

Computing Parabolically Induced Embeddings of Semisimple Complex Lie Algebras in Weyl Algebras

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Abstract. An arbitrary proper parabolic subalgebra \mathfrak{p} of a simple complex Lie algebra \mathfrak{g} induces an embedding $\mathfrak{g} \hookrightarrow \mathbb{W}_n$, and more generally an embedding $\mathfrak{g} \hookrightarrow \mathbb{W}_n \otimes \text{End } V$, where \mathbb{W}_n is the Weyl algebra in n variables, n is the dimension of the nilradical of \mathfrak{p} , and V is an arbitrary \mathfrak{p} -module. We give an elementary proof of this known fact, report on a computer program computing the embeddings, and tabulate exceptional Lie algebra embeddings $G_2 \hookrightarrow \mathbb{W}_5$, $F_4 \hookrightarrow \mathbb{W}_{15}$, $E_6 \hookrightarrow \mathbb{W}_{16}$, $E_7 \hookrightarrow \mathbb{W}_{27}$, $E_8 \hookrightarrow \mathbb{W}_{57}$ arising in this fashion.
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1. Introduction

Let \mathfrak{g} be a semisimple Lie algebra. In this article we present a computational technique for realizing generalized Verma \mathfrak{g} -modules via differential operators. For simple $\mathfrak{g} \neq sp(2n)$, the resulting differential operator realizations of \mathfrak{g} provide candidates for differential operator realizations with minimal number of variables (see [6]). At the same time, these differential operator realizations serve as the first computational step in the “ \mathcal{F} -method” for computing generalized Verma module branching rules.

Our computational technique is contained in the proof of Proposition 3.7. The proposition is known, see [1]. The contribution of the work presented here consists in giving a computational proof, well suited for computer realization, and writing a computer program to tabulate the corresponding exceptional Lie algebra differential operator realizations (Theorem 1.1 and Table 1 below).

Let $\mathfrak{p}', \mathfrak{n}' \subset \mathfrak{g}$ be two Lie subalgebra of \mathfrak{g} such that \mathfrak{n}' is nilpotent and \mathfrak{g} is the direct vector space sum of \mathfrak{n}' and \mathfrak{p}' . Let V be a representation of \mathfrak{p}' . It is then known that there exists a Lie algebra homomorphism mapping \mathfrak{g} to the Lie algebra $\mathbb{W}_n \otimes \text{End } V$, where \mathbb{W}_n is the Weyl algebra of differential operators with

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polynomial coefficients in $n = \dim \mathfrak{n}'$ variables. This is explained, for example, in [6] in the remarks before Lemma 3.3 there. In the particular case that V is a trivial representation, this map may be identified with a homomorphism from \mathfrak{g} into \mathbb{W}_n .

Suppose now that \mathfrak{g} is a simple complex Lie algebra. Then a homomorphism of the above form must be an embedding. In the case that $\mathfrak{p}' = \mathfrak{p}$ is a parabolic subalgebra with reductive Levi part \mathfrak{l} and $\mathfrak{n}' = \mathfrak{n}_-$ is the subalgebra opposite to the nilradical of \mathfrak{p} , in Proposition 3.7 we construct the corresponding embedding

$$\mathfrak{g} \hookrightarrow \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l}). \quad (1)$$

We report a computer program written by the author computes the embeddings (1), as described in Section 4. The program and its source code are free and open source. We have provided an online interface to our program, [11], which allows an user to compute arbitrary examples of (1) within computational range. The dimension of $V_\lambda(\mathfrak{l})$ is restricted by computational power; $\dim V_\lambda(\mathfrak{l}) \leq 1000$ is within practical limits on our system (when running on a 32 bit personal computer). The program is written in the programming language C++, within the project “vector partition function”. Table 1 was generated by our C++ program. In Section 5 we illustrate the details of the construction for G_2 , as printed out by our program.

1.1. Relationship to minimal realizations of simple Lie algebras. If we specialize our computational technique to the case of parabolic subalgebra of maximal dimension and $\lambda = 0$, we obtain a case that is key to the study of minimal embeddings of $\mathfrak{g} \neq \mathfrak{sp}(2n)$ into Weyl algebras. We recall that an embedding of a Lie algebra in the Weyl algebra of n variables \mathbb{W}_n is minimal if it uses the minimal number of variables n . In the case of an exceptional Lie algebra and proper parabolic subalgebra of maximal dimension, we get the embeddings

$$E_8 \hookrightarrow \mathbb{W}_{57}, \quad E_7 \hookrightarrow \mathbb{W}_{27}, \quad E_6 \hookrightarrow \mathbb{W}_{16}, \quad F_4 \hookrightarrow \mathbb{W}_{15}, \quad G_2 \hookrightarrow \mathbb{W}_5. \quad (2)$$

[6] proves that a minimal realization $\mathfrak{sl}(n+1) \hookrightarrow \mathbb{W}_n$ is achieved by the construction presented here and is not achieved for $\mathfrak{sp}(2n)$ (for $\mathfrak{sp}(2n)$, [6] proves the minimal realization is the Shale-Weil representation). If we change the problem to allow localizations $\overline{\mathbb{W}}_k$ of \mathbb{W}_k , [7], [6] solve the problem of minimal realizations in $\overline{\mathbb{W}}_k$. Except for $\mathfrak{sl}(n+1)$, the realizations in $\overline{\mathbb{W}}_k$ use a smaller number of variables than the construction discussed in the present paper. However, if using \mathbb{W}_n rather than localizations, [7], [6] do not resolve the question of whether $\mathfrak{so}(n) \hookrightarrow \mathbb{W}_{n-2}$ and the embeddings (2) are minimal realizations; up to our knowledge, this remains an open question.

Embeddings (2) are presented in the following theorem.

Theorem 1.1. *Let $\mathfrak{g} \subset \mathbb{W}_n \otimes \text{id}$ be the Lie algebra generated by the differential operators indicated in the last column of Table 1, where n is the number indicated in the second column. Then \mathfrak{g} is isomorphic to the exceptional Lie algebra indicated in the first column of the table.*

While the number of variables in the realizations (2) were stated in [6] and [7], the present article is the only one (up to our knowledge) to explicitly present differential operator realizations of the exceptional Lie algebras F_4, E_6, E_7, E_8 in the stated number of variables.

We note that the results of Section 3 are valid over a field of arbitrary characteristic - the only division operation required by Section 3 is the division by a monic polynomial in (5). So long as the structure constants with respect to a Chevalley-Weyl basis of the Lie algebra do not vanish (which does happen in characteristic 2 and 3 for G_2 and in characteristic 2 for F_4), all algorithms described in Section 4 carry through directly. Furthermore, the Lie algebra generated over \mathbb{Z} by the differential operators in Theorem 1.1 is spanned by differential operators with integral coefficients. Clearly those involve only finitely many primes, and therefore Theorem 1.1 remains valid over all fields of sufficiently large finite characteristic.

1.2. Relationship to the “ \mathcal{F} -method”. The technique presented here was originally intended as a tool for solving the generalized Verma module branching problem over a semisimple subalgebra as formulated in [13]. Our technique provides the starting differential operators used in the “ \mathcal{F} -method” for computing branchings of generalized Verma modules, see [8, §2.4]. The software described in the present paper was used in part of the computation of certain families of generalized Verma module branchings of $so(7)$ over G_2 , [12].

2. Notation and preliminaries

\mathfrak{g} denotes a semisimple Lie algebra over the complex numbers. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g})} \mathfrak{g}^\alpha$ be the corresponding root space decomposition of \mathfrak{g} . Let $g_\alpha \in \mathfrak{g}^\alpha$ be a Chevalley-Weyl basis of \mathfrak{g} corresponding to this decomposition. In a Chevalley-Weyl basis, the structure constants of \mathfrak{g} are integral.

We denote by $\mathfrak{b}, \mathfrak{p}$ respectively a fixed Borel and a fixed parabolic subalgebra, such that $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}$. We denote by \mathfrak{l} the reductive Levi part of \mathfrak{p} that contains \mathfrak{h} , by \mathfrak{n} the nilradical of \mathfrak{p} and by \mathfrak{n}_- the subalgebra generated by the root spaces opposite to the root spaces that generate \mathfrak{n} .

For a dominant integral weight $\lambda \in \mathfrak{h}^*$ we denote by $V_\lambda(\mathfrak{l})$ the irreducible finite-dimensional representation of \mathfrak{l} with highest weight λ . We equip $V_\lambda(\mathfrak{l})$ with trivial action of \mathfrak{n} and denote by $M_\lambda(\mathfrak{g}, \mathfrak{p})$ the corresponding generalized Verma module induced by the action of \mathfrak{p} on $V_\lambda(\mathfrak{l})$, i.e.,

$$M_\lambda(\mathfrak{g}, \mathfrak{p}) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_\lambda(\mathfrak{l}).$$

The PBW theorem implies the vector space isomorphism

$$M_\lambda(\mathfrak{g}, \mathfrak{p}) \simeq U(\mathfrak{n}_-) \otimes V_\lambda(\mathfrak{l}). \quad (3)$$

Let \mathbb{W}_n denote the Weyl algebra in n variables over \mathbb{Q} where $\dim \mathfrak{n} = n$. We denote its generators by $\partial_1, \dots, \partial_n, x_1, \dots, x_n$, and as usual we define its

relations to be those of an associative algebra with identity together with

$$\begin{aligned}x_i x_j &= x_j x_i, \\ \partial_i \partial_j &= \partial_j \partial_i, \\ \partial_j x_i - x_i \partial_j &= [\partial_j, x_i] = \delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Denote by \mathbb{S}_n the symmetric algebra generated by $1, x_1, \dots, x_n$. Then \mathbb{S}_n is a \mathbb{W}_n -module under the action \cdot given by applying the differential operators from \mathbb{W}_n on the elements of \mathbb{S}_n , i.e.,

$$\begin{aligned}\partial_i \cdot x_j &:= \delta_{ij} \\ x_j \cdot x_i &:= x_j x_i.\end{aligned}$$

3. Differential operator realizations of generalized Verma modules

The statements in this section are contained in essence in [1]. We present however our own constructive proofs, as they are the basis of our computer realization of the differential operator realizations of generalized Verma modules.

Lemma 3.1. *Let $a, b \in U(\mathfrak{g})$. Then*

$$\begin{aligned}ab^m &= \sum_{k=0}^m \binom{m}{k} (-1)^k b^{m-k} (\text{ad } b)^k(a) \\ b^m a &= \sum_{k=0}^m \binom{m}{k} (\text{ad } b)^k(a) b^{m-k}.\end{aligned}$$

Proof. (Dixmier) Let L_b and R_b stand respectively for left and right multiplication by b in $U(\mathfrak{g})$. As $\text{ad } b = L_b - R_b$ and L_b and R_b commute, the lemma follows from the Newton binomial formula. ■

Definition 3.2. For integers b_1, \dots, b_n , we say that a polynomial $p(a_1, \dots, a_n)$ is (b_1, \dots, b_n) -compatible if $p(a_1, \dots, a_n)$ is divisible by $a_k(a_k - 1) \dots (a_k - b_k + 1)$ whenever $b_k \geq 0$.

The main motivation for the definition of (b_1, \dots, b_n) -compatible polynomials is the following lemma.

Lemma 3.3. *Let I be a finite indexing set. For each $i \in I$, let (b_{i1}, \dots, b_{in}) be an n -tuple of integers and $p_i(a_1, \dots, a_n)$ be a (b_{i1}, \dots, b_{in}) -compatible polynomial (Definition 3.2). Suppose $g : \mathbb{S}_n \rightarrow \mathbb{S}_n$ is a linear operator such that*

$$g(x_1^{a_1} \dots x_n^{a_n}) = \sum_i p_i(a_1, \dots, a_n) x_1^{a_1 - b_{i1}} \dots x_n^{a_n - b_{in}}$$

for all monomials $x_1^{a_1} \dots x_n^{a_n}$. Then there exists a unique element $\omega \in \mathbb{W}_n$ such that the action of g on \mathbb{S}_n equals the \cdot -action of ω on \mathbb{S}_n .

Proof. By linearity, it suffices to prove the Lemma in the case where I has only one element, i.e.,

$$g(x_1^{a_1} \dots x_n^{a_n}) = p(a_1, \dots, a_n)x_1^{a_1-b_1} \dots x_n^{a_n-b_n}.$$

Let $\mathbb{Q}[a_1, \dots, a_n]$ denote the polynomials in the variables a_i .

Case 1. Suppose first $b_k \leq 0$ for all k . Define the \mathbb{Q} -linear map

$$\begin{aligned} \omega_{b_1, \dots, b_n} : \mathbb{Q}[a_1, \dots, a_n] &\rightarrow \mathbb{W}_n \\ a_1^{m_1} \dots a_n^{m_n} &\xrightarrow{\omega_{b_1, \dots, b_n}} \prod_{i=1}^n (x_i^{-b_i} (x_i \partial_i)^{m_i}). \end{aligned} \tag{4}$$

Direct check shows that $\omega_{b_1, \dots, b_n}(p(a_1, \dots, a_n))$ has the same action \mathbb{S}_n as g .

Case 2. Suppose $b_k > 0$ for some k . By the requirements of the lemma the expression

$$p'(a_1, \dots, a_n) := \frac{p(a_1, \dots, a_n)}{\prod_{b_k > 0} a_k(a_k - 1) \dots (a_k - b_k + 1)} \tag{5}$$

is a polynomial. Define

$$\bar{b}_k = \begin{cases} b_k & \text{if } b_k \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Direct check shows that the operator

$$\omega := \omega_{\bar{b}_1, \dots, \bar{b}_n}(p'(a_1, \dots, a_n)) \left(\prod_{b_k > 0} \partial_k^{b_k} \right) \tag{6}$$

has the same action on \mathbb{S}_n as g . The uniqueness of ω follows from the fact that ω is uniquely determined by its action on monomials in the x_i 's. ■

Given data $(\{(p_i, b_{11}, \dots, b_{1n})\}_{i \in I})$ as in Lemma 3.3, let

$$\omega(\{(p_i, b_{11}, \dots, b_{1n})\}_{i \in I}) \tag{7}$$

denote the operator constructed in Lemma 3.3.

We will need the following observation:

Lemma 3.4. *If $p(a_1, \dots, a_n)$ is (b_1, \dots, b_n) -compatible, then $\binom{a_j}{c_j} p(a_1, \dots, a_j - c_j, \dots, a_n)$ is $(b_1, \dots, b_j + c_j, \dots, b_n)$ -compatible.*

Let u_1, \dots, u_n be a basis of \mathfrak{n}_- consisting of Chevalley-Weyl generators. We assume there is an order $<$ on these generators compatible with their labeling, that is,

$$u_1 < u_2 < \dots < u_n. \tag{8}$$

We pose no restrictions on $<$; in particular the order need not be compatible with any particular order of the weights of the generators. Changing the order causes the algorithm of the present paper to produce different differential operator realizations of the simple Lie algebras. The explicit form of the differential operators given in Table 1 depends on $<$.

For each generator u_i , let β_i be the root corresponding to u_i .

Lemma 3.5. *Let $g \in \mathfrak{g}$ and $j \in \{0, \dots, n\}$. There exists a finite indexing set $I(g)$, such that for each $i \in I(g)$ there exist integers (b_{i1}, \dots, b_{in}) , a (b_{i1}, \dots, b_{in}) -compatible polynomial $p_i(a_1, \dots, a_n)$ (Definition 3.2) and an element $f_i \in U(\mathfrak{g})$, such that f_i is a Chevalley-Weyl generator lying outside of \mathfrak{n}_- or equal to 1 and such that*

$$u_1^{a_1} \dots u_j^{a_j} g u_{j+1}^{a_{j+1}} u_n^{a_n} = \sum_{i \in I(g)} p_i(a_1, \dots, a_n) u_1^{a_1 - b_{i1}} \dots u_n^{a_n - b_{in}} f_i. \tag{9}$$

Proof. It suffices to prove our claim in the case when g is a Chevalley-Weyl generator. We proceed by double induction on the weight of g and on the index j . We suppose we have proved by induction hypothesis our statement for all weights smaller than the weight of g . We suppose by second induction hypothesis that for all indices larger than j we have proved our statement for weights smaller than or equal to the weight of g . We have three cases for the induction step.

The first case is that $g = u_{j+1}$ or $g = u_j$; this case is trivial and there is nothing to prove.

The second case is that $g = u_l$ with $l < j$. We apply Lemma 3.1 with respect to $u_j^{a_j} g$ to obtain that our starting expression is a linear combination of $u_1^{a_1} \dots u_{j-1}^{a_{j-1}} u_l u_j^{a_j} \dots u_n^{a_n}$ and monomials of the form

$$\binom{a_j}{c_j} u_1^{a_1} \dots u_{j-1}^{a_{j-1}} f u_j^{a_j - c_j} \dots u_n^{a_n}$$

with the weight of f strictly smaller than the weight of u_l . By induction hypothesis these monomials can be written in the form (9). Here, the compatibility of the coefficients follows from Lemma 3.4. Continuing in this fashion, we may commute u_l past $u_{j-1}^{a_{j-1}}$, and so on, until u_l is positioned next to $u_l^{a_l}$ and so our induction step holds in the second case as well.

The final case is that g is a Chevalley-Weyl generator that is not among the generators u_1, \dots, u_{j+1} . We apply Lemma 3.1 with respect to $g u_{j+1}^{a_{j+1}}$ to get that our starting expression is a linear combination of monomials of the form

$$\binom{a_{j+1}}{c_{j+1}} u_1^{a_1} \dots u_{j+1}^{a_{j+1} - c_{j+1}} f u_{j+2}^{a_{j+2}} \dots u_n^{a_n},$$

where f is a Chevalley-Weyl generator that is either equal to g or is of strictly smaller weight. In both cases the induction step follows from the induction hypothesis and Lemma 3.4. ■

Let $m_1, \dots, m_{\dim V_\lambda(\mathfrak{l})}$ be a basis of $V_\lambda(\mathfrak{l})$. By (3) and the Poincare-Birkhoff-Witt theorem we can write every element in $M_\lambda(\mathfrak{g}, \mathfrak{p})$ uniquely as a linear combination of monomials the form

$$u_1^{a_1} \dots u_n^{a_n} \otimes_{U(\mathfrak{p})} m_i.$$

Define the linear map

$$\begin{aligned} \varphi : U(\mathfrak{n}_-) &\rightarrow \mathbb{S}_n \\ u_1^{a_1} \dots u_n^{a_n} &\xrightarrow{\varphi} x_1^{a_1} \dots x_n^{a_n}. \end{aligned} \tag{10}$$

Define the linear map

$$\begin{aligned} \varphi_\lambda : M_\lambda(\mathfrak{g}, \mathfrak{p}) &\rightarrow \mathbb{S}_n \otimes V_\lambda(\mathfrak{l}) \\ u_1^{a_1} \dots u_n^{a_n} \otimes_{U(\mathfrak{p})} m_i &\xrightarrow{\varphi_\lambda} \varphi(u_1^{a_1} \dots u_n^{a_n}) \otimes m_i = x_1^{a_1} \dots x_n^{a_n} \otimes m_i. \end{aligned}$$

By (3) the map φ_λ is a vector space isomorphism between $M_\lambda(\mathfrak{g}, \mathfrak{p})$ and $\mathbb{S}_n \otimes V_\lambda(\mathfrak{l})$. We can therefore identify $\text{End}(M_\lambda(\mathfrak{g}, \mathfrak{p}))$ with $\text{End}(\mathbb{S}_n \otimes V_\lambda(\mathfrak{l}))$ via the map φ'_λ given by

$$\begin{aligned} \varphi'_\lambda : \text{End}(M_\lambda(\mathfrak{g}, \mathfrak{p})) &\rightarrow \text{End}(\mathbb{S}_n \otimes V_\lambda(\mathfrak{l})) \\ a &\xrightarrow{\varphi'_\lambda} \varphi_\lambda \circ a \circ \varphi_\lambda^{-1}. \end{aligned} \tag{11}$$

We are now in a position to prove the following.

Proposition 3.6. *Let \mathfrak{g} be a semisimple Lie algebra with parabolic subalgebra \mathfrak{p} .*

- (a) *There exists a linear map $\Phi_0 : U(\mathfrak{g}) \rightarrow \mathbb{W}_n \otimes \text{id}$ which makes the following diagram commutative.*

$$\begin{array}{ccc} \text{End } M_0(\mathfrak{g}, \mathfrak{p}) & \xrightarrow{\varphi'_0} & \text{End } \mathbb{S}_n \otimes \text{id} & \simeq & \mathbb{S}_n \\ \uparrow & & \uparrow & & \\ U(\mathfrak{g}) & \xrightarrow{\Phi_0} & \mathbb{W}_n \otimes \text{id} & \simeq & \mathbb{W}_n \end{array}$$

- (b) Φ_0 is a Lie algebra homomorphism.

Proof. (a) Let $g \in \mathfrak{g}$. By Lemma 3.5, there exists data as described in the Lemma such that (9) holds. For each $g \in \mathfrak{g}$, let $I(g)$ be the set given by Lemma 3.5, and let

$$J(g) := \{i \in I \mid f_i = 1\}$$

i.e., the subset corresponding to the summands in (9) that end on 1. Using Lemma 3.5 we can define the element $\omega(\{(p_i, b_{i1}, \dots, b_{in})\}_{i \in J(g)})$ via (7), and therefore we can define the linear map $\Phi_0 : \mathfrak{g} \rightarrow \mathbb{W}_n \otimes \text{id}$ by

$$\Phi_0(g) = \omega(\{(p_i, b_{i1}, \dots, b_{in})\}_{i \in J(g)}) \otimes \text{id}.$$

Let m_1 be a non-zero vector of the one-dimensional vector space $V_0(\mathfrak{l})$. The action of $\varphi'_0(g)$ on $\mathbb{S}_n \otimes m_1$ is computed on each monomial $x_1^{a_1} \dots x_n^{a_n} \otimes m_1$ by applying the map φ_0 to $gu_1^{a_1} \dots u_n^{a_n} \cdot m_1$. We have that the Chevalley-Weyl generators lying outside of \mathfrak{n} have zero action on m_1 . Together with Lemma 3.5, this implies that

$$\begin{aligned} \varphi_0(gu_1^{a_1} \dots u_n^{a_n} \cdot m_1) &= \varphi_0 \left(\sum_{i \in I(g)} p_i(a_1, \dots, a_n) u_1^{a_1 - b_{i1}} \dots u_n^{a_n - b_{in}} f_i \cdot m_1 \right) \\ &= \varphi_0 \left(\sum_{i \in J(g)} p_i(a_1, \dots, a_n) u_1^{a_1 - b_{i1}} \dots u_n^{a_n - b_{in}} \cdot m_1 \right) \\ &= \sum_{i \in J(g)} p_i(a_1, \dots, a_n) x_1^{a_1 - b_{i1}} \dots x_n^{a_n - b_{in}} \otimes m_1 \\ &= \Phi_0(g) \cdot (x_1^{a_1} \dots x_n^{a_n} \otimes m_1) \\ &= \Phi_0(g) \cdot \varphi_0(u_1^{a_1} \dots u_n^{a_n} \otimes_{U(\mathfrak{p})} m_1). \end{aligned}$$

The above proves (a).

- (b) As $M_0(\mathfrak{g}, \mathfrak{p})$ is a (Lie algebra) representation of \mathfrak{g} , the image of \mathfrak{g} in $\text{End } M_0(\mathfrak{g}, \mathfrak{p})$ is a Lie algebra, and by (a) so is the image of \mathfrak{g} in $\text{End } \mathbb{S}_n \otimes \text{id}$. ■

We finish this section with an interpretation of the action of \mathfrak{g} as a Lie subalgebra of $\mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$. We recall that, given associative (and therefore Lie) algebras A_1, A_2 , we have that $A_1 \otimes A_2$ is an associative (and therefore Lie) algebra with $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$. Further, given two A_i -modules M_i , $i = 1, 2$, then $M_1 \otimes M_2$ can be equipped with a $A_1 \otimes A_2$ -module structure via

$$(a_1 \otimes a_2) \cdot (m_1 \otimes m_2) := (a_1 \cdot m_1) \otimes (a_2 \cdot m_2). \quad (12)$$

In particular, $\mathbb{S}_n \otimes V_\lambda(\mathfrak{l})$ is a $\mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ -module; the associative algebra $\mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ is sometimes called a ‘‘Weyl algebra with matrix coefficients’’.

Proposition 3.7. *There is a Lie algebra homomorphism $\Phi_\lambda: U(\mathfrak{g}) \rightarrow \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ which makes the following diagram commutative:*

$$\begin{array}{ccc} \text{End } M_\lambda(\mathfrak{g}, \mathfrak{p}) & \xrightarrow{\varphi'_\lambda} & \text{End } (\mathbb{S}_n \otimes V_\lambda(\mathfrak{l})) \\ \uparrow \cdot & & \uparrow (12) \\ U(\mathfrak{g}) & \xrightarrow{\Phi_\lambda} & \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l}) \end{array}$$

where we are using (12) with respect to the \cdot action of \mathbb{W}_n on \mathbb{S}_n .

Remark. The proposition remains valid if we replace $V_\lambda(\mathfrak{l})$ with an arbitrary \mathfrak{p} -module V and φ'_λ by an operator defined similarly to (11).

Proof. Let $g \in \mathfrak{g}$. Let $\{(f_i, p_i, b_{i1}, \dots, b_{in})\}_{i \in I}$ be the data whose existence is given by Lemma 3.4. We can therefore define $\Phi_\lambda: U(\mathfrak{g}) \rightarrow \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ using Lemma 3.4 so that the last equality in the following computation holds for an arbitrary vector $v \in V_\lambda(\mathfrak{l})$.

$$\begin{aligned} \varphi_\lambda(gu_1^{a_1} \dots u_n^{a_n} \cdot v) &= \varphi_\lambda \left(\sum_{i \in I(g)} p_i(a_1, \dots, a_n) u_1^{a_1 - b_{i1}} \dots u_n^{a_n - b_{in}} \otimes_{U(\mathfrak{p})} (f_i \cdot v) \right) \\ &= \sum_{i \in I(g)} p_i(a_1, \dots, a_n) x_1^{a_1 - b_{i1}} \dots x_n^{a_n - b_{in}} \otimes (f_i \cdot v) \\ &=: \Phi_\varphi(g)(x_1^{a_1} \dots x_n^{a_n} \otimes v). \end{aligned}$$

The rest of the proof is analogous to that of Proposition 3.6 and we omit it. ■

4. An informal algorithm description

In the present section we describe an algorithm that explicitly computes the homomorphism Φ_λ from Proposition 3.7. The homomorphism is described by giving the image of each of the simple Chevalley-Weyl generators of \mathfrak{g} .

For a simple complex Lie algebra \mathfrak{g} , the only input data needed to compute an embedding $\mathfrak{g} \hookrightarrow \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ is the data $\mathfrak{g}, \lambda, \mathfrak{p}$. More precisely, \mathfrak{g} can be described to a computer program by the type of the Lie algebra, λ by a vector given as the sequence of its fundamental coordinates and \mathfrak{p} by a vector of zeroes and ones indicating which simple roots are crossed out.

The root systems of each simple Lie algebra can be generated, for example, using [3, page 162]. The structure constants of a semisimple Lie algebra relative to a Chevalley-Weyl basis (in particular, their signs), can be computed with a uniform method in both the simply and non-simply laced case according to [14, pages 47-51] as implemented in our C++ program, [11].

We suppose $V_\lambda(\mathfrak{l})$ is constructed, in particular we have a vector space basis $m_1, \dots, m_{\dim V_\lambda(\mathfrak{l})}$ of $V_\lambda(\mathfrak{l})$ and we know the action of each $g \in \mathfrak{g}$ as an element of $V_\lambda(\mathfrak{l}) \otimes V_\lambda(\mathfrak{l})^* \simeq \text{End } V_\lambda(\mathfrak{l})$. An algorithm for computing the actions of g on $V_\lambda(\mathfrak{l})$ from the data $\mathfrak{g}, \lambda, \mathfrak{p}$ can be extracted from [10] and [9]; a detailed algorithm description is in preparation, [5]. The algorithm is already implemented and tested in [11].

Let g_i denote the Chevalley-Weyl generators computed using the preceding algorithms. We represent (non-uniquely) a monomial

$$g_{i_1}^{p_1} \cdots g_{i_k}^{p_k}$$

in $U(\mathfrak{g})$ as a list of generator indices and a corresponding list of exponents, i.e., as the *monomial data*

$$((i_1, \dots, i_k), (p_1, \dots, p_k)) \quad i_s \in \mathbb{Z}, p_k \text{ lie in the base ring.} \quad (13)$$

We allow the exponents to belong to any base ring defined in our program that contains¹ \mathbb{Z} . We represent (non-uniquely) an arbitrary element

$$\sum a_{(i_1, \dots, i_k), (p_1, \dots, p_k)} g_{i_1}^{p_1} \cdots g_{i_k}^{p_k}$$

in $U(\mathfrak{g})$ as a collection of monomial data (13) and a list of coefficients lying in the base ring, i.e., as the *element data*

$$((q_1, \dots, q_s), (\mathbf{mon}_1, \dots, \mathbf{mon}_s)) \quad \mathbf{mon}_t \text{ of the form (13), } q_t \text{ lie in the base ring.} \quad (14)$$

4.1. An iterative procedure implementing Lemma 3.5. Let $<$ be a total order on the Chevalley-Weyl generators g_i . For the time being we assume this order to be arbitrary; later on we will fix $<$ to be an extension of the order of the generators of \mathfrak{n}_- given in (8).

Definition 4.1. Suppose that the monomial data (13) giving the monomial $g_{i_1}^{p_1} \cdots g_{i_k}^{p_k}$ has the following property: if there is a generator $g_{i_1}^{p_1}$ that appears to the left of a generator $g_{i_2}^{p_2}$ and at the same time $g_{i_1} > g_{i_2}$, then both p_1 and p_2 are not non-negative integers. Then we say that the monomial data (13) is reduced relative to $<$.

Suppose each monomial data in the element data (14) is reduced relative to $<$. Then we say that the element data (14) is reduced relative to $<$.

Let $u \in U(\mathfrak{g})$. Suppose u is represented by element data X which is not reduced relative to $<$. We introduce an iterative procedure that, after consecutive

¹To achieve this we use C++ templates.

application, aims to transform X to element data X' that is reduced relative to $<$ and represents the same element. The procedure is in essence an implementation of the proof of Lemma 3.5.

We treat each monomial data **mon** participating in (14) separately. Suppose **mon** is not reduced relative to $<$. We remove **mon** from the data (14). Then, we transform **mon** so no two consecutive generator indices i_j, i_{j+1} are equal by adding the corresponding exponents and removing one of the indices. If after this transformation **mon** is reduced relative to $<$ we add it back to the data (14). Otherwise, we find the first generator index i_j for which $g_{i_j} > g_{i_{j+1}}$ and for which either

- p_j is a positive integer and $g_{i_{j+1}}$ is nilpotent or
- p_{j+1} is a positive integer and g_{i_j} is nilpotent.

We then apply Lemma 3.1, choosing the first identity if p_j is a positive integer and the second identity otherwise. The right hand side of the identities in Lemma 3.1 provides us with new set monomials $\mathbf{mon}_1, \dots, \mathbf{mon}_k$ and a new set of coefficients; we multiply those coefficients by the initial coefficient of **mon**, and add back to the data (14).

4.2. Algorithm description. We are ready to implement Proposition 3.7. We recall that we have a fixed semisimple Lie algebra \mathfrak{g} , a highest weight λ , and a nilradical \mathfrak{n}_- . We recall from Section 3 that u_1, \dots, u_n are the Chevalley-Weyl generators of \mathfrak{n}_- .

From now on fix the order $<$ so that $u_i < g_j$ for any Chevalley-Weyl generator $g_j \notin \mathfrak{n}_-$ and so that the order is an extension of (8), that is $u_1 < u_2 < \dots < u_n$. Let $g_{\pm 1}, \dots, g_{\pm k}$ be the simple Chevalley-Weyl generators of \mathfrak{g} . Consider the action of left multiplication by $g_{\pm i}$ on a generic monomial in $U(\mathfrak{n}_-)$, i.e., consider the element

$$g_{\pm i} u_1^{a_1} \dots u_n^{a_n}. \tag{15}$$

The proof of Lemma 3.5 shows that the consecutive application of the iterative procedure described after Definition 4.1 transform the element data corresponding to (15) to element data that is reduced relative to $<$. The so computed data represents the sum (9) from the proof of Lemma 3.5. In particular, we can read off the set of integers $(b_{i_1}, \dots, b_{i_n})$ and $(b_{i_1}, \dots, b_{i_n})$ -compatible polynomials $p_i(a_1, \dots, a_n)$ whose existence is asserted by Lemma 3.5.

Fix now the base ring to be $\mathbb{Q}[a_1, \dots, a_n]$. We recall that the monomials participating in (9) are of the form

$$p(a_1, \dots, a_n) u_1^{a_1 - b_1} \dots u_n^{a_n - b_n} f,$$

where f is some Chevalley-Weyl generator of \mathfrak{g} lying outside of \mathfrak{n}_- and the polynomial p is (b_1, \dots, b_n) -compatible (Definition 3.2). First, we apply Lemma 3.3 to the first multiplicand $p(a_1, \dots, a_n) u_1^{a_1 - b_1} \dots u_n^{a_n - b_n}$ - for (5) we use polynomial division, and for the remainder of the Lemma we use a standard implementation of \mathbb{W}_n . If $\lambda = 0$, we replace f by id; else we replace f by its action on $V_\lambda(\mathfrak{l})$ and take the tensor product with the differential operator computed in the preceding

step. We collect all summands to complete the computation of the homomorphism Φ_λ from Proposition 3.7.

4.3. Proof of main theorem and implementation comments. To apply the algorithm in Section 4 we fix an order $<$ on the Chevalley generators of \mathfrak{n}_- by declaring that $u_1 < u_2$ if the weight β_1 of u_1 is less than the weight β_2 of u_2 , where we are comparing β_1 and β_2 in the graded lexicographic order in with respect to their simple basis coordinates.

Proof of Theorem 1.1. We run the algorithm described in Section 4 as implemented in [11] to generate Table 1. \blacksquare

We are including the number of rational number arithmetic operations made by our program during the computation of Table 1. We are also including computation times, however those are not indicative as we have not made any special attempts to optimize our code for speed. All computations were carried out on a low-end 32-bit laptop computer (1.6 Ghz).

\mathfrak{g}	\mathfrak{p}	Number of arithmetic operations need to generate the embeddings	Running time
E_8	last root crossed	28,754,380	52.10 seconds
E_7	last root crossed	2,431,419	4.50 seconds
E_6	last root crossed	487,021	0.99 seconds
F_4	first (long) root crossed	374,377	0.82 seconds
F_4	last (short) root crossed	469,892	1.07 seconds
G_2	first (short) root crossed	22,185	0.06 seconds
G_2	last (long) root crossed	14,072	0.04 seconds

We finish this section by noting there is a simple consistency check for our computer computations. Let $\mathfrak{g}' \subset \mathbb{W}_n \otimes \text{End } V_\lambda(\mathfrak{l})$ be the Lie algebra generated by the differential operators corresponding to simple generators as constructed above. As asserted by Proposition 3.7, we have that $\mathfrak{g} \simeq \mathfrak{g}'$. In particular we can check that $\dim \mathfrak{g}' = \dim \mathfrak{g}$: this makes sense as the computer code that computes the differential operator Lie algebra \mathfrak{g}' is independent from the code that carries out our algorithm. In this way we are testing two independent computer programs against one another, and the output of both programs against the already known dimension $\dim \mathfrak{g}$. All printouts in Table 1 have passed such a consistency check.

5. Example: the embedding $G_2 \hookrightarrow \mathbb{W}_5$

In the present section we illustrate the computation of $G_2 \hookrightarrow \mathbb{W}_5$, as reported by our computer program. A basis of G_2 is computed in Table 2. The elements h_i are dual to the simple roots of G_2 , and the elements g_i together with $h_1, \frac{1}{3}h_2$ form a Chevalley-Weyl basis of G_2 . The generators g_i are indexed according to the

index of the corresponding root space, and the root spaces are indexed according to their order in the simple coordinate graded lexicographic order.

Both proper parabolic subalgebras of G_2 are maximal. We carry out the computations for the parabolic subalgebra obtained by crossing out the short root. A basis for the nilradical \mathfrak{n}_- is given by $g_{-6}, g_{-5}, g_{-4}, g_{-3}, g_{-1}$. An arbitrary monomial in $U(\mathfrak{n}_-)$ is then a multiple of

$$u := g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5}.$$

Applying consecutively the identities of Lemma 3.1 according to Section 4, we can compute that

$$\begin{aligned} g_1 u &= -3a_4 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-1} g_{-1}^{a_5} g_{-2} + a_5 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5-1} h_1 \\ &+ g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5} g_1 + (-3a_3^2 + 3a_3) g_{-6}^{a_1+1} g_{-5}^{a_2} g_{-4}^{a_3-2} g_{-3}^{a_4} g_{-1}^{a_5} \\ &+ (9a_4^2 a_5^2 - 9a_4 a_5^2 - 9a_4^2 a_5 + 9a_4 a_5) g_{-6}^{a_1+1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-2} g_{-1}^{a_5-2} \\ &- a_2 g_{-6}^{a_1} g_{-5}^{a_2-1} g_{-4}^{a_3+1} g_{-3}^{a_4} g_{-1}^{a_5} \\ &+ (-3a_4 a_5^2 + 9a_4 a_5^2 - 6a_4 a_5) g_{-6}^{a_1} g_{-5}^{a_2+1} g_{-4}^{a_3} g_{-3}^{a_4-1} g_{-1}^{a_5-3} \\ &- 2a_3 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3-1} g_{-3}^{a_4+1} g_{-1}^{a_5} \\ &+ (3a_4 a_5^2 - 3a_4 a_5) g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3+1} g_{-3}^{a_4-1} g_{-1}^{a_5-2} \\ &+ (-a_5^2 - 3a_4 a_5 + a_5) g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5-1} \\ g_{-1} u &= (-3a_4^2 + 3a_4) g_{-6}^{a_1+1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-2} g_{-1}^{a_5} - 3a_3 g_{-6}^{a_1} g_{-5}^{a_2+1} g_{-4}^{a_3-1} g_{-3}^{a_4} g_{-1}^{a_5} \\ &- 2a_4 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3+1} g_{-3}^{a_4-1} g_{-1}^{a_5} + g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5+1} \\ g_2 u &= g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5} g_2 - a_1 g_{-6}^{a_1-1} g_{-5}^{a_2+1} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5} \\ &+ (-2a_4^3 + 6a_4^2 - 4a_4) g_{-6}^{a_1+1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-3} g_{-1}^{a_5} \\ &+ (-a_4^2 + a_4) g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3+1} g_{-3}^{a_4-2} g_{-1}^{a_5} + a_4 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-1} g_{-1}^{a_5+1} \\ g_{-2} u &= g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5} g_{-2} - a_2 g_{-6}^{a_1+1} g_{-5}^{a_2-1} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5} \\ &+ (-3a_4 a_5^2 + 3a_4 a_5) g_{-6}^{a_1+1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4-1} g_{-1}^{a_5-2} \\ &+ (a_5^3 - 3a_5^2 + 2a_5) g_{-6}^{a_1} g_{-5}^{a_2+1} g_{-4}^{a_3} g_{-3}^{a_4} g_{-1}^{a_5-3} \\ &+ (-a_5^2 + a_5) g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3+1} g_{-3}^{a_4} g_{-1}^{a_5-2} \\ &+ a_5 g_{-6}^{a_1} g_{-5}^{a_2} g_{-4}^{a_3} g_{-3}^{a_4+1} g_{-1}^{a_5-1}. \end{aligned}$$

Applying the map φ_0 to the above expressions amounts to renaming the generators $g_{-6}, g_{-5}, g_{-4}, g_{-3}, g_{-1}$ to x_1, \dots, x_5 , replacing the remaining generators by 0, and finally tensoring on the right with m_1 . We recall from Lemma 3.5 that if one of the so obtained monomials is of the form $p(a_1, \dots, a_5) x_1^{a_1-b_1} \dots x_5^{a_5-b_5}$ and $b_i > 0$, then $p(a_1, \dots, a_5)$ is divisible by $a_i(a_i - 1) \dots (a_i - b_i + 1)$. Now we can apply (6) and (4) from Lemma 3.3 to get that

$$\begin{aligned} \Phi_0(g_1) &= (-3x_2 \partial_4 \partial_5^3 + 9x_1 \partial_4^2 \partial_5^2 + 3x_3 \partial_4 \partial_5^2 - x_5 \partial_5^2 \\ &\quad - 3x_4 \partial_4 \partial_5 - 3x_1 \partial_3^2 - 2x_4 \partial_3 - x_3 \partial_2) \otimes \text{id} \\ \Phi_0(g_{-1}) &= (-3x_1 \partial_4^2 - 2x_3 \partial_4 - 3x_2 \partial_3 + x_5) \otimes \text{id} \\ \Phi_0(g_2) &= (-2x_1 \partial_4^3 - x_3 \partial_4^2 + x_5 \partial_4 - x_2 \partial_1) \otimes \text{id} \\ \Phi_0(g_{-2}) &= (x_2 \partial_5^3 - 3x_1 \partial_4 \partial_5^2 - x_3 \partial_5^2 + x_4 \partial_5 - x_1 \partial_2) \otimes \text{id}. \end{aligned}$$

Table 1. Elements of \mathbb{W}_n generating the exceptional Lie algebras. Each entry of the table is a simple Chevalley-Weyl generator, given in the order $g_1, g_{-1}, g_2, g_{-2}, \dots$. Here, $g_{\pm i}$ stands for the Chevalley-Weyl generator of the simple root whose i^{th} coordinate is non-zero. The simple roots are ordered in the same order as the one implied by [4, page 59]. For G_2 and F_4 there are two entries as G_2 and F_4 each have two non-isomorphic parabolic subalgebras of maximal dimension. For G_2, F_4 , the first entry in the table corresponds to first root being crossed out (the first root is short for G_2 and long for F_4).

\mathfrak{g}	n	element of $\mathbb{W}_n \simeq \mathbb{W}_n \otimes \text{id}$
G_2	5	$(-3x_2\partial_4\partial_5^3 + 9x_1\partial_4^2\partial_5^2 + 3x_3\partial_4\partial_5^2 - x_5\partial_5^2 - 3x_4\partial_4\partial_5 - 3x_1\partial_3^2 - 2x_4\partial_3 - x_3\partial_2) \otimes \text{id}$
		$(-3x_1\partial_4^2 - 2x_3\partial_4 - 3x_2\partial_3 + x_5) \otimes \text{id}$
G_2	5	$(-2x_1\partial_4^3 - x_3\partial_4^2 + x_5\partial_4 - x_2\partial_1) \otimes \text{id}$
		$(x_2\partial_5^3 - 3x_1\partial_4\partial_5^2 - x_3\partial_5^2 + x_4\partial_5 - x_1\partial_2) \otimes \text{id}$
G_2	5	$(-3x_5\partial_4 - 3x_1\partial_3^2 - 2x_4\partial_3 - x_3\partial_2) \otimes \text{id}$
		$(-x_4\partial_5 - 3x_1\partial_4^2 - 2x_3\partial_4 - 3x_2\partial_3) \otimes \text{id}$
G_2	5	$(-x_5\partial_5^2 - x_4\partial_4\partial_5 - 2x_1\partial_4^3 - x_3\partial_4^2 - x_2\partial_1) \otimes \text{id}$
		$(-x_1\partial_2 + x_5) \otimes \text{id}$
F_4	15	$(-x_{15}\partial_{15}^2 - x_{14}\partial_{14}\partial_{15} - x_{13}\partial_{13}\partial_{15} - x_{12}\partial_{12}\partial_{15} - x_{11}\partial_{11}\partial_{15} + x_1\partial_7\partial_{10}\partial_{15} - x_7\partial_7\partial_{15} + x_1\partial_7\partial_{12}\partial_{14} + x_{10}\partial_{12}\partial_{14} + 2x_1\partial_9^2\partial_{14} + x_8\partial_9\partial_{14} + x_5\partial_7\partial_{14} + x_1\partial_7\partial_{13}^2 + x_{10}\partial_{13}^2 + 4x_1\partial_9\partial_{11}\partial_{13} - x_6\partial_9\partial_{13} - x_4\partial_7\partial_{13} - 2x_1\partial_{11}^2\partial_{12} + x_6\partial_{11}\partial_{12} - x_3\partial_7\partial_{12} - x_5\partial_{11}^2 - x_4\partial_9\partial_{11} - x_2\partial_7\partial_{10} - x_3\partial_6^2 - x_2\partial_1) \otimes \text{id}$
		$(-x_1\partial_2 + x_{15}) \otimes \text{id}$
F_4	15	$(x_{15}\partial_{14} - x_{12}\partial_{10} - x_1\partial_8^2 - x_9\partial_8 - x_7\partial_5 - x_3\partial_2) \otimes \text{id}$
		$(x_{14}\partial_{15} - x_{10}\partial_{12} - x_1\partial_9^2 - x_8\partial_9 - x_5\partial_7 - x_2\partial_3) \otimes \text{id}$
F_4	15	$(2x_{14}\partial_{13} - x_{13}\partial_{12} - x_{11}\partial_9 - x_8\partial_6 - 2x_5\partial_4 - x_4\partial_3) \otimes \text{id}$
		$(x_{13}\partial_{14} - 2x_{12}\partial_{13} - x_9\partial_{11} - x_6\partial_8 - x_4\partial_5 - 2x_3\partial_4) \otimes \text{id}$
F_4	15	$(x_{13}\partial_{11} + 2x_{12}\partial_9 - 2x_1\partial_7\partial_8 + 2x_{10}\partial_8 - x_9\partial_7 - x_8\partial_5 - x_6\partial_4) \otimes \text{id}$
		$(x_{11}\partial_{13} + x_9\partial_{12} + 2x_1\partial_9\partial_{10} + x_8\partial_{10} - 2x_7\partial_9 - 2x_5\partial_8 - x_4\partial_6) \otimes \text{id}$

\mathfrak{g} n element of $\mathbb{W}_n \simeq \mathbb{W}_n \otimes \text{id}$

	$(-x_5\partial_{12}^2 - x_{13}\partial_{12} - x_3\partial_{10}^2 - x_{11}\partial_{10} - x_7\partial_6 - x_2\partial_1) \otimes \text{id}$
	$(-x_5\partial_{13}^2 - x_{12}\partial_{13} - x_3\partial_{11}^2 - x_{10}\partial_{11} - x_6\partial_7 - x_1\partial_2) \otimes \text{id}$
	$(-x_7\partial_{13}^2 - x_{14}\partial_{13} - x_1\partial_6^2 - x_{10}\partial_9 - x_6\partial_5 - x_3\partial_2) \otimes \text{id}$
	$(-x_7\partial_{14}^2 - x_{13}\partial_{14} - x_1\partial_{10}^2 - x_9\partial_{10} - x_5\partial_6 - x_2\partial_3) \otimes \text{id}$
F_4	$(-x_{15}\partial_{14} - 2x_5\partial_{11}\partial_{12} + x_4\partial_{10}\partial_{11} - x_{13}\partial_{11} - x_{12}\partial_{10} - x_9\partial_8 - 2x_5\partial_4 - x_4\partial_3) \otimes \text{id}$
	$(-x_{14}\partial_{15} + x_4\partial_{12}\partial_{13} - x_{11}\partial_{13} - 2x_3\partial_{11}\partial_{12} - x_{10}\partial_{12} - x_8\partial_9 - x_4\partial_5 - 2x_3\partial_4) \otimes \text{id}$
	$(2x_7\partial_{11}\partial_{15}^2 + 2x_6\partial_{10}\partial_{15}^2 + 2x_5\partial_9\partial_{15}^2 - x_{15}\partial_{15}^2 - 2x_7\partial_{13}\partial_{14}\partial_{15} + 2x_4\partial_9\partial_{14}\partial_{15} - x_{14}\partial_{14}\partial_{15}$ $- 2x_5\partial_{12}\partial_{13}\partial_{15} - 2x_4\partial_{10}\partial_{13}\partial_{15} - x_{13}\partial_{13}\partial_{15} + 2x_4\partial_{11}\partial_{12}\partial_{15} - x_{12}\partial_{12}\partial_{15} - 4x_3\partial_{10}\partial_{11}\partial_{15} - 2x_{11}\partial_{11}\partial_{15}$ $- 4x_1\partial_9\partial_{10}\partial_{15} - 2x_{10}\partial_{10}\partial_{15} - 2x_9\partial_9\partial_{15} + 2x_3\partial_9\partial_{14}^2 + 4x_2\partial_9\partial_{13}\partial_{14} + x_{11}\partial_{13}\partial_{14} + 2x_3\partial_{11}\partial_{12}\partial_{14} + 4x_1\partial_9\partial_{12}\partial_{14}$ $+ x_{10}\partial_{12}\partial_{14} - 2x_8\partial_9\partial_{14} - 2x_2\partial_{10}\partial_{13}^2 + 2x_2\partial_{11}\partial_{12}\partial_{13} - 2x_1\partial_{10}\partial_{12}\partial_{13} + x_9\partial_{12}\partial_{13} + 2x_8\partial_{10}\partial_{13} + 2x_1\partial_{11}\partial_{12}^2 - 2x_8\partial_{11}\partial_{12}$ $- 2x_3\partial_7\partial_{10} - 2x_2\partial_7\partial_9 - 2x_1\partial_6\partial_9 - x_{11}\partial_7 - x_{10}\partial_6 - x_9\partial_5 - x_8\partial_4) \otimes \text{id}$
	$(-2x_7\partial_{11} - 2x_6\partial_{10} - 2x_5\partial_9 - x_4\partial_8 + x_{15}) \otimes \text{id}$
	$(-x_{16}\partial_{16}^2 - x_{15}\partial_{15}\partial_{16} - x_{14}\partial_{14}\partial_{16} - x_{13}\partial_{13}\partial_{16} - x_{12}\partial_{12}\partial_{16} - x_{11}\partial_{11}\partial_{16} - x_{10}\partial_{10}\partial_{16} - x_9\partial_9\partial_{16} - x_8\partial_8\partial_{16} - x_6\partial_6\partial_{16}$ $- x_4\partial_4\partial_{16} + x_7\partial_9\partial_{15} + x_5\partial_6\partial_{15} + x_3\partial_4\partial_{15} + x_7\partial_{11}\partial_{14} + x_5\partial_8\partial_{14} - x_2\partial_4\partial_{14} - x_7\partial_{12}\partial_{13} - x_5\partial_{10}\partial_{13} - x_1\partial_4\partial_{13}$ $- x_3\partial_8\partial_{12} - x_2\partial_6\partial_{12} + x_3\partial_{10}\partial_{11} - x_1\partial_6\partial_{11} + x_2\partial_9\partial_{10} + x_1\partial_8\partial_9) \otimes \text{id}$
	$x_{16} \otimes \text{id}$
	$(-x_{14}\partial_{13} - x_{12}\partial_{11} - x_{10}\partial_8 - x_2\partial_1) \otimes \text{id}$
	$(-x_{13}\partial_{14} - x_{11}\partial_{12} - x_8\partial_{10} - x_1\partial_2) \otimes \text{id}$
	$(x_{16}\partial_{15} - x_9\partial_7 - x_6\partial_5 - x_4\partial_3) \otimes \text{id}$
E_6	$(x_{15}\partial_{16} - x_7\partial_9 - x_5\partial_6 - x_3\partial_4) \otimes \text{id}$
16	$(x_{15}\partial_{14} - x_{11}\partial_9 - x_8\partial_6 - x_3\partial_2) \otimes \text{id}$
	$(x_{14}\partial_{15} - x_9\partial_{11} - x_6\partial_8 - x_2\partial_3) \otimes \text{id}$
	$(x_{14}\partial_{12} + x_{13}\partial_{11} - x_6\partial_4 - x_5\partial_3) \otimes \text{id}$
	$(x_{12}\partial_{14} + x_{11}\partial_{13} - x_4\partial_6 - x_3\partial_5) \otimes \text{id}$
	$(x_{12}\partial_{10} + x_{11}\partial_8 + x_9\partial_6 + x_7\partial_5) \otimes \text{id}$
	$(x_{10}\partial_{12} + x_8\partial_{11} + x_6\partial_9 + x_5\partial_7) \otimes \text{id}$

\mathfrak{g} n element of $\mathbb{W}_n \simeq \mathbb{W}_n \otimes id$

	$(-x_{22}\partial_{21} - x_{20}\partial_{19} - x_{18}\partial_{17} - x_{16}\partial_{14} - x_{13}\partial_{11} - x_2\partial_1) \otimes id$
	$(-x_{21}\partial_{22} - x_{19}\partial_{20} - x_{17}\partial_{18} - x_{14}\partial_{16} - x_{11}\partial_{13} - x_1\partial_2) \otimes id$
	$(-x_{24}\partial_{23} - x_{22}\partial_{20} - x_{21}\partial_{19} - x_{10}\partial_8 - x_7\partial_6 - x_5\partial_4) \otimes id$
	$(-x_{23}\partial_{24} - x_{20}\partial_{22} - x_{19}\partial_{21} - x_8\partial_{10} - x_6\partial_7 - x_4\partial_5) \otimes id$
	$(-x_{24}\partial_{22} - x_{23}\partial_{20} - x_{17}\partial_{15} - x_{14}\partial_{12} - x_{11}\partial_9 - x_3\partial_2) \otimes id$
	$(-x_{22}\partial_{24} - x_{20}\partial_{23} - x_{15}\partial_{17} - x_{12}\partial_{14} - x_9\partial_{11} - x_2\partial_3) \otimes id$
	$(-x_{25}\partial_{24} - x_{20}\partial_{18} - x_{19}\partial_{17} - x_{12}\partial_{10} - x_9\partial_7 - x_4\partial_3) \otimes id$
	$(-x_{24}\partial_{25} - x_{18}\partial_{20} - x_{17}\partial_{19} - x_{10}\partial_{12} - x_7\partial_9 - x_3\partial_4) \otimes id$
	$(-x_{26}\partial_{25} - x_{18}\partial_{16} - x_{17}\partial_{14} - x_{15}\partial_{12} - x_7\partial_5 - x_6\partial_4) \otimes id$
	$(-x_{25}\partial_{26} - x_{16}\partial_{18} - x_{14}\partial_{17} - x_{12}\partial_{15} - x_5\partial_7 - x_4\partial_6) \otimes id$
	$(-x_{27}\partial_{26} - x_{16}\partial_{13} - x_{14}\partial_{11} - x_{12}\partial_9 - x_{10}\partial_7 - x_8\partial_6) \otimes id$
	$(-x_{26}\partial_{27} - x_{13}\partial_{16} - x_{11}\partial_{14} - x_9\partial_{12} - x_7\partial_{10} - x_6\partial_8) \otimes id$
E_7	$(-x_{27}\partial_{27}^2 - x_{26}\partial_{26}\partial_{27} - x_{25}\partial_{25}\partial_{27} - x_{24}\partial_{24}\partial_{27} - x_{23}\partial_{23}\partial_{27} - x_{22}\partial_{22}\partial_{27} - x_{21}\partial_{21}\partial_{27} - x_{20}\partial_{20}\partial_{27} - x_{19}\partial_{19}\partial_{27} - x_{18}\partial_{18}\partial_{27}$ $-x_{17}\partial_{17}\partial_{27} - x_{16}\partial_{16}\partial_{27} - x_{15}\partial_{15}\partial_{27} - x_{14}\partial_{14}\partial_{27} - x_{12}\partial_{12}\partial_{27} - x_{10}\partial_{10}\partial_{27} - x_8\partial_8\partial_{27} - x_{13}\partial_{16}\partial_{26} - x_{11}\partial_{14}\partial_{26} - x_9\partial_{12}\partial_{26}$ $-x_7\partial_{10}\partial_{26} - x_6\partial_8\partial_{26} + x_{13}\partial_{18}\partial_{25} + x_{11}\partial_{17}\partial_{25} + x_9\partial_{15}\partial_{25} - x_5\partial_{10}\partial_{25} - x_4\partial_8\partial_{25} - x_{13}\partial_{20}\partial_{24} - x_{11}\partial_{19}\partial_{24} + x_7\partial_{15}\partial_{24}$ $+x_5\partial_{12}\partial_{24} - x_3\partial_8\partial_{24} + x_{13}\partial_{22}\partial_{23} + x_{11}\partial_{21}\partial_{23} + x_6\partial_{15}\partial_{23} + x_4\partial_{12}\partial_{23} - x_9\partial_{19}\partial_{22} - x_7\partial_{17}\partial_{22} - x_5\partial_{14}\partial_{22}$ $-x_2\partial_8\partial_{22} + x_9\partial_{20}\partial_{21} + x_7\partial_{18}\partial_{21} + x_5\partial_{16}\partial_{21} - x_1\partial_8\partial_{21} - x_6\partial_{17}\partial_{20} + x_2\partial_{10}\partial_{20} + x_6\partial_{18}\partial_{19} + x_4\partial_{16}\partial_{19}$ $+x_1\partial_{10}\partial_{19} - x_3\partial_{14}\partial_{18} - x_2\partial_{12}\partial_{18} + x_3\partial_{16}\partial_{17} - x_1\partial_{12}\partial_{17} + x_2\partial_{15}\partial_{16} + x_1\partial_{14}\partial_{15}) \otimes id$ $x_{27} \otimes id$

element of $\mathbb{W}_n \simeq \mathbb{W}_n \otimes \text{id}$

\mathfrak{g} n

	$(-x_{51}\partial_{50} - x_{49}\partial_{48} - x_{47}\partial_{46} - x_{45}\partial_{43} - x_{42}\partial_{40} - x_{39}\partial_{36} - x_{23}\partial_{20} - x_{19}\partial_{17} - x_{16}\partial_{14} - x_{13}\partial_{12} - x_{11}\partial_{10} - x_9\partial_8) \otimes \text{id}$
	$(-x_{50}\partial_{51} - x_{48}\partial_{49} - x_{46}\partial_{47} - x_{43}\partial_{45} - x_{40}\partial_{42} - x_{36}\partial_{39} - x_{20}\partial_{23} - x_{17}\partial_{19} - x_{14}\partial_{16} - x_{12}\partial_{13} - x_{10}\partial_{11} - x_8\partial_9) \otimes \text{id}$
	$(-x_{53}\partial_{52} - x_{51}\partial_{49} - x_{50}\partial_{48} - x_{38}\partial_{35} - x_{34}\partial_{32} - x_{1}\partial_{28}\partial_{29} - x_{31}\partial_{29} - x_{30}\partial_{28} - x_{27}\partial_{25} - x_{24}\partial_{21} - x_{11}\partial_9 - x_{10}\partial_8 - x_7\partial_6) \otimes \text{id}$
	$(-x_{52}\partial_{53} - x_{49}\partial_{51} - x_{48}\partial_{50} - x_{35}\partial_{38} - x_{32}\partial_{34} - x_{1}\partial_{30}\partial_{31} - x_{29}\partial_{31} - x_{28}\partial_{30} - x_{25}\partial_{27} - x_{21}\partial_{24} - x_9\partial_{11} - x_8\partial_{10} - x_6\partial_7) \otimes \text{id}$
	$(-x_{53}\partial_{51} - x_{52}\partial_{49} - x_{46}\partial_{44} - x_{43}\partial_{41} - x_{40}\partial_{37} - x_{36}\partial_{33} - x_{26}\partial_{23} - x_{22}\partial_{19} - x_{18}\partial_{16} - x_{15}\partial_{13} - x_{10}\partial_7 - x_8\partial_6) \otimes \text{id}$
	$(-x_{51}\partial_{53} - x_{49}\partial_{52} - x_{44}\partial_{46} - x_{41}\partial_{43} - x_{37}\partial_{40} - x_{33}\partial_{36} - x_{23}\partial_{26} - x_{19}\partial_{22} - x_{16}\partial_{18} - x_{13}\partial_{15} - x_7\partial_{10} - x_6\partial_8) \otimes \text{id}$
	$(-x_{54}\partial_{53} - x_{49}\partial_{47} - x_{48}\partial_{46} - x_{41}\partial_{38} - x_{37}\partial_{34} - x_{33}\partial_{30} - x_{29}\partial_{26} - x_{25}\partial_{22} - x_{21}\partial_{18} - x_{13}\partial_{11} - x_{12}\partial_{10} - x_6\partial_5) \otimes \text{id}$
	$(-x_{53}\partial_{54} - x_{47}\partial_{49} - x_{46}\partial_{48} - x_{38}\partial_{41} - x_{34}\partial_{37} - x_{30}\partial_{33} - x_{26}\partial_{29} - x_{22}\partial_{25} - x_{18}\partial_{21} - x_{11}\partial_{13} - x_{10}\partial_{12} - x_5\partial_6) \otimes \text{id}$
	$(-x_{55}\partial_{54} - x_{47}\partial_{45} - x_{46}\partial_{43} - x_{44}\partial_{41} - x_{34}\partial_{31} - x_{1}\partial_{27}\partial_{29} - x_{32}\partial_{29} - x_{30}\partial_{27} - x_{28}\partial_{25} - x_{18}\partial_{15} - x_{16}\partial_{13} - x_{14}\partial_{12} - x_5\partial_4) \otimes \text{id}$
	$(-x_{54}\partial_{55} - x_{45}\partial_{47} - x_{43}\partial_{46} - x_{41}\partial_{44} - x_{31}\partial_{34} - x_{1}\partial_{30}\partial_{32} - x_{29}\partial_{32} - x_{27}\partial_{30} - x_{25}\partial_{28} - x_{15}\partial_{18} - x_{13}\partial_{16} - x_{12}\partial_{14} - x_4\partial_5) \otimes \text{id}$
	$(-x_{56}\partial_{55} - x_{45}\partial_{42} - x_{43}\partial_{40} - x_{41}\partial_{37} - x_{38}\partial_{34} - x_{35}\partial_{32} - x_{27}\partial_{24} - x_{25}\partial_{21} - x_{22}\partial_{18} - x_{19}\partial_{16} - x_{17}\partial_{14} - x_4\partial_3) \otimes \text{id}$
	$(-x_{55}\partial_{56} - x_{42}\partial_{45} - x_{40}\partial_{43} - x_{37}\partial_{41} - x_{34}\partial_{38} - x_{32}\partial_{35} - x_{24}\partial_{27} - x_{21}\partial_{25} - x_{18}\partial_{22} - x_{16}\partial_{19} - x_{14}\partial_{17} - x_3\partial_4) \otimes \text{id}$
	$(-x_{57}\partial_{56} - x_{42}\partial_{39} - x_{40}\partial_{36} - x_{37}\partial_{33} - x_{34}\partial_{30} - x_{1}\partial_{27}\partial_{28} - x_{32}\partial_{28} - x_{31}\partial_{27} - x_{29}\partial_{25} - x_{26}\partial_{22} - x_{23}\partial_{19} - x_{20}\partial_{17} - x_3\partial_2) \otimes \text{id}$
	$(-x_{56}\partial_{57} - x_{39}\partial_{42} - x_{36}\partial_{40} - x_{33}\partial_{37} - x_{30}\partial_{34} - x_{1}\partial_{31}\partial_{32} - x_{28}\partial_{32} - x_{27}\partial_{31} - x_{25}\partial_{29} - x_{22}\partial_{26} - x_{19}\partial_{23} - x_{17}\partial_{20} - x_2\partial_3) \otimes \text{id}$

E_8 57

\mathfrak{g} n element of $\mathbb{W}_n \simeq \mathbb{W}_n \otimes \text{id}$

$$\begin{aligned}
& (-x_{57}\partial_{57}^2 - x_{56}\partial_{56}\partial_{57} - x_{55}\partial_{55}\partial_{57} - x_{54}\partial_{54}\partial_{57} - x_{53}\partial_{53}\partial_{57} - x_{52}\partial_{52}\partial_{57} - x_{51}\partial_{51}\partial_{57} - x_{50}\partial_{50}\partial_{57} - x_{49}\partial_{49}\partial_{57} - x_{48}\partial_{48}\partial_{57} \\
& - x_{47}\partial_{47}\partial_{57} - x_{46}\partial_{46}\partial_{57} - x_{45}\partial_{45}\partial_{57} - x_{44}\partial_{44}\partial_{57} - x_{43}\partial_{43}\partial_{57} - x_{42}\partial_{42}\partial_{57} - x_{41}\partial_{41}\partial_{57} - x_{40}\partial_{40}\partial_{57} + x_{1}\partial_{20}\partial_{39}\partial_{57} \\
& - x_{38}\partial_{38}\partial_{57} - x_{37}\partial_{37}\partial_{57} - x_{1}\partial_{23}\partial_{36}\partial_{57} - x_{35}\partial_{35}\partial_{57} - x_{34}\partial_{34}\partial_{57} + x_{1}\partial_{26}\partial_{33}\partial_{57} - x_{32}\partial_{32}\partial_{57} - x_{31}\partial_{31}\partial_{57} - x_{1}\partial_{29}\partial_{30}\partial_{57} \\
& - x_{29}\partial_{29}\partial_{57} - x_{26}\partial_{26}\partial_{57} - x_{23}\partial_{23}\partial_{57} - x_{20}\partial_{20}\partial_{57} - x_{1}\partial_{20}\partial_{42}\partial_{56} - x_{39}\partial_{42}\partial_{56} + x_{1}\partial_{23}\partial_{40}\partial_{56} - x_{36}\partial_{40}\partial_{56} - x_{1}\partial_{26}\partial_{37}\partial_{56} \\
& - x_{33}\partial_{37}\partial_{56} + x_{1}\partial_{29}\partial_{34}\partial_{56} - x_{30}\partial_{34}\partial_{56} - 2x_{1}\partial_{31}\partial_{32}\partial_{56} - x_{28}\partial_{32}\partial_{56} - x_{27}\partial_{31}\partial_{56} - x_{25}\partial_{29}\partial_{56} - x_{22}\partial_{36}\partial_{56} - x_{19}\partial_{23}\partial_{56} \\
& - x_{17}\partial_{20}\partial_{56} + x_{1}\partial_{20}\partial_{45}\partial_{55} + x_{39}\partial_{45}\partial_{55} - x_{1}\partial_{23}\partial_{43}\partial_{55} + x_{36}\partial_{43}\partial_{55} + x_{1}\partial_{26}\partial_{41}\partial_{55} + x_{33}\partial_{41}\partial_{55} - x_{1}\partial_{29}\partial_{38}\partial_{55} \\
& + x_{30}\partial_{38}\partial_{55} + 2x_{1}\partial_{31}\partial_{35}\partial_{55} + x_{28}\partial_{35}\partial_{55} - x_{24}\partial_{31}\partial_{55} - x_{21}\partial_{29}\partial_{55} - x_{18}\partial_{36}\partial_{55} - x_{16}\partial_{23}\partial_{55} - x_{14}\partial_{20}\partial_{47}\partial_{54} \\
& - x_{39}\partial_{47}\partial_{54} + x_{1}\partial_{23}\partial_{46}\partial_{54} - x_{36}\partial_{46}\partial_{54} - x_{1}\partial_{26}\partial_{44}\partial_{54} - x_{33}\partial_{44}\partial_{54} + 2x_{1}\partial_{32}\partial_{38}\partial_{54} + x_{27}\partial_{38}\partial_{54} - 2x_{1}\partial_{34}\partial_{35}\partial_{54} \\
& + x_{25}\partial_{35}\partial_{54} + x_{24}\partial_{34}\partial_{54} + x_{21}\partial_{32}\partial_{54} - x_{15}\partial_{26}\partial_{54} - x_{13}\partial_{23}\partial_{54} - x_{12}\partial_{20}\partial_{54} + x_{1}\partial_{20}\partial_{49}\partial_{53} + x_{39}\partial_{49}\partial_{53} - x_{1}\partial_{23}\partial_{48}\partial_{53} \\
& + x_{36}\partial_{48}\partial_{53} + x_{1}\partial_{29}\partial_{44}\partial_{53} - x_{30}\partial_{44}\partial_{53} - 2x_{1}\partial_{32}\partial_{41}\partial_{53} - x_{27}\partial_{41}\partial_{53} + 2x_{1}\partial_{35}\partial_{37}\partial_{53} - x_{24}\partial_{37}\partial_{53} + x_{22}\partial_{35}\partial_{53} \\
& + x_{18}\partial_{32}\partial_{53} + x_{15}\partial_{29}\partial_{53} - x_{11}\partial_{23}\partial_{53} - x_{10}\partial_{20}\partial_{53} - x_{1}\partial_{20}\partial_{51}\partial_{52} - x_{39}\partial_{51}\partial_{52} + x_{1}\partial_{23}\partial_{50}\partial_{52} - x_{36}\partial_{50}\partial_{52} - 2x_{1}\partial_{31}\partial_{44}\partial_{52} \\
& - x_{28}\partial_{44}\partial_{52} + 2x_{1}\partial_{34}\partial_{41}\partial_{52} - x_{25}\partial_{41}\partial_{52} - 2x_{1}\partial_{37}\partial_{38}\partial_{52} - x_{22}\partial_{38}\partial_{52} - x_{21}\partial_{37}\partial_{52} - x_{18}\partial_{34}\partial_{52} - x_{15}\partial_{31}\partial_{52} - x_{9}\partial_{23}\partial_{52} \\
& - x_{8}\partial_{20}\partial_{52} + x_{1}\partial_{26}\partial_{48}\partial_{51} + x_{33}\partial_{48}\partial_{51} - x_{1}\partial_{29}\partial_{46}\partial_{51} + x_{30}\partial_{46}\partial_{51} + 2x_{1}\partial_{32}\partial_{43}\partial_{51} + x_{27}\partial_{43}\partial_{51} - 2x_{1}\partial_{35}\partial_{40}\partial_{51} \\
& + x_{24}\partial_{40}\partial_{51} + x_{19}\partial_{35}\partial_{51} + x_{16}\partial_{32}\partial_{51} + x_{13}\partial_{29}\partial_{51} + x_{11}\partial_{26}\partial_{51} - x_{7}\partial_{20}\partial_{51} - x_{1}\partial_{26}\partial_{49}\partial_{50} - x_{33}\partial_{49}\partial_{50} + x_{1}\partial_{29}\partial_{47}\partial_{50} \\
& - x_{30}\partial_{47}\partial_{50} - 2x_{1}\partial_{32}\partial_{45}\partial_{50} - x_{27}\partial_{45}\partial_{50} + 2x_{1}\partial_{35}\partial_{42}\partial_{50} - x_{24}\partial_{42}\partial_{50} + x_{17}\partial_{35}\partial_{50} + x_{14}\partial_{32}\partial_{50} + x_{12}\partial_{29}\partial_{50} + x_{10}\partial_{26}\partial_{50} \\
& + x_{7}\partial_{23}\partial_{50} + 2x_{1}\partial_{31}\partial_{46}\partial_{49} + x_{28}\partial_{46}\partial_{49} - 2x_{1}\partial_{34}\partial_{43}\partial_{49} + x_{25}\partial_{43}\partial_{49} + 2x_{1}\partial_{38}\partial_{40}\partial_{49} + x_{21}\partial_{40}\partial_{49} - x_{19}\partial_{38}\partial_{49} \\
& - x_{16}\partial_{34}\partial_{49} - x_{13}\partial_{31}\partial_{49} + x_{9}\partial_{26}\partial_{49} - x_{6}\partial_{20}\partial_{49} - 2x_{1}\partial_{31}\partial_{47}\partial_{48} - x_{28}\partial_{47}\partial_{48} + 2x_{1}\partial_{34}\partial_{45}\partial_{48} - x_{25}\partial_{45}\partial_{48} - 2x_{1}\partial_{38}\partial_{42}\partial_{48} \\
& - x_{21}\partial_{42}\partial_{48} - x_{17}\partial_{38}\partial_{48} - x_{14}\partial_{34}\partial_{48} - x_{12}\partial_{31}\partial_{48} + x_{8}\partial_{26}\partial_{48} + x_{6}\partial_{23}\partial_{48} + 2x_{1}\partial_{37}\partial_{43}\partial_{47} + x_{22}\partial_{43}\partial_{47} - 2x_{1}\partial_{40}\partial_{41}\partial_{47} \\
& + x_{19}\partial_{41}\partial_{47} + x_{18}\partial_{40}\partial_{47} + x_{16}\partial_{37}\partial_{47} - x_{11}\partial_{31}\partial_{47} - x_{9}\partial_{29}\partial_{47} - x_{5}\partial_{20}\partial_{47} - 2x_{1}\partial_{37}\partial_{45}\partial_{46} - x_{22}\partial_{45}\partial_{46} + 2x_{1}\partial_{41}\partial_{42}\partial_{46} \\
& - x_{18}\partial_{42}\partial_{46} + x_{17}\partial_{41}\partial_{46} + x_{14}\partial_{37}\partial_{46} - x_{10}\partial_{31}\partial_{46} - x_{8}\partial_{29}\partial_{46} + x_{5}\partial_{23}\partial_{46} + 2x_{1}\partial_{40}\partial_{44}\partial_{45} - x_{19}\partial_{44}\partial_{45} + x_{15}\partial_{40}\partial_{45} \\
& + x_{13}\partial_{37}\partial_{45} + x_{11}\partial_{34}\partial_{45} + x_{9}\partial_{32}\partial_{45} - x_{4}\partial_{20}\partial_{45} - 2x_{1}\partial_{42}\partial_{43}\partial_{44} - x_{17}\partial_{43}\partial_{44} - x_{16}\partial_{42}\partial_{44} - x_{14}\partial_{40}\partial_{44} - x_{7}\partial_{31}\partial_{44} \\
& - x_{6}\partial_{29}\partial_{44} - x_{5}\partial_{26}\partial_{44} - x_{15}\partial_{42}\partial_{43} + x_{12}\partial_{37}\partial_{43} + x_{10}\partial_{34}\partial_{43} + x_{8}\partial_{32}\partial_{43} + x_{4}\partial_{23}\partial_{43} - x_{13}\partial_{41}\partial_{42} - x_{11}\partial_{38}\partial_{42} - x_{9}\partial_{35}\partial_{42} \\
& - x_{3}\partial_{20}\partial_{42} - x_{12}\partial_{40}\partial_{41} + x_{7}\partial_{34}\partial_{41} + x_{6}\partial_{32}\partial_{41} - x_{4}\partial_{26}\partial_{41} - x_{10}\partial_{38}\partial_{40} - x_{8}\partial_{35}\partial_{40} + x_{3}\partial_{23}\partial_{40} - x_{2}\partial_{20}\partial_{39} - x_{7}\partial_{37}\partial_{38} \\
& + x_{5}\partial_{32}\partial_{38} + x_{4}\partial_{29}\partial_{38} - x_{6}\partial_{35}\partial_{37} - x_{3}\partial_{26}\partial_{37} + x_{2}\partial_{23}\partial_{36} - x_{5}\partial_{34}\partial_{35} - x_{4}\partial_{31}\partial_{35} + x_{3}\partial_{29}\partial_{34} - x_{2}\partial_{26}\partial_{33} - x_{3}\partial_{31}\partial_{32} \\
& + x_{2}\partial_{29}\partial_{30} - x_{2}\partial_{1}) \otimes \text{id}
\end{aligned}$$

 $(-x_1\partial_2 + x_{57}) \otimes \text{id}$

Table 2: The Lie bracket pairing table of G_2 . Generators are indexed according to root spaces, and root spaces are indexed according to graded lexicographic order. The first simple root is short.

roots	ε - notation	(\bullet, \bullet)	g_{-6}	g_{-5}	g_{-4}	g_{-3}	g_{-2}	g_{-1}	h_1	h_2	g_1	g_2	g_3	g_4	g_5	g_6
$(-3, -2)$	$\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3$	g_{-6}	0	0	0	0	0	0	0	$3g_{-6}$	0	0	$-g_{-4}$	g_{-3}	$-g_{-2}$	$-h_1 - 2/3h_2$
$(-3, -1)$	$-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3$	g_{-5}	0	0	0	0	g_{-6}	0	$3g_{-5}$	$-3g_{-5}$	g_{-4}	0	0	g_{-3}	$-h_1 - 1/3h_2$	g_5
$(-2, -1)$	$\varepsilon_2 - \varepsilon_3$	g_{-4}	0	0	0	$-3g_{-6}$	0	0	g_{-4}	0	$2g_{-3}$	0	0	$-g_{-1}$	0	g_3
$(-1, -1)$	$\varepsilon_1 - \varepsilon_3$	g_{-3}	0	0	$3g_{-6}$	0	0	$3g_{-5}$	$-g_{-4}$	$3g_{-3}$	$3g_{-2}$	$-g_{-1}$	$-h_1 - h_2$	0	0	$-g_4$
$(0, -1)$	$2\varepsilon_1 - \varepsilon_2 - \varepsilon_3$	g_{-2}	0	0	0	0	0	g_{-3}	$-3g_{-2}$	$6g_{-2}$	0	$-1/3h_2$	0	0	0	g_5
$(-1, 0)$	$-\varepsilon_1 + \varepsilon_2$	g_{-1}	0	$-g_{-6}$	$-3g_{-5}$	$-2g_{-4}$	$-g_{-3}$	0	$2g_{-1}$	$-3g_{-1}$	$-h_1$	0	$3g_2$	$2g_3$	0	0
$(0, 0)$	0	h_1	0	$-3g_{-5}$	$-g_{-4}$	g_{-3}	$3g_{-2}$	$-2g_{-1}$	0	0	$2g_1$	$-3g_2$	$-g_3$	g_4	$3g_5$	0
$(0, 0)$	0	h_2	$-3g_{-6}$	$3g_{-5}$	0	$-3g_{-3}$	$-6g_{-2}$	$3g_{-1}$	0	0	$-3g_1$	$6g_2$	$3g_3$	0	$-3g_5$	$3g_6$
$(1, 0)$	$\varepsilon_1 - \varepsilon_2$	g_1	0	$-g_{-4}$	$-2g_{-3}$	$-3g_{-2}$	0	h_1	$-2g_1$	$3g_1$	0	g_3	$2g_4$	0	0	0
$(0, 1)$	$-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$	g_2	$-g_{-5}$	0	0	$g_{-1} + h_2$	$1/3h_2$	0	$3g_2$	$-6g_2$	0	0	0	0	g_6	0
$(1, 1)$	$-\varepsilon_1 + \varepsilon_3$	g_3	g_{-4}	0	$2g_{-1}$	$h_1 + h_2$	g_1	$-3g_2$	g_3	$-3g_3$	$-2g_4$	0	0	0	0	0
$(2, 1)$	$-\varepsilon_2 + \varepsilon_3$	g_4	$-g_{-3}$	$g_{-1} + h_2$	$2h_1 + h_2$	$2g_1$	0	$-2g_3$	$-g_4$	0	$-3g_5$	0	0	0	0	0
$(3, 1)$	$\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3$	g_5	g_{-2}	0	g_1	0	0	$-g_4$	$-3g_5$	$3g_5$	0	$-g_6$	0	0	0	0
$(3, 2)$	$-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3$	g_6	$h_1 + 2/3h_2$	g_2	$-g_3$	g_4	$-g_5$	0	0	$-3g_6$	0	0	0	0	0	0

References

- [1] Conze, N., *Algèbres d'opérateurs différentiels et quotients des algèbres enveloppantes*, Bull. Soc. Math. France **102** (1974), 379–415.
- [2] Dixmier, J., “Algèbres Enveloppantes”, Gauthier-Villars Editeur, 1974.
- [3] de Graaf, W., “Lie algebras: theory and algorithms”, Elsevier, 2000.
- [4] Humphreys, J., “Introduction to Lie algebras and representation theory”, Springer-Verlag, New York, 1972.
- [5] Jackson, S., and T. Milev, *Computing irreducible finite dimensional complex Lie algebra representations*, in preparation.
- [6] Joseph, A., *Minimal realizations and spectrum generating algebras*, Comm. Math. Phys. **36** (1974), 325–338.
- [7] —, *A minimal orbit in a simple Lie algebra and its associated maximal ideal*, Ann. Sci. Ecole Norm. Sup. **9** (1976), 1–29.
- [8] Kobayashi, T., Ørsted, B., Somberg, P., and V. Soucek, *Branching laws for Verma modules and applications in parabolic geometry. I*.
<http://arxiv.org/abs/1305.6040>.
- [9] Littelmann, P., *Paths and root operators in representation theory*, Ann. of Math. **142** (1995), 499–525.
- [10] —, *Cones, crystals, and patterns*, Transf. Groups **3** (1998), 145–179.
- [11] Milev, T., Vector partition function project,
<http://sourceforge.net/projects/vectorpartition/>, January 2014.
Source code, revision 1951,
<http://sourceforge.net/p/vectorpartition/code/1951/tree/trunk/>.
2009–2014.
- [12] Milev, T., and P. Somberg, *The F-method and a branching problem for generalized Verma modules associated to $(G_2, so(7))$* , Arch. Math. (Brno) **49** (2013), 317–332.
- [13] —, *The branching problem for generalized Verma modules, with application to the pair $(so(7), LieG_2)$* , J. Algebra Appl. **13** (2014), DOI: 10.1142/S0219498814500340.
- [14] Samelson, H., “Notes on Lie algebras”, Springer, 2nd edition, 1990.

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