0$-left invariant vector field $\chi \in H^0(\mathcal{G}_0, (\mathcal{T}_{\mathcal{G}})_{(2)\bar{0}})$ such that*

$$\overline{\chi} \in \Big(\bigwedge(\mathfrak{g}_{\bar{1}})^* \otimes \mathfrak{g}/\mathfrak{h}\Big)^{\mathcal{H}_0}, \tag{20}$$

and such that for $w = \beta^{-1}(\mathrm{id}) + \chi$, where $\beta^{-1}(\mathrm{id}) = \sum \varepsilon^i X_i$ is from the proof of Proposition 5.9.b, we have

$$[Y, w] \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}}) \otimes \mathfrak{h}, \;\; Y \in \mathfrak{h}_{\bar{1}}. \tag{21}$$

6. An application

As above let $\mathcal{G}$ be a Lie supergroup and $\mathcal{H}$ be a Lie subsupergroup, $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie superalgebras of $\mathcal{G}$ and $\mathcal{H}$, respectively, and $\mathcal{M} := \mathcal{G}/\mathcal{H}$. Consider the map

$$\rho : \mathfrak{g}_{\bar{0}} \to H^0(\mathrm{pt}, \mathcal{T}_{\mathcal{G}_0\backslash\mathcal{G}})$$

induced by the action of $\mathcal{G}_0$ on $\mathcal{M}$. (Here $\mathcal{T}_{\mathcal{G}_0\backslash\mathcal{G}}$ is the sheaf of vector fields on $\mathcal{G}_0\backslash\mathcal{G}$.) Let us describe its kernel. For $X \in \mathfrak{g}_{\bar{0}}$ and $f \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0\backslash\mathcal{G}})$, we have:

$$\begin{aligned} X(f)(Y)(e) = \;& \tfrac{\mathrm{d}}{\mathrm{d}\,t}|_{t=0} f\big(\mathrm{Ad}(\exp(-tX))Y\big)\big(\exp(tX)\big) = \\ & \tfrac{\mathrm{d}}{\mathrm{d}\,t}|_{t=0} f\big(\mathrm{Ad}(\exp(-tX))Y\big)(e), \end{aligned}$$

where $Y = Y_1 \cdots Y_r$, $Y_i \in \mathfrak{g}_{\bar{1}}$ and t is an even parameter. A vector field X is in $\mathrm{Ker}\,\rho$ if and only if $X(f)(Y)(e) = 0$ for all f and Y. Hence,

$$\mathrm{Ker}\,\rho = \mathrm{Ker}(\mathrm{ad}\,|_{\mathfrak{g}_{\bar{1}}}),$$

where ad is the adjoint representation of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}$.

Furthermore, denote

$$\begin{aligned} A :=\;& \mathrm{Ker}\,\big(\mathcal{G}_0 \ni g \mapsto \bar{l}_g : \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{H}\big); \\ \mathfrak{a} :=\;& \mathrm{Ker}\,\big(\mathfrak{g} \ni X \mapsto H^0(\mathcal{G}_0/\mathcal{H}_0, \mathcal{T}_{\mathcal{G}/\mathcal{H}})\big). \end{aligned}$$

Here $\bar{l}_g$ is the automorphism of $\mathcal{G}/\mathcal{H}$ induced by the left translation l_g. The pair $(A, \mathfrak{a})$ is a super Harish-Chandra pair. An action of $\mathcal{G}$ on $\mathcal{M}$ is called *effective* if the corresponding to $(A, \mathfrak{a})$ Lie supergroup is trivial. As in the case of Lie groups any action of a Lie supergroup can be factored to be effective.

Theorem 6.1. *Assume that the action of $\mathcal{G}$ on $\mathcal{M}$ is effective. If*

$$[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}} \cap \operatorname{Ker}(\operatorname{ad}|_{\mathfrak{g}_{\bar{1}}}),$$

then $\mathcal{M}$ is split.

Proof. Let us show that in this case the vector field $w = \sum \varepsilon^i X_i + 0 = \sum \varepsilon^i X_i$ from Proposition 5.9.b is a (left invariant) split grading operator on $\mathcal{M}$ using Theorem 5.10.

The condition (20) is satisfied trivially, because $\chi = 0$. Let us check the condition (21). We have:

$$[Y, v] = \sum Y(\varepsilon^i) X_i - \sum \varepsilon^i [Y, X_i].$$

Since $[\mathfrak{g}_{\bar{1}}, \mathfrak{h}_{\bar{1}}] \subset \mathfrak{h}_{\bar{0}}$, we get

$$\sum \varepsilon^i [Y, X_i] \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0 \backslash \mathcal{G}}) \otimes \mathfrak{h}.$$

Hence, we have to show that

$$\sum Y(\varepsilon^i) X_i \in H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0 \backslash \mathcal{G}}) \otimes \mathfrak{h}.$$

Assume that $X_1, \dots X_k$ is a basis of $\mathfrak{h}_{\bar{1}}$, $X_1, \dots X_k, X_{k+1}, \dots X_m$ is a basis of $\mathfrak{g}_{\bar{1}}$ and (ε^i) is the system of global odd $\mathcal{G}_0$-left invariant coordinates corresponding to this basis such that $\sum \varepsilon^i X_i$ is as in Proposition 5.9.b. In particular, $\varepsilon^i(\gamma_{\mathfrak{g}}(X_j)) = \delta^i_j$, because $\sum (\varepsilon^i \circ \gamma_{\mathfrak{g}}) \otimes X_i$ is the identity operator in $\mathfrak{g}^*_{\bar{1}} \otimes \mathfrak{g}_{\bar{1}}$.

Let us take $Z \in \operatorname{Ker} \rho$. Clearly, $Z(\varepsilon^i) = 0$ and $X_j(\varepsilon^i)$ is again a $\mathcal{G}_0$-left invariant function on $\mathcal{G}$. By (8), we also have:

$$\varepsilon^i(X_{i_1} \cdots Z \cdots X_{i_k}) = 0.$$

Furthermore, by definition of ε^i, we get that $\varepsilon^i \circ \gamma_{\mathfrak{g}} \in \mathfrak{g}^*_{\bar{1}}$. Hence,

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X_{i_1} \wedge \cdots \wedge X_{i_k})) = 0,$$

if $k > 1$. Summing up all these observations we see that

$$\varepsilon^i(\gamma_{\mathfrak{g}}(X) \cdot Y) = \varepsilon^i(\gamma_{\mathfrak{g}}(X \wedge Y)) + 0,$$

where $Y \in \mathfrak{h}$ and $X \in \bigwedge \mathfrak{g}_{\bar{1}}$. Now we can conclude that

$$\sum Y(\varepsilon^i) X_i = -Y \in \mathfrak{h} \subset H^0(\mathrm{pt}, \mathcal{O}_{\mathcal{G}_0 \backslash \mathcal{G}}) \otimes \mathfrak{h}.$$

The proof is complete. ■

Example 6.2. Consider the super-grassmannian $\mathbf{Gr}_{m|n,k|l}$. It is a $\mathrm{GL}_{m|n}$-homogeneous space, see [9] for more details. Hence, $\mathbf{Gr}_{m|n,k|l} \simeq \mathrm{GL}_{m|n}/\mathcal{H}$ for a certain $\mathcal{H}$. (See, for example, [15].) It the case $k = 0$ or $k = m$, the following holds $[(\mathfrak{gl}_{m|n})_{\bar{1}}, \mathfrak{h}_{\bar{1}}] = 0$. Therefore, by Theorem 6.1, the super-grassmannian is split.

In [9] it was shown that the super-grassmannian $\mathrm{GL}_{m|n,k|l}$ is not split if and only if $0 < k < m$ and $0 < l < n$. (This fact also follows from results in [6] and [13] about non-projectivity of super-grassmannian.)

Finally, let us recall a result proved in [14]:

Theorem 6.3. *If a complex homogeneous supermanifold $\mathcal{M}$ is split, then there is a Lie supergroup $\mathcal{G}$ with $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = 0$, where $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}} = \operatorname{Lie} \mathcal{G}$, such that $\mathcal{G}$ acts on $\mathcal{M}$ transitively.*

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Elizaveta Vishnyakova
Université du Luxembourg
6, rue Richard Coudenhove-Kalergi
L-1359 Luxembourg
vishnyakovae@googlemail.com

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