

## Rigidity of Formal Characters of Lie algebras of type $A$

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**Abstract.** For a complex simple Lie algebra of type  $A$ , given a family of elements  $f_\lambda \in \mathbb{Z}[\Lambda]$ ,  $\lambda \in \Lambda^+$ , we show that  $f_\lambda$  is just the formal character of the Weyl module  $V(\lambda)$  if  $f_\lambda$  satisfies two natural conditions.

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### 1. Introduction

Let  $\mathfrak{g}_l$  be a complex simple Lie algebra of type  $A_l$ , and let  $V(\lambda)$  be the Weyl module. The formal character  $\text{ch}_\lambda$  of  $V(\lambda)$  is determined by the Weyl character formula or by other methods, such as the Freudenthal formula. The Littlewood-Richardson coefficient,  $c_{\mu,\nu}^\lambda$  defined the multiplicity of  $V(\lambda)$  in  $V(\mu) \otimes V(\nu)$ , can be determined according to the formal character  $\text{ch}_\lambda$ . In this paper we shall consider the following problem:

*Given a family of elements  $f_\lambda \in \mathbb{Z}[\Lambda]$ ,  $\lambda \in \Lambda^+$ , when are they equal to  $\text{ch}_\lambda$ ?*

We prove that two natural conditions are enough to do so. One condition says it should contain right information about the Weyl module for proper subalgebras of  $\mathfrak{g}_l$ , that is to say,  $f_\lambda$  for all  $\beta = \lambda - \mu$ ,  $m_\lambda(\mu) = n_\lambda(\mu)$ , if  $|\text{Supp}(\beta)| < l$ . The other says the number  $n_{\mu,\nu}^\lambda$ , defined similarly to  $c_{\mu,\nu}^\lambda$ , should satisfy  $n_{\mu,\nu}^\lambda = n_{\lambda,-w_0\nu}^\mu$ .

The motivation for this problem comes from the modular representation theory of linear algebraic groups in positive characteristic, especially the Lusztig conjecture. In Lusztig's theory, the  $\text{ch} L(\lambda)$  is given by the Kazhdan-Lusztig polynomials and  $\text{ch} V(\mu)$ 's. The first condition on  $\text{ch} L(\lambda)$  can be proved by an induction on rank. So we want to find out some natural condition to ensure the conjecture. Hence we consider the problem in complex field at first, the most simple case.

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## 2. Notation

Let  $\mathfrak{g}_l$  be a complex simple Lie algebra of type  $A_l$ , and let  $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\}$  be the set of simple roots. Let  $\Lambda$  be the set of weights, and  $\omega_1, \omega_2, \dots, \omega_l$  the set of fundamental dominant weights. Then the set of dominant weights is denoted by  $\Lambda^+$ . Let  $\text{ch}_\lambda$ ,  $\lambda \in \Lambda^+$ , be the formal character of the Weyl module  $V(\lambda)$ , they form a free  $\mathbb{Z}$ -module of the commutative ring  $\mathbb{Z}[\Lambda]$ , with base  $\{e(\lambda), \lambda \in \Lambda\}$ , and multiplication  $e(\lambda) * e(\mu) = e(\lambda + \mu)$ . Let  $W$  be the Weyl group, action on  $\mathbb{Z}[\Lambda]$  naturally as  $\sigma e(\lambda) = e(\sigma\lambda)$ . Set

$$\mathbb{Z}[\Lambda]^W = \{f \in \mathbb{Z}[\Lambda] \mid wf = f, w \in W\}.$$

Let  $W_\lambda$  be the  $W$ -orbit of  $\lambda$  and  $h(\lambda) = \sum_{x \in W_\lambda} e(x)$ . Let  $\Pi(\lambda)$  be the set of saturated weights of weight  $\lambda$  and  $\Pi^+(\lambda) = \Pi(\lambda) \cap \Lambda^+$ . It is well known that

$$\text{ch}_\lambda = \sum_{x \in \Pi(\lambda)} m_\lambda(x) e(x) = \sum_{\mu \in \Pi^+(\lambda)} m_\lambda(\mu) h(\mu), \quad \lambda \in \Lambda^+,$$

forms a basis of  $\mathbb{Z}[\Lambda]^W$ .

Recall that  $w_0$  is the longest element of  $W$  with the action  $w_0\omega_i = -\omega_{l-i}$ . Let  $c_{\mu,\nu}^\lambda$  be the Littlewood-Richardson coefficient. According to the complete reducibility of  $\mathfrak{g}_l$ -modules and

$$\dim \text{hom}(U \otimes V, W) = \dim \text{hom}(U, W \otimes V^*),$$

we have

$$[V(\mu) \otimes V(\lambda) : V(\nu)] = [V(\nu) \otimes V(\lambda)^* : V(\mu)] = [V(\nu) \otimes V(-w_0\lambda) : V(\mu)].$$

Hence

$$c_{\mu,\lambda}^\nu = c_{\nu,-w_0\lambda}^\mu.$$

## 3. Main results

**3.1.** Let  $\beta = \sum_{i=1}^l k_i \alpha_i$ , define

$$\text{Supp}(\beta) = \{\alpha_i \mid k_i \neq 0\}.$$

For  $\lambda \in \Lambda^+$ , set  $f_\lambda = \sum_{\mu \in \Pi^+(\lambda)} n_\lambda(\mu) h(\mu) = \sum_{x \in \Pi(\lambda)} n_\lambda(x) e(x) \in \mathbb{Z}[\Lambda]^W$  satisfied  $n_\lambda(\lambda) = 1, n_\lambda(x) = 0$ , if  $x \notin \Pi(\lambda)$ . Then  $f_\lambda$  is also a basis of  $\mathbb{Z}[\Lambda]^W$ . Hence there exists unique  $n_{\mu,\nu}^\lambda$ , such that

$$f_\mu * f_\nu = \sum_{\lambda \in \Pi^+(\mu+\nu)} n_{\mu,\nu}^\lambda f_\lambda.$$

By the definition of  $f_\lambda$ 's, we have

$$n_{\mu,\nu}^{\mu+\nu} = 1; n_{\mu,\nu}^\lambda = n_{\nu,\mu}^\lambda.$$

**Theorem 3.1.** Let  $f'_\lambda$ s,  $\lambda \in \Lambda^+$  be a basis of  $\mathbb{Z}[\Lambda]^W$  of the following form

$$f_\lambda = \sum_{x \in \Pi(\lambda)} n_\lambda(x)e(x), n_\lambda(\lambda) = 1.$$

If these  $f'_\lambda$ s satisfy the following two conditions:

(1) for all  $\beta = \lambda - \mu, m_\lambda(\mu) = n_\lambda(\mu)$ , if  $|\text{Supp}(\beta)| < l$ .

(2)  $n_{\mu,\nu}^\lambda = n_{\lambda,-w_0\nu}^\mu$ .

Then  $f_\lambda = ch_\lambda, n_{\mu,\nu}^\lambda = c_{\mu,\nu}^\lambda$ .

The two conditions in theorem 3.1 are very natural. The first one comes from restricting  $V(\lambda)$  to subalgebra with smaller rank and the second one is because the tensor-hom adjunction. So it is surprised that  $ch_\lambda, n_{\mu,\nu}^\lambda$  can be determined completely only by two conditions. Hence we call this property as the rigidity of formal characters or representation ring of Lie algebra.

Firstly, we need a lemma.

**Lemma 3.2.** For  $f_\lambda$ , the numbers  $n_{\mu,\nu}^\lambda$  are uniquely determined by  $n_\lambda(\mu)$ , and vice versa.

**Proof.** Because  $f_\mu * f_\nu \in \mathbb{Z}[\Lambda]^W$  and  $f_\lambda, \lambda \in \Lambda^+$  form the basis of  $\mathbb{Z}[\Lambda]^W$ , so  $f_\mu * f_\nu$  can be written in terms of a linear combination of base elements  $f_\lambda, \lambda \in \Lambda^+$ , i.e.

$$f_\mu * f_\nu = \sum_{\lambda \in \Pi^+(\mu+\nu)} n_{\mu,\nu}^\lambda f_\lambda.$$

Therefore all these  $n_{\mu,\nu}^\lambda$  are uniquely determined.

Now suppose that we know  $n_{\mu,\nu}^\lambda$  already, and will determine  $n_\lambda(\mu)$  recursively according to a suitable partial order in  $\Lambda^+$  as follow.

Let  $\lambda, \mu \in \Lambda^+$ , define  $\mu < \lambda$  if  $\mu \prec \lambda$  or  $\lambda - \mu \in \Lambda^+$ .

For the fundamental weights  $\omega_i$  and 0, their saturated weight set  $\Pi(\omega_i)$  only contains one dominate weight. So

$$f_{\omega_i} = h(\omega_i), f_0 = h(0) = e(0).$$

Suppose that  $\lambda \in \Lambda^+$  does not be a fundamental dominate weight or 0, then there exist  $\mu, \nu \in \Lambda^+$ , such that

$$\lambda = \mu + \nu, \mu < \lambda, \nu < \lambda.$$

Then

$$f_\mu * f_\nu = f_\lambda + \sum_{s \in \Pi^+(\lambda), s \neq \lambda} n_{\mu,\nu}^s f_s,$$

i.e.

$$f_\lambda = f_\mu * f_\nu - \sum_{x \in \Pi^+(\lambda), x \neq \lambda} n_{\mu,\nu}^x f_x.$$

Therefore we can get a precise expression for  $f_\lambda$  because all  $f_\mu, f_\nu, f_s, n_{\mu,\nu}^s$  can be determined recursively as  $\mu < \lambda, \nu < \lambda, s < \lambda$ . The lemma is proved. ■

**Remark.** This lemma doesn't work for type  $B_l$ . For example, for  $B_2$ , we can't determine  $(1, 0)_{(0,0)}$  by using the above method because there is no way to decompose  $(1, 0)$  into a sum of two non-trivial dominate weights.

**3.2.** From the equation  $f_\lambda = f_\mu * f_\nu - \sum_{x \in \Pi^+(\lambda), s \neq \lambda} n_{\mu, \nu}^s f_s$ , we have more precise expression.

$$\begin{aligned} f_\lambda &= \sum_{t \in \Pi(\lambda)} n_\lambda(t) e(t) \\ &= \sum_{y \in \Pi(\mu)} n_\mu(y) e(y) * \sum_{z \in \Pi(\nu)} n_\nu(z) e(z) - \sum_{s \in \Pi^+(\lambda), s \neq \lambda} n_{\mu, \nu}^s \sum_{x \in \Pi(s)} n_s(x) e(x) \\ &= \sum_{y \in \Pi(\mu), z \in \Pi(\nu)} n_\mu(y) n_\nu(z) e(y+z) - \sum_{s, x \in \Pi(\lambda), s \neq \lambda} n_{\mu, \nu}^s n_s(x) e(x) \\ &= \sum_{t \in \Pi(\lambda)} \left( \sum_{y \in \Pi(\mu), z \in \Pi(\nu), y+z=t} n_\mu(y) n_\nu(z) \right) e(t) - \\ &\quad - \sum_{t \in \Pi(\lambda)} \left( \sum_{s \in \Pi(\lambda), s \neq \lambda} n_{\mu, \nu}^s n_s(t) \right) e(t). \end{aligned}$$

Hence

$$n_\lambda(t) = \sum_{y \in \Pi(\mu), z \in \Pi(\nu), y+z=t} n_\mu(y) n_\nu(z) - \sum_{s \in \Pi(\lambda), s \neq \lambda} n_{\mu, \nu}^s n_s(t), \tag{1}$$

and then

$$n_{\mu, \nu}^t = \sum_{y \in \Pi(\mu), z \in \Pi(\nu), y+z=t} n_\mu(y) n_\nu(z) - \sum_{s \in \Pi(\lambda), s \neq t} n_{\mu, \nu}^s n_s(t). \tag{2}$$

Note from the two formulas that  $n_\lambda(t)$  is only depended on those numbers

$$\begin{aligned} n_\mu(y), & \quad \mu - y \preceq \lambda - t, \mu < \lambda; \\ n_\nu(z), & \quad \nu - z \preceq \lambda - t, \nu < \lambda; \\ n_s(t), n_{\mu, \nu}^s, & \quad t \preceq s \prec \lambda \end{aligned}$$

and  $n_{\mu, \nu}^t$  is only depended on those numbers

$$\begin{aligned} n_\mu(y), & \quad \mu - y \preceq \lambda - t, \mu < \lambda; \\ n_\nu(z), & \quad \nu - z \preceq \lambda - t, \nu < \lambda; \\ n_s(t), n_{\mu, \nu}^s, & \quad t \prec s \preceq \lambda. \end{aligned}$$

We can substitute  $n_{\mu, \nu}^s, s \succ t$ , in (2) for those in (1), and obtain

$$n_\lambda(t) = n_{\mu, \nu}^t + g(n_\mu(y), n_\nu(z), n_s(x)), \tag{3}$$

where  $g(n_\mu(y), n_\nu(z), n_s(x))$  is determined by  $n_\mu(y), n_\nu(z), n_s(x), t \preceq x, s \preceq \lambda$  and  $s - x \prec \lambda - t$ .

**3.3.** We now prove the theorem by induction on  $\Lambda^+$  with the partial order “ $<$ ” and on  $\Pi(\lambda)$  with the partial order “ $\prec$ ”.

It is only necessary to prove  $n_\lambda(\mu) = m_\lambda(\mu), \lambda \in \Lambda^+, \mu \in \Pi^+(\lambda)$ .

Firstly, if  $\lambda = \omega_i$  or 0, their saturated weight set  $\Pi(\lambda)$  only contains one dominate weight. So the theorem holds by the definition of  $f_\lambda$ .

Suppose that  $\lambda \in \Lambda^+$ , not be fundamental weights  $\omega_i$  or 0. Then  $n_\lambda(\lambda) = 1 = m_\lambda(\lambda)$ . Let  $\mu \in \Pi^+(\lambda), \beta = \lambda - \mu$ . Consider the three cases as follows:

(1) when  $|\text{Supp}(\beta)| < l$ , we have  $n_\lambda(\mu) = m_\lambda(\mu)$  by the first condition in theorem.

(2) when  $|\text{Supp}(\beta)| = 1$  and  $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_l = \omega_1 + \omega_l$ , we have  $\mu = \lambda - \beta = \lambda - \omega_1 - \omega_l$ , thus  $\lambda_1 \geq 1, \lambda_l \geq 1$ . Moreover,  $\lambda = (\lambda - \omega_1) + \omega_1, \lambda - \omega_1 \in \Pi^+(\lambda)$ .

By the second condition of theorem, we have

$$n_{\omega_1, \lambda - \omega_1}^\mu = n_{\omega_1, \lambda - \omega_1}^{\lambda - \omega_1 - \omega_l} = n_{\lambda - \omega_1 - \omega_l, -w_0\omega_1}^{\lambda - \omega_1} = n_{\lambda - \omega_1 - \omega_l, \omega_l}^{\lambda - \omega_1} = 1.$$

The last equation holds because

$$\lambda - \omega_1 - \omega_l + \omega_l - (\lambda - \omega_1) = 0.$$

We also have

$$c_{\omega_1, \lambda - \omega_1}^\mu = c_{\omega_1, \lambda - \omega_1}^{\lambda - \omega_1 - \omega_l} = c_{\lambda - \omega_1 - \omega_l, -w_0\omega_1}^{\lambda - \omega_1} = c_{\lambda - \omega_1 - \omega_l, \omega_l}^{\lambda - \omega_1} = 1.$$

Therefore,

$$n_{\omega_1, \lambda - \omega_1}^\mu = c_{\omega_1, \lambda - \omega_1}^\mu.$$

For  $\omega_1 < \lambda, \lambda - \omega_1 < \lambda$ , according to equation (3) and induction hypothesis, we have

$$\begin{aligned} n_\lambda(\mu) &= n_{\omega_1, \lambda - \omega_1}^\mu + g(n_{\omega_1}(y), n_{\lambda - \omega_1}(z), n_s(x)) \\ &= c_{\omega_1, \lambda - \omega_1}^\mu + g(c_{\omega_1}(y), c_{\lambda - \omega_1}(z), c_s(x)) = c_\lambda(\mu). \end{aligned}$$

(3) When  $|\text{Supp}(\beta)| = 1$  and  $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_l + \beta_1, \beta_1 \succ 0$ , we have  $\mu = \lambda - \omega_1 - \omega_l - \beta_1$ . If  $\lambda_i > 0, \lambda_j = 0$  and  $j < i$ , let  $\nu = \lambda - \omega_i$ , then  $\lambda = \nu + \omega_i$ , and

$$n_{\omega_i, \nu}^\mu = n_{\omega_i, \lambda - \omega_i}^{\lambda - \omega_1 - \omega_l - \beta_1} = n_{\lambda - \omega_1 - \omega_l - \beta_1, -w_0\omega_i}^{\lambda - \omega_i} = n_{\lambda - \omega_1 - \omega_l - \beta_1, \omega_{l-i}}^{\lambda - \omega_i} = n_{\mu, \omega_{l-i}}^\nu.$$

For

$$\begin{aligned} &\mu + \omega_{l-i} - \nu \\ &= \lambda - \omega_1 - \omega_l - \beta_1 + \omega_{l-i} - \lambda + \omega_i \\ &= -(\omega_1 + \omega_l) - (\omega_{l-i} + \omega_i) - \beta_1 \\ &= -\sum_{k=1}^l \alpha_k + \sum_{k=1}^{\min(l-i, i)} \sum_{j=k}^{l-k} k\alpha_k - \beta_1, \end{aligned}$$

let  $\mu + \omega_{l-i} - \nu = \sum_{j=1}^l k_j\alpha_j$ , then  $k_1 \leq 0$ .

- (i) if  $k_1 < 0$ , then  $\nu \notin \Pi(\mu + \omega_{l-i})$ , we have  $n_{\mu, \omega_{l-i}}^\nu = 0 = c_{\mu, \omega_{l-i}}^\nu$ .  
(ii) if  $k_1 = 0$ , then  $|\text{Supp}(\mu + \omega_{l-i} - \nu)| < l$ , we have by the first condition

$$n_{\mu, \omega_{l-i}}^\nu = c_{\mu, \omega_{l-i}}^\nu.$$

Moreover,

$$c_{\omega_i, \nu}^\mu = c_{\omega_i, \lambda - \omega_i}^{\lambda - \omega_1 - \omega_l - \beta_1} = c_{\lambda - \omega_1 - \omega_l - \beta_1, -w_0 \omega_i}^{\lambda - \omega_i} = c_{\lambda - \omega_1 - \omega_l - \beta_1, \omega_{l-i}}^{\lambda - \omega_i} = c_{\mu, \omega_{l-i}}^\nu.$$

Therefore, we have also

$$n_{\omega_i, \nu}^\mu = c_{\omega_i, \nu}^\mu.$$

Similarly, by the induction hypothesis and equation (3) we have

$$n_\lambda(\mu) = c_\lambda(\mu).$$

The theorem is proved.

**3.4.** One of the conditions in theorem is about  $m_\lambda(\mu)$ , and the other is about  $n_{\mu, \nu}^\lambda$ . There exists difficulty to generalize the theorem to Lie algebras of other type. Only the two conditions in theorem are not enough to determined  $m_\lambda(\mu)$ . We need to find out a suitable condition. As said before, the motivation for this problem in this paper comes from the modular representation theory of linear algebraic groups in positive characteristic. It is more difficulty to find out natural conditions in this case, for the second condition does not hold in general. We will consider these problems in the future.

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