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On a Geometry Associated with Free Pseudo-Product Fundamental Graded Lie Algebras

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Abstract. We calculate the prolongations of the underlying fundamental graded Lie algebras of free pseudo-product fundamental graded Lie algebras. Also we show that the Lie algebra of infinitesimal automorphisms of a differential system associated with the Hilbert-Cartan equation in a wider sense is isomorphic to the prolongation of the underlying fundamental graded Lie algebra of some free pseudo-product fundamental graded Lie algebra.
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1. Introduction

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a graded Lie algebra (GLA) over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and μ a positive integer. The GLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a fundamental graded Lie algebra (FGLA) if the following conditions hold: (i) $\mathfrak{m}$ is finite dimensional; (ii) $\mathfrak{g}_{-1} \neq \{0\}$, and $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$. Moreover an FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is said to be of the μ-th kind if $\mathfrak{g}_{-\mu} \neq \{0\}$, and $\mathfrak{g}_p = \{0\}$ for all $p < -\mu$. It is well known that every FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is prolonged to a GLA $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ satisfying the following conditions: (i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$; (ii) for $X \in \mathfrak{g}(\mathfrak{m})_p$ ($p \geqq 0$), $[X, \mathfrak{m}] = \{0\}$ implies $X = 0$; (iii) $\mathfrak{g}(\mathfrak{m})$ is maximum among GLAs satisfying conditions (i) and (ii) above. The GLA $\mathfrak{g}(\mathfrak{m})$ is called the prolongation of $\mathfrak{m}$. Note that $\mathfrak{g}(\mathfrak{m})_0$ is the Lie algebra of all the derivations of $\mathfrak{m}$ as a GLA.

In the previous paper [24], we introduced the notion of free pseudo-product FGLAs $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ of type (n, m, μ), which was inspired by the study of the prolongation of free FGLAs in [16]. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA, and let $\mathfrak{e}$ and $\mathfrak{f}$ be nonzero subspaces of $\mathfrak{g}_{-1}$. Then $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ is called a pseudo-product FGLA if the following conditions hold: (i) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$; (ii) $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = \{0\}$.

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Let $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ be a pseudo-product FGLA, and $\mathfrak{g}(\mathfrak{m})=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}(\mathfrak{m})_p$ the prolongation of $\mathfrak{m}$. Moreover let $\mathfrak{g}_0$ be the Lie algebra of all derivations of $\mathfrak{m}$ as a GLA preserving $\mathfrak{e}$ and $\mathfrak{f}$. Also for $p \geqq 1$ we set $\mathfrak{g}_p=\{\, X\in\mathfrak{g}(\mathfrak{m})_p : [X,\mathfrak{g}_k]\subset \mathfrak{g}_{p+k}$ for all $k<0\,\}$ inductively. Then the direct sum $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ becomes a graded subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$.

Let $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ be a pseudo-product FGLA of the μ-th kind, where $\mu\geqq 2$. The pseudo-product FGLA $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ is called a free pseudo-product fundamental graded Lie algebra of type (n,m,μ) if the following conditions hold: (i) $\dim\mathfrak{e}=n$ and $\dim\mathfrak{f}=m$; (ii) Let $(\mathfrak{m}';\mathfrak{e}',\mathfrak{f}')$ be a pseudo-product FGLA of the μ-th kind and φ a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$ such that $\varphi(\mathfrak{e})\subset\mathfrak{e}'$ and $\varphi(\mathfrak{f})\subset\mathfrak{f}'$. Then φ can be extended uniquely to a GLA epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

In the previous paper [24], we investigated the prolongation of a free pseudo-product fundamental graded Lie algebra $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ of type (n,m,μ). If $\mu=2$, then the prolongation of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ is isomorphic to a finite dimensional simple graded Lie algebra (SGLA) of type $(A_{n+m},\{\alpha_n,\alpha_{n+1}\})$ (resp. $((\mathrm{AI})_{n+m},\{\alpha_n,\alpha_{n+1}\})$) in case $\mathbb{K}=\mathbb{C}$ (resp. $\mathbb{K}=\mathbb{R}$) (cf. Remark 5.4). If $\mu=3$, then the prolongation $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ has the following model space: $\mathfrak{g}_{-3}=S^2(V)\otimes W\oplus V\otimes S^2(W)$, $\mathfrak{g}_{-2}=V\otimes W$, $\mathfrak{e}=V$, $\mathfrak{f}=W$, $\mathfrak{g}_0=\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$, $\mathfrak{g}_k=\{0\}$ $(k\geqq 1)$, where V (resp. W) is an n-dimensional (resp. an m-dimensional) vector space. Here the bracket operations on $\mathfrak{g}$ are defined by tensor products in a natural manner (For the details, see Example 5.1).

The first purpose of this paper is to classify the prolongations of the underlying FGLA $\mathfrak{m}$ of free pseudo-product FGLAs $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ of type (n,m,μ) (Theorem 5.6). In particular, the prolongation of the underlying FGLA $\mathfrak{m}$ of a free pseudo-product FGLA $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ of type $(1,1,3)$ is a finite dimensional SGLA of type $(G_2,\{\alpha_1\})$ (resp. $(\mathrm{G},\{\alpha_1\})$) in case $\mathbb{K}=\mathbb{C}$ (resp. $\mathbb{K}=\mathbb{R}$). Also the prolongation $\mathfrak{g}(\mathfrak{m})$ of the underlying FGLA $\mathfrak{m}$ of a free pseudo-product FGLA $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ of type $(1,m,3)$ $(m\geqq 2)$ is as follows: $\mathfrak{g}(\mathfrak{m})_{-3}=S^2(V)\otimes W\oplus V\otimes S^2(W)$, $\mathfrak{g}(\mathfrak{m})_{-2}=V\otimes W$, $\mathfrak{g}(\mathfrak{m})_{-1}=V\oplus W$, $\mathfrak{g}(\mathfrak{m})_0=\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)\oplus V^*\otimes W$, $\mathfrak{g}(\mathfrak{m})_1=V^*$, $\mathfrak{g}(\mathfrak{m})_k=\{0\}$ $(k\geqq 2)$, where V (resp. W) is a 1-dimensional (resp. an m-dimensional) vector space. Note that on this $\mathfrak{g}(\mathfrak{m})$ there exists another important gradation $\mathfrak{g}(\mathfrak{m})=\bigoplus_{p\in\mathbb{Z}}\mathfrak{a}_p$ as follows: $\mathfrak{a}_k=\{0\}$ for $k\leqq -6$ or $k\geqq 2$, $\mathfrak{a}_{-5}=V\otimes S^2(W)$, $\mathfrak{a}_{-4}=S^2(V)\otimes W$, $\mathfrak{a}_{-3}=V\otimes W$, $\mathfrak{a}_{-2}=W$, $\mathfrak{a}_{-1}=V\oplus V^*\otimes W$, $\mathfrak{a}_0=\mathfrak{gl}(V)\oplus\mathfrak{gl}(W)$, $\mathfrak{a}_1=V^*$; then $\mathfrak{g}(\mathfrak{m})=\bigoplus_{p\in\mathbb{Z}}\mathfrak{a}_p$ becomes a pseudo-product GLA of irreducible type.

D. Hilbert [7] and E. Cartan [3], [4] studied the following underdetermined ordinary differential equation, which is called the Hilbert-Cartan equation:

$$\frac{dv}{dx}=\left(\frac{d^2u}{dx^2}\right)^2$$

(also see [13]). To give a good illustration of regular differential systems, K. Yamaguchi [19] showed that a differential system (R,D) associated with the Hilbert-Cartan equation is a standard differential system of type $\mathfrak{m}_6$, where $\mathfrak{m}_6$ is the negative part of a finite dimensional SGLA of type $(G_2, \{\alpha_1, \alpha_2\})$. Furthermore he showed that the Lie algebra of the infinitesimal automorphisms of (R, D) is isomorphic to the 14-dimensional exceptional simple Lie algebra G_2. In Section 6 we consider a little generalization of the above situation. Namely we consider a differential system (R, D) associated with the following underdetermined ordinary differential equations, which is called the Hilbert-Cartan equation in a wider sense:

$$\frac{dv_{ab}}{dx} = \frac{d^2u_a}{dx^2}\frac{d^2u_b}{dx^2} \quad (1 \leqq a \leqq b \leqq m).$$

We show that the differential system (R, D) is a standard differential system of type $\bigoplus_{p<0} \mathfrak{a}_p$ and the Lie algebra of infinitesimal automorphisms of (R, D) is isomorphic to the prolongation $\mathfrak{g}(\mathfrak{m})$ of the underlying FGLA of a free pseudo-product FGLA of type $(1, m, 3)$.

In the paper [2] E. Cartan showed that the symmetry algebra of the following overdetermined involutive system of second order is the exceptional simple Lie algebra G_2:

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{3}\left(\frac{\partial^2 z}{\partial y^2}\right)^3, \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)^2.$$

In [20], [21], and [22], K. Yamaguchi clarified the above results in the modern procedures. Precisely starting from the standard differential system of type $\mathfrak{m}_5$, he obtained the above differential equation and showed that the symmetry algebra is the exceptional simple Lie algebra G_2 by using the realization lemma and the first reduction theorem, where $\mathfrak{m}_5$ is a free pseudo-product FGLA of type $(1, 1, 3)$. In Section 7 we similarly see that the following system of partial differential equations is associated with a free pseudo-product FGLA of type $(1, m, 3)$ $(m \geqq 2)$.

$$\begin{cases} \dfrac{\partial^2 Z_{ab}}{\partial X_a \partial X_b} = \dfrac{1}{3}\left(\dfrac{\partial^2 Z_{ab}}{\partial Y_a \partial Y_b}\right)^3, \ \dfrac{\partial^2 Z_{ab}}{\partial X_a \partial Y_b} = \dfrac{1}{2}\left(\dfrac{\partial^2 Z_{ab}}{\partial Y_a \partial Y_b}\right)^2, \\ \dfrac{\partial^2 Z_{ab}}{\partial Y_a \partial Y_b} = \dfrac{\partial^2 Z_{cd}}{\partial Y_c \partial Y_d}, \ \dfrac{\partial Z_{ab}}{\partial X_e} = \dfrac{\partial Z_{ab}}{\partial Y_e} = 0 \\ \dfrac{\partial Z_{ab}}{\partial X_a} = \dfrac{\partial Z_{bb}}{\partial X_b}, \ \dfrac{\partial Z_{ab}}{\partial X_b} = \dfrac{\partial Z_{aa}}{\partial X_a}, \ \dfrac{\partial Z_{ab}}{\partial Y_a} = \dfrac{\partial Z_{bb}}{\partial Y_b}, \ \dfrac{\partial Z_{ab}}{\partial Y_b} = \dfrac{\partial Z_{aa}}{\partial Y_a}, \\ (1 \leqq a \leqq b \leqq m, 1 \leqq c \leqq d \leqq m, e \neq a, b). \end{cases} \tag{GA}$$

In recent years there have been a lot of interest on Tanaka prolongation theory in relation to the study of rigidity of geometric structures on standard differential systems associated with FGLAs. In particular, nondegenerate pseudo-product structures considered in this paper have been investigated in great generality under the name of multicontact structures (see [5], [6], [10], etc.). Although the theory of multicontact structures is available for our paper, we construct this paper within the framework of the geometric theory of differential equations studied by N. Tanaka, T. Morimoto and K. Yamaguchi. In the future we would like to contribute somewhat to the study of multicontact structures.

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2. Preliminaries

All vector spaces are considered over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. We denote by $\mathbb{C}V$ the complexification of a real vector space V. For a Lie algebra $\mathfrak{g}$ we denote by $\mathrm{Aut}(\mathfrak{g})$ (resp. $\mathrm{Int}(\mathfrak{g})$) the group of all the automorphisms (resp. the inner automorphisms) of $\mathfrak{g}$.

2.1. Graded Lie algebras. Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a graded Lie algebra (GLA). A GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called transitive if for $X \in \mathfrak{g}_p$ $(p \geqq 0)$, $[X, \mathfrak{g}_-] = 0$ implies $X = 0$, where $\mathfrak{g}_-$ is the negative part $\bigoplus_{p<0} \mathfrak{g}_p$ of $\mathfrak{g}$. A GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called irreducible (resp. completely reducible) if the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible (resp. completely reducible). If there exists an element E of $\mathfrak{g}_0$ such that $[E, X] = pX$ for all $p \in \mathbb{Z}$ and $X \in \mathfrak{g}_p$, then E is called the characteristic element of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. For a GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ we denote by $\mathrm{Der}(\mathfrak{g})_0$ the Lie algebra of all the derivations preserving the gradation $(\mathfrak{g}_p)$ of $\mathfrak{g}$.

Now we define fundamental graded Lie algebras. A GLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called a *fundamental graded Lie algebra* (FGLA) if the following conditions hold: (i) $\dim \mathfrak{m} < \infty$; (ii) $\mathfrak{g}_{-1} \neq \{0\}$ and $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$. An FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is said to be of the μ-th kind (or of depth μ) if $\mathfrak{g}_{-\mu} \neq 0$ and $\mathfrak{g}_p = 0$ for all $p < -\mu$. An FGLA $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is called nondegenerate if for $X \in \mathfrak{g}_{-1}$, $[X, \mathfrak{g}_{-1}] = \{0\}$ implies $X = 0$.

Next, following [14], we introduce the notion of the prolongations of FGLAs. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA. A GLA $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ is called the prolongation of $\mathfrak{m}$ if the following conditions hold: (i) $\mathfrak{g}(\mathfrak{m})_p = \mathfrak{g}_p$ for all $p < 0$; (ii) $\mathfrak{g}(\mathfrak{m})$ is a transitive GLA; (iii) If $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ is a GLA satisfying conditions (i) and (ii) above, then $\mathfrak{h} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{h}_p$ can be embedded in $\mathfrak{g}(\mathfrak{m})$ as a GLA. For the construction of $\mathfrak{g}(\mathfrak{m})$, see [14, §5]. Note that $\mathfrak{g}(\mathfrak{m})_0$ is isomorphic to $\mathrm{Der}(\mathfrak{m})_0$.

Let $\mathfrak{m}$ and $\mathfrak{g}(\mathfrak{m})$ be as above. Assume that we are given a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}(\mathfrak{m})_0$. We define subspaces $\mathfrak{g}_p$ $(p \geqq 1)$ of $\mathfrak{g}(\mathfrak{m})_p$ inductively as follows: $\mathfrak{g}_p = \{\, X \in \mathfrak{g}(\mathfrak{m})_p : [X, \mathfrak{g}_k] \subset \mathfrak{g}_{p+k}$ for all $k < 0 \,\}$. If we put $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$, then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ becomes a graded Lie subalgebra of $\mathfrak{g}(\mathfrak{m})$, which is called the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$.

2.2. Differential systems. By a differential system (M, D), we mean a subbundle D of the tangent bundle $T(M)$ of a manifold M. For two differential systems (M_1, D_1) and (M_2, D_2) a diffeomorphism φ of M_1 onto M_2 is called an isomorphism of (M_1, D_1) onto (M_2, D_2) if $\varphi_*(D_1) = D_2$.

For a differential system (M, D), we define a sequence of k-th weak derived sheaves $\{\partial^{(k)}\mathcal{D}\}_{k\geqq 0}$ inductively by setting $\partial^{(0)}\mathcal{D} = \underline{D}$ (the sheaf of the germs of local sections of D) and

$$\partial^{(k)}\mathcal{D} = \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$$

for $k \geqq 1$. By setting $\mathcal{D}^{-p} = \partial^{(p-1)}\mathcal{D}$, we have $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p, q < 0$. Furthermore the Cauchy-Cartan characteristic system $\mathrm{Ch}(D)$ of (M, D) is defined at each $x \in M$ by

$$\mathrm{Ch}(D)(x) = \{\, X \in D(x) : X\rfloor(d\omega)_x \equiv 0 \quad (\mathrm{mod}\ D^{\perp}(x)) \quad \text{for all } \omega \in \mathcal{D}^{\perp}_x \,\},$$

where $D^{\perp}$ is the annihilator subbundle of D in $T^*(M)$ and $\mathcal{D}^{\perp}$ is the sheaf of the germs of local sections of $D^{\perp}$. We say that (M, D) is bracket generating if $\bigcup_{p<0} \mathcal{D}^p = \underline{T(M)}$. We say also (M, D) is regular if there exist subbundles $\{D^p\}_{p<0}$ of $T(M)$ such that $\mathcal{D}^p = \underline{D^p}$ for all $p < 0$.

Let (M, D) be a differential system such that D is bracket generating and regular. There exists a positive integer μ such that $D^{-\mu} = T(M)$. The minimum of such integers is called the depth of D. For each $x \in M$ we put

$$\mathfrak{m}(x) = \bigoplus_{p<0} \mathfrak{g}_p(x),$$

where $\mathfrak{g}_p(x) = D^p(x)/D^{p+1}(x)$. $\mathfrak{m}(x)$ has a naturally defined bracket operation and becomes an FGLA, which is called the symbol algebra at x of (M, D). Let $\mathfrak{m}$ be an FGLA. (M, D) is said to be of type $\mathfrak{m}$ if it is bracket generating and regular and if $\mathfrak{m}(x)$ is isomorphic to $\mathfrak{m}$ as a GLA for all $x \in M$.

Example 2.1. Let $\mathfrak{m}$ be an FGLA and $M(\mathfrak{m})$ a simply connected Lie group with Lie algebra $\mathfrak{m}$. Identifying $\mathfrak{m}$ with the Lie algebra of left invariant vector fields on $M(\mathfrak{m})$, $\mathfrak{g}_{-1}$ induces a left invariant subbundle $D_{\mathfrak{m}}$ of $T(M(\mathfrak{m}))$:

$$D_{\mathfrak{m}}(x) = \{\, X_x \in T_x(M(\mathfrak{m})) : X \in \mathfrak{g}_{-1} \,\}.$$

Then $(M(\mathfrak{m}), D_{\mathfrak{m}})$ becomes a differential system of type $\mathfrak{m}$, which is called a standard differential system of type $\mathfrak{m}$.

Let (M, D) be a differential system and $\Gamma(D)$ the space consisting of sections of D. A vector field X on M is called an infinitesimal automorphism of (M, D) if $[X, \Gamma(D)] \subset \Gamma(D)$. We denote by $\mathcal{A}(M, D)$ the Lie algebra consisting of all the infinitesimal automorphisms of (M, D). Let $\mathfrak{m}$ be an FGLA. As well known, $\mathcal{A}(M(\mathfrak{m}), D_{\mathfrak{m}})$ is isomorphic to $\mathfrak{g}(\mathfrak{m})$ when the prolongation of $\mathfrak{m}$ is finite dimensional (see [19, §2.3]).

3. Finite dimensional simple graded Lie algebras

In this section, following [19], we describe the classification of finite dimensional simple graded Lie algebras.

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a finite dimensional simple graded Lie algebra (SGLA) of the μ-th kind over $\mathbb{C}$ such that the negative part $\mathfrak{g}_-$ is an FGLA. Let $\mathfrak{h}$ be a

Cartan subalgebra of $\mathfrak{g}_0$; then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $E \in \mathfrak{h}$, where E is the characteristic element of $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. Let Φ be a root system of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Phi$, we denote by $\mathfrak{g}^\alpha$ the root space corresponding to α. We set $\mathfrak{h}_\mathbb{R} = \{\, h \in \mathfrak{h} : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in \Phi \,\}$ and let $(h_1, \ldots, h_l)$ be a basis of $\mathfrak{h}_\mathbb{R}$ such that $h_1 = E$. We define the set of positive roots Φ^+ as the set of roots which are positive with respect to the lexicographical ordering in $\mathfrak{h}_\mathbb{R}^*$ determined by the basis $(h_1, \ldots, h_l)$ of $\mathfrak{h}_\mathbb{R}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi^+$ be the corresponding simple root system. In what follows, we assume that the ordering of $(\alpha_1, \ldots, \alpha_l)$ is as in the table of [1]. We set $\Phi_p = \{\, \alpha \in \Phi : \mathfrak{g}^\alpha \subset \mathfrak{g}_p \,\}$ $(p \in \mathbb{Z})$. Then there exists a subset Δ_1 of Δ such that $\Phi_p = \{\, \alpha = \sum_{i=1}^{l} m_i\alpha_i \in \Phi : \sum_{\alpha_i\in\Delta_1} m_i = p \,\}$ for all $p \in \mathbb{Z}$. If $\mathfrak{g}$ has the Dynkin diagram of type X_l $(X = A, \ldots, G)$, then the SGLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of type (X_l, Δ_1). Here we remark that for an automorphism $\bar{\mu}$ of the Dynkin diagram, an SGLA of type (X_l, Δ_1) is isomorphic to that of type $(X_l, \bar{\mu}(\Delta_1))$. We will identify a simple GLA of type (X_l, Δ_1) with that of type $(X_l, \bar{\mu}(\Delta_1))$.

Similarly we can describe the gradations of real simple Lie algebras by utilizing the Satake diagrams. For the details, see [19, §3].

4. Graded Lie algebras $\boldsymbol{W(1,m;\mathbb{K})}$ and $\boldsymbol{K(n;\mathbb{K})}$ of Cartan type

Let $A(k)$ denote the monoid (under addition) of all k-tuples of non-negative integers. We put $\mathfrak{A}(k) = \mathbb{K}[x_1, \ldots, x_k]$. For any k-tuple $\boldsymbol{s} = (s_1, \ldots, s_k)$ of positive integers, we denote by $\mathfrak{A}(k;\boldsymbol{s})_p$ the subspace of $\mathfrak{A}(k)$ spanned by polynomials

$$x^\alpha = x_1^{\alpha_1} \cdots x_k^{\alpha_k} \qquad (\alpha = (\alpha_1, \ldots, \alpha_k) \in A(k),\ \sum_{i=1}^{k} \alpha_i s_i = p).$$

Let $W(k;\mathbb{K})$ be the Lie algebra consisting of all the operators of the form:

$$\sum_{i=1}^{k} P_i \frac{\partial}{\partial x_i} \quad (P_i \in \mathfrak{A}(k)). \tag{4.1}$$

For a k-tuple $\boldsymbol{s} = (s_1, \ldots, s_k)$ of positive integers, we denote by $W(k;\boldsymbol{s};\mathbb{K})_p$ the subspaces of $W(k;\mathbb{K})$ consisting of those operators (4.1) such that the polynomials P_i are contained in $\mathfrak{A}(k;\boldsymbol{s})_{p+s_i}$; then $W(k;\boldsymbol{s};\mathbb{K}) = \bigoplus_{p\in\mathbb{Z}} W(k;\boldsymbol{s};\mathbb{K})_p$ is a transitive GLA. Now we assume that $\boldsymbol{s}$ is the $(1+m)$-tuple $(1, \underbrace{2, \cdots, 2}_{m})$. We denote by $W(1,m;\mathbb{K}) = \bigoplus_{p\in\mathbb{Z}} W(1,m;\mathbb{K})_p$ the GLA $W(1+m;\boldsymbol{s};\mathbb{K}) = \bigoplus_{p\in\mathbb{Z}} W(1+m;\boldsymbol{s};\mathbb{K})_p$. If $m \geqq 2$, then $W(1,m;\mathbb{K})$ is the prolongation of the negative part $W(1,m;\mathbb{K})_-$ of $W(1,m;\mathbb{K})$ (cf. [11]). Also $W(1,m;\mathbb{K})_-$ is isomorphic to the contact algebra of the first order of bidegree $(1,m)$ (cf. [17, p. 134]).

We denote by $K(n;\mathbb{K})$ the subalgebra of $W(2n+1;\mathbb{K})$ consisting of operators that multiply the form $\omega_K = dx_{2n+1} - \sum_{i=1}^{n} x_{i+n} dx_i$ by an element of the algebra $\mathfrak{A}(2n+1)$. We set $K(n;\mathbb{K})_p = W(2n+1;\boldsymbol{t};\mathbb{K})_p \cap K(n;\mathbb{K})$, where $\boldsymbol{t}$ is a $(2n+1)$-tuple of integers $(1,\dots,1,2)$. Then $K(n;\mathbb{K}) = \bigoplus_{p \geqq -2} K(n;\mathbb{K})_p$ is a transitive irreducible GLA such that $K(n;\mathbb{K})$ is the prolongation of $K(n;\mathbb{K})_-$ (cf. [9], [11]). Note that the prolongation of $W(1,1;\mathbb{K})_-$ is isomorphic to $K(1;\mathbb{K})$.

5. Free pseudo-product fundamental graded Lie algebras

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be an FGLA, and let $\mathfrak{e}$ and $\mathfrak{f}$ be nonzero subspaces of $\mathfrak{g}_{-1}$. The triplet $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ is called a pseudo-product FGLA if the following conditions hold: (i) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$; (ii) $[\mathfrak{e},\mathfrak{e}] = [\mathfrak{f},\mathfrak{f}] = \{0\}$. For a given pseudo-product FGLA $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ we say that the pair $(\mathfrak{e},\mathfrak{f})$ is the pseudo-product structure of the FGLA $\mathfrak{m}$. Let $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ be a pseudo-product FGLA. We denote by $\mathfrak{g}_0$ the Lie algebra of all the derivations of $\mathfrak{m}$ preserving the gradation of $\mathfrak{m}$, $\mathfrak{e}$ and $\mathfrak{f}$: $\mathfrak{g}_0 = \{\, D \in \mathrm{Der}(\mathfrak{m})_0 : D(\mathfrak{e}) \subset \mathfrak{e}, D(\mathfrak{f}) \subset \mathfrak{f} \,\}$. The prolongation $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of $(\mathfrak{m},\mathfrak{g}_0)$ is called the prolongation of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$.

We say that two pseudo-product FGLAs $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ and $(\mathfrak{m}';\mathfrak{e}',\mathfrak{f}')$ are isomorphic if there exists a GLA isomorphism φ of $\mathfrak{m}$ onto $\mathfrak{m}'$ such that $\varphi(\mathfrak{e}) = \mathfrak{e}'$ and $\varphi(\mathfrak{f}) = \mathfrak{f}'$.

Let $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ be a pseudo-product FGLA of the μ-th kind, where $\mu \geqq 2$. The triplet $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ is called a free pseudo-product FGLA of type (n,m,μ) if the following conditions hold: (i) $\dim \mathfrak{e} = n$ and $\dim \mathfrak{f} = m$; (ii) Let $(\mathfrak{m}';\mathfrak{e}',\mathfrak{f}')$ be a pseudo-product FGLA of the μ-th kind and φ a surjective linear mapping of $\mathfrak{g}_{-1}$ onto $\mathfrak{g}'_{-1}$ such that $\varphi(\mathfrak{e}) \subset \mathfrak{e}'$ and $\varphi(\mathfrak{f}) \subset \mathfrak{f}'$. Then φ can be extended uniquely to a GLA epimorphism of $\mathfrak{m}$ onto $\mathfrak{m}'$.

Let m, n and μ be positive integers such that $\mu \geqq 2$. There exists a unique free pseudo-product FGLA of type (n,m,μ) up to isomorphism ([24, Proposition 8.2]). Let $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ be a free pseudo-product FGLA of type (n,m,μ), and $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ the prolongation of $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$. Then the mapping $\Phi : \mathfrak{g}_0 \ni D \mapsto (D|\mathfrak{e}, D|\mathfrak{f}) \in \mathfrak{gl}(\mathfrak{e}) \times \mathfrak{gl}(\mathfrak{f})$ is a Lie algebra isomorphism. In particular, the underlying FGLA $\mathfrak{m}$ of the free pseudo-product FGLA $(\mathfrak{m};\mathfrak{e},\mathfrak{f})$ is non-degenerate. By [15, Lemma 1.14], we see that $\mathfrak{g}$ is finite dimensional (see also [24, Lemma 8.1]). In [6] a general result was proved with the multicontact structure in place of the pseudo-product structure.

A transitive GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is called a pseudo-product GLA if there are given nonzero subspaces $\mathfrak{e}$ and $\mathfrak{f}$ of $\mathfrak{g}_{-1}$ satisfying the following conditions: (i) The negative part $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$ is a pseudo-product FGLA with a pseudo-product structure $(\mathfrak{e},\mathfrak{f})$; (ii) $[\mathfrak{g}_0,\mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0,\mathfrak{f}] \subset \mathfrak{f}$. The pair $(\mathfrak{e},\mathfrak{f})$ is called the pseudo-product structure of the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$. If the $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible, then the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ is said to be of irreducible type. Clearly the prolongation of a pseudo-product FGLA becomes a pseudo-product GLA. Also the prolongation of a free pseudo-product FGLA is of irreducible type.

Example 5.1. Let V and W be finite dimensional vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ such that $\dim V = n \geqq 1$ and $\dim W = m \geqq 1$, and let μ be a positive integer with $\mu \geqq 2$. We set $\mathfrak{g}_{-1} = V \oplus W$, $\mathfrak{g}_{-2} = V \otimes W$, $\mathfrak{g}_{-p} = \bigoplus_{k=1}^{p-1} S^k(V) \otimes S^{p-k}(W)$ $(3 \leqq p \leqq \mu)$, $\mathfrak{p}(V,W,\mu) = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$. The bracket operation of $\mathfrak{p}(V,W,\mu)$ is defined as follows: $[\mathfrak{g}_p, \mathfrak{g}_q] = \{0\}$ $(p,q \leqq -2$ or $-\mu \leqq p,q \leqq -1, p+q < -\mu)$, $[V,V] = [W,W] = \{0\}$, $[v,w] = -[w,v] = v \otimes w$, $[v, \alpha \otimes \beta] = -[\alpha \otimes \beta, v] = v \circledcirc \alpha \otimes \beta$, $[\alpha \otimes \beta, w] = -[w, \alpha \otimes \beta] = \alpha \otimes \beta \circledcirc w$, where $v \in V$, $w \in W$, $\alpha \in S^k(V)$ and $\beta \in S^{p-k}(W)$ $(3 \leqq p \leqq \mu-1, 1 \leqq k \leqq p-1)$. Equipped with this bracket operation, $\mathfrak{p}(V,W,\mu)$ becomes a pseudo-product FGLA of the μ-th kind with pseudo-product structure (V,W). Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{p}(V,W,\mu); V, W)$. Then $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(V) \oplus \mathfrak{gl}(W)$. Let $\widehat{\mathfrak{m}} = \bigoplus_{p<0} \widehat{\mathfrak{g}}_p$ be a free pseudo-product FGLA of type (n,m,μ) with pseudo-product structure $(\mathfrak{e}, \mathfrak{f})$. There exists a GLA epimorphism φ of $\widehat{\mathfrak{m}}$ onto $\mathfrak{m}$ such that $\varphi(\mathfrak{e}) = V$ and $\varphi(\mathfrak{f}) = W$. By [24, Proposition 8.3], if $\mu \leqq 3$, then $(\mathfrak{p}(V,W,\mu); V, W)$ is free.

Example 5.2. We give a matrix representation of a free pseudo-product FGLA of type $(n,m,2)$. We set $\mathfrak{g} = \mathfrak{sl}(n+m+1, \mathbb{K})$ and define subspaces $\mathfrak{g}_p$ $(p = -2, \ldots, 2)$ of $\mathfrak{g}$ as follows:

$$\mathfrak{g}_{-2} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A & 0 & 0 \end{bmatrix} : A \in M(m,n) \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & \eta & 0 \end{bmatrix} : \xi \in M(1,n), \eta \in M(m,1) \right\},$$

$$\mathfrak{g}_{0} = \left\{ \begin{bmatrix} A & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & C \end{bmatrix} : A \in \mathfrak{gl}(n,\mathbb{K}), \beta \in \mathbb{K}, C \in \mathfrak{gl}(m,\mathbb{K}), \operatorname{tr} A + \beta + \operatorname{tr} C = 0 \right\},$$

$$\mathfrak{g}_k = \{ {}^tX : X \in \mathfrak{g}_{-k} \} \ (k \geqq 1),$$

where $M(p,q)$ denotes the set of $p \times q$ $\mathbb{K}$-valued matrices. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is a finite dimensional SGLA of type $(A_{n+m}, \{\alpha_n, \alpha_{n+1}\})$ (resp. $((\mathrm{AI})_{n+m}, \{\alpha_n, \alpha_{n+1}\})$) in case $\mathbb{K} = \mathbb{C}$ (resp. $\mathbb{K} = \mathbb{R}$). We set

$$\mathfrak{e} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \xi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \xi \in M(1,n) \right\}, \quad \mathfrak{f} = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \eta & 0 \end{bmatrix} : \eta \in M(m,1) \right\}.$$

Then $(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p; \mathfrak{e}, \mathfrak{f})$ is a free pseudo-product FGLA of type $(n,m,2)$. If $n = 1$, then $\mathfrak{m}$ is isomorphic to the negative part of $W(1,m;\mathbb{K})$.

As for the prolongation of a free pseudo-product FGLA, we already know the following results ([24, Theorem 8.1 (2)]).

Theorem 5.3. *Let* $(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p; \mathfrak{e}, \mathfrak{f})$ *be a free pseudo-product FGLA of type* (n, m, μ) *over* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$*. Let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be the prolongation of* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$*. If* $\mathfrak{g}_1 \neq \{0\}$*, then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is a finite dimensional SGLA of type* $(A_{n+m}, \{\alpha_n, \alpha_{n+1}\})$ *(resp.* $((\mathrm{AI})_{n+m}, \{\alpha_n, \alpha_{n+1}\})$*) in case* $\mathbb{K} = \mathbb{C}$ *(resp.* $\mathbb{K} = \mathbb{R}$*).*

Remark 5.4. Although we assume that the ground field is $\mathbb{C}$ in the paper [24], as is easily observed, all the results hold good even in the case that the ground field is $\mathbb{R}$ by replacing the appeared complex simple Lie algebras with their normal real forms.

Lemma 5.5. *Let* $(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p; \mathfrak{e}, \mathfrak{f})$ *be a free pseudo-product FGLA of type* (n, m, μ) *over* $\mathbb{R}$*. Let* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *be the prolongation of* $\mathfrak{m}$*. If the* $\mathfrak{g}(\mathfrak{m})_0$*-module* $\mathfrak{g}(\mathfrak{m})_{-1}$ *is irreducible, then the* $\mathbb{C}\mathfrak{g}(\mathfrak{m})_0$*-module* $\mathbb{C}\mathfrak{g}(\mathfrak{m})_{-1}$ *is irreducible,*

Proof. Assume that the $\mathbb{C}\mathfrak{g}(\mathfrak{m})_0$-module $\mathbb{C}\mathfrak{g}(\mathfrak{m})_{-1}$ is reducible. Let U be a nontrivial $\mathbb{C}\mathfrak{g}(\mathfrak{m})_0$-submodule of $\mathbb{C}\mathfrak{g}(\mathfrak{m})_{-1}$. Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$. Since U is a $\mathbb{C}\mathfrak{g}_0$-submodule, $U = \mathbb{C}\mathfrak{e}$ or $U = \mathbb{C}\mathfrak{f}$. By [8, p. 64], we have $\mathbb{C}\mathfrak{g}(\mathfrak{m})_{-1} = U \oplus \overline{U}$. Since $\overline{\mathbb{C}\mathfrak{e}} = \mathbb{C}\mathfrak{e}$ and $\overline{\mathbb{C}\mathfrak{f}} = \mathbb{C}\mathfrak{f}$, we get a contradiction. Thus the $\mathbb{C}\mathfrak{g}(\mathfrak{m})_0$-module $\mathbb{C}\mathfrak{g}(\mathfrak{m})_{-1}$ is irreducible. ■

The main purpose of the present section is to prove the following.

Theorem 5.6. *Let* $(\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p; \mathfrak{e}, \mathfrak{f})$ *be a free pseudo-product FGLA of type* (n, m, μ) *over* $\mathbb{K} = \mathbb{C}$ *or* $\mathbb{R}$*. Assume that* $n \leqq m$*. Let* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *be the prolongation of* $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ *and* $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ *the prolongation of* $\mathfrak{m}$*. If* $\dim \mathfrak{g}(\mathfrak{m}) = \infty$*, then* $\mu = 2$*. More precisely, we obtain the following:*

(1) *If* $n = m = 1$ *and* $\mu = 2$*, then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic to a finite dimensional SGLA of type* $(A_2, \{\alpha_1, \alpha_2\})$ *(resp.* $((\mathrm{AI})_2, \{\alpha_1, \alpha_2\})$*), and* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to the contact algebra* $K(1; \mathbb{K})$*.*

(2) *If* $n = 1$*,* $m \geqq 2$ *and* $\mu = 2$*, then* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic to a finite dimensional SGLA of type* $(A_{m+1}, \{\alpha_1, \alpha_2\})$*, (resp.* $((\mathrm{AI})_{m+1}, \{\alpha_1, \alpha_2\})$*) and* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to* $W(1, m; \mathbb{K})$*.*

(3) *If* $m \geqq 2$*,* $n \geqq 2$ *and* $\mu = 2$*, then* $\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}$ *and* $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ *is isomorphic to a finite dimensional SGLA of type* $(A_{m+n}, \{\alpha_m, \alpha_{m+1}\})$ *(resp.* $((\mathrm{AI})_{m+n}, \{\alpha_m, \alpha_{m+1}\})$*) in case* $\mathbb{K} = \mathbb{C}$ *(resp.* $\mathbb{K} = \mathbb{R}$*).*

(4) *If* $n \geqq 2$*,* $m \geqq 2$ *and* $\mu \geqq 3$*, then* $\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}_0$ *and* $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$*.*

(5) *If* $m = n = 1$ *and* $\mu = 3$, *then* $\mathfrak{g}(\mathfrak{m})$ *is a finite dimensional SGLA of type* $(G_2, \{\alpha_1\})$ *(resp.* $(\mathrm{G}, \{\alpha_1\})$*) in case* $\mathbb{K} = \mathbb{C}$ *(resp.* $\mathbb{K} = \mathbb{R}$*).*

(6) *If* $n = 1$, $m \geqq 2$ *and* $\mu = 3$, *then* $\mathfrak{g}(\mathfrak{m})_0/\mathfrak{g}_0$ *(resp.* $\mathfrak{g}(\mathfrak{m})_1$*) is isomorphic to* $\mathrm{Hom}(\mathfrak{e}, \mathfrak{f})$ *(resp.* $\mathfrak{e}^*$*) as a* $\mathfrak{g}_0$*-module, and* $\mathfrak{g}(\mathfrak{m})_2 = \{0\}$.

(7) *If* $n = 1$, $m \geqq 2$ *and* $\mu \geqq 4$, *then* $\mathfrak{g}(\mathfrak{m})_0/\mathfrak{g}_0$ *is isomorphic to* $\mathrm{Hom}(\mathfrak{e}, \mathfrak{f})$ *as a* $\mathfrak{g}_0$*-module and* $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$.

(8) *If* $n = m = 1$ *and* $\mu \geqq 4$, *then* $\mathfrak{g}(\mathfrak{m})_0$ *is isomorphic to* $\mathfrak{gl}(\mathfrak{e} \oplus \mathfrak{f})$ *and* $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$.

Proof. First note that in the real case the complexification $\mathbb{C}\mathfrak{p}(V, W; \mu)$ of $\mathfrak{p}(V, W; \mu)$ is equal to $\mathfrak{p}(\mathbb{C}V, \mathbb{C}W; \mu)$. Hence on account of Lemma 5.5, we can prove the real version of the theorem as in the complex case. So we assume that $\mathbb{K} = \mathbb{C}$. The results for the case $\dim \mathfrak{g}(\mathfrak{m}) = \infty$ are due to [24, Theorem 8.1 (1)]. Also the results for the case $\mu = 2$ are due to the results of §4, Example 5.2 and [19, Theorem 5.2].

We assume that $\mu \geqq 3$ and $\dim \mathfrak{g}(\mathfrak{m}) < \infty$. Considering $\mathfrak{g}(\mathfrak{m})_0$ as a subspace of $\mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1})$, we get

$$\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}(\mathfrak{m})_0 \cap (\mathfrak{gl}(\mathfrak{e}) \oplus \mathfrak{gl}(\mathfrak{f})) \oplus \mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}) \oplus \mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{f}, \mathfrak{e}).$$

Let $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_m)$) be a basis of $\mathfrak{e}$ (resp. $\mathfrak{f}$), and let $(e_1^*, \dots, e_n^*)$ (resp. $(f_1^*, \dots, f_m^*)$) be the dual basis of $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_m)$). Let D be an element of $\mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{e}, \mathfrak{f})$. The element D can be denoted as follows:

$$D = \sum_{i=1}^{n} \sum_{a=1}^{m} \alpha_{ia} f_a \otimes e_i^* \qquad (\alpha_{ia} \in \mathbb{C}).$$

Then $0 = [D, [e_i, e_j]] = [[D, e_i], e_j] + [e_i, [D, e_j]] = \sum_{a=1}^{m} \alpha_{ia}[f_a, e_j] + \alpha_{ja}[e_i, f_a]$. If $\dim \mathfrak{e} \geqq 2$, then $D = 0$ and hence $\mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}) = \{0\}$. Similarly if $\dim \mathfrak{f} \geqq 2$, then $\mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{f}, \mathfrak{e}) = \{0\}$. Hence if $\dim \mathfrak{e} \geqq 2$ and $\dim \mathfrak{f} \geqq 2$, then $\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}_0$, which proves (4).

We assume that $\mu \geqq 3$, $\dim \mathfrak{g}(\mathfrak{m}) < \infty$, $\dim \mathfrak{e} = 1$ and $\dim \mathfrak{f} \geqq 2$. In this case, $\mathfrak{g}_1 = \{0\}$. Let $\tilde{\mathfrak{m}} = \bigoplus_{p<0} \tilde{\mathfrak{g}}_p$ be a universal FGLA $b(\mathfrak{g}_{-1}, \mu)$ of the μ-th kind (cf. [14, pp. 14–17]). For nonzero elements $\lambda \in \mathfrak{e}^*$ and $f \in \mathfrak{f}$, there exists an element $\tilde{D}_{\lambda,f}$ of $\mathrm{Der}(\tilde{\mathfrak{m}})_0$ such that $\tilde{D}_{\lambda,f}(X) = \lambda(X)f$ and $\tilde{D}_{\lambda,f}(Y) = 0$ $(X \in \mathfrak{e}, Y \in \mathfrak{f})$. Since $\tilde{D}_{\lambda,f}([\mathfrak{e}, \mathfrak{e}] + [\mathfrak{f}, \mathfrak{f}]) = 0$, $\tilde{D}_{\lambda,f}$ induces an element $D_{\lambda,f}$ of $\mathfrak{g}(\mathfrak{m})_0 \cap \mathrm{Hom}(\mathfrak{e}, \mathfrak{f})$ such that $D_{\lambda,f}(X) = \lambda(X)f$, $D_{\lambda,f}(Y) = 0$ $(X \in \mathfrak{e}, Y \in \mathfrak{f})$. Thus

$$\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}_0 \oplus \sum_{\lambda \in \mathfrak{e}^*, f \in \mathfrak{f}} \mathbb{C} D_{\lambda,f}$$

and $\mathfrak{g}(\mathfrak{m})_0/\mathfrak{g}_0$ is isomorphic to $\mathrm{Hom}(\mathfrak{e}, \mathfrak{f})$. Considering $\mathfrak{g}(\mathfrak{m})_1$ as a subspace of

$\mathrm{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}(\mathfrak{m})_0)$, we get

$$\begin{aligned}
\mathfrak{g}(\mathfrak{m})_1 &= \mathfrak{g}(\mathfrak{m})_1^{(1)} \oplus \mathfrak{g}(\mathfrak{m})_1^{(2)} \oplus \mathfrak{g}(\mathfrak{m})_1^{(3)} \oplus \mathfrak{g}(\mathfrak{m})_1^{(4)},\\
\mathfrak{g}(\mathfrak{m})_1^{(1)} &= \mathfrak{g}(\mathfrak{m})_1 \cap (\mathrm{Hom}(\mathfrak{e}, \mathfrak{sl}(\mathfrak{f})) \oplus \mathrm{Hom}(\mathfrak{f}, \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}))^{(1)}),\\
\mathfrak{g}(\mathfrak{m})_1^{(2)} &= \mathfrak{g}(\mathfrak{m})_1 \cap (\mathrm{Hom}(\mathfrak{e}, \mathfrak{gl}(\mathfrak{e})) \oplus \mathrm{Hom}(\mathfrak{e}, \mathbb{C}1_{\mathfrak{f}}) \oplus \mathrm{Hom}(\mathfrak{f}, \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}))^{(2)}),\\
\mathfrak{g}(\mathfrak{m})_1^{(3)} &= \mathfrak{g}(\mathfrak{m})_1 \cap (\mathrm{Hom}(\mathfrak{e}, \mathrm{Hom}(\mathfrak{e}, \mathfrak{f})),\\
\mathfrak{g}(\mathfrak{m})_1^{(4)} &= \mathfrak{g}(\mathfrak{m})_1 \cap (\mathrm{Hom}(\mathfrak{f}, \mathfrak{gl}(\mathfrak{e})) \oplus \mathrm{Hom}(\mathfrak{f}, \mathfrak{gl}(\mathfrak{f})),
\end{aligned}$$

where $1_{\mathfrak{f}}$ is the identity transformation on $\mathfrak{f}$,

$$\begin{aligned}
\mathrm{Hom}(\mathfrak{f}, \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}))^{(1)} &= \{ \sum_{a,b=1}^{m} \alpha_{ab} f_a \otimes e^* \otimes f_b^* : \alpha_{ab} \in \mathbb{C}, \sum_{a=1}^{m} \alpha_{aa} = 0 \},\\
\mathrm{Hom}(\mathfrak{f}, \mathrm{Hom}(\mathfrak{e}, \mathfrak{f}))^{(2)} &= \{ \sum_{a=1}^{m} \alpha_{aa} f_a \otimes e^* \otimes f_a^* : \alpha_{aa} \in \mathbb{C} \}.
\end{aligned}$$

Since $\mathfrak{g}(\mathfrak{m})_1^{(4)} \subset \mathfrak{g}_1$, we see that $\mathfrak{g}(\mathfrak{m})_1^{(4)} = 0$. Let D_1 be an element of $\mathfrak{g}(\mathfrak{m})_1^{(1)}$; then D_1 has the following form:

$$D_1 = \sum_{i,j=1}^{m} \alpha_{ij} f_i \otimes f_j^* \otimes e_1^* + \sum_{i,j=1}^{m} \beta_{ij} f_i \otimes e_1^* \otimes f_j^*.$$

Since

$$\begin{aligned}
0 &= [D_1, (\mathrm{ad}\, e_1)^{\mu} f_a] = \sum_{i=1}^{m} (\mu\alpha_{ia} - \beta_{ia})(\mathrm{ad}\, e_1)^{\mu-1} f_i,\\
0 &= [D_1, (\mathrm{ad}\, e_1)^{\mu-2}(\mathrm{ad}\, f_a)^2 e_1] = 2(\mu-2) \sum_{i=1}^{m} \mu\alpha_{ia} (\mathrm{ad}\, e_1)^{\mu-3} (\mathrm{ad}\, f_i) e_1,
\end{aligned}$$

we see that $\mu\alpha_{ia} - \beta_{ia} = 0$ and $\alpha_{ia} = 0$ for all a, i. Thus $D_1 = 0$ and hence $\mathfrak{g}(\mathfrak{m})_1^{(1)} = 0$. Let D_2 be an element of $\mathfrak{g}(\mathfrak{m})_1^{(2)}$; then D_2 has the following form:

$$D_2 = \alpha e_1 \otimes e_1^* \otimes e_1^* + \beta \sum_{i=1}^{n} f_i \otimes f_i^* \otimes e_1^* + \gamma \sum_{i=1}^{m} f_i \otimes e_1^* \otimes f_i^* \quad (\alpha, \beta, \gamma \in \mathbb{C}).$$

Since

$$\begin{aligned}
0 &= [D_2, (\mathrm{ad}\, e_1)^{\mu} f_a] = \left(\frac{\mu(\mu-1)}{2} + \mu\beta - \gamma \right) (\mathrm{ad}\, e_1)^{\mu-1} f_a,\\
0 &= [D_2, (\mathrm{ad}\, e_1)^{\mu-2} (\mathrm{ad}\, f_a)(\mathrm{ad}\, e_1) f_a] = \left(\frac{(\mu-2)(\mu-1)}{2} \alpha + 2(\mu-2)\beta \right) (\mathrm{ad}\, e_1)^{\mu-1} f_a,\\
0 &= [D_2, (\mathrm{ad}\, f_a)^{\mu-3} (\mathrm{ad}\, e_1)^3 f_a] = (3\alpha + 3\beta + (2\mu-7))(\mathrm{ad}\, f_a)^{\mu-3} (\mathrm{ad}\, e_1)^2 f_a,
\end{aligned}$$

we get $0 = \frac{\mu(\mu-1)}{2} + \mu\beta - \gamma = \frac{(\mu-2)(\mu-1)}{2}\alpha + 2(\mu-2)\beta = 3\alpha + 3\beta + (2\mu-7)$. Then we can see that $D_2 = 0$ if $\mu \geqq 4$. Therefore if $\mu \geqq 4$, then $\mathfrak{g}(\mathfrak{m})_1^{(2)} = \{0\}$. On the other hand, if $\mu = 3$, then

$$D_2 = \beta(-2e_1 \otimes e_1^* \otimes e_1^* + \sum_{i=1}^{m} f_i \otimes f_i^* \otimes e_1^* - 3 \sum_{i=1}^{m} f_i \otimes e_1^* \otimes f_i^*).$$

Let D_3 be an element of $\mathfrak{g}(\mathfrak{m})_1^{(3)}$; then D_3 has the following form:

$$D_3 = \sum_{k=1}^{n} \alpha_k f_k \otimes e_1^* \otimes e_1^*.$$

Since $0 = [D_3, (\mathrm{ad}\, f_1)^{\mu-3}(\mathrm{ad}\, e_1)^3 f_1] = \sum_k \alpha_k (\mathrm{ad}\, f_k)(\mathrm{ad}\, f_1)^{\mu-3}(\mathrm{ad}\, e_1) f_1$, we see that $D_3 = 0$. Hence $\mathfrak{g}(\mathfrak{m})_1^{(3)} = 0$. Therefore if $\mu \geqq 4$, then $\mathfrak{g}(\mathfrak{m})_1 = \{0\}$. We consider the case $\mu = 3$. For any $D \in \mathfrak{g}(\mathfrak{m})_1$, there exists an element λ of $\mathfrak{e}^*$ such that $[D, X] = \lambda(X)(2J_{\mathfrak{e}} - J_{\mathfrak{f}})$, $[D, Y] = -3D_{\lambda,Y}$ $(X \in \mathfrak{e}, Y \in \mathfrak{f})$, where $J_{\mathfrak{e}}$ (resp. $J_{\mathfrak{f}}$) is an element of $\mathfrak{g}_0$ such that $[J_{\mathfrak{e}}, X] = -X$ and $[J_{\mathfrak{e}}, Y] = 0$ (resp. $[J_{\mathfrak{f}}, X] = 0$ and $[J_{\mathfrak{f}}, Y] = -Y$) for $X \in \mathfrak{e}$ and $Y \in \mathfrak{f}$. Conversely, for an element λ of $\mathfrak{e}^*$ there exists an element G_λ of $\mathfrak{g}(\mathfrak{m})_1$ such that $[G_\lambda, X] = \lambda(X)(2J_{\mathfrak{e}} - J_{\mathfrak{f}})$, $[G_\lambda, Y] = -3D_{\lambda,Y}$ $(X \in \mathfrak{e}, Y \in \mathfrak{f})$. Thus

$$\mathfrak{g}(\mathfrak{m})_1 = \sum_{\lambda \in \mathfrak{e}^*} \mathbb{C} G_\lambda$$

and $\mathfrak{g}(\mathfrak{m})_1$ is isomorphic to $\mathfrak{e}^*$. Finally we investigate $\mathfrak{g}(\mathfrak{m})_2$. Let D be an element of $\mathfrak{g}(\mathfrak{m})_2$, and let λ be a nonzero element of $\mathfrak{e}^*$. There exist $\alpha, \beta_a \in \mathbb{C}$ such that $[D, e_1] = \alpha G_\lambda$, $[D, f_a] = \beta_a G_\lambda$. Then $[D, (\mathrm{ad}\, e_1)^3 f_a] = 10\alpha\lambda(e_1)[e_1, f_a]$ and $[D, [f_a, [e_1, [e_1, f_a]]]] = 5\beta_a \lambda(e_1)[e_1, f_a]$. Thus $D = 0$ and hence we see that $\mathfrak{g}(\mathfrak{m})_2 = \{0\}$. This proves (6) and (7). If $\dim \mathfrak{e} = \dim \mathfrak{f} = 1$ and $\mu = 3$, then $\mathfrak{g}(\mathfrak{m})_1 \neq \{0\}$ and the $\mathfrak{g}(\mathfrak{m})_0$-module $\mathfrak{g}_{-1}$ is irreducible, so $\mathfrak{g}(\mathfrak{m})$ is a finite dimensional SGLA of type $(G_2, \{\alpha_1\})$. This proves (5). Similarly we can prove (8). ∎

Corollary 5.7. *Under the assumption of Theorem 5.6, if* $\mathfrak{g}(\mathfrak{m})_2 \neq \{0\}$ *and* $\mu \geqq 3$, *then* $\mathfrak{g}(\mathfrak{m})$ *is isomorphic to a finite dimensional SGLA of type* $(G_2, \{\alpha_1\})$ *(resp.* $(G, \{\alpha_1\})$*) in case* $K = \mathbb{C}$ *(resp.* $\mathbb{K} = \mathbb{R}$*).*

6. Hilbert-Cartan equation in a wider sense

Using the methods in [19, §1.3, pp. 420–423] we calculate the symbol algebra of a differential system (R, D) which is associated with the following underdetermined ordinary differential equation, which is called the Hilbert-Cartan equation in a wider sense:

$$\frac{dv_{ab}}{dx} = \frac{d^2 u_a}{dx^2} \frac{d^2 u_b}{dx^2} \quad (1 \leqq a \leqq b \leqq m, m \geqq 2). \tag{HC}$$

Note that under the regularity condition $u_1'' \cdots u_m'' \neq 0$ the solution $(u_a(x), v_{ab}(x))$ of (HC) needs to satisfy the following compatibility condition:

$$2u_a'' u_b'' v_{ab}'' = (u_b'')^2 v_{aa}'' + (u_a'')^2 v_{bb}'' \quad (1 \leqq a \leqq b \leqq m). \tag{CC}$$

We consider a submanifold R defined by (HC) and (CC) in the space J^2 of 2-jets for $(\frac{m(m+1)}{2} + m)$-unknown and 1-independent variables with coordinate system $(x, u_a, v_{ab}, u_a', v_{ab}', u_a'', v_{ab}'')$:

$$R = \left\{ \begin{array}{l} v_{ab}' = u_a'' u_b'',\ 2u_a'' u_b'' v_{ab}'' = (u_b'')^2 v_{aa}'' + (u_a'')^2 v_{bb}''\ (1 \leqq a \leqq b \leqq m) \\ u_1'' \cdots u_m'' \neq 0 \end{array} \right\}.$$

Our differential system (R, D) is obtained by restricting to R the canonical contact system on J^2:

$$D = \{\, \omega'_{ab} = \theta'_a = \pi'_a = \eta'_a = 0\},$$

where

$$\begin{cases} \omega'_{ab} = dv_{ab} - (u''_a u''_b)dx, \\ \theta'_a = du_a - u'_a dx, \\ \pi'_a = du'_a - u''_a dx, \\ \eta'_a = d((u''_a)^2) - v''_{aa}dx = 2u''_a du''_a - v''_{aa}dx. \end{cases}$$

We take $(x, u_a, v_{ab}, p_a, r_a, t_a)$ as a coordinate system on R, where $p_a = u'_a$, $r_a = u''_a$ and $t_a = \frac{1}{2}(u''_a)^{-1}v''_{aa}$. Then (R, D) is given on this coordinate system by

$$D = \{\, \omega_{ab} = \theta_a = \pi_a = \eta_a = 0\,\},$$

where $\omega_{ab} = dv_{ab} - r_a r_b dx$, $\theta_a = du_a - p_a dx$, $\pi_a = dp_a - r_a dx$ and $\eta_a = dr_a - t_a dx$. First we calculate

$$\begin{cases} d\omega_{ab} = r_a dx \wedge \eta_b + r_b dx \wedge \eta_a \\ d\theta_a = dx \wedge \pi_a \\ d\pi_a = dx \wedge \eta_a \\ d\eta_a = dx \wedge dt_a. \end{cases} \tag{6.1}$$

Next, we calculate the weak derived system $\{\partial^{(k)}D\}$. To locate the derived system ∂D, we look at the equation (6.1) modulo the ideal spanned by ω_{ab}, θ_a, π_a, η_a:

$$d\omega_{ab} \equiv d\theta_a \equiv d\pi_a \equiv 0\ , d\eta_a \equiv dx \wedge dt_a \qquad (\text{mod } \omega_{ab}, \theta_a, \pi_a, \eta_a).$$

Hence

$$\partial D = \{\omega_{ab} = \theta_a = \pi_a = 0\,\}.$$

To locate $\partial^{(2)}D$ we proceed to look at $d\omega_{ab}$, $d\theta_a$ and $d\pi_a$ modulo 1-forms ω_{ab}, θ_a, π_a and 2-forms $\eta_a \wedge \eta_b$. Putting $\tilde{\omega}_{ab} = \omega_{ab} - r_b\pi_a - r_a\pi_b$, we have

$$\partial D = \{\, \tilde{\omega}_{ab} = \theta_a = \pi_a = 0\,\}$$

and

$$\begin{cases} d\tilde{\omega}_{ab} = \pi_a \wedge dr_b + \pi_b \wedge dr_a \\ \qquad\quad = \pi_a \wedge \eta_b + t_b\pi_a \wedge dx + \pi_b \wedge \eta_a + t_a\pi_b \wedge dx \equiv 0, \\ d\theta_a = dx \wedge \pi_a \equiv 0, \\ d\pi_a = dx \wedge dr_a = dx \wedge \eta_a, \qquad (\text{mod } \omega_{ab}, \theta_a, \pi_a, \eta_a \wedge \eta_b). \end{cases}$$

Hence we get

$$\partial^{(2)}D = \{\, \tilde{\omega}_{ab} = \theta_a = 0\,\}.$$

To proceed, we put $\bar{\omega}_{ab} = \tilde{\omega}_{ab} + t_a\theta_b + t_b\theta_a$. Then we have

$$\partial^{(2)}D = \{\bar{\omega}_{ab} = \theta_a = 0\,\},$$

and

$$\begin{cases} d\bar{\omega}_{ab} = \pi_a \wedge \eta_b + \zeta_a \wedge \theta_b + \pi_b \wedge \eta_a + \zeta_b \wedge \theta_a, \\ d\theta_a = \xi \wedge \pi_a, \\ d\pi_a = \xi \wedge \eta_a, \\ d\eta_a = \xi \wedge \zeta_a, \end{cases} \tag{6.2}$$

where $\xi = dx$ and $\zeta_a = dt_a$. Hence we get

$$d\bar{\omega}_{ab} \equiv 0, \quad d\theta_a = \xi \wedge \pi_a, \quad (\text{mod } \bar{\omega}_{ab}, \theta_a, \pi_a \wedge \pi_b, \pi_a \wedge \eta_b, \eta_a \wedge \eta_b).$$

This implies that

$$\partial^{(3)} D = \{ \bar{\omega}_{ab} = 0 \}.$$

Furthermore we have

$$d\bar{\omega}_{ab} = \zeta_a \wedge \theta_b + \zeta_b \wedge \theta_a \quad (\text{mod } \bar{\omega}_{ab}, \theta_a \wedge \theta_b, \theta_a \wedge \pi_b, \theta_a \wedge \eta_b, \pi_a \wedge \pi_b, \pi_a \wedge \eta_b, \eta_a \wedge \eta_b).$$

Hence we get

$$\partial^{(4)} D = T(R).$$

Thus we see that (R, D) is a regular differential system of type $\mathfrak{m}_5$, where $\mathfrak{m}_5$ is an FGLA of the 5-th kind, whose Maurer-Cartan equation is given by (6.2). Furthermore (R, D) is locally isomorphic with the standard differential system of type $\mathfrak{m}_5$. Now we give a model space of the FGLA $\mathfrak{m}_5$. Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be the underlying FGLA of a free pseudo-product FGLA $(\mathfrak{p}(V, W, 3); V, W)$ of type $(1, m, 3)$ defined in Example 5.1, $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ the prolongation of $(\mathfrak{p}(V, W, 3); V, W)$, and $\mathfrak{g}(\mathfrak{m})$ the prolongation of $\mathfrak{m}$. Then by Theorem 5.6,

$$\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}_0 \oplus \mathfrak{g}(\mathfrak{m})_{0,-1}, \ \mathfrak{g}_0 \cong \mathfrak{gl}(V) \oplus \mathfrak{gl}(W),$$
$$\mathfrak{g}(\mathfrak{m})_{0,-1} \cong \mathrm{Hom}(V, W), \ \mathfrak{g}(\mathfrak{m})_1 \cong V^*, \ \mathfrak{g}(\mathfrak{m})_p = \{0\} \ (p \geqq 2).$$

Let J be an element of $\mathfrak{g}(\mathfrak{m})_0$ such that $[J, v] = -v, [J, w] = -2w$ for $v \in V$ and $w \in W$. We set $\mathfrak{a}_k = \{ X \in \mathfrak{g}(\mathfrak{m}) : [J, X] = kX \}$. Then $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{k=-5}^{1} \mathfrak{a}_k$, $\mathfrak{a}_{-5} = V \otimes S^2(W)$, $\mathfrak{a}_{-4} = S^2(V) \otimes W$, $\mathfrak{a}_{-3} = V \otimes W$, $\mathfrak{a}_{-2} = W$, $\mathfrak{a}_{-1} = V \oplus \mathfrak{g}(\mathfrak{m})_{0,-1}$, $\mathfrak{a}_0 = \mathfrak{g}_0$, $\mathfrak{a}_1 = \mathfrak{g}(\mathfrak{m})_1$, $\mathfrak{a}_k = \{0\}$ $(k \geqq 2)$. Putting $\mathfrak{e}' = V$ and $\mathfrak{f}' = \mathfrak{g}(\mathfrak{m})_{0,-1} = V^* \otimes W$, we see that $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{k=-5}^{1} \mathfrak{a}_k$ is a pseudo-product GLA of irreducible type with pseudo-product structure $(\mathfrak{e}', \mathfrak{f}')$ and $\mathfrak{m}_5$ is isomorphic to $\mathfrak{a}_- = \bigoplus_{k<0} \mathfrak{a}_k$. Note that $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{k=-5}^{1} \mathfrak{a}_k$ is the prolongation of $\mathfrak{a}_- = \bigoplus_{k<0} \mathfrak{a}_k$. Indeed, for $k < 0$ we put $\mathfrak{k}_k = \{X \in \mathfrak{a}_{-1} : [X, \mathfrak{a}_k] = \{0\} \}$. Then $\mathfrak{k}_k$ is an $\mathfrak{a}_0$-submodule of $\mathfrak{a}_{-1}$ and hence $\mathfrak{k}_{-2} = \mathfrak{g}(\mathfrak{m})_{0,-1}$ and $\mathfrak{k}_{-4} = V$. Let $\check{\mathfrak{a}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{a}}_p$ be the prolongation of $\mathfrak{a}_-$. Then $[\check{\mathfrak{a}}_0, \mathfrak{k}_k] \subset \mathfrak{k}_k$ and hence $\check{\mathfrak{a}}_0 = \mathfrak{g}_0$. Now suppose that $\check{\mathfrak{a}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{a}}_p$ is a finite dimensional SGLA. Since $\mathfrak{a}_{-1} \cong V \oplus (V^* \otimes W)$ and $\check{\mathfrak{a}}_0 \cong \mathfrak{gl}(V) \oplus \mathfrak{gl}(W)$, an SGLA $\check{\mathfrak{a}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{a}}_p$ is of type $((\mathrm{AI})_l, \{\alpha_1, \alpha_2\})$. Since $\mathfrak{a}_-$ is of the 5-th kind, this is a contradiction. By [23, Theorem 3.3], $\check{\mathfrak{a}}_1 = \mathfrak{a}_1$ and $\check{\mathfrak{a}}_k = 0$ $(k \geqq 2)$, and hence $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{k=-5}^{1} \mathfrak{a}_k$ is the prolongation of $\mathfrak{a}_-$. From this fact it follows that the Lie algebra $\mathcal{A}(R, D)$ of infinitesimal automorphisms of (R, D) is isomorphic to $\mathfrak{g}(\mathfrak{m})$ (cf. [19, pp. 419–420]).

7. Partial differential equations associated with free pseudo-product FGLAs

In this section we show that the system of partial differential equations (GA) is associated with a free pseudo-product FGLA of type $(1, m, 3)$ $(m \geqq 2)$.

7.1. Realization of the standard differential systems. Using the methods in [14, §2] we realize the standard differential system of type $\mathfrak{p}(V, W, 3)$.

Let $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ be a free pseudo-product FGLA $\mathfrak{p}(V, W, 3)$, and u^p the projection of $\mathfrak{m}$ onto $\mathfrak{g}_p$ with respect to the decomposition $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$. Let $M(\mathfrak{m})$ be a simply connected Lie group with Lie algebra $\mathfrak{m}$ and $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system. Furthermore let ξ be the Maurer-Cartan form on $M(\mathfrak{m})$, and ξ_j the $\mathfrak{g}_j$-component of ξ with respect to the decomposition $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$; then $D_{\mathfrak{m}}$ is defined by the equation $\xi_p = 0$ $(p \leqq -2)$. Let f be the diffeomorphism of $M(\mathfrak{m})$ onto $\mathfrak{m}$ defined in [14, §2.3]. If we put $\eta = (f^{-1})^* \xi$ and $\eta_p = (f^{-1})^* \xi_p$, then $\widehat{D}_{\mathfrak{m}}$ is defined by the equations $\eta_p = 0$ $(p \leqq -2)$. We induce a Lie group structure on $\mathfrak{m}$ by requiring $f : M(\mathfrak{m}) \to \mathfrak{m}$ to be an isomorphism. With this Lie group structure, $\mathfrak{m}$ is a simply connected Lie group with Lie algebra $\mathfrak{m}$ and an $\mathfrak{m}$-valued 1-form η is the Maurer-Cartan form on the Lie group $\mathfrak{m}$. In what follows we will adopt $(\mathfrak{m}, \widehat{D}_{\mathfrak{m}})$ as the standard differential system of type $\mathfrak{m}$ and also denote by $(M(\mathfrak{m}), D_{\mathfrak{m}})$ the standard differential system $(\mathfrak{m}, \widehat{D}_{\mathfrak{m}})$.

Let $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_m)$) be a basis of V (resp. W). We define a coordinate system $\{x_i, y_a, z_{ia}, w_{ij,a}, v_{i,ab}\}$ on $M(\mathfrak{m})$ as follows:

$$\begin{cases} u^{-1} = \displaystyle\sum_{i=1}^{n} x_i e_i + \sum_{a=1}^{m} y_a f_a, \quad u^{-2} = \sum_{i=1}^{n} \sum_{a=1}^{m} z_{ia} e_i \otimes f_a, \\ u^{-3} = \displaystyle\sum_{a=1}^{m} \sum_{i \leqq j} w_{ij,a} e_i \odot e_j \otimes f_a + \sum_{i=1}^{n} \sum_{a \leqq b} t_{i,ab} e_i \otimes f_a \odot f_b. \end{cases}$$

By virtue of the formula given by N. Tanaka in [14, §2.3], we obtain

$$D_{\mathfrak{m}} = \{ \theta_{ij,a} = \omega_{i,ab} = \pi_{i,a} = 0 \ (1 \leqq i \leqq j \leqq n, 1 \leqq a \leqq b \leqq m) \},$$

where

$$\begin{cases} \theta_{ij,a} = d\widehat{w}_{ij,a} + (z_{ja} - \frac{1}{2} x_j y_a) dx_i + (z_{ia} - \frac{1}{2} x_i y_a) dx_j \\ (1 \leqq i < j \leqq n, 1 \leqq a \leqq m), \\ \theta_{ii,a} = d\widehat{w}_{ii,a} + (z_{ia} - \frac{1}{2} x_i y_a) dx_i \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m), \\ \omega_{i,ab} = d\widehat{t}_{i,ab} - (z_{ia} + \frac{1}{2} x_i y_a) dy_b - (z_{ib} + \frac{1}{2} x_i y_b) dy_a \\ (1 \leqq i \leqq n, 1 \leqq a < b \leqq m), \\ \omega_{i,aa} = d\widehat{t}_{i,aa} - (z_{ia} + \frac{1}{2} x_i y_a) dy_a \ (1 \leqq i \leqq n, 1 \leqq a \leqq m), \\ \pi_{i,a} = dz_{ia} - \frac{1}{2}(x_i dy_a - y_a dx_i) \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m), \end{cases}$$

and $\widehat{w}_{ij,a} = w_{ij,a} - \frac{2}{3}x_i z_{ja} - \frac{2}{3}x_j z_{ia} + \frac{1}{3}x_i x_j y_a$ $(1 \leqq i < j \leqq n, 1 \leqq a \leqq m)$, $\widehat{w}_{ii,a} = w_{ii,a} - \frac{2}{3}x_i z_{ia} + \frac{1}{6}x_i^2 y_a$ $(1 \leqq i \leqq n, 1 \leqq a \leqq m)$, $\widehat{t}_{i,ab} = t_{i,ab} + \frac{2}{3}y_a z_{ib} + \frac{2}{3}y_b z_{ia} + \frac{1}{3}y_a y_b x_i$ $(1 \leqq i \leqq n, 1 \leqq a < b \leqq m)$, $\widehat{t}_{i,aa} = t_{i,aa} + \frac{2}{3}y_a z_{ia} + \frac{1}{6}(y_a)^2 x_i$ $(1 \leqq i \leqq n, 1 \leqq a \leqq m)$. If we further put $\tilde{z}_{ia} = z_{ia} - \frac{1}{2}x_i y_a$ $(1 \leqq i \leqq n, 1 \leqq a \leqq m)$, $\tilde{t}_{i,ab} = \widehat{t}_{i,ab} - \tilde{z}_{ia}y_b - \tilde{z}_{ib}y_a - x_i y_a y_b$ $(1 \leqq i \leqq n, 1 \leqq a < b \leqq m)$, $\tilde{t}_{i,aa} = \widehat{t}_{i,aa} - \tilde{z}_{ia}y_a - \frac{1}{2}x_i y_a^2$ $(1 \leqq i \leqq n, 1 \leqq a \leqq m)$, then

$$\begin{cases} \theta_{ij,a} = d\widehat{w}_{ij,a} + \tilde{z}_{ja}dx_i + \tilde{z}_{ia}dx_j \quad (1 \leqq i < j \leqq n, 1 \leqq a \leqq m), \\ \theta_{ii,a} = d\widehat{w}_{ii,a} + \tilde{z}_{ia}dx_i (= \widehat{\theta}_{ii,a}) \quad (1 \leqq i \leqq m, 1 \leqq a \leqq n), \\ \omega_{i,ab} = d\tilde{t}_{i,ab} + y_b d\tilde{z}_{ia} + y_a d\tilde{z}_{ib} + y_a y_b dx_i \quad (1 \leqq i \leqq n, 1 \leqq a < b \leqq m), \\ \omega_{i,aa} = d\tilde{t}_{i,aa} + y_a d\tilde{z}_{ia} + \dfrac{1}{2}y_a^2 dx_i \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m), \\ \pi_{i,a} = d\tilde{z}_{ia} + y_a dx_i \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m). \end{cases}$$

We set $u_{ij,a} = \frac{1}{2}\widehat{w}_{ij,a}$ $(i \neq j)$, $u_{ii,a} = \widehat{w}_{ii,a}$, $p_{ia} = -\tilde{z}_{ia}$, $v_{i,aa} = 2\tilde{t}_{i,aa}$, $v_{i,ab} = \tilde{t}_{i,ab}$ $(a \neq b)$ and $r_a = y_a$; then

$$\begin{cases} \theta_{ij,a} = du_{ij,a} - \dfrac{1}{2}(p_{ja}dx_i + p_{ia}dx_j) \quad (1 \leqq i \leqq j \leqq n, 1 \leqq a \leqq m), \\ \omega_{i,ab} = dv_{i,ab} - r_a dp_{ib} - r_b dp_{ia} + r_a r_b dx_i \quad (1 \leqq i \leqq n, 1 \leqq a \leqq b \leqq m), \\ \pi_{i,a} = dp_{ia} - r_a dx_i \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m). \end{cases}$$

Note that

$$\omega_{i,ab} \equiv dv_{i,ab} - r_a r_b dx_i \pmod{\pi_{ia}, \pi_{ib}}.$$

We have

$$\begin{cases} d\theta_{ij,a} = \dfrac{1}{2}(dx_i \wedge \pi_{ja} + dx_j \wedge \pi_{ia}) \quad (1 \leqq i \leqq j \leqq n, 1 \leqq a \leqq m), \\ d\omega_{i,ab} = \pi_{ib} \wedge dr_a + \pi_{ia} \wedge dr_b \quad (1 \leqq i \leqq n, 1 \leqq a \leqq b \leqq m), \\ d\pi_{i,a} = dx_i \wedge dr_a \quad (1 \leqq i \leqq n, 1 \leqq a \leqq m), \end{cases}$$

and

$$\partial D_{\mathfrak{m}} = \{\, \theta_{ij,a} = \omega_{i,ab} = 0 \ (1 \leqq i \leqq j \leqq n, 1 \leqq a \leqq b \leqq m) \,\}.$$

7.2. Partial differential equations associated with free pseudo-product fundamental graded Lie algebra of type $(1, m, 3)$. From now on we assume that $n = 1$ and $m \geqq 2$. We use the notation as in the previous subsection. It is convenient to set $\omega_{ab} = \omega_{1,ab}$, $\theta_a = \theta_{11,a}$, $\pi_a = \pi_{1,a}$, $x = x_1$.

For $x \in M(\mathfrak{m})$ we define R_x as the collection of subspaces v of codimension $\frac{m(m+1)}{2}$ in each tangent space $T_x(M(\mathfrak{m}))$ at $x \in M(\mathfrak{m})$ which contains the fiber $\partial D_{\mathfrak{m}}(x)$ of the derived system $\partial D_{\mathfrak{m}}$ of $D_{\mathfrak{m}}$:

$R_x = \{v \in \mathrm{Gr}(T_x(M(\mathfrak{m})), 3m+1) : v \supset \partial D_{\mathfrak{m}}(x)\}$, where $\mathrm{Gr}(T_x(M(\mathfrak{m})), 3m+1)$ denotes the Grassmann manifold consisting of $(3m+1)$-dimensional subspaces of $T_x(M(\mathfrak{m}))$. We set

$$R = \bigcup_{x \in M(\mathfrak{m})} R_x \subset J(M(\mathfrak{m}), 3m+1),$$

where $J(M(\mathfrak{m}), 3m+1)$ is the Grassmann bundle over $M(\mathfrak{m})$ consisting of $(3m+1)$-dimensional contact elements of $M(\mathfrak{m})$ (cf.[17, §1.5]). Let pr_{-3} be the projection $\mathfrak{m}$ onto $\mathfrak{g}_{-3}$ with respect to the decomposition $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$, and $\mathrm{pr}_{-3,+}$ (resp. $\mathrm{pr}_{-3,-}$) the projection $\mathfrak{g}_{-3}$ onto $S^2(V) \otimes W$ (resp. $V \otimes S^2(W)$) with respect to the decomposition $\mathfrak{g}_{-3} = S^2(V) \otimes W \oplus V \otimes S^2(W)$. We denote by R^0_x the subset of R_x consisting of elements v of R such that

$$\mathrm{pr}_{-3,+} |(\mathrm{pr}_{-3}((L_x)_*^{-1}(v)) : \mathrm{pr}_{-3}((L_x)_*^{-1}(v)) \to S^2(V) \otimes W$$

is an isomorphism; then $R^0 = \bigcup_{x \in M(\mathfrak{m})} R^0_x$ is an open submanifold of R. Let $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ be the prolongation of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{f})$ and $\mathfrak{g}(\mathfrak{m}) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}(\mathfrak{m})_p$ the prolongation of $\mathfrak{m}$. Then $\mathfrak{g}(\mathfrak{m})_0 = \mathfrak{g}_0 \oplus \mathfrak{g}(\mathfrak{m})_{0,-1}$. The mapping $\tau : \mathfrak{g}(\mathfrak{m})_{0,-1} \to \mathrm{Hom}(S^2(V) \otimes W, V \otimes S^2(W))$ defined by $\tau(A) = \mathrm{ad}\, A | S^2(V) \otimes W$ is a monomorphism as a $\mathfrak{g}_0$-module. Note that the image $\tau(\mathfrak{g}(\mathfrak{m})_{0,-1})$ is an irreducible $\mathfrak{g}_0$-submodule of $\mathrm{Hom}(S^2(V) \otimes W, V \otimes S^2(W))$ isomorphic to $V^* \otimes W$. Furthermore we denote by κ the mapping of R^0 into $\mathrm{Hom}(S^2(V) \otimes W, V \otimes S^2(W))$ defined by

$$\kappa(v) = (\mathrm{pr}_{-3,-}) \circ (\mathrm{pr}_{-3,+} \,|\, \mathrm{pr}_{-3}((L_x)_*^{-1}(v)))^{-1}.$$

We denote by $\widehat{R}_x$ the subset of R^0_x consisting of the elements v such that $\kappa(v) \in \tau(\mathfrak{g}(\mathfrak{m})_{0,-1})$; then $\widehat{R} = \bigcup_{x \in M(\mathfrak{m})} \widehat{R}_x$ is a submanifold of R. We define two differential systems D^1 and D^2 on $\widehat{R}$ as follows: D^1 is the canonical system obtained by the Grassmannian construction and D^2 is the lift of $D_{\mathfrak{m}}$. More precisely

$$D^1(v) = \nu_*^{-1}(v) \supset D^2(v) = \nu_*^{-1}(D_{\mathfrak{m}}(\nu(v))),$$

where ν is the natural projection of $\widehat{R}$ onto $M(\mathfrak{m})$: $\nu(v) = x$ if $v \in \widehat{R}_x$. We introduce a fiber coordinate system on $\widehat{R}$ by $\varpi_{ab} = \omega_{ab} + \lambda_a \theta_b + \lambda_b \theta_a$, where

$$D^1 = \{\, \varpi_{ab} = 0 \,\}, \quad \partial D_{\mathfrak{m}} = \{\, \omega_{ab} = \theta_a = 0 \}.$$

Here $(x, y_a, p_a, v_{ab}, u_a, \lambda_a)$ constitutes a coordinate system on $\widehat{R}$. Since

$$\begin{aligned}
&d\varpi_{ab} = \pi_a \wedge (dr_b - \lambda_b dx) + \pi_b \wedge (dr_a - \lambda_a dx) + d\lambda_a \wedge \theta_b + d\lambda_b \wedge \theta_a,\\
&\mathrm{Ch}(D^1) = \{\, \varpi_{ab} = \pi_a = dr_a - \lambda_a dx = d\lambda_a = \theta_a = 0 \},\\
&D^2 = \{\, \varpi_{ab} = \pi_a = \omega_{ab} = \theta_a = 0 \,\},\ \partial D^2 = \{\, \varpi_{ab} = \omega_{ab} = \theta_a = 0 \,\},
\end{aligned}$$

we get the following:

(R1) D^1 and D^2 is a differential system of codimension $\frac{m(m+1)}{2}$ and $\frac{m(m+1)}{2} + m$ respectively.

(R2) $\partial D^2 \subset D^1$.

(R3) $\mathrm{Ch}(D^1)$ is a subbundle of D^2 of codimension $2m$.

(R4) $\mathrm{Ch}(D^1)(v) \cap \mathrm{Ch}(D^2)(v) = \{0\}$ for all $v \in \widehat{R}$.

Now we calculate

$$\varpi_{ab} = \omega_{ab} + \lambda_a \theta_b + \lambda_b \theta_a = dZ'_{ab} - P'_a dX_b - P'_b dX_a - Q'_a dY_b - Q'_b dY_a,$$

where

$$\begin{cases} Z'_{ab} = v_{ab} - r_a p_b - r_b p_a + r_a r_b x + \lambda_a u_b + \lambda_b u_a \\ \qquad - \dfrac{1}{2}(\lambda_a r_b + \lambda_b r_a) x^2 + \dfrac{1}{3} x^3 \lambda_a \lambda_b, \\ P'_a = u_a - p_a x - \dfrac{1}{2} r_a x^2 + \dfrac{1}{2} \lambda_a x^3, \\ Q'_a = -p_a + r_a x - \dfrac{1}{2} x^2 \lambda_a, \\ X_a = \lambda_a, \\ Y_a = r_a - x\lambda_a. \end{cases}$$

We solve

$$\begin{cases} r_a = Y_a + X_a x, \\ p_a = -Q'_a + Y_a x + \dfrac{1}{2} X_a x^2, \\ u_a = P'_a - Q'_a x + \dfrac{1}{2} Y_a x^2 + \dfrac{1}{6} X_a x^3, \\ v_{ab} = Z'_{ab} - Y_a Q'_b - Y_b Q'_a - X_a P'_b - X_b P'_a + Y_a Y_b x \\ \qquad + \dfrac{1}{2} X_a Y_b x^2 + \dfrac{1}{2} X_b Y_a x^2 + X_a X_b x^3. \end{cases}$$

From these results, we get

$$\begin{cases} \pi_a = -dQ'_a + x dY_a + \dfrac{1}{2} x^2 dX_a, \\ \theta_a = dP'_a - \dfrac{1}{2} x^2 dY_a - \dfrac{1}{3} x^3 dX_a + x\pi_a. \end{cases}$$

Then

$$D^1 = \{ dZ'_{ab} - P'_a dX_b - P'_b dX_a - Q'_a dY_b - Q'_b dY_a = 0 \}$$

and $(X_a, Y_a, Z'_{ab}, P'_a, Q'_a)$ constitutes a coordinate system on $R' = \widehat{R}/\operatorname{Ch}(D^1)$. Also D^1 drops down to R'. Putting $t = x$, $P_a = P'_a$, $Q_a = Q'_a$, $Z_{ab} = Z'_{ab}$ $(a \neq b)$, and $Z_{aa} = 2Z'_{aa}$, we obtain

$$D^2 = \{ \Omega_{ab} = \Theta_a = \Pi_a = 0 \},$$

where

$$\begin{cases} \Omega_{ab} = dZ_{ab} - P_a dX_b - P_b dX_a - Q_a dY_b - Q_b dY_a \ (a \neq b), \\ \Omega_{aa} = dZ_{aa} - P_a dX_a - Q_a dY_a, \\ \Theta_a = dP_a - \dfrac{1}{2} t^2 dY_a - \dfrac{1}{3} t^3 dX_a, \\ \Pi_a = dQ_a - t dY_a - \dfrac{1}{2} t^2 dX_a. \end{cases}$$

Now we introduce the third gradation $(\mathfrak{b}_p)$ of $\mathfrak{g}(\mathfrak{m})$ as follows: $\mathfrak{g}(\mathfrak{m}) = \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$, $\mathfrak{b}_{-2} = V \otimes S^2(W)$, $\mathfrak{b}_{-1} = (V^* \otimes W) \oplus W \oplus (V \otimes W) \oplus (S^2(V) \otimes W)$,

$\mathfrak{b}_0 = V \oplus (V \otimes V^*) \oplus (W \otimes W^*) \oplus V^*$. Putting $\mathfrak{e}'' = (V^* \otimes W) \oplus W$ and $\mathfrak{f}'' = (V \otimes W) \oplus (S^2(V) \otimes W)$, the GLA $\mathfrak{g}(\mathfrak{m}) = \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$ becomes an irreducible transitive GLA such that $(\mathfrak{b}_-; \mathfrak{e}'', \mathfrak{f}'')$ is a pseudo-product FGLA of the second kind, where $\mathfrak{b}_- = \mathfrak{b}_{-2} \oplus \mathfrak{b}_{-1}$. Let A_0 be a Lie group consisting of automorphisms of $\mathfrak{g}(\mathfrak{m})$ preserving the gradation $(\mathfrak{a}_p)$; then A_0 is a closed Lie subgroup of $\mathrm{Aut}(\mathfrak{g}(\mathfrak{m}))$ with Lie algebra $\mathfrak{a}_0$. We put

$$G = \mathrm{Int}(\mathfrak{g}(\mathfrak{m}))A_0.$$

Then G is a Lie group with Lie algebra $\mathfrak{g}(\mathfrak{m})$. Also let G' (resp. A', B') be the subgroup consisting of elements of G preserving the filtration $(\bigoplus_{p \geqq k} \mathfrak{g}(\mathfrak{m})_p)_{k \in \mathbb{Z}}$ (resp. $(\bigoplus_{p \geqq k} \mathfrak{a}_p)_{k \in \mathbb{Z}}$, $(\bigoplus_{p \geqq k} \mathfrak{b}_p)_{k \in \mathbb{Z}}$); then G', A' and B' are closed subgroups of G with Lie algebras $\bigoplus_{p \geqq 0} \mathfrak{g}(\mathfrak{m})_p$, $\bigoplus_{p \geqq 0} \mathfrak{a}_p$ and $\bigoplus_{p \geqq 0} \mathfrak{b}_p$ respectively. We denote by $M_\mathfrak{g}$ (resp. $M_\mathfrak{a}$, $M_\mathfrak{b}$) the homogeneous space G/G' (resp. G/A', G/B'). Also we denote by π_g (resp. π_b) the natural projection of $M_\mathfrak{a}$ onto $M_\mathfrak{g}$ (resp. $M_\mathfrak{b}$). Furthermore identifying $\mathfrak{g}(\mathfrak{m})$ with the Lie algebra of left invariant vector fields on G, $\bigoplus_{p \geqq -1} \mathfrak{g}_p$, $\bigoplus_{p \geqq -1} \mathfrak{a}_p$ and $\bigoplus_{p \geqq -1} \mathfrak{b}_p$ induce G-invariant differential systems $D_\mathfrak{g}$, $D_\mathfrak{a}$ and $D_\mathfrak{b}$ on $M_\mathfrak{g}$, $M_\mathfrak{a}$ and $M_\mathfrak{b}$ respectively. Also the inclusion mapping of $\mathfrak{m}$ into $\mathfrak{g}(\mathfrak{m})$ induces an local isomorphism $i_\mathfrak{g}$ of $(M(\mathfrak{m}), D_\mathfrak{m})$ into $(M_\mathfrak{g}, D_\mathfrak{g})$. As in [22, p. 371], by applying the realization lemma ([17, Lemma 1.5]) for $(M_\mathfrak{a}, D^1_\mathfrak{a}, \pi_g, M_\mathfrak{g})$ and by taking account of (R1)–(R4), we have a local isomorphism of $(M_\mathfrak{a}, D^1_\mathfrak{a}, D^2_\mathfrak{a})$ into $(\widehat{R}, D^1, D^2)$, where $D^1_\mathfrak{a} = \partial^{(3)} D_\mathfrak{a}$ and $D^2_\mathfrak{a} = \partial D_\mathfrak{a}$. Similarly $\mathfrak{e}' \oplus \bigoplus_{p \geqq 0} \mathfrak{a}_p$ and $\mathfrak{f}'' \oplus \mathfrak{b}_0$ induce G-invariant completely integrable differential systems $E_\mathfrak{a}$ and $F_\mathfrak{b}$ on $M_\mathfrak{a}$ and $M_\mathfrak{b}$ respectively. Since $\mathrm{Ker}(\pi_b)_* = E_\mathfrak{a} = \mathrm{Ch}(D^1_\mathfrak{a})$, the differential system (R', D^1) is locally isomorphic to $(M_\mathfrak{b}, D_\mathfrak{b})$. We shall identify (R', D^1) with $(M_\mathfrak{b}, D_\mathfrak{b})$. We set $M'_\mathfrak{b} = M_\mathfrak{b}/F_\mathfrak{b}$ and denote by π'_b the projection of $M_\mathfrak{b}$ onto $M'_\mathfrak{b}$; then (X_a, Y_a, Z_{ab}) constitutes a coordinate system on $M'_\mathfrak{b}$.

Now recall that the Grassmann bundle and the canonical system. Following [17, §1.5], we define the Grassmann bundle $J(M'_\mathfrak{b}, 2m)$ over $M'_\mathfrak{b}$:

$$J(M'_\mathfrak{b}, 2m) = \bigcup_{x \in M'_\mathfrak{b}} J_x, \quad J_x = \mathrm{Gr}(T_x(M'_\mathfrak{b}), 2m).$$

$J(M'_\mathfrak{b}, 2m)$ is endowed with the canonical subbundle C of $T(J(M'_\mathfrak{b}, 2m))$ as follows: Let ρ^1_0 be the projection of $J(M'_\mathfrak{b}, 2m)$ onto $M'_\mathfrak{b}$. For $v \in J(M'_\mathfrak{b}, 2m)$ we set

$$C^1(v) = (\rho^1_0)^{-1}_*(v) \subset T_v(J(M'_\mathfrak{b}, 2m)).$$

Then C^1 is called the canonical system on $J(M'_\mathfrak{b}, 2m)$. We have an inhomogeneous Grassmann coordinate of $J(M'_\mathfrak{b}, 2m)$ as follows: We consider the open set U of $J(M'_\mathfrak{b}, 2m)$ consisting of the elements v of $J(M'_\mathfrak{b}, 2m)$ such that

$$dX_1 \wedge \cdots \wedge dX_m \wedge dY_1 \wedge \cdots \wedge dY_m | v \neq 0$$

Then we have a coordinate system $(X_a, Y_a, Z_{ab}, P^c_{ab}, Q^c_{ab})$ on U defined by

$$dZ_{ab}|u = \sum_{c=1}^{m} P^c_{ab} dX_c|u + \sum_{c=1}^{m} Q^c_{ab} dY_c|u,$$

where $u \in U$. Clearly C^1 is given in the coordinate system by

$$C^1 = \{\, \widehat{\Omega}_{ab} = 0 \,\},$$

where

$$\widehat{\Omega}_{ab} = dZ_{ab} - \sum_{c=1}^{m} P^c_{ab} dX_c - \sum_{c=1}^{m} Q^c_{ab} dY_c.$$

Applying the realization lemma for $(M_{\mathfrak{b}}, D_{\mathfrak{b}}, \pi_b, M'_{\mathfrak{b}})$ we have a local imbedding φ of $M_{\mathfrak{b}}$ into $J(M'_{\mathfrak{b}}, 2m)$ such that $\varphi_*^{-1}(C^1) = D_{\mathfrak{b}}$ and

$$R' = M_{\mathfrak{b}} = \{\, P^c_{ab} = \delta_{ac} P^b_{bb} + \delta_{bc} P^a_{aa}, Q^c_{ab} = \delta_{ac} Q^b_{bb} + \delta_{bc} Q^a_{aa} \ (a \neq b) \,\}.$$

We next define the second order prolongation $J^2(M'_{\mathfrak{b}}, 2m)$ of $J(M'_{\mathfrak{b}}, 2m)$ (cf. [18, §2]). For each $u \in J(M'_{\mathfrak{b}}, 2m)$, let J^2_u be the set of all $2m$-dimensional integral elements v of $(J(M'_{\mathfrak{b}}, 2m), C^1)$ at u such that $v \cap Q^1(u) = \{0\}$, where $Q^1 = \mathrm{Ker}(\rho^1_0)_*$. Then $J^2(M'_{\mathfrak{b}}, 2m)$ is defined by

$$J^2(M'_{\mathfrak{b}}, 2m) = \bigcup_{u \in J^1(M'_{\mathfrak{b}}, 2m)} J^2_u.$$

Let ρ^2_1 be the projection of $J^2(M'_{\mathfrak{b}}, 2m)$ onto $J^1(M'_{\mathfrak{b}}, 2m)$. We set $\rho = \rho^2_1 \circ \rho^1_0$. The canonical system C^2 is defined by

$$C^2(v^2) = (\rho^2_1)_*^{-1}(v^2) \quad \text{for } v^2 \in J^2(M'_{\mathfrak{b}}, 2m).$$

Then we have a coordinate system $(X_a, Y_a, Z_{ab}, P^c_{ab}, Q^c_{ab}, R^{cd}_{ab}, S^{cd}_{ab}, T^{cd}_{ab})$ on $\rho^{-1}(U)$ defined by

$$dP^c_{ab}|u = \sum_{d=1}^{m} R^{cd}_{ab} dX_d|u + \sum_{d=1}^{m} S^{cd}_{ab} dY_d|u,$$
$$dQ^c_{ab}|u = \sum_{d=1}^{m} S^{cd}_{ab} dX_d|u + \sum_{c=1}^{m} T^{cd}_{ab} dY_d|u,$$

where $u \in U$. Clearly C^2 is given in the coordinate system by

$$C^2 = \{\, \widehat{\Omega}_{ab} = \widehat{\Theta}^c_{ab} = \widehat{\Pi}^c_{ab} = 0 \,\},$$

where

$$\begin{cases} \widehat{\Theta}^c_{ab} = dP^c_{ab} - \sum_{d=1}^{m} R^{cd}_{ab} dX_d - \sum_{d=1}^{m} S^{cd}_{ab} Y_d, \\ \widehat{\Pi}^c_{ab} = dQ^c_{ab} - \sum_{d=1}^{m} S^{cd}_{ab} dX_d - \sum_{d=1}^{m} T^{cd}_{ab} dY_d. \end{cases}$$

Applying the realization lemma for $(M_{\mathfrak{a}}, D^2_{\mathfrak{a}}, \pi_b, M_{\mathfrak{b}})$ there exists a local imbedding ψ of $(M_{\mathfrak{a}}, D^2_{\mathfrak{a}})$ into $(J^2(M'_{\mathfrak{b}}, 2m), C^2)$. From the above results $\widehat{R}$ is given by the following equations:

$$\begin{cases} R^{ab}_{ab} = \dfrac{1}{3}(T^{ab}_{ab})^3, \ S^{ab}_{ab} = \dfrac{1}{2}(T^{ab}_{ab})^2, \\ R^{cd}_{ab} = S^{cd}_{ab} = T^{cd}_{ab} = 0, \ T^{ab}_{ab} = T^{cd}_{cd}, \\ P^{b}_{ab} = P^{a}_{aa}, \ P^{a}_{ab} = P^{b}_{bb}, \ P^{e}_{ab} = 0, \\ Q^{b}_{ab} = Q^{a}_{aa}, \ Q^{a}_{ab} = Q^{b}_{bb}, \ Q^{e}_{ab} = 0, \end{cases}$$

where $1 \leqq a \leqq b \leqq m, \ \{c,d\} \neq \{a,b\}, e \neq a,b.$ Thus we may consider that $\widehat{R}$ is realized by the system of partial differential equations (GA).

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