Journal of Lie Theory
Volume **25** (2015) 477–484

Direct Limit of Parafree Lie Algebras

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Communicated by J. Faraut

Abstract. We consider direct limits of parafree Lie algebras and we prove that the direct limit of any directed system of parafree Lie algebras exists in the variety of parafree Lie algebras.
Mathematics Subject Classification 2010: 17B99.
Key Words and Phrases: Parafree Lie Algebras, Free Lie Algebras, Direct Limit.

1. Introduction and the Main Result

In 1967 G. Baumslag introduced a new kind of groups which can be characterized in terms of their descending central series. In his papers [2], [3], [4] Baumslag has investigated the properties of such groups and he has called them parafree groups. Parafree Lie algebras were derived by translating the notion of parafree groups to Lie algebras. In [5] and [6] H. Baur has translated the formal arguments used for parafree groups to parafree Lie algebras and he has proved the existence of these Lie algebras. Baumslag's and Baur's works have given a start for studies in the theory of parafree Lie algebras. In the literature many questions about parafree Lie algebras are waiting answers. In this work we discuss direct limits of parafree Lie algebras and prove that the direct limit of any directed system of parafree Lie algebras exists. The technique of proving is analogous to that in [7]. Here is our main result:

Theorem 1.1. *i) The direct limit of parafree Lie algebras exists.*
ii) The direct limit of any directed system of parafree Lie algebras is a parafree Lie algebra.

2. Notations and Definitions

Let L be a Lie algebra over a field k. The descending central series

$$L = \gamma_1(L) \supseteq \gamma_2(L) \supseteq .... \supseteq \gamma_n(L) \supseteq ...$$

is defined inductively by $\gamma_2(L) = [L, L]$, $\gamma_{n+1}(L) = [\gamma_n(L), L]$, $n \geq 1$. If n is the smallest integer satisfying $\gamma_n(L) = 0$, then L is called nilpotent of

ISSN 0949–5932 / $2.50

degree n . Let $n_1, n_2, \ldots, n_k$ be a sequence of positive integers with $n_i \geq 1$ for $i = 1, 2, \ldots, k$. We define polycentral series of L, relative to this sequence, inductively by

$$L_{n_1,n_2,\ldots,n_i} = \gamma_{n_i}\left(\gamma_{n_{i-1}}(\ldots(\gamma_{n_1}(L))\ldots)\right)$$

for $i \leq k$. In case $n_1 = n_2 = \ldots = n_i = 2$ we write $\delta^i(L) = L_{n_1,n_2,\ldots,n_i}$ and call it the i-th term of the derived series of L. If m is the smallest integer satisfying $\delta^m(L) = \{0\}$ then L is called solvable of degree m .

Definition 2.1. A Lie algebra L is called residually nilpotent if

$$\bigcap_{n=1}^{\infty} \gamma_n(L) = \{0\},$$

equivalently, given any non-trivial element $u \in L$, there exists an ideal J of L such that $u \notin J$ with L/J nilpotent.

Now we define the notion of parafree Lie algebras. Firstly, we associate with the descending central series of L its descending central sequence:

$$L/\gamma_2(L)\ , L/\gamma_3(L)\ , \ldots$$

We say that two Lie algebras L and H have the same descending central sequence if $L/\gamma_n(L) \cong H/\gamma_n(H)$ for every $n \geq 1$.

Definition 2.2. A Lie algebra is called parafree over a set X if,

i) L is residually nilpotent, and

ii) L has the same descending central sequence as a free Lie algebra generated by the set X .

The cardinality of X is called the rank of L . A subset Y of a parafree Lie algebra L is called a paragenerating set if Y generates L modulo $\gamma_2(L)$.

3. Direct Limits in Parafree Lie Algebras

Definition 3.1. Let I be a set with a partial order $\leq$. Then I is called a directed set if for any elements $i, j \in I$, there exists an element $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 3.2. A system $\left(\{P_\alpha\}_{\alpha \in I}; \{\phi_{\alpha\beta}\}_{\alpha \leq \beta}\right)$ of parafree Lie algebras P_α is called a directed system, if

i) I is a directed set;

ii) if $\alpha, \beta \in I, \alpha \leq \beta$, then $\phi_{\alpha\beta} : P_\alpha \longrightarrow P_\beta$ is a Lie algebra monomorphism such that $\phi_{\beta\gamma}\phi_{\alpha\beta} = \phi_{\alpha\gamma}$ for $\alpha \leq \beta \leq \gamma$;

iii) $\phi_{\alpha\alpha} = id_{P_\alpha}$.

Definition 3.3. A direct limit of a directed system $\left(\{P_\alpha\}_{\alpha\in I}\,;\{\phi_{\alpha\beta}\}_{\alpha\leq\beta}\right)$ of parafree Lie algebras is a pair $\left(P;\{\phi_\alpha\}_{\alpha\in I}\right)$, where P is a parafree Lie algebra and each $\phi_\alpha : P_\alpha \longrightarrow P$ is a Lie algebra monomorphism such that

i) $\phi_\beta\phi_{\alpha\beta} = \phi_\alpha$ for $\alpha, \beta \in I$,

ii) if $\psi_\alpha : P_\alpha \to Q$, where Q is a parafree Lie algebra and ψ_α is a Lie algebra monomorphism such that $\psi_\beta\phi_{\alpha\beta} = \psi_\alpha$ for $\alpha \leq \beta$, then there exists a unique Lie algebra monomorphism $\psi : P \to Q$ such that $\psi_\alpha = \psi\phi_\alpha$ for $\alpha \in I$.

In the sequel, let $\left(\{P_\alpha\}_{\alpha\in I}\,;\{\phi_{\alpha\beta}\}_{\alpha\leq\beta}\right)$ be a fixed directed system of finitely generated disjoint parafree Lie algebras and Q be the disjoint union of the parafree Lie algebras P_α, i.e. $Q = \underset{\alpha\in I}{\cup} P_\alpha$, where $\alpha, \beta \in I, \alpha \neq \beta$ and $P_\alpha \cap P_\beta = \emptyset$.

Definition 3.4. Define a binary relation $\sim$ on Q by: $x \sim y$, $x \in P_\alpha, y \in P_\beta$, if there exists $\gamma \in I, \alpha, \beta \leq \gamma$ such that $\phi_{\alpha\gamma}(x) = \phi_{\beta\gamma}(y)$.

It is easy to show that $\sim$ is an equivalence relation. Obviously, " $\sim$ " satisfies reflexivity and antisymmetry. Now we prove it also satisfies transitivity. Let $x \sim y, y \sim z, x \in P_\alpha, y \in P_\beta$ and $z \in P_\gamma$. Then there exists $l_1 \in I$ such that $\alpha, \beta \leq l_1$ and $\phi_{\alpha l_1}(x) = \phi_{\beta l_1}(y)$; there exists $l_2 \in I$ such that $\beta, \gamma \leq l_2$ and $\phi_{\beta l_2}(y) = \phi_{\gamma l_2}(z)$. Let $l_1, l_2 \leq l_3$. We get

$$\phi_{\alpha l_3}(x) = \phi_{l_1 l_3}\phi_{\alpha l_1}(x) = \phi_{l_1 l_3}\phi_{\beta l_1}(y) = \phi_{\beta l_3}(y)$$

and

$$\phi_{\gamma l_3}(z) = \phi_{l_2 l_3}\phi_{\gamma l_2}(z) = \phi_{l_2 l_3}\phi_{\beta l_2}(y) = \phi_{\beta l_3}(y).$$

That is, $\phi_{\alpha l_3}(x) = \phi_{\gamma l_3}(z)$. Hence $x \sim z$.

Proposition 3.5. *If $x \sim y, x \in P_\alpha$ and $y \in P_\beta$, then for all $\gamma \in I$, $\alpha, \beta \leq \gamma$,*

$$\phi_{\alpha\gamma}(x) = \phi_{\beta\gamma}(y).$$

Proof. Since $x \sim y$, then there exists γ_1 such that $\alpha, \beta \leq \gamma_1$ and $\phi_{\alpha\gamma_1}(x) = \phi_{\beta\gamma_1}(y)$. For $\gamma, \gamma_1 \in I$, there exists γ_2 such that $\gamma, \gamma_1 \leq \gamma_2$ and

$$\phi_{\alpha\gamma_2}(x) = \phi_{\gamma_1\gamma_2}\phi_{\alpha\gamma_1}(x) = \phi_{\gamma_1\gamma_2}\phi_{\beta\gamma_1}(y) = \phi_{\beta\gamma_2}(y).$$

Since $\gamma \leq \gamma_2$, then

$$\phi_{\gamma\gamma_2}\phi_{\beta\gamma}(y) = \phi_{\beta\gamma_2}(y) = \phi_{\alpha\gamma_2}(x) = \phi_{\gamma\gamma_2}\phi_{\alpha\gamma}(x).$$

Because $\phi_{\gamma\gamma_2}$ is a monomorphism we have $\phi_{\beta\gamma}(y) = \phi_{\alpha\gamma}(x)$. ■

Proposition 3.6. *For all $\alpha \leq \alpha_1$, $x \in P_\alpha$, then $x \sim \phi_{\alpha\alpha_1}(x)$.*

Proof. Let $\alpha \leq \alpha_1 \leq \alpha_2$, then $\phi_{\alpha\alpha_2}(x) = \phi_{\alpha_1\alpha_2}\phi_{\alpha\alpha_1}(x)$. Hence $x \sim \phi_{\alpha\alpha_1}(x)$. ∎

Proposition 3.7. *If* $x_\alpha \sim y_\alpha$, $x_\alpha, y_\alpha \in P_\alpha$, *then* $x_\alpha = y_\alpha$.

Proof. Since $x_\alpha \sim y_\alpha$, there exists $\beta \in I$ such that $\alpha \leq \beta$ and $\phi_{\alpha\beta}(x_\alpha) = \phi_{\alpha\beta}(y_\alpha)$. By the monomorphism of $\phi_{\alpha\beta}$, we obtain $x_\alpha = y_\alpha$. ∎

Definition 3.8. Let $P = \{\overline{x} : x \in Q\}$ where $\overline{x} = \{y \in Q : y \sim x\}$. Define binary operations on P by:

$$[\overline{x},\overline{y}] = \overline{[\phi_{\alpha\gamma}(x), \phi_{\beta\gamma}(y)]}, \alpha, \beta \leq \gamma,$$

$$\overline{x}+\overline{y} = \overline{\phi_{\alpha\gamma}(x) + \phi_{\beta\gamma}(y)},$$

where $x \in P_\alpha$, $y \in P_\beta$; $[\overline{x}, \overline{y}]$ is defined in P if there exists $\alpha \in I$, $x_\alpha \in \overline{x} \cap P_\alpha$ and $y_\alpha \in \overline{y} \cap P_\alpha$, such that $[x_\alpha, y_\alpha]$ and $x_\alpha + y_\alpha$ are defined in P_α.

Lemma 3.9. *If* $[\overline{x}, \overline{y}]$ *and* $\overline{x}+\overline{y}$ *are defined in* $P, x_\beta \in \overline{x} \cap P_\beta$ *and* $y_\beta \in \overline{y} \cap P_\beta$, *then* $[x_\beta, y_\beta]$ *and* $x_\beta + y_\beta$ *are defined in* P_β *and* $[\overline{x}, \overline{y}] = \overline{[x_\beta, y_\beta]}$, $\overline{x}+\overline{y} = \overline{x_\beta + y_\beta}$. *In particular,* $[-,-]$ *and* $+$ *as defined above are well defined.*

Proof. Since $[\overline{x}, \overline{y}]$ is defined in P, there exists $\alpha \in I, x_\alpha \in \overline{x} \cap P_\alpha$ and $y_\alpha \in \overline{y} \cap P_\alpha$ such that $[x_\alpha, y_\alpha]$ is defined in P_α. Let $\gamma \in I$ with $\alpha, \beta \leq \gamma$. Then $[\phi_{\alpha\gamma}(x_\alpha), \phi_{\alpha\gamma}(y_\alpha)]$ is defined in P_α. Since $x_\alpha \sim x_\beta$, $y_\alpha \sim y_\beta$ and $\alpha, \beta \leq \gamma$, we have

$$\phi_{\alpha\gamma}(x_\alpha) = \phi_{\beta\gamma}(x_\beta) \text{ and } \phi_{\alpha\gamma}(y_\alpha) = \phi_{\beta\gamma}(y_\beta).$$

Hence

$$\phi_{\alpha\gamma}([x_\alpha, y_\alpha]) = [\phi_{\alpha\gamma}(x_\alpha), \phi_{\alpha\gamma}(y_\alpha)] = [\phi_{\beta\gamma}(x_\beta, \phi_{\beta\gamma}(y_\beta)] = \phi_{\beta\gamma}([x_\beta, y_\beta])$$

which implies that $[x_\beta, y_\beta]$ is defined in P_β (because $\phi_{\beta\gamma}$ is a Lie algebra monomorphism), and $[x_\alpha, y_\alpha] \sim [x_\beta, y_\beta]$. Therefore

$$[\overline{x}, \overline{y}] = \overline{[x_\alpha, y_\alpha]} = \overline{[x_\beta, y_\beta]}.$$

Applying similar arguments to $\overline{x}+\overline{y}$ we obtain $\overline{x}+\overline{y} = \overline{x_\alpha + y_\alpha} = \overline{x_\beta + y_\beta}$. ∎

Lemma 3.10. *For all* $\alpha, \beta \in I$, $0_\alpha \sim 0_\beta$, *where* 0_α *and* 0_β *are the zeros of* P_α *and* P_β *respectively.*

Proof. $\phi_{\alpha\gamma}(0_\alpha) = 0_\gamma = \phi_{\beta\gamma}(0_\beta)$. Therefore $0_\alpha \sim 0_\beta$. ∎

Define $\overline{0_\alpha} = 0$.

Proof of Theorem 1.1:

Proof. Let $(\{P_\alpha\}_{\alpha\in I};\{\phi_{\alpha\beta}\}_{\alpha\leq\beta})$ be a directed system of parafree Lie algebras and P be as in Definition 3.8.

i) Suppose that $x_\alpha, y_\alpha \in P_\alpha$. Consider the mapping $\phi_\alpha : P_\alpha \to P$ defined by $\phi_\alpha(x_\alpha) = \overline{x_\alpha}$. Then we have

$$\phi_\alpha\left([x_\alpha, y_\alpha]\right) = \overline{[x_\alpha, y_\alpha]} = [\overline{x_\alpha}, \overline{y_\alpha}] = \left[\phi_\alpha\left(x_\alpha\right), \phi_\alpha\left(y_\alpha\right)\right].$$

Obviously, $\phi_\alpha\left(0_\alpha\right) = \overline{0_\alpha} = 0$. Thus we prove ϕ_α is a Lie algebra homomorphism. Furthermore, if $\phi_\alpha\left(x_\alpha\right) = \overline{0}$ then $x_\alpha \sim 0$. Thus there is $\beta \in I, \alpha \leq \beta$ such that $\phi_{\alpha\beta}\left(x_\alpha\right) = \phi_{\alpha\beta}\left(0\right)$. Since $\phi_{\alpha\beta}$ is a monomorphism, then $x_\alpha = 0$. Thus ϕ_α is a Lie algebra monomorphism. Now we will prove that $\phi_\alpha, \alpha \in I$, satisfies conditions of Definition 3.3.
Firstly, by Proposition 3.6, for all $x_\alpha \in P_\alpha$, $\alpha \leq \beta$, we have $x_\alpha \sim \phi_{\alpha\beta}\left(x_\alpha\right)$. Hence

$$\phi_\beta\phi_{\alpha\beta}\left(x_\alpha\right) = \overline{\phi_{\alpha\beta}\left(x_\alpha\right)} = \overline{x_\alpha} = \phi_\alpha\left(x_\alpha\right),$$

namely, $\phi_\beta\phi_{\alpha\beta} = \phi_\alpha$. Secondly, suppose that $\left(S;\{\psi_\alpha\}_{\alpha\in I}\right)$ is a directed system of parafree Lie algebras such that $\psi_\beta\phi_{\alpha\beta} = \psi_\alpha$ for all $\alpha \leq \beta$. Define $\psi : P \to S$ by

$$\psi\left(\overline{x_\alpha}\right) = \psi_\alpha\left(x_\alpha\right), \alpha \in I.$$

We show that ψ is well defined. If $\overline{x_\alpha} = \overline{x_\beta}$ then $x_\alpha \sim x_\beta$, which implies that there is $\gamma \in I$, $\gamma \geq \alpha, \beta$ such that $\phi_{\alpha\gamma}\left(x_\alpha\right) = \phi_{\beta\gamma}\left(x_\beta\right)$. Therefore

$$\psi_\gamma\phi_{\alpha\gamma}\left(x_\alpha\right) = \psi_\gamma\phi_{\beta\gamma}\left(x_\beta\right)$$

and

$$\psi_\alpha\left(x_\alpha\right) = \psi_\beta\left(x_\beta\right), \psi\left(\overline{x_\alpha}\right) = \psi\left(\overline{x_\beta}\right).$$

Therefore ψ is well defined. It is clear that ψ is a Lie algebra monomorphism. Now we prove that ψ is the unique Lie algebra monomorphism such that $\psi\phi_\alpha = \psi_\alpha$, $\alpha \in I$. Suppose that $\phi : P \to S$ is a Lie algebra monomorphism such that $\phi\phi_\alpha = \psi_\alpha$, for any $\alpha \in I$, then

$$\phi\left(\overline{x_\alpha}\right) = \phi\phi_\alpha\left(x_\alpha\right) = \psi_\alpha\left(x_\alpha\right) = \psi\left(\overline{x_\alpha}\right).$$

Therefore, ψ is the unique Lie algebra monomorphism satisfying the condition. Hence the direct limit of the directed system $(\{P_\alpha\}_{\alpha\in I};\{\phi_{\alpha\beta}\}_{\alpha\leq\beta})$ exists and is equal to $(P;\{\phi_\alpha\}_{\alpha\in I})$.

We next prove (ii).
We are going to prove that the set $P = \{\overline{x} : x \in Q\}$ is a parafree Lie algebra with operations $[,]$ and $+$, where the operations $[,]$ and $+$ are as in Definition 3.8. Straightforward calculations show that P is a Lie algebra. In order to prove parafreeness of P , we need the equality

$$\gamma_n\left(P\right) = \bigcup_{\alpha\in I}\gamma_n\left(P_\alpha\right). \tag{1}$$

Consider the set $K = \{\overline{w_\alpha} : \alpha \in I,\, w_\alpha \in \gamma_n(P_\alpha)\}$. Obviously $K \subset \gamma_n(P)$. Conversely, let $w \in \gamma_n(P)$. Then w can be written as a linear combination of elements of the form

$$[[[\overline{u}_1,\overline{u}_2],...],\overline{u_n}] \tag{2}$$

where $\overline{u}_i \in P$, We consider the following cases.

a) If $u_1,...,u_n \in P_k$, for any $k \in I$, then by definition

$$[[[\overline{u}_1,\overline{u}_2],...],\overline{u_n}] = \overline{[[...,[u_1,u_2],...],u_n]}.$$

Clearly $[[[u_1,u_2],...],u_n] \in \gamma_n(P_k)$. Hence $[[[\overline{u}_1,\overline{u}_2],...],\overline{u_n}] \in K$. Therefore we get the equality (1). Since for all $\alpha \in I$, P_α are parafree then

$$\bigcap_{n=1}^{\infty} \gamma_n(P) = \bigcap_{n=1}^{\infty}\Big(\bigcup_{\alpha\in I}\gamma_n(P_\alpha)\Big) = \bigcup_{\alpha\in I}\Big(\bigcap_{n=1}^{\infty}\gamma_n(P_\alpha)\Big) = 0.$$

b) Now let $u_i \in P_{\alpha_i}$, i=1,...,n, $\alpha_i \in I$. Then there exists $\beta \in I$ such that $\beta \geq \max\{\alpha_i : \alpha_i \in I, i = 1,...,n\}$ and

$$[[...[\overline{u}_1,\overline{u}_2],...],\overline{u_n}] = \overline{[[...[\varphi_{\alpha_{1\beta}}(u_1),\varphi_{\alpha_{2\beta}}(u_2)]...],\varphi_{\alpha_{n\beta}}(u_n)]}.$$

Clearly $[[...[\varphi_{\alpha_{1\beta}}(u_1),\varphi_{\alpha_{2\beta}}(u_2)]...],\varphi_{\alpha_{n\beta}}(u_n)] \in \gamma_n(P_\beta)$. Thus $[[...[\overline{u}_1,\overline{u}_2],...],\overline{u_n}] \in K$. Hence any linear combination of the elements of the form (2) belongs to K. Therefore we get the equality (1). Since for all $\alpha \in I$, P_α is parafree then

$$\bigcap_{n=1}^{\infty} \gamma_n(P) = \bigcap_{n=1}^{\infty}\Big(\bigcup_{\alpha\in I}\gamma_n(P_\alpha)\Big) = \bigcup_{\alpha\in I}\Big(\bigcap_{n=1}^{\infty}\gamma_n(P_\alpha)\Big) = 0.$$

Hence P is residually nilpotent.

Now we show that P has the same descending central sequence as a free Lie algebra. Define a binary relation $\approx$ on $\bigcup_{\alpha\in I}(P_\alpha/\gamma_n(P_\alpha))$ by: $x \approx y$, $x = x_\alpha + \gamma_n(P_\alpha) \in (P_\alpha/\gamma_n(P_\alpha))$ and $y = y_\beta + \gamma_n(P_\beta) \in (P_\beta/\gamma_n(P_\beta))$, iff $x_\alpha \sim y_\beta$. Let $\overline{\overline{x_\alpha + \gamma_n(P_\alpha)}}$ be the equivalence class of $x_\alpha + \gamma_n(P_\alpha)$ with respect to $\approx$. Consider the homomorphism ψ from $(P/\gamma_n(P))$ to $\bigcup_{\alpha\in I}(P_\alpha/\gamma_n(P_\alpha))$ defined by

$$\psi : (\overline{x_\alpha} + \gamma_n(P)) \;\to \overline{\overline{x_\alpha + \gamma_n(P_\alpha)}},$$

where $x_\alpha \in P_\alpha$, $\alpha \in I$. For $x_\alpha \in P_\alpha$, $x_\beta \in P_\beta$ and $x_\alpha \sim x_\beta$, let $\overline{x_\alpha} + \gamma_n(P) = \overline{x_\beta} + \gamma_n(P)$. Hence

$$\psi(\overline{x_\alpha} + \gamma_n(P)) = \overline{\overline{x_\alpha + \gamma_n(P_\alpha)}} = \overline{\overline{x_\beta + \gamma_n(P_\beta)}} = \psi(\overline{x_\beta} + \gamma_n(P)).$$

Therefore ψ is well defined. If $x \in \bigcup_{\alpha\in I}(P_\alpha/\gamma_n(P_\alpha))$ then for any $\alpha \in I$,

$$x = \overline{\overline{x_\alpha + \gamma_n(P_\alpha)}} = \psi\left(\overline{x}_\alpha + \gamma_n(P)\right).$$

Hence ψ is onto. Now we compute the kernel of ψ. If $\overline{x_\alpha} + \gamma_n(P) \in ker\psi$, then

$$\psi\left(\overline{x_\alpha} + \gamma_n(P)\right) = \overline{\overline{x_\alpha + \gamma_n(P_\alpha)}} = \overline{\overline{\gamma_n(P_\alpha)}}.$$

Thus $\overline{x}_\alpha = \overline{0}$ and $\overline{x}_\alpha + \gamma_n(P) = \overline{0} + \gamma_n(P) = \gamma_n(P)$. Hence $ker\psi = \gamma_n(P)$, namely ψ is one to one. Hence ψ is an isomorphism and

$$P/\gamma_n(P) \cong \bigcup_{\alpha\in I}\left(P_\alpha/\gamma_n(P_\alpha)\right). \tag{3}$$

Let X_α be a finite paragenerating set of $P_\alpha, \alpha \in I$. By definiton X_α freely generates P_α modulo $\gamma_2(P_\alpha)$. Then X_α freely generates P_α modulo $\gamma_n(P_\alpha)$ [1]. Then

$$P_\alpha/\gamma_n(P_\alpha) \cong F_\alpha/\gamma_n(F_\alpha),$$

where F_α is a free Lie algebra which is freely generated by X_α. Let F be the free product of F_α, $\alpha \in I$. Then by the equality (3) and parafreeness of P_α,

$$P/\gamma_n(P) \cong \bigcup_{\alpha\in I}\left(P_\alpha/\gamma_n(P_\alpha)\right) \cong \bigcup_{\alpha\in I}\left(F_\alpha/\gamma_n(F_\alpha)\right) \cong F/\gamma_n(F).$$

Therefore P is a parafree Lie algebra. ■

Theorem 1.1 has many applications. One of them is the construction of parafree Lie algebras with certain properties. For example union of free Lie algebras of rank two is a parafree Lie algebra. This follows without much difficulty from Theorem 1.1.

Example 3.11. Let I be a directed set and F_i be the free Lie algebra generated by free generators a_i and b_i for $i \in I$. Consider the homomorphism ϕ_{ij} from F_i into F_j defined by

$$\begin{aligned} \phi_{ij} : a_i &\to a_i + u_i, \\ b_i &\to b_j, \end{aligned}$$

where $u_i \in \gamma_2(F_j), i, j \in I, i < j$. Then straightforward calculations show that $\left(\{F_i\}_{i\in I}; \{\phi_{ij}\}_{i<j}\right)$ is a directed system. Now , by taking F_i and ϕ_{ij} instead of P_α and $\phi_{\alpha\beta}$, respectively , in Theorem 1.1, we get that $\bigcup F_i$ is a parafree Lie algebra.

Acknowledgment. The authors would like to thank the referee for useful remarks on a former version of the paper.

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Received February 22, 2014
and in final form August 23, 2014