

## Smallest Complex Nilpotent Orbits with Real Points

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**Abstract.** Let  $\mathfrak{g}$  be a non-compact real simple Lie algebra without complex structure, and denote by  $\mathfrak{g}_{\mathbb{C}}$  the complexification of  $\mathfrak{g}$ . This paper focuses on non-zero nilpotent adjoint orbits in  $\mathfrak{g}_{\mathbb{C}}$  meeting  $\mathfrak{g}$ . We show that the poset consisting of such nilpotent orbits equipped with the closure ordering has the minimum  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ . Furthermore, we determine such  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  in terms of the Dynkin–Kostant classification even in the cases where  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  does not coincide with the minimal nilpotent orbit in  $\mathfrak{g}_{\mathbb{C}}$ . We also prove that the intersection  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  is the union of all minimal nilpotent orbits in  $\mathfrak{g}$ .

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### 1. Introduction and statement of main results

Let  $\mathfrak{g}$  be a non-compact real simple Lie algebra without complex structure. This means that the complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  is simple. Denote by  $\mathcal{N}$  the nilpotent cone of  $\mathfrak{g}_{\mathbb{C}}$  and by  $\mathcal{N}/G_{\mathbb{C}}$  the set of complex nilpotent (adjoint) orbits of the group  $G_{\mathbb{C}} := \text{Int}(\mathfrak{g}_{\mathbb{C}})$  of inner-automorphisms.

By abuse of notation, we write  $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$  for the set consisting of complex nilpotent orbits that meet  $\mathfrak{g}$ . Note that

$$\mathcal{N}/G_{\mathbb{C}} \supset \mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}.$$

The finite sets  $\mathcal{N}/G_{\mathbb{C}}$  and  $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$  are both posets with respect to the closure ordering such that the zero-orbit  $[0]$  is the minimum. We ask what are minimal orbits in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$ . It is well known that  $(\mathcal{N}/G_{\mathbb{C}}) \setminus \{[0]\}$  has the minimum  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ , which is called *the minimal nilpotent orbit in  $\mathfrak{g}_{\mathbb{C}}$* . The minimal nilpotent orbit  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  is the adjoint orbit that goes through a highest root vector with respect to a positive system  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  where  $\mathfrak{h}_{\mathbb{C}}$  is a Cartan subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  (see [6, Chapter 4.3] for the details). In order to investigate  $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$ , we need a positive system  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  of the restricted root system of a maximally split abelian subspace  $\mathfrak{a}$  of  $\mathfrak{g}$  (see Section 2 for the definition of maximally split abelian subspaces of  $\mathfrak{g}$ ).

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Our concern in this paper is with minimal orbits in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$ . Our first main result is here:

**Theorem 1.1.** *The following three conditions on a complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $\mathfrak{g}_{\mathbb{C}}$  with  $\mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{g} \neq \emptyset$  are equivalent:*

- (i)  $\mathcal{O}^{G_{\mathbb{C}}}$  is minimal in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$  with respect to the closure ordering.
- (ii) The dimension of  $\mathcal{O}^{G_{\mathbb{C}}}$  attains its minimum in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$ .
- (iii)  $\mathcal{O}^{G_{\mathbb{C}}} \supset (\mathfrak{g}_{\lambda} \setminus \{0\})$ , where  $\lambda$  is the highest root of  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  and  $\mathfrak{g}_{\lambda}$  is the root space of  $\lambda$  (the dimension of  $\mathfrak{g}_{\lambda}$  is not necessary to be one).

Furthermore, there uniquely exists such  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$ .

The unique complex nilpotent orbit in  $\mathfrak{g}_{\mathbb{C}}$  in Theorem 1.1 will be denoted by  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ . In many cases,  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} = \mathcal{O}_{\min}^{G_{\mathbb{C}}}$ .

Our second main result concerns detailed properties of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  when  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$ .

**Theorem 1.2.** (1) *The following five conditions on  $\mathfrak{g}$  are equivalent:*

- (i)  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$ .
- (ii)  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} = \emptyset$ .
- (iii)  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ .
- (iv) There exists a black node  $\alpha$  in the Satake diagram of  $\mathfrak{g}$  such that  $\alpha$  has some edges connected to the added node in the extended Dynkin diagram of  $\mathfrak{g}_{\mathbb{C}}$ .
- (v)  $\mathfrak{g}$  is isomorphic to one of  $\mathfrak{su}^*(2k)$ ,  $\mathfrak{so}(n-1, 1)$ ,  $\mathfrak{sp}(p, q)$ ,  $\mathfrak{f}_{4(-20)}$  or  $\mathfrak{e}_{6(-26)}$  where  $k \geq 2, n \geq 5$  and  $p, q \geq 1$ .

(2) *If the above equivalent conditions on  $\mathfrak{g}$  are satisfied, then the complex nilpotent orbit  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  is characterized by the weighted Dynkin diagram in Table 1 via the Dynkin–Kostant classification.*

The equivalence between (ii) and (v) in Theorem 1.2 (1) was stated on Brylinski [4, Theorem 4.1] without proof. We provide a proof for the convenience of the readers.

Our work is motivated by the recent progress in the theory of infinite dimensional representations. For an irreducible (admissible) representation  $\pi$  of a real reductive Lie group  $G$  with its Lie algebra  $\mathfrak{g}$ , one can define the associated variety  $\mathcal{AV}(\text{Ann } \pi) (\subset \mathcal{N})$  of the annihilator  $\text{Ann } \pi$  in the enveloping algebra  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ . It is known that there uniquely exists a complex nilpotent orbit  $\mathcal{O}_{\pi}^{G_{\mathbb{C}}}$  meeting  $\mathfrak{g}$  such that  $\mathcal{AV}(\text{Ann } \pi) = \overline{\mathcal{O}_{\pi}^{G_{\mathbb{C}}}}$ . Half the complex dimension of  $\mathcal{O}_{\pi}^{G_{\mathbb{C}}}$  coincides with the Gelfand–Kirillov dimension of  $\pi$ .

Table 1: List of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  for the cases  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$ .

$\mathfrak{g}$	$\dim_{\mathbb{C}} \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$	Weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$
$\mathfrak{su}^*(2k)$	$8k - 8$	$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ \circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \end{array} \quad (k \geq 3)$ $\begin{array}{ccc} 0 & 2 & 0 \\ \circ & \circ & \circ \end{array} \quad (k = 2)$
$\mathfrak{so}(n - 1, 1)$	$2n - 4$	$\begin{array}{ccccccc} 2 & 0 & 0 & \cdots & 0 & 0 \\ \circ & \circ & \circ & \cdots & \circ & \circ \end{array} \quad (n \text{ is odd, } n \geq 5)$ $\begin{array}{ccccccc} & & & & & 0 & \\ & & & & & \circ & \\ 2 & 0 & 0 & \cdots & 0 & \circ & \\ \circ & \circ & \circ & \cdots & \circ & \circ & \\ & & & & & \circ & \end{array} \quad (n \text{ is even, } n \geq 6)$
$\mathfrak{sp}(p, q)$	$4(p + q) - 2$	$\begin{array}{ccccccc} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ \circ & \circ & \circ & \circ & \cdots & \circ & \circ \end{array} \quad (p + q \geq 3, p, q \geq 1)$ $\begin{array}{c} 0 & 2 \\ \circ & \circ \end{array} \quad (p = q = 1)$
$\mathfrak{e}_{6(-26)}$	32	$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ \circ & \circ & \circ & \circ & \circ \\ & &   & & \\ & & \circ & & \\ & & 0 & & \end{array}$
$\mathfrak{f}_{4(-20)}$	22	$\begin{array}{cccc} 0 & 0 & 0 & 1 \\ \circ & \circ & \circ & \circ \end{array}$

If  $\pi$  is a minimal representation in the sense that the annihilator of  $\pi$  is the Joseph ideal ([13]), then  $\mathcal{AV}(\text{Ann } \pi) = \overline{\mathcal{O}_{\min}^{G_{\mathbb{C}}}}$ . Hence, by Vogan [24],  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{p}_{\mathbb{C}} \neq \emptyset$  or equivalently  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} \neq \emptyset$ , by the Kostant–Sekiguchi correspondence, where  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  is a Cartan decomposition of  $\mathfrak{g}$  and  $\mathfrak{p}_{\mathbb{C}}$  denotes the complexification of  $\mathfrak{p}$ . Therefore, minimal representations do not exist for simple Lie groups  $G$  if  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} = \emptyset$  or equivalently, if the Lie algebra  $\mathfrak{g}$  of  $G$  is one of the five simple Lie algebras in Theorem 1.2.

In the cases where  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} = \emptyset$  or equivalently  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$ , there is no minimal representation of  $G$ , however, Hilgert, Kobayashi and Möllers [11] recently constructed the “smallest” irreducible unitary representations  $\pi$  of certain families of reductive Lie groups  $G$ . They proved that  $\mathcal{AV}(\text{Ann } \pi) = \overline{\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}}$  for their representations  $\pi$ . Therefore,  $\pi$  attains the minimum of the Gelfand–Kirillov dimensions of infinite dimensional irreducible representations of  $G$ . They constructed an  $L^2$ -model of such representations on a Lagrangian subvariety of minimal nilpotent  $G$ -orbits in  $\mathfrak{g}$ . Our results here were used in [11, Section 2.1.3] in their proof that their representation  $\pi$  attains the minimum of the Gelfand–Kirillov dimension of all infinite dimensional irreducible (admissible) representations of  $G$ .

Our work is also related to [16, Corollary 5.9] by Kobayashi and Oshima, on the classification of reductive symmetric pairs  $(\mathfrak{g}, \mathfrak{h})$  for which there exists a  $(\mathfrak{g}, K)$ -module that is discretely decomposable as an  $(\mathfrak{h}, H \cap K)$ -module in the sense of [15].

With applications to representation theory in mind, we also study the intersection  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  as a union of real nilpotent (adjoint) orbits in  $\mathfrak{g}$  in this

paper.

We denote the nilpotent cone for  $\mathfrak{g}$  by

$$\mathcal{N}(\mathfrak{g}) := \mathcal{N} \cap \mathfrak{g}$$

and by  $\mathcal{N}(\mathfrak{g})/G$  the set of real nilpotent (adjoint) orbits in  $\mathfrak{g}$  by the group  $G := \text{Int}(\mathfrak{g})$  of inner-automorphisms. The set  $\mathcal{N}(\mathfrak{g})/G$  is a poset with respect to the closure ordering, where the zero-orbit  $[0]$  in  $\mathfrak{g}$  is the minimum in  $\mathcal{N}(\mathfrak{g})/G$ .

For each real nilpotent orbit  $\mathcal{O}^G$  in  $\mathfrak{g}$ , there exists the unique complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $\mathfrak{g}_{\mathbb{C}}$  which contains  $\mathcal{O}^G$ . Then  $\mathcal{O}^G$  is a real form of  $\mathcal{O}^{G_{\mathbb{C}}}$ . The correspondence  $\mathcal{O}^G$  to  $\mathcal{O}^{G_{\mathbb{C}}}$  gives a surjective map

$$\mathcal{N}(\mathfrak{g})/G \twoheadrightarrow \mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}} (\subset \mathcal{N}/G_{\mathbb{C}}).$$

We remark that this map needs not be injective. It is known that for a complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$ , the intersection  $\mathcal{O}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  splits into finitely many real nilpotent orbits.

Our third main result is a characterization of minimal orbits in  $(\mathcal{N}(\mathfrak{g})/G) \setminus \{[0]\}$  as real forms of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ . More precisely, we prove the next theorem:

**Theorem 1.3.** *The following four conditions on a real nilpotent orbit  $\mathcal{O}^G$  in  $\mathfrak{g}$  are equivalent:*

- (i)  $\mathcal{O}^G$  is minimal in  $(\mathcal{N}(\mathfrak{g})/G) \setminus \{[0]\}$  with respect to the closure ordering.
- (ii) The dimension of  $\mathcal{O}^G$  attains its minimum in  $(\mathcal{N}(\mathfrak{g})/G) \setminus \{[0]\}$ .
- (iii)  $\mathcal{O}^G$  is contained in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .
- (iv)  $\mathcal{O}^G$  is a real form of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .

*In particular, the real dimension of such  $\mathcal{O}^G$  is equal to the complex dimension of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .*

We say that a real nilpotent orbit  $\mathcal{O}^G$  in  $\mathfrak{g}$  is *minimal* if  $\mathcal{O}^G$  satisfies the equivalent conditions in Theorem 1.3. In this sense, the intersection  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  is the disjoint union of all minimal real nilpotent orbits in  $\mathfrak{g}$ .

Our fourth main result is to determine the number of minimal real nilpotent orbits in  $\mathfrak{g}$  as follows:

**Theorem 1.4.** *For a non-compact real simple Lie algebra  $\mathfrak{g}$  without complex structure,*

$$\begin{aligned} \#\{ \text{minimal real nilpotent orbits in } \mathfrak{g} \} \\ = \begin{cases} 1 & \text{if } (\mathfrak{g}, \mathfrak{k}) \text{ is of non-Hermitian type,} \\ 2 & \text{if } (\mathfrak{g}, \mathfrak{k}) \text{ is of Hermitian type,} \end{cases} \end{aligned}$$

where  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  is a Cartan decomposition of  $\mathfrak{g}$ .

Note that all of five real simple Lie algebras  $\mathfrak{g}$  in Theorem 1.2 are of non-Hermitian type, and therefore there uniquely exists a minimal real nilpotent orbit in such  $\mathfrak{g}$ .

In our proof of Theorem 1.4, we study  $MA$ -orbits in a highest root space  $\mathfrak{g}_\lambda$  of  $\mathfrak{g}$  (see Section 6 for the notation of a group  $MA$  and more details).

Our results on real nilpotent orbits (Theorems 1.3 and 1.4) yield those on  $K_{\mathbb{C}}$ -orbits on the nilpotent cone  $\mathcal{N}(\mathfrak{p}_{\mathbb{C}})$  via the Kostant–Sekiguchi correspondence. To be precise, we fix some notation.

Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be a Cartan decomposition of  $\mathfrak{g}$ , denote its complexification by  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ , and  $K_{\mathbb{C}}$  the connected complex subgroup of  $\text{Int}(\mathfrak{g}_{\mathbb{C}})$  with its Lie algebra  $\mathfrak{k}_{\mathbb{C}}$ . We define the nilpotent cone for  $\mathfrak{p}_{\mathbb{C}}$  by

$$\mathcal{N}(\mathfrak{p}_{\mathbb{C}}) := \mathcal{N} \cap \mathfrak{p}_{\mathbb{C}},$$

on which  $K_{\mathbb{C}}$  acts with finitely many orbits. The Kostant–Sekiguchi correspondence is a bijection between two finite sets

$$\mathcal{N}(\mathfrak{g})/G \xleftrightarrow{1:1} \mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}}$$

which preserves the closure ordering (Barbasch–Sepanski [3]).

Let us denote by  $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/G_{\mathbb{C}}$  the subset of  $\mathcal{N}/G_{\mathbb{C}}$  consisting of complex nilpotent orbits in  $\mathfrak{g}_{\mathbb{C}}$  meeting  $\mathfrak{p}_{\mathbb{C}}$ . Recall that a complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $\mathfrak{g}_{\mathbb{C}}$  meets  $\mathfrak{g}$  if and only if it meets  $\mathfrak{p}_{\mathbb{C}}$  (Sekiguchi [23, Proposition 1.11]). Thus  $\mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/G_{\mathbb{C}}$  coincides with  $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$  as a subset of  $\mathcal{N}/G_{\mathbb{C}}$ .

Further, for each nilpotent  $K_{\mathbb{C}}$ -orbit  $\mathcal{O}^{K_{\mathbb{C}}}$  in  $\mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}}$ , there uniquely exists a complex nilpotent  $G_{\mathbb{C}}$ -orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  containing  $\mathcal{O}^{K_{\mathbb{C}}}$ , and the correspondence  $\mathcal{O}^{K_{\mathbb{C}}}$  to  $\mathcal{O}^{G_{\mathbb{C}}}$  gives a surjective map

$$\mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}} \twoheadrightarrow \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/G_{\mathbb{C}} (\subset \mathcal{N}/G_{\mathbb{C}}).$$

Thus, the Kostant–Sekiguchi correspondence gives the following commutative diagram:

$$\begin{array}{ccc} \mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}} & = & \mathcal{N}_{\mathfrak{p}_{\mathbb{C}}}/G_{\mathbb{C}} \\ \uparrow & & \uparrow \\ \mathcal{N}(\mathfrak{g})/G & \xleftrightarrow{1:1} & \mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}} \end{array}$$

Therefore, we have next two corollaries to Theorems 1.3 and 1.4:

**Corollary 1.5.** *For a non-compact real simple Lie algebra  $\mathfrak{g}$  without complex structure and its Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ , the following conditions on a nilpotent  $K_{\mathbb{C}}$ -orbit  $\mathcal{O}^{K_{\mathbb{C}}}$  in  $\mathfrak{p}_{\mathbb{C}}$  are equivalent:*

- (i)  $\mathcal{O}^{K_{\mathbb{C}}}$  is minimal in  $(\mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}}) \setminus \{[0]\}$  with respect to the closure ordering.
- (ii) The dimension of  $\mathcal{O}^{K_{\mathbb{C}}}$  attains its minimum in  $(\mathcal{N}(\mathfrak{p}_{\mathbb{C}})/K_{\mathbb{C}}) \setminus \{[0]\}$ .
- (iii)  $\mathcal{O}^{K_{\mathbb{C}}}$  is contained in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .

We say that a nilpotent  $K_{\mathbb{C}}$ -orbit  $\mathcal{O}^{K_{\mathbb{C}}}$  is *minimal* if  $\mathcal{O}^{K_{\mathbb{C}}}$  satisfies the equivalent conditions on Corollary 1.5.

**Corollary 1.6.**

$$\#\{ \text{minimal nilpotent } K_{\mathbb{C}}\text{-orbits in } \mathfrak{p}_{\mathbb{C}} \} = \begin{cases} 1 & \text{if } (\mathfrak{g}, \mathfrak{k}) \text{ is of non-Hermitian type,} \\ 2 & \text{if } (\mathfrak{g}, \mathfrak{k}) \text{ is of Hermitian type.} \end{cases}$$

**Remark 1.7.** • A part of our main results, e.g. Theorem 1.2 (2) and Theorem 1.4, could be proved by using the classification of real nilpotent orbits in  $\mathfrak{g}$  (see Remark 5.6 for more details). In this paper, our proof does not rely on the classification of real nilpotent orbits.

- Theorem 1.4 and Corollary 1.6 should be known to experts. In particular, the claim of [16, Proposition 2.2] includes Corollary 1.6. For the sake of completeness, we give a proof of Theorem 1.4 in this paper.

The paper is organized as follows. In Section 2, we recall the definition of weighted Dynkin diagrams of complex nilpotent orbits in complex semisimple Lie algebras and some well-known facts for a highest root of a restricted root system of  $(\mathfrak{g}, \mathfrak{a})$ . We prove Theorems 1.1 and 1.3 in Section 3. In Section 4, we give a proof of the first claim of Theorem 1.2. We determine the weighted Dynkin diagrams of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  in Section 5. Finally, we give a proof of Theorem 1.4 in Section 6.

## 2. Preliminary results

### 2.1. Weighted Dynkin diagrams of complex nilpotent orbits.

Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex semisimple Lie algebra. In this subsection, we recall the definition of weighted Dynkin diagrams of complex nilpotent orbits in  $\mathfrak{g}_{\mathbb{C}}$ .

Let us fix a Cartan subalgebra  $\mathfrak{h}_{\mathbb{C}}$  of  $\mathfrak{g}_{\mathbb{C}}$ . We denote by  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  the root system for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Then the root system  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  becomes a subset of the dual space  $\mathfrak{h}^*$  of

$$\mathfrak{h} := \{ H \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(H) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \}.$$

We write  $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  for the Weyl group of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  acting on  $\mathfrak{h}$ . Take a positive system  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  of the root system  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Then a closed Weyl chamber

$$\mathfrak{h}_+ := \{ H \in \mathfrak{h} \mid \alpha(H) \geq 0 \text{ for any } \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \}$$

becomes a fundamental domain of  $\mathfrak{h}$  for the action of  $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ .

Let  $\Pi$  be the simple system of  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Then for each  $H \in \mathfrak{h}$ , we define a map by

$$\Psi_H : \Pi \rightarrow \mathbb{R}, \quad \alpha \mapsto \alpha(H).$$

We call  $\Psi_H$  the weighted Dynkin diagram corresponding to  $H \in \mathfrak{h}$ , and  $\alpha(H)$  the weight on a node  $\alpha \in \Pi$  of the weighted Dynkin diagram. Since  $\Pi$  is a basis of  $\mathfrak{h}^*$ , the map

$$\Psi : \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R}), \quad H \mapsto \Psi_H$$

is bijective. Furthermore,

$$\mathfrak{h}_+ \rightarrow \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad H \mapsto \Psi_H$$

is also bijective.

A triple  $(H, X, Y)$  is said to be an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}_{\mathbb{C}}$  if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H \quad (H, X, Y \in \mathfrak{g}_{\mathbb{C}}).$$

For any  $\mathfrak{sl}_2$ -triple  $(H, X, Y)$  in  $\mathfrak{g}_{\mathbb{C}}$ , the elements  $X$  and  $Y$  are nilpotent in  $\mathfrak{g}_{\mathbb{C}}$ , and  $H$  is hyperbolic in  $\mathfrak{g}_{\mathbb{C}}$ , i.e.  $\text{ad}_{\mathfrak{g}_{\mathbb{C}}} H \in \text{End}(\mathfrak{g}_{\mathbb{C}})$  is diagonalizable with only real eigenvalues.

Combining the Jacobson–Morozov theorem with the results of Kostant [17], for each complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$ , there uniquely exists an element  $H_{\mathcal{O}}$  of  $\mathfrak{h}_+$  with the following property: There exists  $X, Y \in \mathcal{O}^{G_{\mathbb{C}}}$  such that  $(H_{\mathcal{O}}, X, Y)$  is an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}_{\mathbb{C}}$ . Furthermore, by the results of Malcev [18], the following map is injective:

$$\mathcal{N}/G_{\mathbb{C}} \hookrightarrow \mathfrak{h}_+, \quad \mathcal{O}^{G_{\mathbb{C}}} \mapsto H_{\mathcal{O}},$$

where  $\mathcal{N}/G_{\mathbb{C}}$  denotes the set of all complex nilpotent orbits in  $\mathfrak{g}_{\mathbb{C}}$ . For each complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$ , the weighted Dynkin diagram corresponding to  $H_{\mathcal{O}}$  is called the weighted Dynkin diagram of  $\mathcal{O}^{G_{\mathbb{C}}}$ . Dynkin [9] classified all such weighted Dynkin diagrams for each complex simple Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  as a classification of three dimensional simple subalgebras of  $\mathfrak{g}_{\mathbb{C}}$  (see also [2] for more details). In particular, by his results, any weight of the weighted Dynkin diagram of  $\mathcal{O}^{G_{\mathbb{C}}}$  is given by 0, 1 or 2 for any complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$ .

In the rest of this subsection, we suppose that  $\mathfrak{g}_{\mathbb{C}}$  is simple. Let  $\phi$  be the highest root of  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Then the minimal nilpotent orbit in  $\mathfrak{g}_{\mathbb{C}}$  can be written by

$$\mathcal{O}_{\min}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot ((\mathfrak{g}_{\mathbb{C}})_{\phi} \setminus \{0\}),$$

where  $(\mathfrak{g}_{\mathbb{C}})_{\phi}$  is the root space of  $\phi$  in  $\mathfrak{g}_{\mathbb{C}}$ . We denote the coroot of  $\phi$  by  $H_{\phi}$ . That is,  $H_{\phi}$  is the unique element in  $\mathfrak{h}$  with

$$\alpha(H_{\phi}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle} \quad \text{for any } \alpha \in \mathfrak{h}^*,$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathfrak{h}^*$  induced by the Killing form  $B_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$ . Since  $\phi$  is dominant,  $H_{\phi}$  is in  $\mathfrak{h}_+$ . Furthermore,  $H_{\phi}$  is the hyperbolic element corresponding to  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  since we can find  $X_{\phi} \in \mathfrak{g}_{\phi}$ ,  $Y_{\phi} \in \mathfrak{g}_{-\phi}$  such that  $(H_{\phi}, X_{\phi}, Y_{\phi})$  is an  $\mathfrak{sl}_2$ -triple. Therefore, the weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  is

$$\Psi_{H_{\phi}} : \Pi \rightarrow \mathbb{R}_{\geq 0}, \quad \alpha \mapsto \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle}. \tag{2.1}$$

In particular, for the cases where  $\text{rank } \mathfrak{g}_{\mathbb{C}} \geq 2$  i.e.  $\mathfrak{g}_{\mathbb{C}}$  is not isomorphic to  $\mathfrak{sl}(2, \mathbb{C})$ , we observe that the weight on  $\alpha$  of the weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  is 1 [resp. 0] if and only if the nodes  $\alpha$  and  $-\phi$  are connected [resp. disconnected] by some edges in the extended Dynkin diagram of  $\mathfrak{g}_{\mathbb{C}}$ . The weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  for each simple  $\mathfrak{g}_{\mathbb{C}}$  can be found in [6, Chapter 5.4 and 8.4] (see also Table 2 in Section 4).

## 2.2. Highest roots of restricted root systems.

In this subsection we recall some well-known facts, which will be used for proofs of Theorems 1.1 and 1.3, for a highest root of a restricted root system of real semisimple Lie algebra without proof.

Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex simple Lie algebra and  $\mathfrak{g}$  a non-compact real form of  $\mathfrak{g}$  with a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . We fix a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$ , which is called a *maximally split abelian subspace* of  $\mathfrak{g}$ , and write  $\Sigma(\mathfrak{g}, \mathfrak{a})$  for the restricted root system for  $(\mathfrak{g}, \mathfrak{a})$ . For each restricted root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ , we denote by  $A_{\xi} \in \mathfrak{a}$  the coroot of  $\xi$ .

Then the lemma below holds:

**Lemma 2.1.** *For any restricted root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  and any non-zero root vector  $X_{\xi}$  in  $\mathfrak{g}_{\xi}$ , there exists  $Y_{\xi} \in \mathfrak{g}_{-\xi}$  such that  $(A_{\xi}, X_{\xi}, Y_{\xi})$  is an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ .*

We fix an ordering on  $\mathfrak{a}$  and write  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  for the positive system of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  corresponding to the ordering on  $\mathfrak{a}$ . We denote by  $\lambda$  the highest root of  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  with respect to the ordering on  $\mathfrak{a}$ . Next lemma claims that the highest root  $\lambda$  depends only on the positive system  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  but not on the ordering on  $\mathfrak{a}$ :

**Lemma 2.2.** *The highest root  $\lambda$  of  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  is the unique dominant longest root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ .*

The following lemma gives a characterization of the highest root  $\lambda$  of  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ :

**Lemma 2.3.** *Let  $\xi$  be a root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . If  $\xi$  is not the highest root, then for any non-zero root vector  $X_{\xi}$  in  $\mathfrak{g}_{\xi}$ , there exists a positive root  $\eta$  in  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  and a root vector  $X_{\eta} \in \mathfrak{g}_{\eta}$  such that  $[X_{\xi}, X_{\eta}] \neq 0$ . In particular,  $\xi$  is the highest root if and only if  $\xi + \eta \in \mathfrak{a}^*$  is not in  $\Sigma(\mathfrak{g}, \mathfrak{a}) \cup \{0\}$  for any  $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ .*

## 3. Proofs of Theorem 1.1 and Theorem 1.3

We consider the same setting as in Section 2, and fix connected Lie groups  $G_{\mathbb{C}}$  and  $G$  with its Lie algebras  $\mathfrak{g}_{\mathbb{C}}$  and  $\mathfrak{g}$ , respectively.

In this section, we give proofs of Theorems 1.1 and 1.3. To this, we prove the next two lemmas:

**Lemma 3.1.** *Let  $\mathcal{O}'_0$  be a non-zero real nilpotent orbit in  $\mathfrak{g}$ . Then there exists a non-zero highest root vector  $X_{\lambda}$  in  $\mathfrak{g}_{\lambda}$  such that  $X_{\lambda}$  is contained in the closure of  $\mathcal{O}'_0$ .*

**Lemma 3.2.** *For any two non-zero highest root vectors  $X_\lambda, X'_\lambda$  in  $\mathfrak{g}_\lambda$ , there exists  $g \in G_{\mathbb{C}}$  such that  $gX_\lambda = X'_\lambda$ .*

Theorems 1.1 and 1.3 follow from Lemmas 3.1 and 3.2 immediately.

We also remark that Lemma 3.1 implies the next proposition, which will be used in Section 6 to prove Theorem 1.4.

**Proposition 3.3.** *Any  $G$ -orbit in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  meets  $\mathfrak{g}_\lambda \setminus \{0\}$ .*

Let us give proofs of Lemmas 3.1 and 3.2 as follows.

**Proof of Lemma 3.1.** Let us put  $\mathfrak{m} = Z_{\mathfrak{t}}(\mathfrak{a})$ . Then  $\mathfrak{g}$  can be decomposed as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\xi.$$

For each  $X' \in \mathfrak{g}$ , we denote by

$$X' = X'_m + X'_a + \sum_{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})} X'_\xi \quad (X'_m \in \mathfrak{m}, X'_a \in \mathfrak{a}, X'_\xi \in \mathfrak{g}_\xi),$$

and set

$$\Sigma_{X'} := \{ \xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid X'_\xi \neq 0 \}.$$

Here we remark that if  $X'$  is a non-zero nilpotent element in  $\mathfrak{g}$ , then the set  $\Sigma_{X'}$  is not empty. As a first step of the proof, we shall prove that for a non-zero nilpotent element  $X'$  in  $\mathfrak{g}$  and  $\lambda' \in \Sigma_{X'}$ , if  $\lambda'$  is the longest in  $\Sigma_{X'}$ , then the root vector  $X'_{\lambda'}$  is in  $\overline{\mathcal{O}'_0}$  where  $\mathcal{O}'_0$  is the real nilpotent orbit in  $\mathfrak{g}$  through  $X'$ . Let us take  $A' \in \mathfrak{a}$  satisfying that

$$0 < \lambda'(A') \text{ and } \xi(A') < \lambda'(A') \text{ for any } \xi \in \Sigma_{X'} \setminus \{\lambda'\}.$$

Note that such  $A'$  exists since  $\lambda'$  is the longest in  $\Sigma_{X'}$ . Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(\text{ad}_{\mathfrak{g}} kA')X' \text{ for each } k \in \mathbb{N}.$$

Then  $X'_k$  is in  $\overline{\mathcal{O}'_0}$  for each  $k$  since  $\overline{\mathcal{O}'_0}$  is stable by positive scalars. Furthermore, since

$$\lim_{k \rightarrow \infty} X'_k = \lim_{k \rightarrow \infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A') - \lambda'(A'))} X'_\xi = X'_{\lambda'},$$

we obtain that  $X'_{\lambda'}$  is in  $\overline{\mathcal{O}'_0}$ . To complete the proof, we only need to show that there exists  $X' \in \overline{\mathcal{O}'_0}$  such that  $\lambda \in \Sigma_{X'}$  since  $\lambda$  is the longest in  $\Sigma(\mathfrak{g}, \mathfrak{a})$  by Lemma 2.2. Let us put

$$\begin{aligned} \Sigma_{\overline{\mathcal{O}'_0}} &:= \{ \xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \text{there exists } X' \in \overline{\mathcal{O}'_0} \text{ such that } X'_\xi \neq 0 \} \\ &= \bigcup_{X' \in \overline{\mathcal{O}'_0}} \Sigma_{X'}. \end{aligned}$$

Then the Weyl group  $W(\mathfrak{g}, \mathfrak{a})$  acts on  $\Sigma_{\overline{\mathcal{O}'_0}}$  since  $W(\mathfrak{g}, \mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$  and  $K$  acts on  $\overline{\mathcal{O}'_0}$  where  $K$  is the analytic subgroup of  $G$  with respect to  $\mathfrak{k}$ . In particular, we can and do choose a dominant root  $\lambda_0$  which is the longest one in  $\Sigma_{\overline{\mathcal{O}'_0}}$ . Then we can find a root vector  $X'_{\lambda_0}$  in  $\mathfrak{g}_{\lambda_0} \cap \overline{\mathcal{O}'_0}$  by using the fact proved above. We assume that  $\lambda_0 \neq \lambda$ . Then by Lemma 2.3, we can find  $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$  and  $X_\eta \in \mathfrak{g}_\eta$  such that  $[X'_{\lambda_0}, X_\eta] \neq 0$ . Since  $\lambda_0$  is dominant and  $\eta$  is a positive root, a root  $\lambda_0 + \eta$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is more longer than  $\lambda_0$ . Furthermore, by considering  $X'' := \exp(\text{ad}_{\mathfrak{g}} X_\eta)X'_{\lambda_0} \in \overline{\mathcal{O}'_0}$ , we have

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{\mathcal{O}'_0}}.$$

This contradicts the definition of  $\lambda_0$ . Thus  $\lambda_0 = \lambda$ . ■

**Proof of Lemma 3.2.** Fix non-zero highest root vectors  $X_\lambda$  and  $X'_\lambda$ . Let  $A_\lambda$  the coroot of  $\lambda$  in  $\mathfrak{a}$ . Then by Lemma 2.1, we can find  $Y_\lambda$  and  $Y'_\lambda$  in  $\mathfrak{g}_{\mathbb{C}}$  such that  $(A_\lambda, X_\lambda, Y_\lambda)$  and  $(A_\lambda, X'_\lambda, Y'_\lambda)$  are  $\mathfrak{sl}_2$ -triples in  $\mathfrak{g}_{\mathbb{C}}$ , respectively. Thus by applying Malcev's theorem in [18] there exists  $g \in G_{\mathbb{C}}$  such that  $gX_\lambda = X'_\lambda$ . ■

#### 4. Complex nilpotent orbits and real forms

Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex simple Lie algebra and  $\mathfrak{g}$  a non-compact real form of  $\mathfrak{g}_{\mathbb{C}}$ . In this section, we will give a necessary and sufficient condition of  $\mathfrak{g}$  for  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} = \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  including the first claim of Theorem 1.2.

We fix  $G, G_{\mathbb{C}}$  for the connected Lie group with its Lie algebra  $\mathfrak{g}, \mathfrak{g}_{\mathbb{C}}$ , respectively. Let  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  be a Cartan decomposition of  $\mathfrak{g}$ . We fix a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  and its ordering. Let  $\lambda$  be the highest root of the restricted root system  $\Sigma(\mathfrak{g}, \mathfrak{a})$  for  $(\mathfrak{g}, \mathfrak{a})$  with respect to the ordering on  $\mathfrak{a}$ . Then by Theorem 1.1, which was already proved in Section 3, the complex nilpotent orbit

$$\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot (\mathfrak{g}_\lambda \setminus \{0\})$$

is the minimum in  $(\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}) \setminus \{[0]\}$ .

We extend  $\mathfrak{a}$  and its ordering to a Cartan subalgebra  $\mathfrak{h}_{\mathbb{C}}$  of  $\mathfrak{g}_{\mathbb{C}}$  and an ordering on it. Let  $\phi$  be the highest root of the root system  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  with respect to the ordering on  $\mathfrak{h}_{\mathbb{C}}$ . We recall that the complex nilpotent orbit

$$\mathcal{O}_{\min}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot ((\mathfrak{g}_{\mathbb{C}})_\phi \setminus \{0\})$$

is the minimum in  $(\mathcal{N}/G_{\mathbb{C}}) \setminus \{[0]\}$ .

Then the next proposition, including the first claim of Theorem 1.2, holds:

**Proposition 4.1.** *The following conditions on a non-compact real simple Lie algebra  $\mathfrak{g}$  without complex structure are equivalent:*

- (i)  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .
- (ii)  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} = \emptyset$ .

- (iii)  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{p}_{\mathbb{C}} = \emptyset$  where  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$  is the complexification of a Cartan decomposition of  $\mathfrak{g}$ .
- (iv)  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ .
- (v) The highest root  $\phi$  of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  defined above is not a real root.
- (vi) The weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  does not match the Satake diagram of  $\mathfrak{g}$  (see Section 4 for the notation).
- (vii) There exists a node  $\alpha$  of Dynkin diagram of  $\mathfrak{g}_{\mathbb{C}}$  such that  $\alpha$  is black in the Satake diagram of  $\mathfrak{g}$  and has some edges connected to the added node in the extended Dynkin diagram of  $\mathfrak{g}_{\mathbb{C}}$ .
- (viii) There exists an infinite-dimensional (non-holomorphic) irreducible  $(\mathfrak{g}_{\mathbb{C}}, G_U)$ -module  $X$  such that  $X$  is discretely decomposable as a  $(\mathfrak{g}, K)$ -module, where  $G_U$  is a connected compact real form of  $G_{\mathbb{C}}$  (See [15, Section 1.2] for the definition of the discrete decomposability).
- (ix) There exists an infinite-dimensional (non-holomorphic) irreducible  $(\mathfrak{g}_{\mathbb{C}}, G_U)$ -module  $X$  such that  $X$  is discretely decomposable as a  $(\mathfrak{k}_{\mathbb{C}}, K)$ -module, where  $G_U$  is a connected compact real form of  $G_{\mathbb{C}}$ .
- (x)  $\text{pr}_{\mathfrak{p}_{\mathbb{C}}}(\mathcal{O}_{\min}^{G_{\mathbb{C}}})$  is contained in the nilpotent cone  $\mathcal{N}(\mathfrak{p}_{\mathbb{C}}) := \mathcal{N} \cap \mathfrak{p}_{\mathbb{C}}$  in  $\mathfrak{p}_{\mathbb{C}}$ , where  $\text{pr}_{\mathfrak{p}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{p}_{\mathbb{C}}$  denotes the projection with respect to the decomposition  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ .
- (xi)  $\text{pr}_{\mathfrak{k}_{\mathbb{C}}}(\mathcal{O}_{\min}^{G_{\mathbb{C}}})$  is contained in the nilpotent cone  $\mathcal{N} \cap \mathfrak{k}_{\mathbb{C}}$  in  $\mathfrak{k}_{\mathbb{C}}$ , where  $\text{pr}_{\mathfrak{k}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{k}_{\mathbb{C}}$  denotes the projection with respect to the decomposition  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}_{\mathbb{C}}$ .
- (xii)  $\mathfrak{g}$  is isomorphic to one of the following simple Lie algebras

$$\mathfrak{su}^*(2k), \mathfrak{so}(n, 1), \mathfrak{sp}(p, q), \mathfrak{e}_{6(-26)} \text{ and } \mathfrak{f}_{4(-20)},$$

for  $k \geq 2$ ,  $n \geq 5$  and  $p, q \geq 1$ .

The equivalences among (i), (ii) and (iii) follow from the definition of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  and [23, Proposition 1.11].

By the list of the Satake diagrams of non-compact real simple Lie algebras (see also Table 2 for the Satake diagram of each  $\mathfrak{g}$ ) and the extended Dynkin diagrams of complex simple Lie algebras, one can easily check the equivalence (vii)  $\Leftrightarrow$  (xii).

The equivalences among (v), (viii), (x), (xi) and (xii) were proved by [16, Lemma 4.6 and Theorem 5.2] in a more general setting. In particular, the equivalences (v)  $\Leftrightarrow$  (viii)  $\Leftrightarrow$  (x) and (v)  $\Leftrightarrow$  (ix)  $\Leftrightarrow$  (xi) can be obtained by applying their results for the symmetric pairs  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g})$  and  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ , respectively (see also [16, Remark 4.5] for the discrete decomposability of a representation of  $G$  with respect to a symmetric pair  $(G, H)$  and its associated pair  $(G, H^a)$ ).

In this section, we give a proof of the remaining equivalences, namely, the equivalences among (ii), (iv), (v), (vi) and (vii).

Note that the equivalence (ii)  $\Leftrightarrow$  (v), which will be proved in Section 4 of this paper, is used in a proof of [16, Corollary 5.9].

**Remark 4.2.** The equivalences among (ii), (xi) and (xii) in Proposition 4.1 were stated on Brylinski’s paper [4] without proof. It should be noted that Brylinski [4] also claimed that the following condition on  $\mathfrak{g}$  is also equivalent to the condition (ii):

- $K_{\mathbb{C}}$  has a Zariski open orbit in  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ , where  $K_{\mathbb{C}}$  is the analytic subgroup of  $G_{\mathbb{C}}$  with  $\text{Lie } K_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}}$ .

**4.1. Satake diagrams and weighted Dynkin diagrams of complex nilpotent orbits.**

In order to explain the notation in (vi), we first recall the definition of the Satake diagram of a real form  $\mathfrak{g}$  of  $\mathfrak{g}_{\mathbb{C}}$  briefly. All facts which will be used for the definition of the Satake diagrams can be found in [1] or [21]. Throughout this subsection,  $\mathfrak{g}_{\mathbb{C}}$  can be a general complex semisimple Lie algebra and  $\mathfrak{g}$  a general real form of  $\mathfrak{g}_{\mathbb{C}}$ .

We fix a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  of  $\mathfrak{g}$ . Take a maximal abelian subspace  $\mathfrak{a}$  in  $\mathfrak{p}$ , and extend it to a maximal abelian subspace  $\mathfrak{h} = \sqrt{-1}\mathfrak{k} + \mathfrak{a}$  in  $\sqrt{-1}\mathfrak{k} + \mathfrak{p}$ . Then the complexification, denoted by  $\mathfrak{h}_{\mathbb{C}}$ , of  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}_{\mathbb{C}}$ , and  $\mathfrak{h}$  coincides with the real form

$$\{X \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(X) \in \mathbb{R} \text{ for any } \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\}$$

of  $\mathfrak{h}_{\mathbb{C}}$  where  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  is the reduced root system for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Let us denote by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\} \subset \mathfrak{a}^*$$

the restricted root system for  $(\mathfrak{g}, \mathfrak{a})$ . We will denote by  $W(\mathfrak{g}, \mathfrak{a})$ ,  $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  the Weyl group of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ ,  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ , respectively. Fix an ordering on  $\mathfrak{a}$  and extend it to an ordering on  $\mathfrak{h}$ . We write  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ ,  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  for the positive system of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ ,  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  corresponding to the ordering on  $\mathfrak{a}$ ,  $\mathfrak{h}$ , respectively. Then  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$  can be written by

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\}.$$

We denote by  $\Pi$  the fundamental system of  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Then

$$\bar{\Pi} = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Pi\} \setminus \{0\}$$

becomes the simple system of  $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ . Let us denote by  $\Pi_0$  the set of all simple roots in  $\Pi$  whose restrictions to  $\mathfrak{a}$  are zero.

The Satake diagram  $S$  of  $\mathfrak{g}$  consists of the following three data: the Dynkin diagram of  $\mathfrak{g}_{\mathbb{C}}$  with nodes  $\Pi$ , black nodes  $\Pi_0$  in  $S$ , and arrows joining  $\alpha \in \Pi \setminus \Pi_0$  and  $\beta \in \Pi \setminus \Pi_0$  in  $S$  whose restrictions to  $\mathfrak{a}$  are the same.

Second, we define the relation “match” between a weighted Dynkin diagram and a Satake diagram as follows:

**Definition 4.3** ([20, Definition 7.3]). Let  $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$  be a weighted Dynkin diagram (see Section 2 for the definition) and  $S$  the Satake diagram of  $\mathfrak{g}$  with nodes  $\Pi$  defined above. We say that  $\Psi_H$  *matches*  $S$  if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.

**Remark 4.4.** The concept of “match” appeared earlier in Djokovic [7] (weighted Satake diagrams) and Sekiguchi [22, Proposition 1.16].

The following two facts were proved in [20]. In particular, by using Theorem 4.6, one can easily check whether or not a given complex nilpotent orbit meets a given real form.

**Theorem 4.5.** *The bijection  $\Psi$  between  $\mathfrak{h}$  and  $\text{Map}(\Pi, \mathbb{R})$  defined in Section 2 induces a bijection below:*

$$\mathfrak{a} \xleftrightarrow{1:1} \{ \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S \}.$$

**Theorem 4.6** ([20, Proposition 7.8 and Theorem 7.10]). *Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex semisimple Lie algebra and  $\mathfrak{g}$  a real form of  $\mathfrak{g}_{\mathbb{C}}$ . For a complex nilpotent orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  in  $\mathfrak{g}_{\mathbb{C}}$ , the following conditions are equivalent:*

- (i) *The orbit  $\mathcal{O}^{G_{\mathbb{C}}}$  meets  $\mathfrak{g}$ .*
- (ii) *The hyperbolic element  $H_{\mathcal{O}}$  corresponding to  $\mathcal{O}^{G_{\mathbb{C}}}$  is in  $\mathfrak{a}$  (see Section 2 for the notation).*
- (iii) *The weighted Dynkin diagram of  $\mathcal{O}^{G_{\mathbb{C}}}$  matches the Satake diagram of  $\mathfrak{g}$  (see Section 4 for the notation).*

We give examples for Theorem 4.6 as follows:

**Example 4.7.** If  $\mathfrak{g}$  is a split real form of  $\mathfrak{g}_{\mathbb{C}}$ , then all nodes of the Satake diagram of  $\mathfrak{g}$  are white with no arrows. Thus all complex nilpotent orbits in  $\mathfrak{g}_{\mathbb{C}}$  meet  $\mathfrak{g}$  since all weighted Dynkin diagram matches the Satake diagram of  $\mathfrak{g}$ .

**Example 4.8.** If  $\mathfrak{u}$  is a compact real form of  $\mathfrak{g}_{\mathbb{C}}$ , then all nodes of the Satake diagram of  $\mathfrak{u}$  are black. Thus any non-zero complex nilpotent orbit in  $\mathfrak{g}_{\mathbb{C}}$  does not meet  $\mathfrak{u}$  since any non-zero weighted Dynkin diagram does not match the Satake diagram of  $\mathfrak{u}$ .

By the list of the weighted Dynkin diagrams of the minimal nilpotent orbit  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  for simple  $\mathfrak{g}_{\mathbb{C}}$  (cf. [6, Chapter 5.4 and 8.4]) and the list of the Satake diagrams of non-compact real forms  $\mathfrak{g}$ , one can easily check that  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  meets  $\mathfrak{g}$  or not as follows:

**Example 4.9.** In Table 2, we check whether or not the minimal nilpotent orbit  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  in a complex simple Lie algebra  $\mathfrak{g}_{\mathbb{C}}$  meets a non-compact real form  $\mathfrak{g}$ . Here

$n \geq 2, k \geq 1, m \geq 1$  and  $p \geq 1$  in Table 2.

Table 2: List of the weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  and the Satake diagram of  $\mathfrak{g}$ .

$\mathfrak{g}$	Weighted Dynkin diagram of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ on the Satake diagram of $\mathfrak{g}$	$\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$ ?
$\mathfrak{sl}(n, \mathbb{R})$		Yes
$\mathfrak{su}^*(2k)$		No
$\mathfrak{su}(n-p, p)$		Yes
$\mathfrak{su}(k, k)$		Yes
$\mathfrak{so}(2k, 1)$		No
$\mathfrak{so}(2k+1-p, p)$ $2 \leq p \leq k-1$		Yes
$\mathfrak{so}(k+1, k)$		Yes
$\mathfrak{sp}(k, \mathbb{R})$		Yes
$\mathfrak{sp}(k-p, p)$		No
$\mathfrak{sp}(m, m)$		No
$\mathfrak{so}(2k-1, 1)$		No

$\mathfrak{so}(2k - p, p)$ $2 \leq p \leq k - 2$		Yes
$\mathfrak{so}(k + 1, k - 1)$		Yes
$\mathfrak{so}(k, k)$		Yes
$\mathfrak{so}^*(4m)$		Yes
$\mathfrak{so}^*(4m + 2)$		Yes
$\mathfrak{e}_{6(6)}$		Yes
$\mathfrak{e}_{6(2)}$		Yes
$\mathfrak{e}_{6(-14)}$		Yes

$\mathfrak{e}_{6(-26)}$	$\begin{array}{ccccccccc} & 0 & 0 & 0 & 0 & 0 & & & \\ & \circ & \bullet & \bullet & \bullet & \circ & & & \\ & & &   & & & & & \\ & & & \bullet & 1 & & & & \end{array}$	No
$\mathfrak{e}_{7(7)}$	$\begin{array}{ccccccccc} & 0 & 0 & 0 & 0 & 0 & 1 & & \\ & \circ & \circ & \circ & \circ & \circ & \circ & & \\ & & & &   & & & & \\ & & & & \circ & 0 & & & \end{array}$	Yes
$\mathfrak{e}_{7(-5)}$	$\begin{array}{ccccccccc} & 0 & 0 & 0 & 0 & 0 & 1 & & \\ & \bullet & \circ & \bullet & \circ & \circ & \circ & & \\ & & & &   & & & & \\ & & & & \bullet & 0 & & & \end{array}$	Yes
$\mathfrak{e}_{7(-25)}$	$\begin{array}{ccccccccc} & 0 & 0 & 0 & 0 & 0 & 1 & & \\ & \circ & \circ & \bullet & \bullet & \bullet & \circ & & \\ & & & &   & & & & \\ & & & & \bullet & 0 & & & \end{array}$	Yes
$\mathfrak{e}_{8(8)}$	$\begin{array}{ccccccccc} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \circ & \\ & & & & &   & & & \\ & & & & & \circ & 0 & & \end{array}$	Yes
$\mathfrak{e}_{8(-24)}$	$\begin{array}{ccccccccc} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & \circ & \circ & \circ & \bullet & \bullet & \bullet & \circ & \\ & & & & &   & & & \\ & & & & & \bullet & 0 & & \end{array}$	Yes
$\mathfrak{f}_{4(4)}$	$\begin{array}{cccc} & 1 & 0 & 0 & 0 \\ & \circ & \circ & \circ & \circ \\ & & \Rightarrow & \Rightarrow & \end{array}$	Yes
$\mathfrak{f}_{4(-20)}$	$\begin{array}{cccc} & 1 & 0 & 0 & 0 \\ & \bullet & \bullet & \bullet & \circ \\ & & \Rightarrow & \Rightarrow & \end{array}$	No
$\mathfrak{g}_{2(2)}$	$\begin{array}{cc} & 1 & 0 \\ & \circ & \circ \\ & \Rightarrow & \Rightarrow \end{array}$	Yes

### 4.2. Proof of Proposition 4.1.

We consider the same setting as in Section 4 and suppose that  $\mathfrak{g}_{\mathbb{C}}$  is simple and  $\mathfrak{g}$  is not compact. In Proposition 4.1, the equivalence between (ii) and (vi) is obtained by Theorem 4.6. In this subsection, we complete a proof of Proposition 4.1 by proving the equivalence among (ii), (iv), (v) and (vii).

**Proof of the equivalence between (iv) and (v) in Proposition 4.1.** Recall that

$$\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = \#\{ \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid \alpha|_{\mathfrak{a}} = \lambda \}.$$

If  $\phi$  is a real root, then for each root  $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  except for  $\phi$ , we have  $\alpha|_{\mathfrak{a}} \neq \lambda (= \phi|_{\mathfrak{a}})$  since  $\phi$  is the longest root of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Thus  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1$  in this case. Conversely, we assume that  $\phi$  is not a real root. Let us denote by  $\tau$  the anti  $\mathbb{C}$ -linear involution corresponding to  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ . That is,  $\tau$  is the complex

conjugation of  $\mathfrak{g}_{\mathbb{C}}$  with respect to its real form  $\mathfrak{g}$ . Then  $\tau$  induces the involution  $\tau^*$  on  $\mathfrak{h}^*$ , and it preserves  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Since  $\phi|_{\sqrt{-1}\mathfrak{t}} \neq 0$ , we obtain that  $\tau^*\phi \neq \phi$  and  $(\tau^*\phi)|_{\mathfrak{a}} = \phi|_{\mathfrak{a}} = \lambda$ . Hence,  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ . ■

**Proof of the equivalence between (ii) and (v) in Proposition 4.1.** Recall that  $H_{\phi} \in \mathfrak{h}$  is the hyperbolic element corresponding to  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  (see Section 2 for the notation). Thus by Theorem 4.6,  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  meets  $\mathfrak{g}$  if and only if  $H_{\phi}$  is in  $\mathfrak{a}$ . By the definition of  $H_{\phi}$ , the highest root  $\phi$  is real if and only if  $H_{\phi}$  is in  $\mathfrak{a}$ . This completes the proof. ■

**Proof of the equivalence between (vi) and (vii) in Proposition 4.1.** In the case where  $\mathfrak{g}_{\mathbb{C}} \simeq \mathfrak{sl}(2, \mathbb{C})$ , our non-compact real form  $\mathfrak{g}$  must be isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ . Then our claim holds since the Satake diagram of  $\mathfrak{sl}(2, \mathbb{C})$  has no black node and matches any weighted Dynkin diagram. Let us consider the cases where  $\text{rank } \mathfrak{g}_{\mathbb{C}} \geq 2$ . In these case, as we observed in the last of Section 2, for a simple root  $\alpha \in \Pi$ , the weight on  $\alpha$  for the weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  is 1 [resp. 0] if  $\alpha$  has some edges [resp. no edge] connected to the node  $-\phi$  in the extended Dynkin diagram. Then we only need to show that for a pair  $\alpha, \beta \in \Pi$  joined by an arrow on the Satake diagram of  $\mathfrak{g}$ , the node  $\alpha$  has some edges connected to  $-\phi$  if and only if  $\beta$  has some edges connected to  $-\phi$ . By [10, Lemma 2.10], there exists an involution  $\sigma^*$  of  $\mathfrak{h}^*$  such that  $\sigma^*\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ ,  $\sigma^*\Pi = \Pi$  and  $\sigma^*\alpha = \beta$ . Note that  $\sigma^*\phi = \phi$  since  $\phi$  is the unique longest dominant root in  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . Therefore, we have

$$\langle \alpha, -\phi \rangle = \langle \alpha, -\sigma^*\phi \rangle = \langle \sigma^*\alpha, -\phi \rangle = \langle \beta, -\phi \rangle.$$

This completes the proof. ■

### 5. Weighted Dynkin diagrams of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$

Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex simple Lie algebra and  $\mathfrak{g}$  a non-compact real form of  $\mathfrak{g}_{\mathbb{C}}$ . In this section, we determine  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  for each  $\mathfrak{g}$  by describing the weighted Dynkin diagram of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ . Recall that Proposition 4.1 claims that  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} = \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  if and only if  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1$ . Thus our concern is in the cases where  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$  i.e.  $\mathfrak{g}$  is isomorphic to one of  $\mathfrak{su}^*(2k)$ ,  $\mathfrak{so}(n, 1)$ ,  $\mathfrak{sp}(p, q)$ ,  $\mathfrak{e}_{6(-26)}$  or  $\mathfrak{f}_{4(-20)}$ .

We use the same notation as in Section 4 and assume that  $\mathfrak{g}_{\mathbb{C}}$  is simple and  $\mathfrak{g}$  is non-compact. Let us denote by

$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(A) \geq 0 \text{ for any } \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a}) \}.$$

Then  $\mathfrak{a}_+$  is a fundamental domain of  $\mathfrak{a}$  for the action of  $W(\mathfrak{g}, \mathfrak{a})$ . Since

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\},$$

we have  $\mathfrak{a}_+ = \mathfrak{h}_+ \cap \mathfrak{a}$ .

Recall that  $\lambda$  is dominant by Lemma 2.2. Thus the coroot  $A_{\lambda}$  is in  $\mathfrak{a}_+(\subset \mathfrak{h}_+)$ . Therefore,  $A_{\lambda}$  is the hyperbolic element corresponding to  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  since we can find  $X_{\lambda} \in \mathfrak{g}_{\lambda}$ ,  $Y_{\lambda} \in \mathfrak{g}_{-\lambda}$  such that the triple  $(A_{\lambda}, X_{\lambda}, Y_{\lambda})$  is an  $\mathfrak{sl}_2$ -triple

in  $\mathfrak{g}_{\mathbb{C}}$ . Therefore, to determine the weighted Dynkin diagram of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ , we need to compute the weighted Dynkin diagram corresponding to  $A_{\lambda}$ .

Our first purpose of this section is to show the following proposition which gives a formula of  $A_{\lambda}$  by  $H_{\phi}$ , where  $H_{\phi}$  is the hyperbolic element corresponding to  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$  (see Section 2).

**Proposition 5.1.** *We denote by  $\tau$  the anti  $\mathbb{C}$ -linear involution corresponding to  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$ , i.e.  $\tau$  is the complex conjugation of  $\mathfrak{g}_{\mathbb{C}}$  with respect to the real form  $\mathfrak{g}$ . Then*

$$A_{\lambda} = \begin{cases} H_{\phi} & \text{if } \dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1, \\ H_{\phi} + \tau H_{\phi} & \text{if } \dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2. \end{cases}$$

In particular, if  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ , then the weighted Dynkin diagram of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  can be computed by the sum of the weighted Dynkin diagrams corresponding to  $H_{\phi}$ , i.e. the weighted Dynkin diagram of  $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ , and that corresponding to  $\tau H_{\phi}$ .

We compute the weighted Dynkin diagram corresponding to  $A_{\lambda}$  for each  $\mathfrak{g}$  with  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$  in Section 5.

**5.1. Proof of Proposition 5.1.**

Recall that Proposition 4.1 claims that  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1$  if and only if the highest root  $\phi$  of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  is real, i.e.  $\phi|_{\sqrt{-1}\mathfrak{t}} = 0$ . We give a proof of Proposition 5.1 as a sequence of the following two lemmas:

**Lemma 5.2.**

$$A_{\lambda} = \frac{\langle \phi, \phi \rangle}{2\langle \lambda, \lambda \rangle} (H_{\phi} + \tau H_{\phi}),$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $\mathfrak{h}^*$  and on  $\mathfrak{a}^*$  induced by the Killing form  $B_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$ . In particular, if  $\phi$  is a real root, then  $A_{\lambda} = H_{\phi}$ .

**Lemma 5.3.** *Suppose that  $\phi$  is not a real root. Then  $\langle \phi, \phi \rangle = 2\langle \lambda, \lambda \rangle$ .*

**Proof of Lemma 5.2.** We consider  $\mathfrak{h}^*$  as  $\mathfrak{a}^* + \sqrt{-1}\mathfrak{t}^*$ . Then for each  $\xi \in \mathfrak{a}^*$ ,

$$\begin{aligned} \xi\left(\frac{\langle \phi, \phi \rangle}{2\langle \lambda, \lambda \rangle} (H_{\phi} + \tau H_{\phi})\right) &= \frac{\langle \phi, \phi \rangle}{\langle \lambda, \lambda \rangle} \xi(H_{\phi}) \quad (\text{since } \xi(H_{\phi}) = \xi(\tau H_{\phi})) \\ &= \frac{2\langle \xi, \phi \rangle}{\langle \lambda, \lambda \rangle} \quad (\text{by the definition of } H_{\phi}) \\ &= \frac{2\langle \xi, \lambda \rangle}{\langle \lambda, \lambda \rangle} \quad (\text{since } \phi|_{\mathfrak{a}} = \lambda) \\ &= \xi(A_{\lambda}). \end{aligned}$$

This completes the proof. ■

**Proof of Lemma 5.3.** We write  $\tau^*$  for the involution on  $\mathfrak{h}^*$  induced by  $\tau$ . It is enough to show that  $\langle \phi, \tau^* \phi \rangle = 0$  because  $\lambda = \frac{1}{2}(\phi + \tau^* \phi)$ . By [1, Proposition 1.3],  $\tau^*$  is a normal involution of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ , i.e. for each root  $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ , the element

$\alpha - \tau^*\alpha$  is not a root of  $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . In particular, for any root  $\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  with  $\tau^*\alpha \neq \alpha$ , we have  $\langle \alpha, \tau^*\alpha \rangle \leq 0$ . Recall that we are assuming that  $\phi$  is not real. Thus  $\phi \neq \tau^*\phi$ , and then  $\langle \phi, \tau^*\phi \rangle \leq 0$ . The root  $\tau^*\phi$  is in  $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  since the ordering on  $\mathfrak{h}$  is an extension of the ordering on  $\mathfrak{a}$ . Then we also obtain that  $\langle \phi, \tau^*\phi \rangle \geq 0$  since the highest root  $\phi$  is dominant. Therefore,  $\langle \phi, \tau^*\phi \rangle = 0$ . ■

**5.2. Weighted Dynkin diagrams of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ .**

We now determine the weighted Dynkin diagram of  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  for each  $\mathfrak{g}$  with  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ , i.e.  $\mathfrak{g}$  is isomorphic to one of  $\mathfrak{su}^*(2k)$ ,  $\mathfrak{so}(n, 1)$ ,  $\mathfrak{sp}(p, q)$ ,  $\mathfrak{e}_{6(-26)}$  or  $\mathfrak{f}_{4(-20)}$ . By Proposition 5.1, our goal is to compute the weighted Dynkin diagram corresponding to  $A_{\lambda} = H_{\phi} + \tau H_{\phi}$ .

For simplicity, we denote by  $S$  the Satake diagram of  $\mathfrak{g}$ . For each simple root  $\alpha$  in  $\Pi$ , we denote by  $H_{\alpha} \in \mathfrak{h}$  the coroot of  $\alpha$ .

Then the next lemma holds:

**Lemma 5.4.** *The set*

$\{ H_{\alpha} \mid \alpha \text{ is black in } S \} \sqcup \{ H_{\alpha} - H_{\beta} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S \}$   
*becomes a basis of  $\sqrt{-1}\mathfrak{t}$ .*

**Proof.** We denote by

$$\Omega = \{ H_{\alpha} \mid \alpha \text{ is black in } S \} \sqcup \{ H_{\alpha} - H_{\beta} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S \}.$$

It is known that there is no triple  $\{\alpha, \beta, \gamma\}$  in  $\Pi \setminus \Pi_0$  such that  $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}} = \gamma|_{\mathfrak{a}}$  (this fact can be found in [1, Section 2.8]). Thus  $\Omega$  is linearly independent and

$$\sharp\Omega = \sharp\Pi - \sharp\bar{\Pi}.$$

Recall that  $\bar{\Pi}$  is a simple system of the restricted root system  $\Sigma(\mathfrak{g}, \mathfrak{a})$ , we have  $\dim_{\mathbb{R}} \mathfrak{a} = \sharp\bar{\Pi}$ . Since  $\dim_{\mathbb{R}} \mathfrak{h} = \sharp\Pi$  and  $\sqrt{-1}\mathfrak{t}$  is the orthogonal complement space of  $\mathfrak{a}$  in  $\mathfrak{h}$  for the Killing form  $B_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$ , it remains to prove that

$$B_{\mathbb{C}}(H', A) = 0 \quad \text{for any } H' \in \Omega, A \in \mathfrak{a}.$$

Let us take  $\alpha \in \Pi_0$ , i.e.  $\alpha$  is a black node in  $S$ . Since  $\alpha|_{\mathfrak{a}} = 0$ , we have

$$B_{\mathbb{C}}(H_{\alpha}, A) = \frac{2\alpha(A)}{\langle \alpha, \alpha \rangle} = 0 \quad \text{for any } A \in \mathfrak{a}.$$

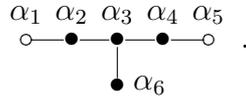
Furthermore, by [10, Lemma 2.10], there exists an involution  $\sigma^*$  of  $\mathfrak{h}^*$  such that  $\sigma^*\alpha = \beta$  for all pair  $\alpha, \beta \in \Pi \setminus \Pi_0$  such that  $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}$ , i.e.  $\alpha$  and  $\beta$  are joined by an arrow in  $S$ . In particular  $|\alpha| = |\beta|$  for such pair. Thus for any  $A \in \mathfrak{a}$ , we have

$$\begin{aligned} B_{\mathbb{C}}(H_{\alpha} - H_{\beta}, A) &= \frac{2\alpha(A)}{\langle \alpha, \alpha \rangle} - \frac{2\beta(A)}{\langle \beta, \beta \rangle} \\ &= 0 \quad (\text{since } \alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}} \text{ and } |\alpha| = |\beta|). \end{aligned}$$

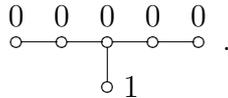
This completes the proof. ■

By using Lemma 5.4, we shall compute the weighted Dynkin diagram corresponding to  $A_\lambda = H_\phi + \tau H_\phi$ . In this paper, we only give the computation for the case  $\mathfrak{g} = \mathfrak{e}_{6(-26)}$  below. For the other  $\mathfrak{g}$  with  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ , we can compute the weighted Dynkin diagram corresponding to  $A_\lambda$  by the same way.

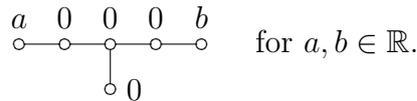
**Example 5.5.** Let  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}) = (\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{e}_{6(-26)})$ . We denote the Satake diagram of  $\mathfrak{e}_{6(-26)}$  by



By Table 2, the weighted Dynkin diagram corresponding to  $H_\phi$  is



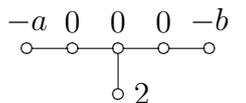
We now compute the weighted Dynkin diagram corresponding to  $A_\lambda = H_\phi + \tau H_\phi$ . By Theorem 4.5, the weighted Dynkin diagram corresponding to  $A_\lambda$  matches the Satake diagram of  $\mathfrak{e}_{6(-26)}$ . Thus we can put the weighted Dynkin diagram corresponding to  $A_\lambda$  as



To determine  $a, b \in \mathbb{R}$ , we also put

$$H_\phi^{\text{im}} := H_\phi - \tau H_\phi \in \sqrt{-1}\mathfrak{t} .$$

Since  $A_\lambda + H_\phi^{\text{im}} = 2H_\phi$ , the weighted Dynkin diagram corresponding to  $H_\phi^{\text{im}}$  can be written by



That is, we have

$$\begin{aligned} \alpha_1(H_\phi^{\text{im}}) &= -a, \\ \alpha_2(H_\phi^{\text{im}}) &= \alpha_3(H_\phi^{\text{im}}) = \alpha_4(H_\phi^{\text{im}}) = 0, \\ \alpha_5(H_\phi^{\text{im}}) &= -b, \\ \alpha_6(H_\phi^{\text{im}}) &= 2. \end{aligned}$$

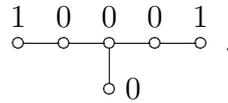
By Lemma 5.4, the set  $\{H_{\alpha_2}, H_{\alpha_3}, H_{\alpha_4}, H_{\alpha_6}\}$  becomes a basis of  $\sqrt{-1}\mathfrak{t}$ . Thus  $H_\phi^{\text{im}} \in \sqrt{-1}\mathfrak{t}$  can be written by

$$H_\phi^{\text{im}} = c_2 H_{\alpha_2} + c_3 H_{\alpha_3} + c_4 H_{\alpha_4} + c_6 H_{\alpha_6} \quad \text{for } c_2, c_3, c_4, c_6 \in \mathbb{R} .$$

Since  $\alpha_i(H_{\alpha_j}) = 2\langle\alpha_i, \alpha_j\rangle/\langle\alpha_j, \alpha_j\rangle$ , by comparing with the Dynkin diagram of  $\mathfrak{e}_{6,\mathbb{C}}$ , we obtain

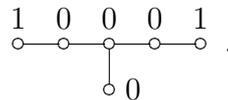
$$\begin{aligned} \alpha_1(H_\phi^{\text{im}}) &= -c_2, \\ \alpha_2(H_\phi^{\text{im}}) &= 2c_2 - c_3, \\ \alpha_3(H_\phi^{\text{im}}) &= -c_2 + 2c_3 - c_4 - c_6, \\ \alpha_4(H_\phi^{\text{im}}) &= -c_3 + 2c_4, \\ \alpha_5(H_\phi^{\text{im}}) &= -c_4, \\ \alpha_6(H_\phi^{\text{im}}) &= -c_3 + 2c_6. \end{aligned}$$

Hence  $a = b = 1$ . Therefore, the weighted Dynkin diagram of  $\mathcal{O}_{\min,\mathfrak{g}}^{G_{\mathbb{C}}}$  for  $\mathfrak{g} = \mathfrak{e}_{6(-26)}$  is



The result of our computations for all  $\mathfrak{g}$  with  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$  is in Table 1 in Section 1.

**Remark 5.6.** The weighted Dynkin diagram of  $\mathcal{O}_{\min,\mathfrak{g}}^{G_{\mathbb{C}}}$  for each  $\mathfrak{g}$  with  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$  can be determined by the classification result of real nilpotent orbits in  $\mathfrak{g}$ . For example, let us consider the case where  $\mathfrak{g} = \mathfrak{e}_{6(-26)}$  as follows. Djokovic [8] proved that there exist only two non-trivial real nilpotent orbits in  $\mathfrak{e}_{6(-26)}$ . The list of real nilpotent orbits in  $\mathfrak{e}_{6(-26)}$  can be found in the table in [6, Chapter 9.6] and the weighted Dynkin diagram of the complexification of each orbit is described in the first column of the table. Recall that the real dimension of a real nilpotent orbit and the complex dimension of its complexification are the same. In a table in [6, Chapter 8.4], the complex dimensions of the complex nilpotent orbits in  $\mathfrak{e}_{6,\mathbb{C}}$  corresponding to the label 1 and 2 can be found as 32 and 48, respectively. Therefore,  $\mathfrak{e}_{6(-26)}$  has a two real nilpotent orbits  $\mathcal{O}_1$  and  $\mathcal{O}_2$  with  $\dim_{\mathbb{R}} \mathcal{O}_1 = 32$  and  $\dim_{\mathbb{R}} \mathcal{O}_2 = 48$ , respectively. In particular,  $\mathfrak{e}_{6(-26)}$  has the unique real nilpotent orbit  $\mathcal{O}_1$  with the minimal positive dimension, and the weighted Dynkin diagram of its complexification is



Furthermore, by the Hasse diagram of complex nilpotent orbits in  $\mathfrak{e}_{6,\mathbb{C}}$ , which can be found in [5, §13.4] we can observe that the complexification of  $\mathcal{O}_1$  is contained in the closure of the complexification of  $\mathcal{O}_2$ . Thus the complexification of  $\mathcal{O}_1$  is minimal in  $\mathcal{N}_{\mathfrak{e}_{6(-26)}/G_{\mathbb{C}}}$  except for the zero-orbit, and hence, the complexification of  $\mathcal{O}_1$  is our  $\mathcal{O}_{\min,\mathfrak{g}}^{G_{\mathbb{C}}}$  in this case.

**6.  $G$ -orbits in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$**

Let  $\mathfrak{g}_{\mathbb{C}}$  be a complex simple Lie algebra and  $\mathfrak{g}$  a non-compact real form of  $\mathfrak{g}_{\mathbb{C}}$ . A proof of Theorem 1.4 is given in this section.

Throughout this section, we take  $G$  for the connected linear Lie group with its Lie algebra  $\mathfrak{g}$  and  $G_{\mathbb{C}}$  the complexification of  $G$ . We also fix a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$  of  $\mathfrak{g}$ , and write  $K$  for the maximal compact subgroup of  $G$  with its Lie algebra  $\mathfrak{k}$ . We take a maximal abelian subspace  $\mathfrak{a}$  of  $\mathfrak{p}$  and fix an ordering on  $\mathfrak{a}$ . Let  $\lambda$  be the highest root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  with respect to the ordering on  $\mathfrak{a}$ . Let us denote by  $M := Z_K(\mathfrak{a})$  and  $A := \exp \mathfrak{a}$ . Then the closed subgroup  $MA$  of  $G$  coincides with  $Z_G(\mathfrak{a})$ . Thus  $MA$  acts on the highest root space  $\mathfrak{g}_{\lambda}$  by the adjoint action. Note that  $M$  is not connected in some cases.

Our purpose in this section is to show the following three propositions:

**Proposition 6.1.** *The map*

$$\begin{aligned} \{ \text{non-zero } MA\text{-orbits in } \mathfrak{g}_{\lambda} \} &\rightarrow \{ G\text{-orbits in } \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g} \} \\ \mathcal{O}^{MA} &\mapsto G \cdot \mathcal{O}^{MA} \end{aligned}$$

*is bijective.*

**Proposition 6.2.** *Suppose that  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ . Then  $\mathfrak{g}_{\lambda} \setminus \{0\}$  becomes a single  $MA$ -orbit.*

**Proposition 6.3.** *Suppose that  $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1$ . Then the following holds:*

- (i) *If  $(\mathfrak{g}, \mathfrak{k})$  is of non-Hermitian type, then  $\mathfrak{g}_{\lambda} \setminus \{0\}$  becomes a single (disconnected)  $MA$ -orbit.*
- (ii) *If  $(\mathfrak{g}, \mathfrak{k})$  is of Hermitian type, then  $\mathfrak{g}_{\lambda} \setminus \{0\}$  splits into two connected  $MA$ -orbits.*

By combining the classification (xii) in Proposition 4.1 with the list of simple Lie algebras of non-Hermitian type, we see that  $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  only if  $(\mathfrak{g}, \mathfrak{k})$  is of non-Hermitian type. Therefore, Theorem 1.4 follows from above propositions immediately.

**6.1. Bijection between the set of non-zero  $MA$ -orbits in  $\mathfrak{g}_{\lambda}$  and the set  $G$ -orbits in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ .**

We prove Proposition 6.1 in this subsection.

By Theorem 1.1, the orbit  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$  can be written by

$$\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot (\mathfrak{g}_{\lambda} \setminus \{0\}),$$

and by Proposition 3.3, any  $G$ -orbit in  $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$  meets  $\mathfrak{g}_{\lambda} \setminus \{0\}$ . Thus the map in Proposition 6.1 is well-defined and surjective. Therefore, the proof of Proposition 6.1 is reduced to show that: For  $X_{\lambda}, X'_{\lambda} \in \mathfrak{g}_{\lambda}$ , if there exists  $g \in G$  such that  $gX_{\lambda} = X'_{\lambda}$ , then there exists  $m \in M$  and  $a \in A$  such that  $maX_{\lambda} = X'_{\lambda}$ .

We prove the claim above dividing into two lemmas below:

**Lemma 6.4.** For  $X_\lambda \in \mathfrak{g}_\lambda \setminus \{0\}$  and  $g \in G$ , if  $gX_\lambda$  is also in  $\mathfrak{g}_\lambda$ , then there exists  $m' \in N_K(\mathfrak{a})$  and  $a \in A$  such that  $m'aX_\lambda = gX_\lambda$ .

**Lemma 6.5.** For  $X_\lambda \in \mathfrak{g}_\lambda \setminus \{0\}$  and  $m' \in N_K(\mathfrak{a})$ , if  $m'X_\lambda$  is also in  $\mathfrak{g}_\lambda$ , then there exists  $m \in M (= Z_K(\mathfrak{a}))$  such that  $mX_\lambda = m'X_\lambda$ .

**Proof of Lemma 6.4.** For simplicity, we put  $X'_\lambda := gX_\lambda$ . Since  $N_G(\mathfrak{a}) = N_K(\mathfrak{a})A$ , it is enough to find  $g' \in G$  such that  $g'X_\lambda = X'_\lambda$  and  $g'\mathfrak{a} = \mathfrak{a}$ . Let  $A_\lambda$  be the coroot of  $\lambda$  in  $\mathfrak{a}$ . Then by Lemma 2.1, there exists  $Y_\lambda, Y'_\lambda \in \mathfrak{g}_{-\lambda}$  such that  $(A_\lambda, X_\lambda, Y_\lambda)$  and  $(A_\lambda, X'_\lambda, Y'_\lambda)$  are both  $\mathfrak{sl}_2$ -triples in  $\mathfrak{g}$ . Since  $g$  is an automorphism of  $\mathfrak{g}$  and  $gX_\lambda = X'_\lambda$ , the triple  $(gA_\lambda, X'_\lambda, gY_\lambda)$  is also an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ . In particular,  $(A_\lambda, X'_\lambda, Y'_\lambda)$  and  $(gA_\lambda, X'_\lambda, gY_\lambda)$  are both  $\mathfrak{sl}_2$ -triples in  $\mathfrak{g}$  with the same nilpotent element. Therefore, by Kostant's theorem for  $\mathfrak{sl}_2$ -triples with the same nilpotent element in a semisimple Lie algebra, there exists an element  $g_1 \in G$  such that

$$g_1(gA_\lambda) = A_\lambda, \quad g_1X'_\lambda = X'_\lambda \text{ and } g_1(gY_\lambda) = Y'_\lambda.$$

Write  $g_2 := g_1 \cdot g$ . Then

$$g_2A_\lambda = A_\lambda, \quad g_2X_\lambda = X'_\lambda \text{ and } g_2Y_\lambda = Y'_\lambda.$$

Recall that  $\mathfrak{a} = \mathbb{R}A_\lambda \oplus \text{Ker } \lambda$ . If we find  $g_3 \in G$  such that

$$g_3(g_2 \text{Ker } \lambda) = \text{Ker } \lambda, \quad g_3A_\lambda = A_\lambda \text{ and } g_3X'_\lambda = X'_\lambda,$$

then we can take  $g'$  as  $g_3 \cdot g_2$ . We shall find such  $g_3$ . Let us denote by  $\mathfrak{l}' = \mathbb{R}\text{-span}\langle A_\lambda, X'_\lambda, Y'_\lambda \rangle$  the subalgebra spanned by the  $\mathfrak{sl}_2$ -triple  $(A_\lambda, X'_\lambda, Y'_\lambda)$ . Then there exists a Cartan involution  $\theta'$  on  $\mathfrak{g}$  preserving  $\mathfrak{l}'$  by Mostow's theorem [19, Theorem 6]. We set

$$\begin{aligned} \mathfrak{g}_0 &:= Z_{\mathfrak{g}}(\mathfrak{l}') \\ &= \{ X \in \mathfrak{g} \mid [X, A_\lambda] = [X, X'_\lambda] = 0 \}, \end{aligned}$$

where the second equation can be obtained by the representation theory of  $\mathfrak{sl}(2, \mathbb{C})$ . We note that  $\mathfrak{g}_0$  is a reductive subalgebra of  $\mathfrak{g}$  since the Cartan involution  $\theta'$  preserves  $\mathfrak{g}_0$ . The subspace  $\text{Ker } \lambda$  of  $\mathfrak{a}$  is contained in  $\mathfrak{g}_0$  since  $[\text{Ker } \lambda, \mathfrak{l}'] = \{0\}$ . In particular,  $\text{Ker } \lambda$  becomes a maximally split abelian subspace of  $\mathfrak{g}_0$ . We have

$$\begin{aligned} [g_2 \text{Ker } \lambda, A_\lambda] &= g_2[\text{Ker } \lambda, A_\lambda] = \{0\}, \\ [g_2 \text{Ker } \lambda, X'_\lambda] &= g_2[\text{Ker } \lambda, X'_\lambda] = \{0\}. \end{aligned}$$

Thus the subspace  $g_2 \text{Ker } \lambda$  of  $g_2\mathfrak{a}$  is also contained in  $\mathfrak{g}_0$  and becomes a maximally split abelian subspace of  $\mathfrak{g}_0$ . Let us write  $G_0$  for the analytic subgroup of  $G$  with its Lie algebra  $\mathfrak{g}_0$ . Recall that any two maximally split abelian subalgebras of  $\mathfrak{g}_0$  are  $G_0$ -conjugate. Then there exists  $g_3 \in G_0 \subset G$  such that

$$g_3(g_2 \text{Ker } \lambda) = \text{Ker } \lambda,$$

and hence  $g_3A_\lambda = A_\lambda$  and  $g_3X'_\lambda = X'_\lambda$ . ■

Let us give a proof of Lemma 6.5 as follows.

**Proof of Lemma 6.5.** We denote the element of  $W(\mathfrak{g}, \mathfrak{a}) = N_K(\mathfrak{a})/Z_K(\mathfrak{a})$  corresponding to  $m' \in N_K(\mathfrak{a})$  by  $w$ . Then  $w\lambda = \lambda$  since  $m'\mathfrak{g}_\lambda = \mathfrak{g}_{w\lambda}$  and  $m'\mathfrak{g}_\lambda \cap \mathfrak{g}_\lambda \neq \{0\}$  by the assumption. By [12, Section 10.4, Lemma B],  $w$  is a product of root reflections  $s_1, s_2, \dots, s_l$  in  $W(\mathfrak{g}, \mathfrak{a})$  with  $s_i\lambda = \lambda$ . We write  $\xi_i$  for the root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  corresponding to  $s_i$  for each  $i = 1, \dots, l$ . Let  $\mathfrak{g}_i$  be the root space of  $\xi_i$ . Since  $s_i\lambda = \lambda$ , each  $\xi_i$  is orthogonal to  $\lambda$  in  $\mathfrak{a}^*$ . We can and do choose  $X_i$  be a non-zero root vector of  $\mathfrak{g}_i$  such that

$$B_{\mathbb{C}}(X_i, \theta X_i) = -\frac{2}{\langle \xi_i, \xi_i \rangle}$$

where  $\theta$  is the Cartan involution of  $\mathfrak{g}$  corresponding to  $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ . Then the element  $k_i = \exp \frac{\pi}{2}(X_i + \theta X_i)$  in  $N_K(\mathfrak{a})$  acts on  $\mathfrak{a}$  as the reflection  $s_i$ . Thus  $m := m'k_l k_{l-1} \cdots k_1$  acts trivially on  $\mathfrak{a}$ . That is,  $m \in Z_K(\mathfrak{a}) = M$ . It remains to prove that  $k_i X_\lambda = X_\lambda$ . Since  $\lambda$  is longest root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  by Lemma 2.2, and  $\xi_i$  is orthogonal to  $\lambda$ , the element  $\xi_i \pm \lambda$  of  $\mathfrak{a}^*$  is not a root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . In particular,  $[X_i, X_\lambda] = 0$  and  $[\theta X_i, X_\lambda] = 0$ . Hence,  $k_i X_\lambda = X_\lambda$  for any  $i$ . Therefore, we obtain that  $mX_\lambda = m'X_\lambda$ . ■

**6.2. MA-orbits in  $\mathfrak{g}_\lambda$  in the cases where  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ .**

In this subsection, we focus on the cases where  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ , i.e.  $\mathfrak{g}$  is isomorphic to one of  $\mathfrak{su}^*(2k)$ ,  $\mathfrak{so}(n-1, 1)$ ,  $\mathfrak{sp}(p, q)$ ,  $\mathfrak{e}_{6(-26)}$  or  $\mathfrak{f}_{4(-20)}$ , and give a proof of Proposition 6.2.

We write  $M_0$  for the identity component of  $M$ . Then  $M_0, M_0A$  are the analytic subgroups of  $G$  with its Lie algebra  $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ ,  $\mathfrak{m} \oplus \mathfrak{a} = Z_{\mathfrak{g}}(\mathfrak{a})$ , respectively.

Since  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ , the space  $\mathfrak{g}_\lambda \setminus \{0\}$  is connected. Thus, to prove Proposition 6.2, it is enough to show the lemma below:

**Lemma 6.6.**  $[\mathfrak{m} \oplus \mathfrak{a}, X_\lambda] = \mathfrak{g}_\lambda$  for any non-zero highest root vector  $X_\lambda$  in  $\mathfrak{g}_\lambda$ . In particular, any  $M_0A$ -orbit in  $\mathfrak{g}_\lambda$  is open.

**Proof of Lemma 6.6.** We fix a non-zero highest root vector  $X_\lambda$  in  $\mathfrak{g}_\lambda$ . Let  $A_\lambda$  be the coroot of  $\lambda$  in  $\mathfrak{a}$ . By Lemma 2.1, there exists  $Y_\lambda \in \mathfrak{g}_{-\lambda}$  such that  $(A_\lambda, X_\lambda, Y_\lambda)$  is an  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$ . By the theory of representations of  $\mathfrak{sl}(2, \mathbb{R})$ , we obtain that

$$[\mathfrak{g}_0, X_\lambda] = \mathfrak{g}_2,$$

where

$$\mathfrak{g}_i := \{ X \in \mathfrak{g} \mid [A_\lambda, X] = iX \} \quad \text{for each } i \in \mathbb{Z}.$$

One can easily check that

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \xi(A_\lambda) = 0\}} \mathfrak{g}_\xi.$$

Furthermore,  $\mathfrak{g}_\lambda = \mathfrak{g}_2$  since  $\lambda$  is a longest root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  (see Lemma 2.2). Thus, for each  $\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})$  with  $\xi(A_\lambda) = 0$ , we have  $\mathfrak{g}_\xi, X_\lambda \subset \mathfrak{g}_2 \cap \mathfrak{g}_{\xi+\lambda} = \mathfrak{g}_\lambda \cap \mathfrak{g}_{\xi+\lambda} = \{0\}$ . Hence  $[\mathfrak{m} \oplus \mathfrak{a}, X_\lambda] = [\mathfrak{g}_0, X_\lambda] = \mathfrak{g}_2 = \mathfrak{g}_\lambda$ . ■

**6.3. MA-orbits in  $\mathfrak{g}_\lambda$  in the cases where  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$ .**

Throughout this subsection, we consider the cases where  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$  and give a proof of Proposition 6.3.

Let us denote by  $\mathfrak{g}_\lambda^+$  and  $\mathfrak{g}_\lambda^-$  the connected components of  $\mathfrak{g}_\lambda \setminus \{0\}$ . Since for any  $t \in \mathbb{R}$  and  $X_\lambda \in \mathfrak{g}_\lambda$ , we have  $(\exp tA_\lambda)X_\lambda = e^{2t}X_\lambda$ ,

where  $A_\lambda$  is the coroot of  $\lambda$  in  $\mathfrak{a}$ ,  $A = \exp \mathfrak{a}$  acts transitively on  $\mathfrak{g}_\lambda^+$ ,  $\mathfrak{g}_\lambda^-$ , respectively.

We ask what is the condition to the existence of  $m \in M$  such that  $m \cdot \mathfrak{g}_\lambda^+ = \mathfrak{g}_\lambda^-$ . The following lemma answers our question:

**Lemma 6.7.** *There exists  $m \in M$  such that  $m \cdot \mathfrak{g}_\lambda^+ = \mathfrak{g}_\lambda^-$  if and only if the type of the restricted root system  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is not  $C$  nor  $BC$ . Here, we consider the root system of type  $A_1$ ,  $B_2$  as  $C_1$ ,  $C_2$ , respectively.*

To prove Lemma 6.7, we use the following fact for a structure of  $M$ .

**Theorem 6.8** (cf. [14, Chapter VII, Section 5]). *For any root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ , we define  $\gamma_\xi \in G_{\mathbb{C}}$  by*

$$\gamma_\xi = \exp \pi \sqrt{-1} A_\xi,$$

where  $A_\xi$  is the coroot of  $\xi$  in  $\mathfrak{a}$ . Let  $F$  be the subgroup of  $G_{\mathbb{C}}$  generated by  $\gamma_\xi$  for all root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . Then all  $\gamma_\xi$  are in  $M$  and  $M = FM_0$ , where  $M_0$  is the identity component of  $M$ .

**Proof of Lemma 6.7.** We first assume that the type of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is not  $C$  and not  $BC$ . Then  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is reduced and the Dynkin diagram of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  satisfying the following property: All nodes of the diagram corresponding to a longest root of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  have some edges with odd multiplicity. This means that for any longest root  $\mu$ , there exists a root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  such that  $2\langle \mu, \xi \rangle / \langle \xi, \xi \rangle$  is odd. In particular, since the highest root  $\lambda$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is a longest root (by Lemma 2.2), we can find a root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  such that  $2\langle \lambda, \xi \rangle / \langle \xi, \xi \rangle$  is odd. Therefore, the element  $\gamma_\xi = \exp \pi \sqrt{-1} A_\xi$  of  $M$  (by Theorem 6.8) acts on  $\mathfrak{g}_\lambda$  as the scalar multiplication of  $-1$ . Thus in this case, we can take  $m = \gamma_\xi$  satisfying that  $m \cdot \mathfrak{g}_\lambda^+ = \mathfrak{g}_\lambda^-$ . Conversely, we suppose that the type of  $\Sigma(\mathfrak{g}, \mathfrak{a})$  is  $C$  or  $BC$ . Then we can observe that for any longest root  $\mu$  and root  $\xi$  of  $\Sigma(\mathfrak{g}, \mathfrak{a})$ ,  $2\langle \mu, \xi \rangle / \langle \xi, \xi \rangle$  is even. Since the highest root  $\lambda$  is longest, all generators  $\gamma_\xi = \exp \pi \sqrt{-1} A_\xi$  of  $F$  act on  $\mathfrak{g}_\lambda$  trivially. Thus all elements of  $M = FM_0$  preserve  $\mathfrak{g}_\lambda^+$  and  $\mathfrak{g}_\lambda^-$ , respectively. ■

By the list of non-compact simple Lie algebras and its restricted root systems, we can obtain the following fact:

**Theorem 6.9.** *Suppose that  $\mathfrak{g}$  is a non-compact real simple Lie algebra with  $\dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1$  (thus,  $\mathfrak{g}$  is not isomorphic to one of  $\mathfrak{su}^*(2k)$ ,  $\mathfrak{so}(n-1, 1)$ ,  $\mathfrak{sp}(p, q)$ ,  $\mathfrak{e}_{6(-26)}$  nor  $\mathfrak{f}_{4(-20)}$ ). Then the type of the restricted root system of  $\mathfrak{g}$  is  $C$  or  $BC$  if and only if  $(\mathfrak{g}, \mathfrak{k})$  is Hermitian. Here, we consider the root system of type  $A_1$ ,  $B_2$  as  $C_1$ ,  $C_2$ , respectively.*

Combining Lemma 6.7 with Theorem 6.9, we obtain Proposition 6.3.

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