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Classification of Lie Superalgebras Supported over a Reductive Lie Algebra with One-Dimensional Center and a Simple Lie Algebra as a First Derived Ideal

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Abstract. It is the aim of this work to provide a concrete list of representatives of the isomorphism classes of finite-dimensional Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ supported over a reductive Lie algebra $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{z}$, where $\mathfrak{m}$ is a simple Lie algebra and $\mathfrak{z}$, the center of $\mathfrak{g}_0$, is one-dimensional. The classification given here does not impose the extra hypothesis that $\mathfrak{g}_1$ be a completely reducible $\mathfrak{g}_0$-module.
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1. introduction

The aim of this work is to classify finite-dimensional Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over an algebraically closed field $\mathbb{F}$ of characteristic zero, for which $\mathfrak{g}_0$ is a reductive Lie algebra of the form $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{z}$, where $\mathfrak{m}$ is a simple Lie algebra and $\mathfrak{z}$, the center of $\mathfrak{g}_0$, is one-dimensional. In general, a Lie superalgebra structure over a supervector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a bilinear map $[\![\cdot,\cdot]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying:

(i) $[\![\mathfrak{g}_i, \mathfrak{g}_j]\!] \subset \mathfrak{g}_{i+j}$, $i, j \in \mathbb{Z}_2$.

(ii) $[\![x, y]\!] = -(-1)^{|x||y|}[\![y, x]\!]$,

and *the super-jacobi identity*

$$(-1)^{|x||z|}[\![[\![x, y]\!], z]\!] + (-1)^{|x||y|}[\![[\![y, z]\!], x]\!] + (-1)^{|y||z|}[\![[\![z, x]\!], y]\!] = 0, \tag{1}$$

for all $x, y, z \in (\mathfrak{g}_0 \cup \mathfrak{g}_1) \setminus \{0\}$, and $|x| = i \in \mathbb{Z}_2$ for any $x \in \mathfrak{g}_i$.

A straightforward computation shows that $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ is a representation space for $\mathfrak{g}_0$, via $[\![\cdot,\cdot]\!]|_{\mathfrak{g}_0 \times \mathfrak{g}_1} : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_1)$, and it is easy to conclude that a Lie superalgebra can be viewed as a triple $(\mathfrak{g}_0, \rho, \Gamma)$ consisting of a Lie algebra $\mathfrak{g}_0$ with the bracket $[\cdot,\cdot]$, a representation $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$ or –as it is usual to say– a $\mathfrak{g}_0$-module $\mathfrak{g}_1$, and a symmetric bilinear map $\Gamma := [\![\cdot,\cdot]\!]|_{\mathfrak{g}_1 \times \mathfrak{g}_1} : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$, satisfying the following identities:

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(J1) $[x, \Gamma(u,v)] = \Gamma(\rho(x)u, v) + \Gamma(u, \rho(x)v), \quad x \in \mathfrak{g}_0, u, v \in \mathfrak{g}_1$.

(J2) $\rho(\Gamma(u,v))(w) + \rho(\Gamma(v,w))(u) + \rho(\Gamma(u,w))(v) = 0, \quad u, v, w \in \mathfrak{g}_1$.

Then, $[\![x+u, y+v]\!] = [x,y] + \rho(x)(v) - \rho(y)u + \Gamma(u,v)$, for any $x, y \in \mathfrak{g}_0$, and any $u, v \in \mathfrak{g}_1$. One usually refers to the triple $(\mathfrak{g}_0, \rho, \Gamma)$ as a Lie superalgebra supported over $\mathfrak{g}_0$.

Two such superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{g}' = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ are isomorphic if and only if there exist a Lie algebra isomorphism $T : \mathfrak{g}_0 \to \mathfrak{g}'_0$ and a $\mathfrak{g}_0$-module isomorphism $S : \mathfrak{g}_1 \to \mathfrak{g}'_1$ such that

$$\rho'(T(x)) \circ S = S \circ \rho(x), \ x \in \mathfrak{g}_0 \tag{2}$$

$$\Gamma'(u,v) = T(\Gamma(S^{-1}(u), S^{-1}(v))), \ u, v \in \mathfrak{g}'_1. \tag{3}$$

A fundamental problem is to classify Lie superalgebras up to isomorphism. The classification of large families has been achieved by making some special assumptions on either $\mathfrak{g}_0$, or on the representation ρ, or on the lattice of ideals of a given Lie superalgebra to be taken as starting point. Thus for example, a classical Lie superalgebra is a simple Lie superalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is completely reducible. V. Kac in [3] classified the classical Lie superalgebras over an algebraically closed field of characteristic zero. As a by-product, simple Lie superalgebras for which $\mathfrak{g}_0$ is a simple Lie algebra were classified. Using the Kac's classification, A. Elduque classified in [2] the Lie superalgebras supported over a reductive Lie algebra $\mathfrak{g}_0$, for which the $\mathfrak{g}_0$-action on $\mathfrak{g}_1$ is completely reducible. It is the aim of this work to provide a concrete list of representatives of the isomorphism classes of Lie superalgebras supported over a reductive Lie algebra $\mathfrak{g}_0$ with one-dimensional center. The classification given here, however, does not need the extra hypothesis on the complete reducibility of ρ.

Examples 1.1. Examples based on a supervector space $V = V_0 \oplus V_1$.

The general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(V_0|V_1)$. This superalgebra is supported over $\mathfrak{g}_0 = \mathfrak{gl}(V_0) \oplus \mathfrak{gl}(V_1)$ and its representation space is $\mathfrak{g}_1 = \mathrm{Hom}(V_1, V_0) \oplus \mathrm{Hom}(V_0, V_1)$ with $\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$ given by $\rho((x,y))(a,b) = (x \circ a - a \circ y, \ y \circ b - b \circ x)$. Finally, $\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is given by $\Gamma((a,b),(a',b')) = (a \circ b' + a' \circ b, \ b \circ a' + b' \circ a)$. Notice that $\mathfrak{gl}(V_0|V_1) \simeq \mathrm{End}(V_0 \oplus V_1)$, as $\mathbb{Z}_2$-graded vector spaces over $\mathbb{F}$.

The Lie superalgebra $\mathrm{Der}(\mathfrak{g})$. For the special case in which $V = V_0 \oplus V_1$ itself is a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, it follows from the super-jacobi identity that for a given $x \in (\mathfrak{g}_0 \cup \mathfrak{g}_1) \setminus \{0\}$, say $x \in \mathfrak{g}_i$, so that $|x| = i$, the map $[\![x, \cdot]\!] \in \mathrm{End}(\mathfrak{g}_0 \oplus \mathfrak{g}_1)$ satisfies: $[\![x, [\![y,z]\!]]\!] = [\![[\![x,y]\!], z]\!] + [\![y_0 + (-1)^i y_1, [\![x,z]\!]]\!]$, for all $y, z \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. More generally, any map $\delta \in \mathrm{End}(\mathfrak{g})$ satisfying $\delta([\![y,z]\!]) = [\![\delta(y), z]\!] + [\![y_0 + (-1)^i y_1, \delta(z)]\!]$ is called a derivation of $\mathfrak{g}$ of degree i $(i = 0, 1)$. We write $(\mathrm{Der}(\mathfrak{g}))_i$ for the $\mathbb{F}$-vector space of derivations of degree i, and let $\mathrm{Der}(\mathfrak{g}) = (\mathrm{Der}(\mathfrak{g}))_0 \oplus (\mathrm{Der}(\mathfrak{g}))_1$ be the $\mathbb{Z}_2$-graded vector space of derivations of $\mathfrak{g}$. A straightforward computation shows that $\mathrm{Der}(\mathfrak{g})$ is a Lie subsuperalgebra of $\mathfrak{gl}(\mathfrak{g}_0|\mathfrak{g}_1)$.

The following notation is used in the next examples. Given a finite dimensional vector space U over $\mathbb{F}$, and a non-degenerate bilinear form $q : U \times U \to \mathbb{F}$, we can define the following invertible linear maps:

$$q^\flat : U \to U^*, \quad u \mapsto q(u, \cdot) \quad \text{and} \quad q^\sharp : U^* \to U, \quad \phi \mapsto q^\sharp(\phi)$$

where $q(q^\sharp(\phi), u) = \phi(u)$, for any $u \in U$.

The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(V_0|V_1)$. Let $q : V_0 \times V_0 \to \mathbb{F}$ and $\omega : V_1 \times V_1 \to \mathbb{F}$ respectively be a symmetric and a skew-symmetric, non-degenerate bilinear forms. Let $\mathfrak{g}_0 = \mathfrak{o}(V_0) \oplus \mathfrak{sp}(V_1)$, the orthogonal and the symplectic Lie algebras associated to q and ω, respectively, and set $\mathfrak{g}_1 = \mathrm{Hom}(V_0, V_1)$. In this case ρ and Γ are defined as:
$\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$; $\rho((x, y))(a) = y \circ a - a \circ x$,
$\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$; $\Gamma(a, a') = (*a \circ a' + *a' \circ a,\, a \circ (*a') + a' \circ (*a))$,
where $*a \in \mathrm{Hom}(V_0, V_1)$ is defined via the isomorphisms $q^\flat : V_0 \to V_0^*$ and $\omega^\sharp : V_1^* \to V_1$ as $*a = q^\sharp \circ a^* \circ \omega^\flat$, and $a^* \in \mathrm{Hom}(V_1^*, V_0^*)$ is the dual of a.

Examples 1.2. Examples associated to an ordinary vector space V.

The Lie superalgebra $C(V)$. For the special case $\dim(V_0) = 2$ of the orthosymplectic Lie superalgebra just defined. In this case

$$\mathfrak{o}(V_0) = \{x \in \mathrm{End}(V_0) \mid q^\flat \circ x + x \circ q^\sharp = 0\}$$

is one-dimensional. Let $x_0 \in \mathfrak{o}(V_0)$ be a generator. Therefore, $\mathfrak{g}_0 = \mathfrak{osp}(V_0|V_1)_0 = \langle x_0 \rangle \oplus \mathfrak{sp}(V_1) \simeq \mathbb{F} \oplus \mathfrak{sp}(V_1)$. In this case it is customary to drop the subindex from V_1 and simply write $\mathfrak{g}_0 = \mathbb{F} \oplus \mathfrak{sp}(V)$. Now $\mathfrak{g}_1 = \mathrm{Hom}(V_0, V) \simeq V_0^* \otimes V$, and it is easy to see that the representation ρ is given by,

$$\rho(\lambda x_0, y)(\phi \otimes v) = \phi \otimes y(v) - \lambda(\phi \circ x_0) \otimes v,$$

for any $\phi \in V_0^*$, $v \in V$, $y \in \mathfrak{sp}(V)$ and $\lambda \in \mathbb{F}$. On the other hand, a straightforward computation shows that the map Γ is given by

$$\begin{aligned}\Gamma(\phi \otimes v,\, \phi' \otimes v') \;&= \big(\omega(v, v')\left(\phi \otimes q^\sharp(\phi') - \phi' \otimes q^\sharp(\phi)\right), \\ &\qquad -q(q^\sharp(\phi), q^\sharp(\phi'))\left(\omega^\flat(v) \otimes v' + \omega^\flat(v') \otimes v\right)\big),\end{aligned}$$

for any $\phi, \phi' \in V_0^*$, and any $v, v' \in V$. The resulting Lie superalgebra is denoted by $C(V)$.

The Lie superalgebra $\mathfrak{sl}(V|\mathbb{F})$. Consider again the Lie superalgebra $\mathfrak{gl}(V_0|V_1)$ of the Examples 1.1 above for the special case $\dim(V_1) = 1$. Thus, $V_1 \simeq \mathbb{F}$, $\mathrm{Hom}(V_0, \mathbb{F}) \simeq V_0^*$ and $\mathrm{Hom}(\mathbb{F}, V_0) \simeq V_0$. Therefore, $\mathfrak{g}_1 = V_0 \oplus V_0^*$ and we shall write $\mathfrak{gl}(V|\mathbb{F})$ instead of $\mathfrak{gl}(V_0|V_1)$, understanding of course that $V = V_0$ is now an arbitrary vector space over $\mathbb{F}$. Clearly $\mathfrak{gl}(V|\mathbb{F})$ is a Lie superalgebra supported over $\mathfrak{gl}(V) \oplus \mathbb{F}$. Now, we shall further restrict ourselves to the Lie subsuperalgebra $\mathfrak{sl}(V|\mathbb{F})$ of $\mathfrak{gl}(V|\mathbb{F})$ consisting of *supertrace*-less elements. Thus, $\mathfrak{g}_0 = \{(x, \mathrm{tr}(x)) \mid x \in \mathfrak{gl}(V)\} \subset \mathfrak{gl}(V) \oplus \mathbb{F}$ in this case. Clearly, $\mathfrak{g}_0 \simeq \mathfrak{gl}(V)$. Finally, ρ and Γ are then given by:

$\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$; $\rho(x)(u, \phi) = ((x - \mathrm{tr}(x)\,\mathrm{id})u,\ \phi \circ (\mathrm{tr}(x) id - x))$,
$\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$; $\Gamma((u, \phi)\ (u', \phi')) = \phi'(\cdot)u + \phi(\cdot)u'$,
respectively.

The Lie superalgebra $P(V)$. Let $q : V \times V \to \mathbb{F}$ be a non-degenerate symmetric bilinear form, and let $\mathfrak{g}_0 = \mathfrak{sl}(V)$ and $\mathfrak{g}_1 = \mathrm{Sym}(V) \oplus \mathrm{Skw}(V)$ where $\mathrm{Sym}(V) \subset \mathrm{End}(V)$ consists of the q-adjoint elements and $\mathrm{Skw}(V) \subset \mathrm{End}(V)$ of the q-skew-adjoint ones. In this case, ρ and Γ are defined as:
$\rho : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_1)$; $\rho(x)((a, b)) = (x \circ a - a \circ {*x},\ {*x} \circ b - b \circ x)$.
$\Gamma : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$; $\Gamma((a, b),\ (a', b')) = a \circ b' + a' \circ b$,
where ${*x} = -(q^\flat)^{-1} x^* q^\flat$.

Trivial extension of a simple Lie algebra. Let $\mathfrak{m}$ be a simple Lie algebra and consider the supervector space $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{m}$, with $\mathfrak{g}_0 = \mathfrak{m}$, and $\mathfrak{g}_1 = \mathfrak{m}$. It is useful to introduce the so called *change of parity map* $\Pi = \left(\begin{smallmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{smallmatrix}\right) : \mathfrak{m} \oplus \mathfrak{m} \to \mathfrak{m} \oplus \mathfrak{m}$, as it makes it easier to distinguish even from odd elements as follows: $x \in \mathfrak{m} = \mathfrak{g}_0$ if and only if $\Pi x \in \mathfrak{g}_1$. Then, we may write $\mathfrak{g}_1 = \mathrm{Im}\Pi|_{\mathfrak{g}_0} = \Pi\mathfrak{m}$, and define a Lie superalgebra structure on $\mathfrak{g} = \mathfrak{m} \oplus \Pi\mathfrak{m}$ by letting $\rho(x)(\Pi y) = \Pi[x, y]$, and $\Gamma \equiv 0$.

The Lie superalgebra $Q(V)$. Take the special case when $\mathfrak{m} = \mathfrak{sl}(V)$ in the previous example. Then, it has the structure of simple Lie superalgebra on the supervector space $\mathfrak{sl}(V) \oplus \Pi\mathfrak{sl}(V)$ by taking ρ as above and letting

$$\Gamma(\Pi y, \Pi y') = y \circ y' + y' \circ y - \frac{2\,\mathrm{tr}(y \circ y')}{\dim(V)}\,\mathrm{id}\,. \tag{4}$$

The resulting Lie superalgebra supported over $\mathfrak{sl}(V)$ with this particular Γ is denoted by $Q(V)$ in order to distinguish it from the trivial extension of $\mathfrak{m} = \mathfrak{sl}(V)$.

Examples 1.3. Extensions of Lie superalgebras by derivations. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{g}_0|\mathfrak{g}_1)$ be a subalgebra of derivations. The extension of $\mathfrak{g}$ by $\mathfrak{h}$ is the Lie superalgebra defined on $\mathfrak{g} \oplus \mathfrak{h}$ in such a way that $\mathfrak{g}$ and $\mathfrak{h}$ are subalgebras and $[\![\delta, x]\!] = \delta(x)$, for any $x \in \mathfrak{g}$, and $\delta \in \mathfrak{h}$.

The Lie superalgebra $P(V) \oplus \langle \partial_0 \rangle$. Let $\partial_0 \in \mathfrak{gl}(P(V)_0|P(V)_1)$ defined on generators through $\partial_0((x, (a, b))) = (a, -b)$. Then ∂_0 is an even derivation and $\mathrm{Der}(P(V)) = P(V) \oplus \langle \partial_0 \rangle$ (see [3], Proposition 5.1.2), and we can consider the extension of $P(V)$ by $\langle \partial_0 \rangle$.

The Lie superalgebra $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$. Let $\mathfrak{m}$ be a simple Lie algebra and consider a pair of derivations $\delta_i \in \mathfrak{gl}(\mathfrak{m}|\Pi\mathfrak{m})_i$, $i = 0, 1$, by letting them defined on generators through the linear maps $\delta_0((x, \Pi y)) = \Pi y$ and $\delta_1((x, \Pi y)) = y$, respectively. Then, we can consider any of the extensions $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$, $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$, and $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0 \rangle$.

The Lie superalgebra $Q(v) \oplus \langle \partial_1 \rangle$. Use the notation ∂_1 for the corresponding odd derivation $\partial_1 \in \mathfrak{gl}(Q(V)_0|Q(V)_1)_1$ defined on generators through $\partial_1((x, \Pi y)) = y$, and consider the extension of $Q(v)$ by $\langle \partial_1 \rangle$.

In the following table we summarize some of the examples just given, together with some extensions by derivations. They are precisely the building blocks of the classification theorem aimed at in this paper.

$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$	$\mathfrak{m}$	$\mathfrak{s}_0$	$\mathfrak{s}_1$	$\rho : \mathfrak{s}_0 \to \mathfrak{gl}(\mathfrak{s}_1)$	$\Gamma : \mathfrak{s}_1 \times \mathfrak{s}_1 \to \mathfrak{s}_0$
$C(V)$	$\mathfrak{sp}(V)$	$\mathfrak{o}(V_0) \oplus \mathfrak{sp}(V)$ $\dim(V_0) = 2$	$\mathrm{Hom}(V_0, V)$	$\rho(\lambda x_0, y)(\phi \otimes v) = \phi \otimes y(v) - \lambda(\phi \circ x_0) \otimes v$	$\Gamma(\phi \otimes v,\ \phi' \otimes v') = \big(\omega(v,v')\big(\phi \otimes q^\sharp(\phi') - \phi' \otimes q^\sharp(\phi)\big),\ -q(q^\sharp(\phi), q^\sharp(\phi'))\big(\omega^\flat(v) \otimes v' + \omega^\flat(v') \otimes v\big)\big)$
$\mathfrak{sl}(V\|\mathbb{F})$	$\mathfrak{sl}(V)$	$\mathfrak{gl}(V)$	$V \oplus V^*$	$\rho(a)(u,\phi) = ((a + \mathrm{tr}(a)\,\mathrm{id})u,\ \phi \circ (\mathrm{tr}(a)\,\mathrm{id} - a))$	$\Gamma((u,\phi),(u',\phi')) = \phi'(\cdot)u + \phi(\cdot)u'$
$P(V) \oplus \langle \partial_0 \rangle$	$\mathfrak{sl}(V)$	$\mathfrak{sl}(V) \oplus \langle \partial_0 \rangle$	$\mathrm{Sym}(V) \oplus \mathrm{Skw}(V)$	$\rho(x)(b,c) = (x \circ b - b \circ {*x},\ {*x} \circ c - c \circ x)$ $\rho(\partial_0)(b,c) = (b, -c)$	$\Gamma((b,c),(b',c')) = b \circ c' + b' \circ c$
$\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$	$\mathfrak{m}$ arbitrary	$\mathfrak{m} \oplus \langle \delta_0 \rangle$	$\Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$	$\rho(x)(\Pi y, \delta_1) = \Pi[x,y]_{\mathfrak{m}}$ $\rho(\delta_0)(\Pi(y), \delta_1) = (\Pi y, -\delta_1)$	$\Gamma((\Pi y, \delta_1),(\Pi y', \delta_1)) = y + y'$
$Q(V)$	$\mathfrak{sl}(V)$	$\mathfrak{sl}(V)$	$\Pi\mathfrak{sl}(V)$	$\rho(x)(\Pi y) = \Pi[xy]_{\mathfrak{sl}(V)}$	$\Gamma(\Pi y, \Pi y') = y \circ y' + y' \circ y - \frac{2\,\mathrm{tr}(y \circ y')}{\dim(V)}\,\mathrm{id}$
$Q(V) \oplus \langle \partial_1 \rangle$	$\mathfrak{sl}(V)$	$\mathfrak{sl}(V)$	$\Pi\mathfrak{sl}(V) \oplus \langle \partial_1 \rangle$	$\rho(x)(\Pi y, \partial_1) = \Pi[x,y]_{\mathfrak{sl}(V)}$	$\Gamma((\Pi y, \partial_1),(\Pi y', \partial_1)) = y \circ y' + y' \circ y - \frac{2\,\mathrm{tr}(y \circ y')}{\dim(V)}\,\mathrm{id} + y + y'$
$P(V)$	$\mathfrak{sl}(V)$	$\mathfrak{sl}(V)$	$\mathrm{Sym}(V) \oplus \mathrm{Skw}(V)$	$\rho(x)(b.c) = (x \circ b - b \circ {*x},\ {*x} \circ c - c \circ x)$	$\Gamma((b,c),(b',c')) = b \circ c' + b' \circ c$
$\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$	$\mathfrak{m}$ arbitrary	$\mathfrak{m}$	$\Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$	$\rho(x)(\Pi y, \delta_1) = \Pi[x,y]_{\mathfrak{m}}$	$\Gamma((\Pi y, \delta_1),(\Pi y', \delta_1)) = y + y'$

Table 1: Building blocks of the classification theorem

The principal results can be summarized in the following statement whose proof is precisely the aim of this work (see Theorem 3.3 and Theorem 4.4).

Theorem 1.4. *Let V, $\mathfrak{s}$, δ_i and ∂_i be as in the table 1 above and as in the examples 1.1, 1.2, and 1.3. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional Lie superalgebra over $\mathbb{F}$, such that $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{z}$ where $\mathfrak{m}$ is a simple Lie algebra and $\mathfrak{z} = \langle z \rangle$ is a one-dimensional center of $\mathfrak{g}_0$. Then $\mathfrak{g}$ is isomorphic to one of the following superalgebras.*

(i) *The semidirect product $\mathfrak{s} \oplus \mathfrak{a}(g)$ of the semisimple Lie superalgebra $\mathfrak{s} \simeq \mathfrak{sl}(V|\mathbb{F})$, $C(V)$, $P(V) \oplus \langle \partial_0 \rangle$, and the ideal $\mathfrak{a}(g)$. Actually, $\mathfrak{a}(g)$ is an $\mathfrak{m} \oplus \mathfrak{z}$-module. The $\mathfrak{m}$-action is trivial, and the $\mathfrak{z}$-action is defined via the linear map $g : \mathfrak{a}(g) \to \mathfrak{a}(g)$, through $\rho(z)(a) = g(a)$, including the cases $\mathfrak{s} = \mathfrak{sl}(V|\mathbb{F})$ and $\mathfrak{s} = C(V)$, for which $g = 0$. Finally, $\Gamma|_{\mathfrak{a}(g)\times\mathfrak{a}(g)} = 0$.*

Furthermore, two such superalgebras $\mathfrak{s} \oplus \mathfrak{a}(g)$ and $\mathfrak{s}' \oplus \mathfrak{a}'(g')$ are isomorphic if and only if $\mathfrak{s} \simeq \mathfrak{s}'$ and if there exist a non-zero scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and a linear map $S : \mathfrak{a}(g) \to \mathfrak{a}'(g')$ such that $g' = \alpha^{-1}(S \circ g \circ S^{-1})$.

(ii) *$\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ where $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0 \rangle$ is a subalgebra and $\langle \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ and $\mathfrak{a}(u; g)$ are $\mathfrak{m} \oplus \langle \delta_0 \rangle$-modules such that the $\mathfrak{m}$-action is trivial, and the action of δ_0 is given by $\rho(\delta_0)(\delta_1) = -\delta_1 + u$ for some $u \in \mathfrak{a}(u; g)$ and $\rho(\delta_0)(a) = g(a)$ for some linear map $g : \mathfrak{a}(u; g) \to \mathfrak{a}(u; g)$. In this case we have $\Gamma|_{\Pi\mathfrak{m}\times\mathfrak{a}(u;g)} = 0$ and $\Gamma|_{\langle\delta_1\rangle\oplus\mathfrak{a}(u;g)\times\langle\delta_1\rangle\oplus\mathfrak{a}(u;g)} = 0$.*

Furthermore, two such superalgebras $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ and $\mathfrak{m}' \oplus \Pi\mathfrak{m}' \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}'(u'; g')$ are isomorphic if and only if $\mathfrak{m} \simeq \mathfrak{m}'$ and if there exist a non-zero scalar $\beta \in \mathbb{F}$, a vector $u_1 \in \mathfrak{a}'(u'; g')$ and a linear map $S : \mathfrak{a}(u; g) \to \mathfrak{a}'(u', g')$ such that $g' = S \circ g \circ S^{-1}$ and $u' = \beta^{-1}(S(u) + (g' + \mathrm{id})u_1)$.

(iii) *$\mathfrak{s} \oplus \mathfrak{z} \oplus \mathfrak{a}(\mathcal{B})$ where $\mathfrak{s} \simeq \mathfrak{m}$, $P(V)$, $Q(V)$, $Q(V) \oplus \langle \partial_1 \rangle$, $\mathfrak{m} \oplus \Pi(\mathfrak{m}) \oplus \langle \delta_1 \rangle$, and $\mathfrak{a}(\mathcal{B})$ is a trivial $\mathfrak{m} \oplus \mathfrak{z}$-module. The Lie algebra bracket on $\mathfrak{g}_0$ is given via the Lie bracket on $\mathfrak{s}_0$ as $[(x, z), (y, z)] = [x, y]_{\mathfrak{s}_0}$; the $\mathfrak{s}_0$-action on $\mathfrak{s}_1$ is given as in $\mathfrak{s}$, and the $\mathfrak{z}$-action on $\mathfrak{s}_1$ is trivial. Finally Γ is given in terms of a symmetric, bilinear, $\mathfrak{g}_0$-invariant, non-zero map*

$$\mathcal{B} \ : \ \mathfrak{s}_1 \oplus \mathfrak{a}(g, \mathcal{B}) \times \mathfrak{s}_1 \oplus \mathfrak{a}(g, \mathcal{B}) \to \mathfrak{z},$$

as $\Gamma(u + a, u' + a') = \Gamma_{\mathfrak{s}}(u, u') + \mathcal{B}(u + a, u' + a')$.

Two such superalgebras $\mathfrak{s} \oplus \mathfrak{z} \oplus a(\mathcal{B})$ and $\mathfrak{s}' \oplus \mathfrak{z} \oplus a'(\mathcal{B}')$ are isomorphic if and only if there exist an isomorphism of Lie superalgebras $\Phi = T \oplus S_1 : \mathfrak{s} \to \mathfrak{s}'$, a non-zero scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and a linear map $S_2 : \mathfrak{a}(\mathcal{B}) \to \mathfrak{a}'(\mathcal{B}')$, such that $\mathcal{B}'(S_1(u) + S_2(a), S_1(u') + S_2(a')) = \alpha\mathcal{B}(u + a, u' + a')$.

(iv) *$\mathfrak{s} \oplus \mathfrak{z} \oplus \mathfrak{a}(g)$ where $\mathfrak{s} \simeq \mathfrak{m}$, $P(V)$, $Q(V)$, $Q(V) \oplus \langle \partial_1 \rangle$, $\mathfrak{m} \oplus \Pi(\mathfrak{m}) \oplus \langle \delta_1 \rangle$, and $\mathfrak{a}(g)$ is an $\mathfrak{m} \oplus \mathfrak{z}$-module for which the $\mathfrak{m}$-action is trivial, and the $\mathfrak{z}$-action is defined via a linear map $g : \mathfrak{a}(g) \to \mathfrak{a}(g)$ through $\rho(z)a = g(a)$. The Lie algebra bracket on $\mathfrak{g}_0$ is given as in (i). The $\mathfrak{s}_0$-action on $\mathfrak{s}_1$ is*

given as in $\mathfrak{s}$, *and the* $\mathfrak{z}$-*action on* $\mathfrak{s}_1$ *is trivial. Finally* Γ *is given by* $\Gamma(u+a, u'+a') = \Gamma_{\mathfrak{s}}(u, u')$.

Two such superalgebras $\mathfrak{s} \oplus \mathfrak{z} \oplus a(g)$ *and* $\mathfrak{s}' \oplus \mathfrak{z} \oplus a'(g')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi = T \oplus S_1 : \mathfrak{s} \to \mathfrak{s}'$, *a non-zero scalar* $\alpha \in \mathbb{F} \setminus \{0\}$, *and a linear map* $S_2 : \mathfrak{a}(g) \to \mathfrak{a}'(g')$, *such that* $g' = \alpha(S_2 \circ g \circ S_2^{-1})$.

(v) $\mathfrak{s} \oplus \mathfrak{z} \oplus \mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *where* $\mathfrak{s} \simeq P(V)$, $Q(V)$, $Q(V) \oplus \langle \partial_1 \rangle$, $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$, *and* $\mathfrak{s}_1 \oplus \mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *and* $\mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *are* $\mathfrak{m} \oplus \mathfrak{z}$-*modules for which the* $\mathfrak{m}$-*action on* $\mathfrak{s}_1$ *is given as in* $\mathfrak{s}$, *and the* $\mathfrak{z}$-*action on* $\mathfrak{s}_1$ *is given by a non-zero linear map* $f : \mathfrak{s}_1 \to \mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *as* $\rho(z)(w) = f(w)$. *The* $\mathfrak{m}$-*action on* $\mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *is given by* $\rho_{\mathfrak{a}} : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C}))$ *and the* $\mathfrak{z}$-*action is trivial. The Lie algebra bracket on* $\mathfrak{g}_0$ *is given as in (i), and* Γ *is given in terms of a bilinear* $\mathfrak{g}_0$-*invariant, non-zero map* $\mathcal{C} : \mathfrak{s}_1 \times \mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C}) \to \mathfrak{z}$ *by* $\Gamma(w+u, w'+u') = \Gamma_{\mathfrak{s}}(w, w') + \mathcal{C}(w, u') + \mathcal{C}(w', u)$.

Two such superalgebras $\mathfrak{s} \oplus \mathfrak{z} \oplus a(f, \rho_{\mathfrak{a}}, \mathcal{C})$ *and* $\mathfrak{s}' \oplus \mathfrak{z} \oplus a'(f', \rho'_{\mathfrak{a}}, \mathcal{C}')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi = T \oplus S_1 : \mathfrak{s} \to \mathfrak{s}'$, *an isomorphism of* $\mathfrak{m}$-*modules* $S_2 : \mathfrak{a}(f, \rho_{\mathfrak{a}}, \mathcal{C}) \to \mathfrak{a}'(f', \rho'_{\mathfrak{a}'}, \mathcal{C}')$, *a non-zero scalar* $\alpha \in \mathbb{F} \setminus \{0\}$ *and a linear map* $S_{21} : \mathfrak{s}_1 \to \mathfrak{a}'(f', \rho'_{\mathfrak{a}'}, \mathcal{C}')$ *such that* $f' = \alpha^{-1}(S_2 \circ f \circ S_1)$, $\mathcal{C}'(\cdot, \cdot) = \alpha\mathcal{C}(S_1^{-1}(\cdot), S_2^{-1}(\cdot))$ *and* $\mathcal{C}'(S_1(w_1), S_{21}(w_2)) + \mathcal{C}'(S_1(w_2), S_{21}(w_1)) = 0$ *for any* $w_1, w_2 \in \mathfrak{s}_1$.

(vi) $\mathfrak{s} \oplus \mathfrak{z} \oplus \mathfrak{a}(f, g)$ *where* $\mathfrak{s} \simeq P(V)$, $Q(V)$, $Q(V) \oplus \langle \partial_1 \rangle$, $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_1 \rangle$, *and* $\mathfrak{s}_1 \oplus \mathfrak{a}(f, g)$ *and* $\mathfrak{a}(f, g)$ *are* $\mathfrak{m} \oplus \mathfrak{z}$-*modules for which the* $\mathfrak{m}$-*action on* $\mathfrak{s}_1$ *is given in* $\mathfrak{s}$ *and the* $\mathfrak{z}$-*action on* $\mathfrak{s}_1$ *is given by a non-zero linear map* $f : \mathfrak{s}_1 \to \mathfrak{a}(f, g)$ *as* $\rho(z)(w) = f(w)$. *The* $\mathfrak{m}$-*action on* $\mathfrak{a}(f, g)$ *is trivial, and the* $\mathfrak{z}$-*action on* $\mathfrak{a}(f, g)$ *is defined by the linear map* $g : \mathfrak{a}(f, g) \to \mathfrak{a}(f, g)$ *by* $\rho(z)a = g(a)$. *The Lie algebra bracket on* $\mathfrak{g}_0$ *is given as in (i), and* Γ *is given as in (ii).*

Two such superalgebras $\mathfrak{s} \oplus \mathfrak{z} \oplus a(f, g)$ *and* $\mathfrak{s}' \oplus \mathfrak{z} \oplus a'(f', g')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi = T \oplus S_1 : \mathfrak{s} \to \mathfrak{s}'$, *two linear maps* $S_2 : \mathfrak{a}(f, g) \to \mathfrak{a}'(f', g')$ *and* $S_{21} : \mathfrak{s}_1 \to \mathfrak{a}'(f', g')$, *and a non-zero scalar* $\alpha \in \mathbb{F} \setminus \{0\}$, *such that* $f' = \alpha^{-1}(S_2 \circ f - \alpha g \circ S_{21}) \circ S_1^{-1}$ *and* $g' = \alpha^{-1}(S_2 \circ g \circ S_2^{-1})$.

We give the proof of this theorem within the following sections. The proof is naturally divided into two main subcases. Namely, the Lie superalgebras for which $\mathrm{Rad}(\mathfrak{g})_0 = \{0\}$, and those for which $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$. These cases are worked out in detail in sections 3 and 4, respectively. But first, in section 2, we deal with the semisimple Lie superalgebras of the form $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$. The main technique we use there to classify them up to isomorphism follows [2], and it consists in determining all the possibilities for *the minimal ideal* I in $\mathfrak{g}$. It turns out, of course (see Lemma 2.1 below), that I is unique and its possibilities are easy to handle in a case-by-case proof of the main theorem. For the sake of completeness and to make our exposition self-contained, we have included an Appendix with the classification of those Lie superalgebras supported over a simple Lie superalgebra

$\mathfrak{g}_0$. The results there are due to A. Elduque (see [2]), and we use them in our classification for those superalgebras having $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$.

2. Semisimple Lie superalgebras $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$

In this section, all semisimple Lie superalgebras of the form $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ where $\mathfrak{m}$ is a simple Lie algebra and $\mathfrak{z}$ is one-dimensional will be described using the classification of simple Lie superalgebras given in [3] and the description of semisimple Lie superalgebras in terms of simple ones provided therein.

Lemma 2.1. *Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra and let $I \subset \mathfrak{g}$ be a minimal ideal. Then, I is unique, and I is a simple Lie superalgebra for which $I_0 \simeq \mathfrak{m}$ or $I_0 \simeq \mathfrak{m} \oplus \mathfrak{z}$, or $I \simeq \mathfrak{m} \oplus \Pi\mathfrak{m}$.*

Proof. Let $I \subset \mathfrak{g}$ be a minimal ideal. Then, $I \simeq \mathfrak{a} \otimes \Lambda(U)$ where $\mathfrak{a}$ is a simple Lie superalgebra and U is a finite-dimensional vector space (see [3], Proposotion 5.1.1). Thus, $I_0 \simeq \mathfrak{a}_0 \otimes \Lambda(U)_0 \oplus \mathfrak{a}_1 \otimes \Lambda(U)_1$. Since $I_0 \subset \mathfrak{g}_0$ is an ideal, it follows that $I_0 \simeq \mathfrak{z}$, $I_0 \simeq \mathfrak{m}$, or $I_0 \simeq \mathfrak{m} \oplus \mathfrak{z}$. Notice that $I_0 \not\simeq \mathfrak{z}$. In fact, if $I_0 \simeq \mathfrak{z}$, then $\mathfrak{a}_0 \simeq \mathbb{F}$ and $U \simeq \mathbb{F}$, but this is not possible because there is no simple Lie superalgebra with this property. Assume that $I_0 \simeq \mathfrak{m}$ and write $\Lambda(U) = \mathbb{F} \oplus N \oplus \Lambda(U)_1$ where $N = \bigoplus_{n\in\mathbb{N}} \Lambda^{2n}(U)$. Then, $\mathfrak{a}_0 \otimes N \oplus \mathfrak{a}_1 \otimes \Lambda(U)_1 = \{0\}$ because it is a nilpotent ideal of $\mathfrak{m}$. Thus, $\mathfrak{a}_0 \simeq \mathfrak{m}$ and $U = \{0\}$, or $\mathfrak{a} \simeq \mathfrak{m}$ and $U \simeq \mathbb{F}$. Hence, I is a simple Lie superalgebra for which $I_0 \simeq \mathfrak{m}$ or $I \simeq \mathfrak{m} \otimes \wedge(\mathbb{F}) \simeq \mathfrak{m} \oplus \Pi\mathfrak{m}$. Now assume that $I_0 \simeq \mathfrak{m} \oplus \mathfrak{z}$. Then, $U = \{0\}$ and $\mathfrak{a}_0 \simeq \mathfrak{m} \oplus \mathfrak{z}$. In fact, if $U \neq \{0\}$, then $\mathfrak{a}_0 \otimes N \oplus \mathfrak{a}_1 \otimes \Lambda(U)_1 \simeq \mathfrak{z}$. Thus, $\mathfrak{a}_1 \simeq \mathfrak{z} \simeq \mathbb{F}$. Again, this is not possible. Finally, if I and J are minimal ideals, then $\mathfrak{m} \subset I \cap J$. Hence, $I = J$. ■

Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra and let $I \subset \mathfrak{g}$ be its minimal ideal, then $\mathfrak{g} \hookrightarrow \mathrm{Der}(I)$, via the monomorphism $x \mapsto \mathrm{ad}(x) = [\![x, \cdot]\!]|_I$. In fact, from the proof of Lemma 2.1 it follows that $I_0 = \mathfrak{m}$ or $I_0 = \mathfrak{m} \oplus \mathfrak{z}$. Hence, since $\ker(\mathrm{ad})_0 \subset Z(I_0) \subset \mathfrak{z}$ it follows that $\ker(\mathrm{ad})_0$ is a solvable ideal of $\mathfrak{g}_0$, and as a consequence $\ker(\mathrm{ad})$ is a solvable ideal of $\mathfrak{g}$. Thus, since $\mathfrak{g}$ is semisimple, it follows that $\ker(\mathrm{ad}) = 0$.

Note that in the previous argument, we use the following result.

Proposition 2.2 ([6], Proposition 2, p. 236)**.** *A Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is solvable if and only if $\mathfrak{g}_0$ is.*

It is well-known that, in general, a semisimple Lie superalgebra cannot be written as a sum of simple Lie superalgebras. However, the following result describes semisimple Lie superalgebras in terms of simple ones as follows.

Theorem 2.3 (see [3], Theorem 6)**.** *Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be finite-dimensional simple Lie superalgebras. Let $U_1, \dots, U_r$ be finite dimensional vector spaces and let $\mathfrak{m} = \bigoplus_{i=1}^r \mathfrak{m}_i \otimes \Lambda(U_i)$. Finally, let $\mathrm{Int}(\mathfrak{m})$ and $\mathrm{Int}(\mathfrak{m}_i)$ be the set of inner derivations* *of $\mathfrak{m}$, and $\mathfrak{m}_i$, respectively. Then,*

$$\begin{aligned}\mathfrak{m} = \mathrm{Int}(\mathfrak{m}) &= \bigoplus_{i=1}^{r} \mathrm{Int}(\mathfrak{m}_i) \otimes \Lambda(U_i) \\ \subset \mathrm{Der}(\mathfrak{m}) &= \bigoplus_{i=1}^{r} (\mathrm{Der}(\mathfrak{m}_i) \otimes \wedge(U_i) \oplus \mathrm{id} \otimes \mathrm{Der}(\Lambda(U_i))).\end{aligned}$$

Let $\mathfrak{g} \subset \mathrm{Der}(\mathfrak{m})$ be a subalgebra containing $\mathfrak{m}$, and let $\mathfrak{g}_i$ be the set of components of elements of $\mathfrak{g}$ in $\mathrm{id} \otimes \mathrm{Der}(\Lambda(U_i))$. Then, $\mathfrak{g}$ is semisimple if and anly if $\Lambda(U_i)$ is $\mathfrak{g}_i$-simple, for all $i = 1, \ldots, r$. Moreover, any finite-dimensional semisimple Lie superalgebra arises in this manner.

The statement of this theorem uses the following:

Definition 2.4. Let A be a superalgebra and suppose $L \subset \mathrm{Der}(A)$. Then, A is L-simple if A contains no non-trivial ideals that are invariant under L.

Proposition 2.5. *Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra. Then, $\mathfrak{g}$ is isomorphic to one of the following superalgebras: $C(V)$, $\mathfrak{sl}(V'|\mathbb{F})$, $P(V') \oplus \langle \partial_0 \rangle$, or $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$ where V and V' are finite-dimensional vector spaces such that $\dim(V) \geq 2$ and $\dim(V') \geq 3$.*

Proof. Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra and let I be its minimal ideal. According to Lemma 2.1 there are three diferent structures for I. Applying Theorem 2.3 to each one of them all the semisimple Lie superalgebras are obtained.

(i) **Case** $I_0 \simeq \mathfrak{m}$. If I is a simple Lie superalgebra for which $I_0 \simeq \mathfrak{m}$ and $I \subset \mathfrak{g} \subset \mathrm{Der}(I)$, then $\mathrm{Der}(I)_0 \neq \mathfrak{m}$. Therefore, $I \simeq P(V)$ where V is a finite-dimensional vector space such that $\dim(V) \geq 3$, (see [3], Proposition 5.1.2). Thus, $P(V) \subset \mathfrak{g} \subset P(V) \oplus \langle \partial_0 \rangle$. Hence, $\mathfrak{g} \simeq P(V) \oplus \langle \partial_0 \rangle$.

(ii) **Case** $I \simeq \mathfrak{m} \oplus \Pi\mathfrak{m}$. Let e be the generator of $\Lambda(\mathbb{F})$, and let $d_0, d_1 \in \mathrm{End}(\Lambda(\mathbb{F}))$ be defined by

$$\begin{aligned} d_0(1) &= 0, \qquad d_0(e) = e. \\ d_1(1) &= 0, \qquad d_1(e) = 1. \end{aligned}$$

Then, $\mathrm{Der}(\Lambda(\mathbb{F})) = \langle d_0, d_1 \rangle$; d_0 is an even derivation and d_1 is an odd derivation. In this case we have:

$$\mathfrak{m} \otimes \Lambda(\mathbb{F}) \subset \mathfrak{g} \subset \mathfrak{m} \otimes \Lambda(\mathbb{F}) \oplus \langle \mathrm{id} \otimes d_0, \mathrm{id} \otimes d_1 \rangle.$$

Notice that $\Lambda(\mathbb{F})$ is not $\langle \mathrm{id} \otimes d_0 \rangle$-simple because $\mathrm{id} \otimes d_0(\langle e \rangle) = \langle e \rangle$. Nevertheless, $\Lambda(\mathbb{F})$ is $\langle \mathrm{id} \otimes d_0, \mathrm{id} \otimes d_1 \rangle$-simple. Then, $\mathfrak{g} \simeq \mathfrak{m} \otimes \Lambda(\mathbb{F}) \oplus \langle \mathrm{id} \otimes d_0, \mathrm{id} \otimes d_1 \rangle$. Hence, $\mathfrak{g} \simeq \mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$, under the identifications $\delta_0 \leftrightarrow \mathrm{id} \otimes d_0$ and $\delta_1 \leftrightarrow \mathrm{id} \otimes d_1$.

(iii) **Case** $I_0 \simeq \mathfrak{m}\oplus\mathfrak{z}$**.** Let I be a simple Lie superalgebra for which $I_0 \simeq \mathfrak{m}\oplus\mathfrak{z}$. Then $I \simeq \mathfrak{sl}(V|\mathbb{F})$ or $I \simeq C(V)$, where V is a finite-dimensional vector space such that $\dim(V) \geq 2$, and in this case $\operatorname{Der}(I) \simeq \operatorname{Int}(I)$ (see [3], Proposition 5.1.2). Thus, if $I \subset \mathfrak{g} \subset \operatorname{Der}(I)$, then $\mathfrak{g} \simeq \mathfrak{sl}(V|\mathbb{F})$ or $\mathfrak{g} \simeq C(V)$. Finally, when $\dim(V) = 2$, $\mathfrak{sl}(V|\mathbb{F}) \simeq C(V)$.

■

3. Lie superalgebras $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{z}\oplus\mathfrak{g}_1$ for which $\operatorname{Rad}(\mathfrak{g})_0 = \{0\}$

Let $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{z}\oplus\mathfrak{g}_1$ be a Lie superalgebra for which $\operatorname{Rad}(\mathfrak{g})_0 = \{0\}$. Then, $\operatorname{Rad}(\mathfrak{g}) \subset \mathfrak{g}_1$ is an $\mathfrak{m}\oplus\mathfrak{z}$-module and $\widetilde{\mathfrak{g}} \simeq \mathfrak{g}/\operatorname{Rad}(\mathfrak{g})$ is a semisimple Lie superalgebra with $\widetilde{\mathfrak{g}}_0 \simeq \mathfrak{m}\oplus\mathfrak{z}$. Hence, $\widetilde{\mathfrak{g}}$ is isomorphic to one of the superalgebras given in Proposition 2.5. Let W be an $\mathfrak{m}$-module such that $\mathfrak{g}_1 = W\oplus\operatorname{Rad}(\mathfrak{g})$. Then, $\widetilde{\mathfrak{g}} \simeq \mathfrak{m}\oplus\mathfrak{z}\oplus W$ as vector spaces because W is not necessarily invariant under the action of $\mathfrak{z}$. Thus, the structure of $\mathfrak{g}$, $(\mathfrak{m}\oplus\mathfrak{z}, \rho, \Gamma)$, can now be written in terms of the structure of $\widetilde{\mathfrak{g}}$, $(\mathfrak{m}\oplus\mathfrak{z}, \widetilde{\rho}, \widetilde{\Gamma})$, and the structure on $\operatorname{Rad}(\mathfrak{g})$, as follows:
The Lie algebra bracket on $\mathfrak{g}_0$ is given by:

$$[\cdot,\cdot] = [\cdot,\cdot]_{\mathfrak{m}\oplus\mathfrak{z}}.$$

The $\mathfrak{m}$-action on $\mathfrak{g}_1$ is given by:

$$\rho(A) = \widetilde{\rho}(A) \oplus \rho_R(A), \quad A \in \mathfrak{m}$$

where $\rho_R(A) = \rho(A)|_{\operatorname{Rad}(\mathfrak{g})}$. The $\mathfrak{z}$-action on $\mathfrak{g}_1$ is given by:

$$\rho(z) = \widetilde{\rho}(z) + f + \rho_R(z)$$

where $\rho_R(z) = \rho(z)|_{\operatorname{Rad}(\mathfrak{g})}$, and $f : W \to \operatorname{Rad}(\mathfrak{g})$ is a linear map. Notice that W is invariant under the action of $\mathfrak{z}$ if and only if $f = 0$. Finally, since $\Gamma(\mathfrak{g}_1, \operatorname{Rad}(\mathfrak{g})) = 0$, then Γ is given by:

$$\Gamma(w_1+u_1, w_2+u_2) = \widetilde{\Gamma}(w_1, w_2), \quad w_1, w_2 \in W,\ u_1, u_2 \in \operatorname{Rad}(\mathfrak{g}).$$

Lemma 3.1. *Let* $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{z}\oplus\mathfrak{g}_1$ *be a Lie superalgebra for which* $\operatorname{Rad}(\mathfrak{g})_0 = \{0\}$. *Then,* $\operatorname{Rad}(\mathfrak{g})$ *is a trivial* $\mathfrak{m}$*-module.*

Proof. Consider the decomposition $\mathfrak{g} = \mathfrak{m}\oplus\mathfrak{z}\oplus W\oplus\operatorname{Rad}(\mathfrak{g})$ given above. Notice that $\Gamma(W,W)$ is an ideal of $\mathfrak{m}\oplus\mathfrak{z}$ and $\Gamma(W,W) \neq 0$. In fact, if $\Gamma(W,W) = 0$, then $\widetilde{\Gamma}(W,W)\oplus W$ is a solvable ideal of $\widetilde{\mathfrak{g}}$. Hence, $\Gamma(W,W) = \mathfrak{m}$ or $\Gamma(W,W) = \mathfrak{m}\oplus\mathfrak{z}$. On the other hand, it follows from the identity (J2) that,

$$\rho(\mathfrak{m})(\operatorname{Rad}(\mathfrak{g})) \subset \rho(\Gamma(W,W))(\operatorname{Rad}(\mathfrak{g})) \subset \rho(\Gamma(\operatorname{Rad}(\mathfrak{g}), W))(W) = 0.$$

Therefore, $\operatorname{Rad}(\mathfrak{g})$ is a trivial $\mathfrak{m}$-module. Moreover, if $\Gamma(W,W) = \mathfrak{m}\oplus\mathfrak{z}$, then $\operatorname{Rad}(\mathfrak{g})$ is a trivial $\mathfrak{m}\oplus\mathfrak{z}$-module, and this is just the case when $\widetilde{\mathfrak{g}}$ is a simple Lie superalgebra. ■

Lemma 3.2. *Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a Lie superalgebra for which* $\mathrm{Rad}(\mathfrak{g})_0 = \{0\}$ *and let $f : W \to \mathrm{Rad}(\mathfrak{g})$ be defined as above. Then,*

(i) If $\widetilde{\mathfrak{g}} \simeq C(V)$, $\mathfrak{sl}(V|\mathbb{F})$ or $P(V) \oplus \langle \partial_0 \rangle$, then $f = 0$.

(ii) If $\widetilde{\mathfrak{g}} \simeq \mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$, then $f|_{\Pi\mathfrak{m}} = 0$ and $f|_{\langle \delta_1 \rangle}$ is an arbitrary linear map.

Proof. Notice that $\rho(\mathfrak{z}) \circ \rho(\mathfrak{m}) = \rho(\mathfrak{m}) \circ \rho(\mathfrak{z})$, and since $\mathrm{Rad}(\mathfrak{g})$ is a trivial $\mathfrak{m}$-module it follows that $\rho(\mathfrak{z})(\rho(\mathfrak{m})(W)) \subset \rho(\mathfrak{m})(W)$. Hence, $\rho(\mathfrak{m})(W) \subset W$ is an $\mathfrak{m} \oplus \mathfrak{z}$-module, and as a consequence $f|_{\rho(\mathfrak{m})(W)} = 0$. On the other hand, since $\rho(\mathfrak{m})(W) = \widetilde{\rho}(\mathfrak{m})(W) = W$ for $\widetilde{\mathfrak{g}} \simeq C(V)$, $\mathfrak{sl}(V|\mathbb{F})$ and $P(V) \oplus \langle \partial_0 \rangle$ (see table 1), it follows that $f = 0$ in these cases.

Finally, assume that $\widetilde{\mathfrak{g}} \simeq \mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle$. Then, $\widetilde{\rho}(\mathfrak{m})(W) = \Pi\mathfrak{m}$. Hence $f|_{\Pi\mathfrak{m}} = 0$, and there are no restrictions for $f|_{\langle \delta_1 \rangle}$. This means that the action of δ_0 in δ_1 is given by $\rho(\delta_0)(\delta_1) = -\delta_1 + u$, for some $u \in \mathrm{Rad}(\mathfrak{g})$.

■

Theorem 3.3. *Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a Lie superalgebra for which $\mathrm{Rad}(\mathfrak{g})_0 = \{0\}$. Then, $\mathfrak{g}$ is isomorphic to one of the following superalgebras:*

(i) The semidirect product $\mathfrak{s} \oplus \mathfrak{a}(g)$ of the semisimple Lie superalgebra $\mathfrak{s} \simeq \mathfrak{sl}(V|\mathbb{F})$, $C(V)$, $P(V) \oplus \langle \partial_0 \rangle$, and the ideal $\mathfrak{a}(g)$. Actually, $\mathfrak{a}(g)$ is an $\mathfrak{m} \oplus \mathfrak{z}$-module. The $\mathfrak{m}$-action is trivial, and the $\mathfrak{z}$-action is defined via the linear map $g : \mathfrak{a}(g) \to \mathfrak{a}(g)$, through $\rho(z)(a) = g(a)$, including the cases $\mathfrak{s} = \mathfrak{sl}(V|\mathbb{F})$ and $\mathfrak{s} = C(V)$, for which $g = 0$. Finally, $\Gamma|_{\mathfrak{a}(g) \times \mathfrak{a}(g)} = 0$.

Furthermore, two such superalgebras $\mathfrak{s} \oplus \mathfrak{a}(g)$ and $\mathfrak{s}' \oplus \mathfrak{a}'(g')$ are isomorphic if and only if $\mathfrak{s} \simeq \mathfrak{s}'$ and if there exist a non-zero scalar $\alpha \in \mathbb{F} \setminus \{0\}$, and a linear map $S : \mathfrak{a}(g) \to \mathfrak{a}'(g')$ such that $g' = \alpha^{-1}(S \circ g \circ S^{-1})$.

(ii) $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ where $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0 \rangle$ is a subalgebra and $\langle \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ and $\mathfrak{a}(u; g)$ are $\mathfrak{m} \oplus \langle \delta_0 \rangle$-modules such that the $\mathfrak{m}$-action is trivial, and the action of δ_0 is given by $\rho(\delta_0)(\delta_1) = -\delta_1 + u$ for some $u \in \mathfrak{a}(u; g)$ and $\rho(\delta_0)(a) = g(a)$ for some linear map $g : \mathfrak{a}(u; g) \to \mathfrak{a}(u; g)$. In this case we have $\Gamma|_{\Pi\mathfrak{m} \times \mathfrak{a}(u;g)} = 0$ and $\Gamma|_{\langle \delta_1 \rangle \oplus \mathfrak{a}(u;g) \times \langle \delta_1 \rangle \oplus \mathfrak{a}(u;g)} = 0$.

Furthermore, two such superalgebras $\mathfrak{m} \oplus \Pi\mathfrak{m} \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}(u; g)$ and $\mathfrak{m}' \oplus \Pi\mathfrak{m}' \oplus \langle \delta_0, \delta_1 \rangle \oplus \mathfrak{a}'(u'; g')$ are isomorphic if and only if $\mathfrak{m} \simeq \mathfrak{m}'$ and if there exist a non-zero scalar $\beta \in \mathbb{F}$, a vector $u_1 \in \mathfrak{a}'(u'; g')$ and a linear map $S : \mathfrak{a}(u; g) \to \mathfrak{a}'(u', g')$ such that $g' = S \circ g \circ S^{-1}$ and $u' = \beta^{-1}(S(u) + (g' + \mathrm{id})u_1)$.

Proof. (i) Let $\Phi = T \oplus \widetilde{S} : \mathfrak{g} \to \mathfrak{g}'$ be an isomorphim of Lie superalgebras where $T : \mathfrak{m} \oplus \mathfrak{z} \to \mathfrak{m}' \oplus \mathfrak{z}$ is an isomorphism of Lie algebras and $\widetilde{S} : \mathfrak{s}_1 \oplus \mathfrak{a}(g) \to \mathfrak{s}'_1 \oplus \mathfrak{a}'(g')$ is an isomorphism of $\mathfrak{m} \oplus \mathfrak{z}$-modules. Since $\mathfrak{a}(g)$ is a trivial $\mathfrak{m}$-module it follows that, $\widetilde{S}(\mathfrak{a}(g)) = \mathfrak{a}'(g')$ and $\widetilde{S}(\mathfrak{s}_1) = \mathfrak{s}'_1$. Hence, $T \oplus \widetilde{S}|_{\mathfrak{s}_1} : \mathfrak{s} \to \mathfrak{s}'$ is an isomorphism of Lie superalgebras and $S = \widetilde{S}|_{\mathfrak{a}(g)} : \mathfrak{a}(g) \to \mathfrak{a}'(g')$ satisfies $S(\rho(z)(r)) = \rho'(T(z))(S(r))$, $r \in \mathfrak{a}(g)$. Hence, since $T = \widetilde{T} \oplus \alpha$ with $\widetilde{T} : \mathfrak{m} \to \mathfrak{m}'$ an isomorphism of Lie algebras and $\alpha \in \mathbb{F} \setminus \{0\}$, it follows that $g' = \alpha^{-1}(S \circ g \circ S^{-1})$. The converse statement follows by construction.

(ii) Let $\Phi = T \oplus \widetilde{S} : \mathfrak{g} \to \mathfrak{g}'$ be an isomorphim of Lie superalgebras, where $T : \mathfrak{m} \oplus \langle \delta_0 \rangle \to \mathfrak{m}' \oplus \langle \delta_0 \rangle$ is an isomorphism of Lie algebras and

$$\widetilde{S} : \Pi\mathfrak{m} \oplus \langle \delta_1 \rangle \oplus \mathfrak{a}(u; g) \to \Pi\mathfrak{m}' \oplus \langle \delta_1 \rangle \oplus \mathfrak{a}'(u'; g')$$

is an isomorphism of $\mathfrak{m} \oplus \mathfrak{z}$-modules. Similar to (i),

$$T \oplus \widetilde{S}|_{\Pi\mathfrak{m}} : \mathfrak{m} \oplus \langle \delta_0 \rangle \oplus \Pi\mathfrak{m} \to \mathfrak{m}' \oplus \langle \delta_0 \rangle \oplus \Pi\mathfrak{m}'$$

is an isomorphism of Lie superalgebras and $\widetilde{S}(\langle \delta_1 \rangle \oplus \mathfrak{a}(u; g)) = \langle \delta_1 \rangle \oplus \mathfrak{a}'(u'; g')$. Moreover, analogous to (i), it follows that $\widetilde{S}(\mathfrak{a}(u; g)) = \mathfrak{a}'(u'; g')$ and $S = \widetilde{S}|_{\mathfrak{a}(u,g)}$ satisfies $g' = \alpha(S \circ g \circ S^{-1})$. For this case a straightforward computation shows that $\alpha = 1$. Finally, $\widetilde{S}(\delta_1) = \beta\delta_1 + u_1$ for some $\beta \in \mathbb{F} \setminus \{0\}$ and $u_1 \in \mathfrak{a}'(u', g')$. Moreover, $\widetilde{S}(\rho(\delta_0)(\delta_1)) = \rho'(T(\delta_0))(\widetilde{S}(\delta_1))$. Hence, $u' = \beta(S(u) + (g' + \mathrm{id})u_1)$. ∎

4. Lie superalgebras $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ for which $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$

Let $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ be a Lie superalgebra for which $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$. Then, $\mathrm{Rad}(\mathfrak{g}) = \mathfrak{z} \oplus U$, where $U \subset \mathfrak{g}_1$ is an $\mathfrak{m} \oplus \mathfrak{z}$-module, and $\widetilde{\mathfrak{g}} \simeq \mathfrak{g}/\mathrm{Rad}(\mathfrak{g})$ is a semisimple Lie superalgebra with $\widetilde{\mathfrak{g}}_0 \simeq \mathfrak{m}$ (see the Appendix). Let W be an $\mathfrak{m}$-module such that $\mathfrak{g}_1 \simeq W \oplus U$. Then, $\widetilde{\mathfrak{g}} \simeq \mathfrak{m} \oplus W$ as vector spaces. Notice that $\rho(\mathfrak{z})(\mathfrak{g}_1) \subset \mathrm{Rad}(\mathfrak{g})_1 = U$, then we set $f = \rho(z)|_W : W \to U$. Hence, W is an $\mathfrak{m} \oplus \mathfrak{z}$-module if and only if $f = 0$. Thus, the structure of $\mathfrak{g}$, $(\mathfrak{m} \oplus \mathfrak{z}, \rho, \Gamma)$, can be written in terms of the structure of $\widetilde{\mathfrak{g}}$, $(\mathfrak{m}, \widetilde{\rho}, \widetilde{\Gamma})$, and the structure on $\mathrm{Rad}(\mathfrak{g})$ as follows:
The Lie algebra bracket on $\mathfrak{g}_0$ is given by,

$$[\cdot, \cdot] = [\cdot, \cdot]_{\mathfrak{m} \oplus \mathfrak{z}}.$$

The $\mathfrak{m}$-action on $\mathfrak{g}_1$ is given by:

$$\rho(A) = \widetilde{\rho}(A) \oplus \rho_R(A), \quad A \in \mathfrak{m},$$

where $\rho_R(A) = \rho(A)|_U$. The $\mathfrak{z}$-action on $\mathfrak{g}_1$ is given by:

$$\rho(z)(w + u) = f(w) + \rho_R(z)(u), \quad w \in W,\ u \in U,$$

where $\rho_R(z) = \rho(z)|_U$. Finally, Γ is given by:

$$\Gamma(w_1 + u_1, w_2 + u_2) = \widetilde{\Gamma}(w_1, w_2) + \mathcal{B}(w_1 + u_1, w_2 + u_2), \quad w_i, w_2 \in W,\ u_1, u_2 \in U,$$

where $\mathcal{B} : W \oplus U \times W \oplus U \to \mathfrak{z}$ is a bilinear, symmetric map.

Remark 4.1. From $(J1)$ it follows that $\mathcal{B}$ is ρ-invariant, i.e.

$$\mathcal{B}(\rho(A)u, v) + \mathcal{B}(u, \rho(A)v) = 0, \quad A \in \mathfrak{m} \oplus \mathfrak{z}, \quad u, v \in \mathfrak{g}_1.$$

Proposition 4.2. *Let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *be a Lie superalgebra for which* $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$. *Using the notation above, we have:*

(i) If $\mathcal{B}|_{U\times U} \neq 0$, *then the* $\mathfrak{z}$*-action on* $\mathfrak{g}_1$ *is trivial, in particular* ρ *is completely reducible.*

(ii) If $\mathcal{B}|_{U\times U} = 0$, *and* $\mathcal{B}|_{W\times U} \neq 0$, *then* $\mathrm{Rad}(\mathfrak{g})$ *is an abelian ideal. Moreover, if* $f \neq 0$, *then* $\mathcal{B}|_{W\times W} = 0$.

(iii) If $\mathcal{B}|_{U\times U} = 0$, $\mathcal{B}|_{W\times U} = 0$, *and* $\mathcal{B}|_{W\times W} \neq 0$, *then* ρ *is completely reducible. Moreover,* U *is a trivial* $\mathfrak{m}\oplus\mathfrak{z}$*-module.*

(iv) If $\mathcal{B} = 0$, *then* U *is a trivial* $\mathfrak{m}$*-module.*

Proof. Since $\mathfrak{z}$ is one-dimensional, we shall assume throughout the proof that $\mathcal{B}$ is $\mathbb{F}$-valued.

(i) Note first that for any $u \in U$, and any $v_1, v_2 \in \mathfrak{g}_1$,

$$\mathcal{B}(u,v_1)\rho(z)(v_2) + [\Gamma(v_1,v_2),u] + \mathcal{B}(u,v_2)\rho(z)(v_1) = 0, \tag{5}$$

which follows from the identity (J2). Then, taking $u_0 \in U$ such that $\mathcal{B}(u_0,u_0) \neq 0$, and $v_1 = v_2 = u = u_0$, it follows that $\rho_R(z)(u_0) = 0$. Hence, using the identity (5) one more time, and taking $v_2 = u = u_0$, it follows that $\rho(z)(v_1) = 0$, $v_1 \in \mathfrak{g}_1$. Thereby, ρ is completely reducible.

(ii) For the first statement, it is enough to prove that the $\mathfrak{z}$-action on U is trivial. Let $w_0 \in W$ and $u_0 \in U$ be such that $\mathcal{B}(w_0,u_0) = 1$. Setting $v_1 = w_0$ and $v_2 = u_0$ in the identity (5), and since $\mathcal{B}|_{U\times U} = 0$, it follows that

$$\mathcal{B}(u,w_0)\rho_R(z)(u_0) + \mathcal{B}(w_0,u_0)\rho_R(z)(u) = 0, \quad u \in U.$$

For the particular case $u = u_0$, it follows that $\rho_R(z)(u_0) = 0$. Hence, $\rho_R(z)(u) = 0$, for any $u \in U$. Thus, $\mathrm{Rad}(\mathfrak{g})$ is an abelian ideal.

On the other hand, for any $w_1, w_2, w_3 \in W$, we have,

$$\mathcal{B}(w_1,w_2)f(w_3) + \mathcal{B}(w_2,w_3)f(w_1) + \mathcal{B}(w_1,w_3)f(w_2) = 0, \tag{6}$$

which follows from the identity (J2). Now, let $w_0 \in W$ be such that $f(w_0) \neq 0$. Putting $w_1 = w_2 = w_3 = w_0$ in (6), it follows that $\mathcal{B}(w_0,w_0) = 0$. Now setting $w_1 = w_2 = w_0$ in the identity (6), it follows that $\mathcal{B}(w_0,w_3) = 0$, for any $w_3 \in W$. Finally, putting $w_1 = w_0$ in the identity (6), it follows that $\mathcal{B}|_{W\times W} = 0$.

(iii) First it will be proved that W is a completely reducible $\mathfrak{m}\oplus\mathfrak{z}$-module. It is enough to prove that the $\mathfrak{z}$-action on W, given by f, is trivial. Let $w_0 \in W$ be such that $\mathcal{B}(w_0,w_0) = 1$. Taking $w_1 = w_2 = w_3 = w_0$ in the identity (6), it follows that $f(w_0) = 0$. Using in the identity (6) one more time, putting $w_1 = w = w_0$, it follows that $f = 0$. Hence, W is a completely reducible $\mathfrak{m}\oplus\mathfrak{z}$-module.

Now it will be proved that U is a completely reducible $m \oplus \mathfrak{z}$-module. It follows from (J2), under the current hypotheses, that

$$\rho_R(\mathfrak{m})(U) \subset \rho_R(\Gamma(W,W))(U) \subset f(\Gamma(W,U))(W) = 0.$$

Hence, U is a trivial $\mathfrak{m}$-module, On the other hand, from (5), it follows that

$$\rho_R(\widetilde{\Gamma}(w_0,w_0))(u) = -\mathcal{B}(w_0,w_0)\rho_R(\mathfrak{z})(u), \ u \in U.$$

Then, since $\mathcal{B}(w_0,w_0) = 1$ and $\widetilde{\Gamma}(w_0,w_0) \in \mathfrak{m}$, it follows that $\rho_R(\mathfrak{z}) = 0$. Thus, U is a trivial $\mathfrak{m} \oplus \mathfrak{z}$-module. In particular, U is completely reducible.

(iv) The same argument used in (iii) shows that U is a trivial $\mathfrak{m}$-module.

■

Corollary 4.3. *Let* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *be a Lie superalgebra for which* $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$.

(i) If ρ *is completely reducible, then the* $\mathfrak{m}$*-action on* U *is trivial.*

(ii) The solvability degree of $\mathrm{Rad}(\mathfrak{g})$ *is at most two.*

Proof. Let $\mathcal{B}$ be the bilinear symmetric map associated to $\mathfrak{g}$.

(i) From Proposition 4.2.(iv), it is enough to assume that $\mathcal{B} \neq 0$. Then, by Proposition 4.2.(i), (ii), and (iii), it follows that the $\mathfrak{z}$-action on U is trivial, and since ρ is completely reducible, i.e. $f = 0$, we obtain that the $\mathfrak{z}$-action on $\mathfrak{g}_1$ is trivial. Hence, from the identity (5) it follows that,

$$\rho_R(\widetilde{\Gamma}(w_1,w_2))(u) = 0, \quad u \in U, \ w_1,w_2 \in W.$$

Let $A \in \mathfrak{m}$. Since $\widetilde{\mathfrak{g}}$ is semisimple, it follows that A can be written as a finite linear combination of products of the form $\widetilde{\Gamma}(w_i,w_j)$ for some $w_i,w_j \in W$. Hence, $\rho_R(A) = 0$. Thereby, the $\mathfrak{m}$-action on U is trivial.

(ii) The first derived ideal of $\mathrm{Rad}(\mathfrak{g})$ satisfies:

$$(\mathrm{Rad}(\mathfrak{g})^{(1)})_0 = \mathrm{Span}\{\mathcal{B}(u,v) | u,v \in U\} \quad \text{and} \quad (\mathrm{Rad}(\mathfrak{g})^{(1)})_1 = \mathrm{Im}(\rho_R(z)).$$

Notice that if $\mathcal{B}|_{U\times U} \neq 0$, then by Proposition 4.2.(i), it follows that $\mathrm{Im}(\rho_R(z)) = 0$. Hence, clearly $\mathrm{Rad}(\mathfrak{g})^{(2)} = 0$. Finally, if $\mathcal{B}|_{U\times U} = 0$, then $(\mathrm{Rad}(\mathfrak{g})^{(1)})_0 = 0$ and as a consequence $\mathrm{Rad}(\mathfrak{g})^{(2)} = 0$.

■

Using Proposition 4.2, Corollary 4.3, and the description of semisimple Lie superalgebras of the type $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{g}_1$, given in the Appendix, the following main result is obtained.

Theorem 4.4. *Let* $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{z} \oplus \mathfrak{g}_1$ *be a Lie superalgebra for which* $\mathrm{Rad}(\mathfrak{g})_0 = \mathfrak{z}$. *Then,* $\mathfrak{g}$ *is isomorphic to one of the following superalgebras.*

(i) $\mathfrak{s}\oplus\mathfrak{z}\oplus\mathfrak{a}(\mathcal{B})$ *where* $\mathfrak{s}\simeq\mathfrak{m}$, $P(V)$, $Q(V)$, $Q(V)\oplus\langle\partial_1\rangle$, $\mathfrak{m}\oplus\Pi(\mathfrak{m})\oplus\langle\delta_1\rangle$, *and* $\mathfrak{a}(\mathcal{B})$ *is a trivial* $\mathfrak{m}\oplus\mathfrak{z}$*-module. The Lie algebra bracket on* $\mathfrak{g}_0$ *is given via the Lie bracket on* $\mathfrak{s}_0$ *as* $[(x,z),(y,z)]=[x,y]_{\mathfrak{s}_0}$*; the* $\mathfrak{s}_0$*-action on* $\mathfrak{s}_1$ *is given as in* $\mathfrak{s}$*, and the* $\mathfrak{z}$*-action on* $\mathfrak{s}_1$ *is trivial. Finally* Γ *is given in terms of a symmetric, bilinear,* $\mathfrak{g}_0$*-invariant, non-zero map*

$$\mathcal{B}:\mathfrak{s}_1\oplus\mathfrak{a}(g,\mathcal{B})\times\mathfrak{s}_1\oplus\mathfrak{a}(g,\mathcal{B})\to\mathfrak{z},$$

as $\Gamma(u+a,u'+a')=\Gamma_{\mathfrak{s}}(u,u')+\mathcal{B}(u+a,u'+a')$.

Two such superalgebras $\mathfrak{s}\oplus\mathfrak{z}\oplus a(\mathcal{B})$ *and* $\mathfrak{s}'\oplus\mathfrak{z}\oplus a'(\mathcal{B}')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi=T\oplus S_1:\mathfrak{s}\to\mathfrak{s}'$*, a non-zero scalar* $\alpha\in\mathbb{F}\setminus\{0\}$*, and a linear map* $S_2:\mathfrak{a}(\mathcal{B})\to\mathfrak{a}'(\mathcal{B}')$*, such that* $\mathcal{B}'(S_1(u)+S_2(a),S_1(u')+S_2(a'))=\alpha\mathcal{B}(u+a,u'+a')$*.*

(ii) $\mathfrak{s}\oplus\mathfrak{z}\oplus\mathfrak{a}(g)$ *where* $\mathfrak{s}\simeq\mathfrak{m}$, $P(V)$, $Q(V)$, $Q(V)\oplus\langle\partial_1\rangle$, $\mathfrak{m}\oplus\Pi(\mathfrak{m})\oplus\langle\delta_1\rangle$, *and* $\mathfrak{a}(g)$ *is an* $\mathfrak{m}\oplus\mathfrak{z}$*-module for which the* $\mathfrak{m}$*-action is trivial, and the* $\mathfrak{z}$*-action is defined via a linear map* $g:\mathfrak{a}(g)\to\mathfrak{a}(g)$ *through* $\rho(z)a=g(a)$*. The Lie algebra bracket on* $\mathfrak{g}_0$ *is given as in (i). The* $\mathfrak{s}_0$*-action on* $\mathfrak{s}_1$ *is given as in* $\mathfrak{s}$*, and the* $\mathfrak{z}$*-action on* $\mathfrak{s}_1$ *is trivial. Finally* Γ *is given by* $\Gamma(u+a,u'+a')=\Gamma_{\mathfrak{s}}(u,u')$*.*

Two such superalgebras $\mathfrak{s}\oplus\mathfrak{z}\oplus a(g)$ *and* $\mathfrak{s}'\oplus\mathfrak{z}\oplus a'(g')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi=T\oplus S_1:\mathfrak{s}\to\mathfrak{s}'$*, a non-zero scalar* $\alpha\in\mathbb{F}\setminus\{0\}$*, and a linear map* $S_2:\mathfrak{a}(g)\to\mathfrak{a}'(g')$*, such that* $g'=\alpha(S_2\circ g\circ S_2^{-1})$*.*

(iii) $\mathfrak{s}\oplus\mathfrak{z}\oplus\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})$ *where* $\mathfrak{s}\simeq P(V)$, $Q(V)$, $Q(V)\oplus\langle\partial_1\rangle$, $\mathfrak{m}\oplus\Pi\mathfrak{m}\oplus\langle\delta_1\rangle$, *and* $\mathfrak{s}_1\oplus\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})$*, and* $\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})$ *are* $\mathfrak{m}\oplus\mathfrak{z}$*-modules for which the* $\mathfrak{m}$*-action on* $\mathfrak{s}_1$ *is given as in* $\mathfrak{s}$*, and the* $\mathfrak{z}$*-action on* $\mathfrak{s}_1$ *is given by a non-zero linear map* $f:\mathfrak{s}_1\to\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})$ *as* $\rho(z)(w)=f(w)$*. The* $\mathfrak{m}$*-action on* $\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})$ *is given by* $\rho_{\mathfrak{a}}:\mathfrak{m}\to\mathfrak{gl}(\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C}))$ *and the* $\mathfrak{z}$*-action is trivial. The Lie algebra bracket on* $\mathfrak{g}_0$ *is given as in (i), and* Γ *is given in terms of a bilinear* $\mathfrak{g}_0$*-invariant, non-zero map* $\mathcal{C}:\mathfrak{s}_1\times\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})\to\mathfrak{z}$ *by* $\Gamma(w+u,w'+u')=\Gamma_{\mathfrak{s}}(w,w')+\mathcal{C}(w,u')+\mathcal{C}(w',u)$*.*

Two such superalgebras $\mathfrak{s}\oplus\mathfrak{z}\oplus a(f,\rho_{\mathfrak{a}},\mathcal{C})$ *and* $\mathfrak{s}'\oplus\mathfrak{z}\oplus a'(f',\rho'_{\mathfrak{a}},\mathcal{C}')$ *are isomorphic if and only if there exist an isomorphism of Lie superalgebras* $\Phi=T\oplus S_1:\mathfrak{s}\to\mathfrak{s}'$*, an isomorphism of* $\mathfrak{m}$*-modules* $S_2:\mathfrak{a}(f,\rho_{\mathfrak{a}},\mathcal{C})\to\mathfrak{a}'(f',\rho'_{\mathfrak{a}'},\mathcal{C}')$*, a non-zero scalar* $\alpha\in\mathbb{F}\setminus\{0\}$*, and a linear map* $S_{21}:\mathfrak{s}_1\to\mathfrak{a}'(f',\rho'_{\mathfrak{a}'},\mathcal{C}')$ *such that* $f'=\alpha^{-1}(S_2\circ f\circ S_1)$*,* $\mathcal{C}'(\cdot,\cdot)=\alpha\mathcal{C}(S_1^{-1}(\cdot),S_2^{-1}(\cdot))$*, and* $\mathcal{C}'(S_1(w_1),S_{21}(w_2))+\mathcal{C}'(S_1(w_2),S_{21}(w_1))=0$*, for any* $w_1,w_2\in\mathfrak{s}_1$*.*

(iv) $\mathfrak{s}\oplus\mathfrak{z}\oplus\mathfrak{a}(f,g)$ *where* $\mathfrak{s}\simeq P(V)$, $Q(V)$, $Q(V)\oplus\langle\partial_1\rangle$, $\mathfrak{m}\oplus\Pi\mathfrak{m}\oplus\langle\delta_1\rangle$, *and* $\mathfrak{s}_1\oplus\mathfrak{a}(f,g)$ *and* $\mathfrak{a}(f,g)$ *are* $\mathfrak{m}\oplus\mathfrak{z}$*-modules for which the* $\mathfrak{m}$*-action on* $\mathfrak{s}_1$ *is given in* $\mathfrak{s}$*, and the* $\mathfrak{z}$*-action on* $\mathfrak{s}_1$ *is given by a non-zero linear map* $f:\mathfrak{s}_1\to\mathfrak{a}(f,g)$ *as* $\rho(z)(w)=f(w)$*. The* $\mathfrak{m}$*-action on* $\mathfrak{a}(f,g)$ *is trivial and the* $\mathfrak{z}$*-action on* $\mathfrak{a}(f,g)$ *is defined by the linear map* $g:\mathfrak{a}(f,g)\to\mathfrak{a}(f,g)$ *by* $\rho(z)a=g(a)$*. The Lie algebra bracket on* $\mathfrak{g}_0$ *is given as in (i), and* Γ *is given as in (ii).*

Two such superalgebras $\mathfrak{s}\oplus\mathfrak{z}\oplus a(f,g)$ and $\mathfrak{s}'\oplus\mathfrak{z}\oplus a'(f',g')$ are isomorphic if and only if there exist an isomorphism of Lie superalgebras $\Phi = T\oplus S_1 : \mathfrak{s}\to\mathfrak{s}'$, two linear maps $S_2 : \mathfrak{a}(f,g)\to\mathfrak{a}'(f',g')$ and $S_{21} : \mathfrak{s}_1\to\mathfrak{a}'(f',g')$, and a non-zero scalar $\alpha\in\mathbb{F}\setminus\{0\}$, such that $f' = \alpha^{-1}(S_2\circ f-\alpha g\circ S_{21})\circ S_1^{-1}$ and $g' = \alpha^{-1}(S_2\circ g\circ S_2^{-1})$.

The statements regarding the isomorphisms between the Lie superalgebras referred to in (i)-(iv) are proved in exactly the same way as in Theorem 3.3.

APENDIX

Lie superalgebras $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ for which $\mathfrak{g}_0$ is a simple Lie algebra

In this section finite dimensional Lie superalgebras $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ for which $\mathfrak{g}_0$ is a simple Lie algebra are described. We also provide a list of those semisimple Lie superalgebras with $\mathfrak{g}_0$ simple (see [2]).

Theorem 4.5. *Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a Lie superalgebra for which $\mathfrak{g}_0$ is a simple Lie algebra and $\mathfrak{g}_1$ is a finite dimensional $\mathfrak{g}_0$-module. Then, $\mathfrak{g}$ has one of the following structures:*

(i) $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ is a semidirect product of the simple Lie algebra $\mathfrak{g}_0$ and the $\mathfrak{g}_0$-module $\mathfrak{g}_1$; in this case $\Gamma=0$.

(ii) $\mathfrak{g}\simeq\mathfrak{s}\oplus\mathfrak{h}$ is a direct sum of two ideals, with $s\neq\mathfrak{g}_0$ a semisimple Lie superalgebra for which $\mathfrak{s}_0\simeq\mathfrak{g}_0$ and $\mathfrak{h}$ is a trivial $\mathfrak{g}_0$-module for which $\Gamma|_{\mathfrak{h}\times\mathfrak{h}}=0$.

Proof. Since $\mathfrak{g}_0$ is a simple Lie algebra, it follows that $\mathrm{Rad}(\mathfrak{g})_0=0$. Then, $\mathrm{Rad}(\mathfrak{g})\subset\mathfrak{g}_1$ is a $\mathfrak{g}_0$-module. Let W be a $\mathfrak{g}_0$-module such that $\mathfrak{g}_1=W\oplus\mathrm{Rad}(\mathfrak{g})$. Then $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})\simeq\mathfrak{g}_0\oplus W$ is a semisimple Lie superalgebra.
Claim. If $W\neq 0$, then $\mathfrak{g}_0\oplus W$ is an ideal of $\mathfrak{g}$ and $\mathrm{Rad}(\mathfrak{g})=\mathrm{Z}(\mathfrak{g})$, the center of $\mathfrak{g}$. In fact, notice that $\Gamma(W,W)$ is an ideal of $\mathfrak{g}_0$ and $\Gamma(W,W)\oplus W$ is an ideal of $\mathfrak{g}_0\oplus W$. Then, since $\mathfrak{g}_0$ is simple and $\mathfrak{g}_0\oplus W$ is semisimple, it follows that $\Gamma(W,W)=\mathfrak{g}_0$. Hence, using (J2) it follows that:

$$\rho(\mathfrak{g}_0)(\mathrm{Rad}(\mathfrak{g}))=\rho(\Gamma(W,W))(\mathrm{Rad}(\mathfrak{g}))\subset\rho(\Gamma(W,\mathrm{Rad}(\mathfrak{g})))(W)=0.$$

This shows that $\mathfrak{g}_0\oplus W$ is an ideal of $\mathfrak{g}$ and $\mathrm{Rad}(\mathfrak{g})\subset\mathrm{Z}(\mathfrak{g})$. Therefore, $\mathrm{Rad}(\mathfrak{g})=\mathrm{Z}(\mathfrak{g})$.
On the other hand, assume that $W=0$. Then $\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})\simeq\mathfrak{g}_0$. In this case there are no restrictions for the $\mathfrak{g}_0$-module $\mathrm{Rad}(\mathfrak{g})$, and since it is an odd ideal of $\mathfrak{g}$, it follows that $\Gamma=0$. Hence $\mathfrak{g}=\mathfrak{g}_0\oplus\mathrm{Rad}(\mathfrak{g})$ is a semidirect product. ∎

The same technique used in section 2 is applied here for finding all semisimple Lie superalgebras $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ for which $\mathfrak{g}_0$ is a simple Lie algebra. In this context the following result holds, whose proof is analogous to the proof of Lemma 2.1.

Lemma 4.6. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra for which $\mathfrak{g}_0$ is a simple Lie algebra and let $I \subset \mathfrak{g}$ be a minimal ideal. Then, I is unique and I is a simple Lie superalgebra for which $I_0 \simeq \mathfrak{g}_0$ or $I \simeq \mathfrak{g}_0 \oplus \Pi\mathfrak{g}_0$.*

Proposition 4.7. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a semisimple Lie superalgebra for which $\mathfrak{g}_0$ is a simple Lie algebra. Then, $\mathfrak{g}$ is a simple Lie algebra or $\mathfrak{g}$ is isomorphic to one of the superalgebras $P(V)$, $Q(V)$, $Q(V) \oplus \langle \partial_1 \rangle$ or $\mathfrak{g}_0 \oplus \Pi\mathfrak{g}_0 \oplus \langle \delta_1 \rangle$ (see table 1) where V is a vector space of dimension greater than two.*

Proof. Let I be the minimal ideal of $\mathfrak{g}$. Then, if I is a simple Lie superalgebra, then I is isomorphic to the simple Lie algebra $\mathfrak{g}_0$ or, using the results given in [3], I is isomorphic to one of the superalgebras $P(V)$ or $Q(V)$ where $\dim(V) \geq 3$. On the other hand, since $I \subset \mathfrak{g} \subset \mathrm{Der}(I)$, Theorem 2.3 can be applied to every possible minimal ideal. Hence, we end up with the following cases:

(i) **Case $I \simeq \mathfrak{g}_0$.** If $I \simeq \mathfrak{g}_0$, then $\mathfrak{g}_0 \subset \mathfrak{g} \subset \mathrm{Der}(\mathfrak{g}_0)$. Hence, $\mathfrak{g} \simeq \mathfrak{g}_0$ is a simple Lie algebra.

(ii) **Case $I \simeq P(V)$.** For this case we have that $\mathrm{Der}(P(V)) = P(V) \oplus \langle \partial_0 \rangle$. Then $P(V) \subset \mathfrak{g} \subset P(V) \oplus \langle \partial_0 \rangle$. Hence, $\mathfrak{g} \simeq P(V)$.

(iii) **Case $I \simeq Q(V)$.** In this case $\mathrm{Der}(Q(V)) = Q(V) \oplus \langle \partial_1 \rangle$. Then, $Q(V) \subset \mathfrak{g} \subset Q(V) \oplus \langle \partial_1 \rangle$. Hence, $\mathfrak{g} \simeq Q(V)$ or $\mathfrak{g} \simeq Q(V) \oplus \langle \partial_1 \rangle$.

(iv) **Case $I \simeq \mathfrak{g}_0 \oplus \Pi\mathfrak{g}_0$.** Using the notation of the proof of Proposition 2.5, we have $\mathrm{Der}(\mathfrak{g}_0 \otimes \Lambda(\mathbb{F})) \simeq \mathfrak{g}_0 \otimes \Lambda(\mathbb{F}) \oplus \langle \mathrm{id} \otimes d_0, \mathrm{id} \otimes d_1 \rangle$. Then, $\mathfrak{g}_0 \otimes \Lambda(\mathbb{F}) \subset \mathfrak{g} \subset \mathfrak{g}_0 \otimes \Lambda(\mathbb{F}) \oplus \langle \mathrm{id} \otimes d_0, \mathrm{id} \otimes d_1 \rangle$. Notice that $\Lambda(\mathbb{F})$ contains no non-trivial ideals invariant by $\langle \mathrm{id} \otimes d_1 \rangle$. Hence, $\mathfrak{g} \simeq \mathfrak{g}_0 \otimes \Lambda(\mathbb{F}) \oplus \langle \mathrm{id} \otimes d_1 \rangle \simeq \mathfrak{g}_o \oplus \Pi\mathfrak{g}_0 \oplus \langle \delta_1 \rangle$. ■

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References

[1] Cheng, S.-J., *Differentiably Simple Lie Superalgebras and Representation of Semisimple Lie Superalgebras*, J. of Algebra **173** (1995), 1–43.

[2] Elduque, A., *Lie Superalgebras with Semisimple Even Part*, J. of Algebra **183** (1996), 649–663.

[3] Kac, V.G., *Lie Superalgebras*, Advances in Mathematics **26** (1977), 8–96.

[4] Salgado, G., and O. A. Sánchez-Valenzuela, *Lie Superalgebras over* $\mathfrak{gl}(2)$, Communications in Algebra **39** (2011), 2114–2136.

[5] —, *Lie Superalgebras Based on* $\mathfrak{gl}_n$ *Associated to the Adjoint Representation and Invariant Geometric Structure Defined on Them*, Communications in Mathematical Physics **241** (2003), 505–518.

[6] Scheunert, M., "The Theory of Lie Superalgebras, an Introduction," Lecture Notes in Mathematics **716**, Springer-Verlag, New York, 1979.

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