

# $C^*$ -Algebras with Norm Controlled Dual Limits and Nilpotent Lie Groups

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**Abstract.** Motivated by the description of the  $C^*$ -algebras of 5-dimensional nilpotent Lie groups as algebras of operator fields defined over their spectra, we introduce the family of  $C^*$ -algebras with norm controlled dual limits and we show by explicit computations that the  $C^*$ -algebras of every 5-dimensional nilpotent Lie group belong to this class.

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## 1. Introduction.

In recent papers, the  $C^*$ -algebra of the Heisenberg groups, of threadlike groups and “ $ax + b$ ”-like groups have been described as algebras of operator fields (see [LT11] and [LL10]). For this description a precise understanding of the topology of the spectrum of these groups was essential (see for instance [ALS07] for the case of threadlike groups). In this paper we study the group  $C^*$ -algebra of all connected nilpotent Lie groups of dimension  $\leq 5$  as an algebra of operator fields. This family of Lie groups has been classified by several authors; a list can be found for instance in [Nie83]. It contains the Heisenberg groups of dimensions 3 and 5 and also the threadlike groups  $F_4$  and  $F_5$ . There are 6 simply connected nilpotent indecomposable Lie groups of dimension 5. Thanks to Kirillov’s orbit picture of the spectrum of a connected simply connected nilpotent Lie group, we have a description of the spectrum of these groups in terms of the structure of the space of its co-adjoint orbits. But the orbit theory is only an algorithm, it does not give us any detail about the result of computations. The topology of the orbit space or the behaviour of the operators  $\pi(F)$ ,  $F \in C^*(G)$  as  $\pi$  varies in the spectrum is different for each of these groups and must be studied case by case.

The paper begins with section 2, where some definitions, methods and results are presented which are needed in the sequel. In section 3, a family of  $C^*$ -algebras, which we call  $C^*$ -algebras with norm controlled dual limits (see Definition 3.3) is introduced. This is a family of separable  $CCR$ -algebras  $A$ ,

for which there exists a finite increasing family  $S_0 \subset S_1 \subset \dots \subset S_d = \widehat{A}$  of closed subsets of the spectrum  $\widehat{A}$  of  $A$ , such that for  $i = 1, \dots, d$  the subsets  $\Gamma_0 = S_0$  and  $\Gamma_i := S_i \setminus S_{i-1}$  have separated relative topologies and which have the property that for every converging sequence  $\bar{\gamma} = ((\gamma_k, \mathcal{H}_k))_k \subset S_i$  with limit set  $L(\bar{\gamma}) \subset S_{i-1}$  there exists a sequence  $(\tilde{\sigma}_{\bar{\gamma},k})_k : CB(S_{i-1}) \mapsto B(\mathcal{H}_k)$  (here  $CB(S_{i-1})$  denotes the  $C^*$ -algebra of continuous bounded operator fields defined over  $S_{i-1}$ ) of linear mappings, which is uniformly bounded in  $k$ , such that for every  $a \in A$  we have  $\lim_{k \rightarrow \infty} \|\gamma_k(a) - \tilde{\sigma}_k(a)\|_{\text{op}} = 0$ , where  $\tilde{\sigma}_k(a) = \tilde{\sigma}_{\bar{\gamma},k}(\widehat{a}|_{S_{i-1}})$ . These  $C^*$ -algebras are then completely determined by the topology of their spectra (in particular by the limit sets  $L(\bar{\gamma})$  of properly converging sequences in  $\widehat{A}$ ) and these mappings  $(\tilde{\sigma}_{\bar{\gamma},k})_k$  (see Corollary 3.6).

We then study the 6 groups of dimension  $\leq 5$  case by case and we show that all of them have  $C^*$ -algebras with norm controlled dual limits. For the Heisenberg and the threadlike groups this has already be shown in the paper [LT11]. There remains then only the 4 groups  $G_{5,2}, G_{5,3}, G_{5,4}$  and  $G_{5,6}$ , which are treated separately in the sections 6,7,8 and 9. Since the structure of the dual space of these groups is different for each of them, we must determine the topology of  $\widehat{G}$  group by group and construct by hand for every limit set  $\bar{\mathcal{O}}$  of a properly converging sequence in  $\mathfrak{g}^*/G$ , these essential mappings  $\sigma_{\bar{\mathcal{O}},k}$ .

To understand these mappings  $\sigma_{\bar{\mathcal{O}},k}$ , one has to recall a theorem of Fell (see [Fel60]), where he shows that in the case of a properly converging net  $\bar{\mathcal{O}} = (\pi_k)_k \subset \widehat{A}$  of a  $C^*$ -algebra  $A$ , with limit set  $L$ , one has that

$$\lim_{k \rightarrow \infty} \|\pi_k(a)\|_{\text{op}} = \sup_{\pi \in L} \|\pi(a)\|_{\text{op}}, \quad a \in A.$$

To implement that theorem we need the mappings  $\tilde{\sigma}_{\bar{\mathcal{O}},k}$ . To explain roughly these mappings we shall take our properly converging sequence  $\bar{\mathcal{O}} = (\mathcal{O}_k)_k \subset S_i$ , with limit set  $L \subset S_{i-1}$ , realize the representations  $\pi_k$  of the co-adjoint orbits  $\mathcal{O}_k$  as  $\pi_k = \text{ind}_{P_k}^G \chi_{\ell_k}$ , for some  $\ell_k \in \mathcal{O}_k$ , choose for our limit sets  $L$ , for every  $k \in \mathbb{N}$ , a partition  $L = \bigcup_j I_j^k$  of  $L$  with small measurable sets  $I_j^k$  and we consider a corresponding partition  $\mathcal{O}_k^j \subset \mathcal{O}_k = \bigcup_j \mathcal{O}_j^k$  where the distance from the  $\mathcal{O}_j^k$  to the corresponding set  $I_j^k$  is going to 0 as  $k$  goes to  $\infty$ . We can find for every  $k, j$  and element  $p_j^k$  in  $I_j^k$ , an element  $g_j^k \in U_j^k := \{g \in G; \text{Ad}^*(g)\ell_k \in \mathcal{O}_j^k\}$ , such that  $\text{Ad}^*(g_j^k)\ell_k = p_j^k$ , and such that  $P_j^k := \text{Ad}(g_j^k)P_k$  is subordinated to  $p_j^k$ .

Let  $\sigma_{k,j} := \text{ind}_{P_j^k}^G \chi_{p_j^k}$ . These data can be chosen in such a way, that the difference of operators  $M_{k,j} \circ (\pi_k(F) - \sigma_{k,j}(F)) \circ M_{k,j}$  goes to 0 uniformly in  $j$  as  $k$  tends to  $\infty$ . Here  $M_{k,j}$  denotes the multiplication operator with the characteristic function of the set  $U_{k,j}$ . It suffices now to disintegrate the representations  $\sigma_{k,j} = \int_{S_j^k} \pi_{k,j,s} d\mu_{k,j}(s)$  over a measure set  $(S_j^k, \mu_{k,j})$  into an integral of irreducible representations  $\pi_{k,j,s}$  contained in  $S_{i-1}$ , to obtain these mappings  $\bar{\sigma}_{\bar{\mathcal{O}},k}$ :

$$\tilde{\sigma}_{\bar{\mathcal{O}},k}(\varphi) = \sum_j M_{k,j} \circ \left( \int_{S_j^k} \varphi(\pi_{k,j,s}) d\mu_{k,j}(s) \right) \circ M_{k,j}, \quad \varphi \in CB(S_{i-1}).$$

The mappings  $\tilde{\sigma}_{\bar{\mathcal{O}},k}$  determine the “control” of the behaviour of the sequences

of operators  $\pi_k(a), a \in C^*(G)$ . In some forthcoming papers, they will also be constructed for all nilpotent Lie groups of dimension 6.

Let  $A$  be a separable CCR  $C^*$ -algebra, let  $\bar{\pi} = ((\pi_k, \mathcal{H}_k))_k$  be a properly converging sequence in  $\widehat{A}$  with limit set  $L$ . Let  $\bar{\pi} : A \rightarrow \mathcal{B}(\sum_k \mathcal{H}_k)$  be the corresponding  $C^*$ -homomorphism, i.e.  $\bar{\pi}(a) = (\pi_0(a), \pi_1(a), \pi_2(a), \dots), a \in A$ . Let  $A_{\bar{\pi}}$  be the image of  $\bar{\pi}$  and  $K_{\bar{\pi}}$  be its kernel. Let  $N_{\bar{\pi}}$  be the closed two-sided ideal of  $A$  consisting of all  $a \in A$  for which  $\lim_k \|\pi_k(a)\|_{\text{op}} = 0$ . Let  $K_L = \bigcap_{\rho \in L} \ker(\rho)$  be the kernel of the closed subset  $L \subset \widehat{A}$ . By the theorem of Fell,  $K_L = N_{\bar{\pi}}$  since for every  $a \in A$

$$\lim_k \|\pi_k(a)\|_{\text{op}} = \sup_{\rho \in L} \|\rho(a)\|_{\text{op}}.$$

This observation allows us to define the homomorphism  $\tilde{\pi} : A/K_L \rightarrow \bar{\pi}(A)/\bar{\pi}(I_{\bar{\pi}})$ . By [CE76], Corollary 3.11, there exists a bounded linear mapping  $\psi : A/K_L \rightarrow \bar{\pi}(A)$ , such that for every  $a \in A$ :  $\psi(a + K_L) + \bar{\pi}(I_{\bar{\pi}}) = \tilde{\pi}(a + K_L)$ .

The operator valued Fourier transform:

$$\mathcal{F}(a)(\pi) = \widehat{a}(\pi) := \pi(a), a \in A, \pi \in \widehat{A},$$

maps  $A/K_L$  isometrically onto the algebra of operator fields  $\mathcal{F}(A)_{|L} = \{\widehat{a}_{|L}, a \in A\}$ . We obtain in this way a bounded linear mapping  $\tilde{\sigma} : \mathcal{F}(A)_{|L} \mapsto \bar{\pi}(A)$ ,

$$\tilde{\sigma}(\widehat{a}_{|L}) = \psi(a + K_L), a \in A,$$

which satisfies the relation  $\bar{\pi}(a) - \tilde{\sigma}(\widehat{a}_{|L}) \in \bar{\pi}(I_{\bar{\pi}})$ , i.e. if we write  $\tilde{\sigma}(\widehat{a}_{|L})$  as a sequence, i. e.

$$\tilde{\sigma}(\widehat{a}_{|L}) = (\tilde{\sigma}_k(\widehat{a}_{|L}) \in \mathcal{K}(\mathcal{H}_k))_k$$

then

$$\lim_k \|\tilde{\sigma}_k(\widehat{a}_{|L}) - \pi_k(a)\|_{\text{op}} = 0.$$

Hence this “norm control”  $\tilde{\sigma}$  always exists in an abstract way. The problem is how to find a precise expression for the mappings  $\tilde{\sigma}_k$ , whenever the algebra  $A$  and the representations  $\pi_k$  are concretely given.

## 2. Preliminaries.

### 2.1. Orbit picture.

In this section we indicate some definitions, methods and results, which will be essentially needed in the sequel.

#### Kirillov theory.

The orbit method developed by Kirillov for a connected simply connected nilpotent Lie group  $G$  tells us that for every irreducible unitary representation  $\pi$  of  $G$  there exists an  $\ell \in \mathfrak{g}^*$  and a polarization  $\mathfrak{p} \subset \mathfrak{g}$  at  $\ell$  (i.e. a subalgebra  $\mathfrak{p}$  of

$\mathfrak{g}$  of dimension  $d = \frac{\dim(\mathfrak{g}) + \dim(\mathfrak{g}(\ell))}{2}$  with the property  $\langle \ell, [\mathfrak{p}, \mathfrak{p}] \rangle = \{0\}$ ) such that  $\pi$  is equivalent to the induced representation  $\pi_{\ell, \mathfrak{p}} = \text{ind}_P^G \chi_\ell$  of the unitary character  $\chi_\ell = e^{-2\pi i \ell \circ \log|_P}$  from  $P = \exp(\mathfrak{p})$  to  $G$ . Furthermore for two linear functionals  $\ell, \ell'$  on  $\mathfrak{g}$  and the Pukanszky polarizations  $\mathfrak{p}$  at  $\ell$  ( resp  $\mathfrak{p}'$  at  $\ell'$ ), the representations  $\pi_{\ell, \mathfrak{p}}$  and  $\pi_{\ell', \mathfrak{p}'}$  are equivalent if and only if  $\ell$  and  $\ell'$  are contained in the same  $G$ -orbit (see [CG90]). Let  $[\pi]$  denote the unitary equivalence class of a unitary representation  $\pi$  of  $G$ . The Kirillov map

$$K : \mathfrak{g}^*/G \rightarrow \widehat{G}; O = \text{Ad}^*(G)\ell \rightarrow [\pi_{\ell, \mathfrak{p}}] = \pi_O,$$

is a homeomorphism of the orbit space  $\mathfrak{g}^*/G$  onto  $\widehat{G}$  (see [LL94]).

**Coadjoint orbits.**

Let  $\mathfrak{g}$  be a nilpotent Lie algebra of dimension  $n$ . In this paper we shall use the following notation: If  $X_1, \dots, X_r$  are any vectors of  $\mathfrak{g}$ ,  $\langle X_1, \dots, X_r \rangle$  will denote the subspace generated by  $X_1, \dots, X_r$ . Let's now fix a Jordan-Hölder basis  $\mathcal{Z} = \{Z_1, \dots, Z_n\}$  of  $\mathfrak{g}$ , i. e. such that  $[Z_i, Z_j] \subset \langle Z_{r+1}, \dots, Z_n \rangle$  where  $r = \max(i, j)$ , for all  $i, j \in \{1, \dots, n\}$ . Let  $\mathfrak{g}_k = \langle Z_k, \dots, Z_n \rangle$  and  $\ell_k = \ell|_{\mathfrak{g}_k}$  for all  $\ell \in \mathfrak{g}^*$ . Denote by  $I_\ell$  the set of Pukanszky jumping indices for  $\ell$  in  $\mathfrak{g}^*$ . This means by definition that

$$j \in I_\ell \Leftrightarrow \mathfrak{g}(\ell) + \mathfrak{g}_{j+1} \subsetneq \mathfrak{g}(\ell) + \mathfrak{g}_j,$$

where  $\mathfrak{g}(\ell)$  is the stabilizer of  $\ell$  in  $\mathfrak{g}$ , i.e.  $\mathfrak{g}(\ell) = \{U \in \mathfrak{g}; \langle \ell, [U, \mathfrak{g}] \rangle = \{0\}\}$ . The number of Pukanszky jumping indices for  $\ell$  in  $\mathfrak{g}$  is equal to the dimension of the coadjoint orbit of  $\ell$  in  $\mathfrak{g}^*$ . Furthermore, for any  $\ell \in \mathfrak{g}^*$ , we have  $I_\ell = I_{\ell'}$  for any  $\ell'$  in the coadjoint orbit of  $\ell$ .

For an index set  $I = \{i_1 < \dots < i_d\}$  let  $\mathfrak{g}_I^*$  be the  $G$ -invariant subset of all the  $\ell$ 's in  $\mathfrak{g}^*$  for which  $I_\ell = I$ . Denote by  $\mathcal{I}^{\mathcal{Z}}$  the collection of all index sets  $I$  for which  $\mathfrak{g}_I^*$  is not empty. For every  $I \in \mathcal{I}^{\mathcal{Z}}$  the subset  $\mathfrak{g}_I^*$  is Zariski locally closed in  $\mathfrak{g}^*$ . For an index set  $I \in \mathcal{I}^{\mathcal{Z}}$  of length  $d_I$ , let  $P_I : \mathfrak{g}^* \rightarrow \mathbb{R}$  be the polynomial function defined by  $P_I(\ell) = \det(M_I(\ell))$ , where  $M_I(\ell) = (a_{i,j}(\ell))_{i,j \in I}$  is the  $d_I \times d_I$  matrix with coefficients

$$a_{i,j}(\ell) := \langle \ell, [Z_i, Z_j] \rangle.$$

This polynomial function is  $G$ -invariant on  $\mathfrak{g}_I^*$ . There exists an order on the set of indices  $\mathcal{I}^{\mathcal{Z}}$  such that

$$\mathfrak{g}_I^* = \{\ell; P_I(\ell) = 0, P_{I'}(\ell) = 0, \forall I' < I\}.$$

For each index set  $I = \{i_1 < \dots < i_d\} \in \mathcal{I}^{\mathcal{Z}}$  we can find a smooth parametrization of all the orbits contained in  $\mathfrak{g}_I^*$ : Let  $d$  be the dimension of the orbits in  $\mathfrak{g}_I^*$ . There exist  $n$  rational functions  $p_j(z, \ell)$  on  $\mathbb{R}^d \times \mathfrak{g}^*$ , which are regular on  $\mathbb{R}^d \times \mathfrak{g}_I^*$ , given by polynomials in  $z \in \mathbb{R}^d$  for fixed  $\ell$ , such that:

- $p_i(z, \ell) = z_j$  for  $i = i_j \in I$ ,

- $p_i((z_1, \dots, z_d), \ell)$  does only depend on  $(z_d, \dots, z_j)$  if  $i_{j-1} < i \leq i_j$ ,
- $Ad^*(G)\ell = \{\sum_{i=1}^n p_i(z, \ell)Z_i, z \in \mathbb{R}^d\}$ .

Let for  $I \in \mathcal{I}^{\mathbb{Z}}$ :

$$\mathfrak{n}_I^* := \{\ell \in \mathfrak{g}^*; \langle \ell, Z_i \rangle = 0, i \in I\}.$$

Then every orbit  $O \subset \mathfrak{g}_I^*$  has a one point intersection with  $\mathfrak{n}_I^*$  given by

$$O \cap \mathfrak{n}_I^* = \{\ell_O = \sum_{i=1}^n p_i(0, \ell)Z_i\}.$$

The mapping  $O \rightarrow \ell_O$  from the orbit space  $(\mathfrak{g}_I^*)/G$  into  $\mathfrak{g}_I^*$  is continuous, since the functions  $\ell \rightarrow p_j(z, \ell)$  are  $G$ -invariant and continuous on  $\mathfrak{g}_I^*$  for fixed  $z \in \mathbb{R}^d$ . (For more details on the parametrization of co-adjoint orbits see [CG90]).

It follows from the trace formula of Kirillov, that for any Schwartz function  $F$  on  $G$  and for any  $I \in \mathcal{I}^{\mathbb{Z}}$ , the function  $\mathfrak{g}_I^*/G \rightarrow \mathbb{C}, O \mapsto \text{tr}(\pi_O(F))$  is continuous (see for instance [CG90]).

**Induced representations.**

**Definition 2.1.** Let  $H = \exp(\mathfrak{h})$  be a closed connected subgroup of a connected nilpotent Lie group  $G = \exp(\mathfrak{g})$  and let  $\chi_\ell : H \rightarrow \mathbb{T}; \chi_\ell(h) = e^{-2\pi i \langle \ell, \log(h) \rangle}, h \in H (\ell \in \mathfrak{g}^*)$ , be a unitary character of  $H$ . The quotient space  $G/H$  has a unique left invariant measure  $d\dot{g}$ . With this measure we can define the Hilbert space  $\mathcal{H} = \mathcal{H}_{H,\ell} = L^2(G/H, \ell)$  by

$$L^2(G/H, \ell) := \{\xi : G \rightarrow \mathbb{C}, \xi \text{ measurable, } \xi(gh) = \chi_\ell(h^{-1})\xi(g), h \in H, g \in G, \|\xi\|_2^2 := \int_{G/H} |\xi(g)|^2 d\dot{g} < \infty\}.$$

The group  $G$  acts by left translation on this space and defines a unitary representation

$$\sigma_{\ell,\mathfrak{h}}(g)\xi(u) := \xi(g^{-1}u), \xi \in L^2(G/H, \ell), g, u \in G,$$

called the induced representation of  $\chi_\ell$  (from  $H$  to  $G$ ).

If  $\mathfrak{h}$  is a polarization at  $\ell$ , then the representation  $\sigma_{\ell,\mathfrak{h}}$  is irreducible and we denote it sometimes by  $\pi_{\ell,\mathfrak{h}}$ . If we take a Malcev basis  $\mathcal{B} = \{X_1, \dots, X_d\}$  of  $\mathfrak{g}$  relative to  $\mathfrak{h}$ , which means that the subspaces  $\mathfrak{g}_j := \text{span}\{X_j, \dots, X_d, \mathfrak{h}\}$  are subalgebras of  $\mathfrak{g}$  and that  $\mathfrak{g} = \bigoplus_{j=1}^d \mathbb{R}X_j \oplus \mathfrak{h}$  is a direct sum, then the mapping  $E_{\mathcal{B}} : \mathbb{R}^d \times H \mapsto G, E_{\mathcal{B}}(t_1, \dots, t_d, h) := \exp(t_1 X_1) \cdots \exp(t_d X_d)h$  is a diffeomorphism and it allows us to identify the Hilbert space  $L^2(G/H, \ell)$  with the space  $L^2(\mathbb{R}^d)$ . We shall consider in the following pages always the representations  $\sigma_{\ell,\mathfrak{h}}$  as representations on the space  $L^2(\mathbb{R}^d)$  (see [CG90]).

It is well known that for  $F \in L^1(G)$  the operator  $\sigma_{\ell,\mathfrak{h}}(F)$  is a kernel operator with kernel function

$$F_{\ell,\mathfrak{h}}(s, t) = \int_H F(sht^{-1})\chi_\ell(h)dh, s, t \in G.$$

If  $H = \exp(\mathfrak{h})$  is a normal subgroup of  $G$ , then the function  $F_{\ell, \mathfrak{h}}$  can be written as

$$F_{\ell, \mathfrak{h}}(s, t) = \widehat{F}^{\mathfrak{h}}(st^{-1}, \text{Ad}^*(t)\ell|_{\mathfrak{h}}), s, t \in G,$$

where

$$\widehat{F}^{\mathfrak{h}}(s, q) := \int_H F(sh)e^{-2\pi i \langle q, \log(h) \rangle} dh, s \in G, q \in \mathfrak{h}^*. \tag{2.1}$$

We denote for a normal subgroup  $H = \exp(\mathfrak{h})$  by  $L_c^1 = L_{c, \mathfrak{h}}^1$  the subspace of  $L^1(G)$  consisting of all  $F$ 's in  $L^1(G)$  for which  $\widehat{F}^{\mathfrak{h}} \in C_c^\infty(G/H \times \mathfrak{h}^*)$ . This space  $L_c^1$  is dense in  $L^1(G)$  and hence it is also dense in  $C^*(G)$ . The functions  $\widehat{F}^{\mathfrak{h}}$  satisfy the covariance condition

$$\widehat{F}^{\mathfrak{h}}(sh, q) = \chi_q(h^{-1})\widehat{F}^{\mathfrak{h}}(s, q), s \in G, h \in H, q \in \mathfrak{h}^*.$$

Let  $g \in G$ , let  $\mathfrak{h}^g := \text{Ad}(g)\mathfrak{h}$ ,  $H^g := gHg^{-1}$  and  $\ell^g := \text{Ad}^*(g)\ell$ . The representations  $\sigma_{\ell, \mathfrak{h}}$  and  $\sigma_{\ell^g, \mathfrak{h}^g}$  are equivalent, the intertwining operator  $T_g$  is given by:

$$T_g \xi(x) := \xi(xg), x \in G, \xi \in L^2(G/H, \chi_\ell). \tag{2.2}$$

Since for every  $F \in L_c^1$  the function  $\widehat{F}^{\mathfrak{h}}$  is smooth with compact support on  $G/H \times \mathfrak{h}^*$ , there exists a function  $\varphi \in C_c(G/H)$  such that

$$|\widehat{F}^{\mathfrak{h}}(s, q)| \leq \|q\| |\varphi(s)|, s \in G, q \in \mathfrak{h}^*.$$

We shall often use in the sequel the following fact.

Let  $F : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$  be a smooth function with compact support for which there exists some continuous function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  with compact support, such that

$$|F(x, y)| \leq \varphi(x - y), x, y \in \mathbb{R}^d.$$

Then by Young's inequality, for the kernel operator  $T_F$  defined on  $L^2(\mathbb{R}^d)$  by

$$T_F(\xi)(x) := \int_{\mathbb{R}^d} F(x, y)\xi(y)dy \text{ for } \xi \in L^2(\mathbb{R}^d), x \in \mathbb{R}^d,$$

its operator norm  $\|T_F\|_{\text{op}}$  is bounded by the  $L^1$ -norm  $\|\varphi\|_1$  of  $\varphi$ .

**Convergence of traces.**

Let  $\mathfrak{g}$  be a nilpotent Lie algebra. We fix a Euclidean scalar product on  $\mathfrak{g}$ . Let  $(\ell_k)_{k \in \mathbb{N}}$  be a converging sequence in  $\mathfrak{g}^*$  with limit  $\ell$ . Let  $(\mathfrak{h}_k)_k$  be a sequence of subalgebras of  $\mathfrak{g}$ , such that  $\dim(\mathfrak{h}_k) = n - d, k \in \mathbb{N}$ , for some  $d \in \mathbb{N}^*$ , and such that  $\langle \ell_k, [\mathfrak{h}_k, \mathfrak{h}_k] \rangle = \{0\}, k \in \mathbb{N}$ . We pick for every  $k \in \mathbb{N}$  a Malcev basis  $\mathcal{Z}_k = \{Z_1^k, \dots, Z_n^k\}$  of  $\mathfrak{g}$ , such that  $\mathfrak{h}_k = \text{span}\{Z_{n-d}^k, \dots, Z_n^k\}$ . Assume (passing if necessary to a subsequence), that the vectors  $Z_j^k$  converge for every  $j = 1, \dots, n$  to a vector  $Z_j$ , such that  $\mathcal{Z} := \{Z_1, \dots, Z_n\}$  is a Malcev basis of  $\mathfrak{g}$  which passes through  $\mathfrak{h} := \text{span}\{Z_{n-d}, \dots, Z_n\}$ . Then  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$ , which is subordinated to  $\ell$ , i.e.  $\langle \ell, [\mathfrak{h}, \mathfrak{h}] \rangle = \{0\}$ . Let  $H_k := \exp(\mathfrak{h}_k), k \in \mathbb{N}$  and  $H = \exp(\mathfrak{h})$ . Let  $\sigma_k := \sigma_{\ell_k, \mathfrak{h}_k}$  and  $\sigma = \sigma_{\ell, \mathfrak{h}}$ . This gives us the following.

**Proposition 2.2.** *The representations  $\sigma_k$  converge weakly to the representation  $\sigma$ . This means that for every  $\xi, \eta \in L^2(\mathbb{R}^d)$ , for every  $a \in C^*(G)$ :*

$$\lim_{k \rightarrow \infty} \langle \sigma_k(a)\xi, \eta \rangle = \langle \sigma(a)\xi, \eta \rangle.$$

**Proof.** Take first  $F \in C_c(G)$  and  $\xi, \eta \in C_c(\mathbb{R}^d)$ . We have then that:

$$\langle \sigma_k(F)\xi, \eta \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \xi(u)F_k(u, v)\bar{\eta}(v)dudv,$$

where

$$F_k(u, v) := \int_{\mathbb{R}^{n-d}} F(E_k(u)E_{k,d}(h)E_k(v)^{-1})\chi_k(E_{k,d}(h))dh, u, v \in \mathbb{R}^n,$$

and where  $E_k = E_{Z_k}, E_{k,d} = E_{k|\mathbb{R}^{n-d}}, k \in \mathbb{N}$ . We get a similar expression for  $k = \infty$ , i.e. for  $E_Z$  etc. It is easy to see that the supports of the functions  $f_k : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{n-d} \rightarrow \mathbb{C}$  defined by

$$f_k(u, v, h) := \xi(u)F(E_k(u)E_{k,d}(h)E_k(v)^{-1})\chi_k(E_{k,d}(h))\bar{\eta}(v),$$

are contained in a common compact set and that the functions converge pointwise to the function

$$f(u, v, h) := \xi(u)F(E(u)E_d(h)E(v)^{-1})\chi(E_d(h))\bar{\eta}(v),$$

where  $E_d = E_{\mathcal{B}|\mathbb{R}^{n-d}}$ . Therefore by Lebegue’s theorem of dominated convergence,

$$\lim_{k \rightarrow \infty} \langle \sigma_k(F)\xi, \eta \rangle = \langle \sigma(F)\xi, \eta \rangle.$$

The proposition now follows from the density of  $C_c(G)$  in  $C^*(G)$  and the density of  $C_c(\mathbb{R}^d)$  in  $L^2(\mathbb{R}^d)$ . ■

**Theorem 2.3.** *Let  $(\pi_k)_k \subset \widehat{G}$  be a sequence which converges to a single limit point  $\pi$ , such that  $\lim_{k \rightarrow \infty} tr(\pi_k(a)) = tr(\pi(a))$  for every  $a$  in the Pedersen ideal  $j(\emptyset)$  of  $C^*(G)$ . Then there exists a subsequence (also indexed by  $k$  for simplicity of notations), a Hilbert space  $\mathcal{H}$ , for every  $k \in \mathbb{N}$  a concrete realization  $(\sigma_k, \mathcal{H})$  of  $\pi_k$  and a concrete realization  $(\sigma, \mathcal{H})$  of  $\pi$  on  $\mathcal{H}$ , such that*

$$\lim_{k \rightarrow \infty} \|\sigma_k(a) - \sigma(a)\|_{op} = 0, a \in C^*(G).$$

**Proof.** We write  $\pi_k \simeq \sigma_k, k \in \mathbb{N}$ , where  $\sigma_k = \text{ind}_{P_k}^G \chi_{\ell_k}, k \in \mathbb{N}$ , where  $\ell_k$  is an element in the Kirillov orbit of  $\pi_k$  and where  $P_k = \exp(\mathfrak{p}_k)$  is a polarization at  $\ell_k$ . Since  $\pi_k$  converges to  $\pi$ , we can assume that  $\lim_{k \rightarrow \infty} \ell_k = \ell$  for some  $\ell$  in the orbit of  $\pi$  and that (passing to a subsequence)  $\lim_{k \rightarrow \infty} \mathfrak{p}_k = \mathfrak{p}$  in the sense of the preceding Proposition 2.2. Then  $\mathfrak{p}$  is a polarization at  $\ell$ , since  $\pi$  is the only limit

of the sequence  $(\pi_k)_k$ . For every  $a$  in the Pedersen ideal  $\in j(\emptyset)$ , the operators  $\sigma_k(a), k \in \mathbb{N}$ , and  $\sigma(a)$  have finite rank. Take now  $a = a^* \in j(\emptyset)$ . Then

$$\begin{aligned} & \|\sigma_k(a) - \sigma(a)\|_{HS}^2 \\ &= \operatorname{tr}(\sigma_k(a)^2) + \operatorname{tr}(\sigma(a^2)) - \operatorname{tr}(\sigma_k(a) \circ \sigma(a)) - \operatorname{tr}(\sigma(a) \circ \sigma_k(a)) \\ &= \operatorname{tr}(\sigma_k(a)^2) + \operatorname{tr}(\sigma(a^2)) - 2\operatorname{tr}(\sigma_k(a) \circ \sigma(a)) \\ &\rightarrow \operatorname{tr}(\sigma(a)^2) + \operatorname{tr}(\sigma(a^2)) - 2\operatorname{tr}(\sigma(a) \circ \sigma(a)) \\ &= 0, \end{aligned}$$

by Proposition 2.2, the continuity of the trace and the fact that  $\sigma(a)$  has finite rank. Hence

$$\lim_{k \rightarrow \infty} \|\sigma_k(a) - \sigma(a)\|_{\text{op}} = 0.$$

The theorem now follows from the density of  $j(\emptyset)$  in  $C^*(G)$ . ■

**2.2. Multiplication operators.** Denote for a measurable subset  $S \subset X$  of a measure space  $(X, \mu)$  the multiplication operator with the indicator of a measurable subset  $S$  on  $L^2(X, \mu)$  by  $M_S$ .

**Proposition 2.4.** *Let  $(X, \mu)$  a measure space, let  $(\sigma_i)_{i \in I}$  be a family of bounded linear operators on the Hilbert space  $\mathcal{H} = L^2(X, \mu)$ , such that  $\|\sigma_i\|_{\text{op}} \leq C, \forall i \in I$ , for some  $C > 0$ . Suppose furthermore that there exists families  $(T_{i,j})_{i \in I} (j = 1, \dots, N)$  and  $(S_i)_{i \in I}$  of measurable subsets of  $X$  such that  $T_{i,j} \cap T_{i',j} = \emptyset, (j = 1, \dots, N), S_i \cap S_{i'} = \emptyset$  whenever  $i \neq i'$ . Then the linear operator*

$$\sigma = \sum_{j=1}^N \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i},$$

*is bounded by  $NC$ .*

**Proof.** Let us write

$$\sigma^j := \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}, \quad j = 1, \dots, N.$$

Then  $\sigma = \sum_{j=1}^N \sigma^j$  and for  $\xi \in L^2(X, \mu), j \in \{1, \dots, N\}$  we then have:

$$\begin{aligned} \|\sigma^j(\xi)\|_2^2 &= \int_X \left| \sum_{i \in I} M_{T_{i,j}} \circ \sigma_i \circ M_{S_i}(\xi)(x) \right|^2 d\mu(x) \\ &= \sum_{i \in I} \int_{T_{i,j}} |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x) \\ &\leq \sum_{i \in I} \int_X |\sigma_i(M_{S_i}(\xi))(x)|^2 d\mu(x) \\ &\leq \sum_{i \in I} C^2 \int_X |M_{S_i}(\xi(x))|^2 d\mu(x) \\ &= C^2 \left( \sum_{i \in I} \int_{S_i} |\xi(x)|^2 d\mu(x) \right) \\ &\leq C^2 \int_X |\xi(x)|^2 d\mu(x) \\ &= C^2 \|\xi\|_2^2. \end{aligned}$$

Hence  $\|\sigma\|_{\text{op}} \leq NC$ . ■

**2.3. Quotient groups.** Let  $G$  be a locally compact group and  $H$  a closed normal subgroup of  $G$ . Then the canonical projection  $P_{G/H} : L^1(G) \rightarrow L^1(G/H)$  defined by:

$$P_{G/H}F(x) := \int_H F(xh)dh, F \in L^1(G), x \in G,$$

is a surjective homomorphism (see [RS00]). Let  $p_{G/H} : G \mapsto G/H$  be the quotient map. Then we can identify  $\widehat{G/H}$  with  $H^\perp$  through the mapping  $\widehat{G/H} \ni \tilde{\pi} \mapsto \tilde{\pi} \circ p_{G/H} \in \widehat{G}$ , where

$$H^\perp = \left\{ \pi \in \widehat{G}, \pi(H) = \mathbb{I}_{\mathcal{H}_\pi} \right\}.$$

Evidently for any  $F \in L^1(G)$  and  $\pi \in H^\perp$ :

$$(\tilde{\pi} \circ p_{G/H})(F) = \tilde{\pi}(P_{G/H}(F)) = \pi(F)$$

and so

$$\ker(P_{G/H}) = \{F \in L^1(G), \pi(F) = 0, \pi \in H^\perp\}.$$

Since for  $\tilde{F} = P_{G/H}(F) \in L^1(G/H)$

$$\|\tilde{F}\|_{C^*(G/H)} = \sup_{\pi \in H^\perp} \|\pi(F)\|_{\text{op}} \leq \|F\|_{C^*(G)},$$

the mapping  $P_{G/H}$  extends then to a surjective  $*$ -homomorphism (also denoted by  $P_{G/H}$ ) of  $C^*(G)$  onto  $C^*(G/H)$ . Let  $I_H$  be the kernel of the (closed ) subset

$H^\perp \subset \widehat{G}$  in  $C^*(G)$ . Then  $C^*(G/H)$  is isomorphic to the quotient  $C^*$ -algebra  $C^*(G)/I_H$ .

**2.4. Stone Weierstrass.** We shall use several times in this paper the following version of the Stone-Weierstrass Theorem:

**Theorem 2.5.** ([Dix77], Proposition 11.1.6)

Let  $A$  be a  $C^*$ -algebra, and  $B$  a sub- $C^*$ -algebra of  $A$ . Suppose that  $A$  is of type I, that every irreducible representation of  $A$  restricts to an irreducible representation of  $B$  and that this restriction mapping  $\widehat{A} \ni \pi \rightarrow \pi|_B \in \widehat{B}$  is injective. Then  $B = A$ .

**2.5. Riemann-Lebesgue Lemma.**

**Definition 2.6.** We say that a net  $(\gamma_i)_{i \in \mathbb{I}}$  in a topological space  $\Gamma$  goes to infinity, if the net contains no converging subnet. In particular, a sequence of orbits  $\overline{\mathcal{O}} = (\mathcal{O}_k)_{k \in \mathbb{N}} \subset \mathfrak{g}^*$  goes to infinity, if for any compact subset  $K \subset \mathfrak{g}^*$  there exists an index  $k_0$  such that  $K \cap \mathcal{O}_k = \emptyset$  whenever  $k \geq k_0$ .

**Proposition 2.7.** (Riemann-Lebesgue Lemma) Let  $A$  be a  $C^*$ -algebra. If a net  $(\pi_k)_k \subset \widehat{A}$  goes to infinity, then  $\lim_{k \rightarrow \infty} \|\pi_k(a)\|_{\text{op}} = 0$  for all  $a \in A$ .

**Proof.** We know from [Dix77] Proposition 3.3.7, that for every  $c > 0$  and  $a \in A$ , the subset  $\{\pi \in \widehat{A}; \|\pi(a)\|_{\text{op}} \geq c\}$  is quasi-compact. This shows that  $\lim_{k \rightarrow \infty} \|\pi_k(a)\|_{\text{op}} = 0$ , if the net  $(\pi_k)_k$  goes to infinity. ■

### 3. A special class of $C^*$ -algebras.

**Definition 3.1.** Let  $A$  be a  $C^*$ -algebra with spectrum  $\widehat{A}$ . We choose for every  $\gamma \in \widehat{A}$  a representation  $(\pi_\gamma, \mathcal{H}_\gamma)$  in the equivalence class  $\gamma$ . Let  $l^\infty(\widehat{A})$  be the algebra of all bounded operator fields defined over  $\widehat{A}$  by

$$l^\infty(\widehat{A}) := \left\{ \phi = (\phi(\pi_\gamma) \in \mathcal{B}(\mathcal{H}_\gamma))_{\gamma \in \widehat{A}}, \|\phi\|_\infty := \sup_\gamma \|\phi(\pi_\gamma)\|_{\text{op}} < \infty \right\}.$$

We define for  $a \in A$  its Fourier transform  $\mathcal{F}(a) = \widehat{a}$  by:

$$\mathcal{F}(a)(\gamma) = \widehat{a}(\gamma) := \pi_\gamma(a), \quad \gamma \in \widehat{A}.$$

Then  $\widehat{a}$  is a bounded field of operators over  $\widehat{A}$ , i.e.  $\widehat{a} \in l^\infty(\widehat{A})$ . The mapping

$$\mathcal{F} : A \rightarrow l^\infty(\widehat{A}); \quad a \mapsto \widehat{a}$$

is an isometric  $*$ -homomorphism.

**Definition 3.2.** Let  $S$  be a topological space. We say that  $S$  is locally compact of step  $\leq d$  if there exists a finite increasing family  $S_0 \subset S_1 \subset \dots \subset S_d = S$  of

closed subsets of  $S$ , such that the subsets  $\Gamma_0 = S_0$  and  $\Gamma_i := S_i \setminus S_{i-1}$ ,  $i = 1, \dots, d$ , are locally compact and Hausdorff in their relative topologies. Let  $A$  be a separable CCR  $C^*$ -algebra, i.e. for every irreducible representation  $(\pi, \mathcal{H})$  of  $A$ , the image of  $\pi$  is the algebra of compact operators  $\mathcal{K}(\mathcal{H})$ . We assume that the spectrum  $\widehat{A}$  of  $A$  is a locally compact space of step  $\leq d$  and that for every  $i \in \{0, \dots, d\}$  there is a Hilbert space  $\mathcal{H}_i$  and for every  $\gamma \in \Gamma_i$  a concrete realization  $(\pi_\gamma, \mathcal{H}_i)$  of  $\gamma$  on the Hilbert space  $\mathcal{H}_i$ . The set  $S_0$  is the collection  $\mathcal{X}$  of all characters of  $A$ .

For a closed subset  $C \subset S$ , denote by  $CB(C)$  the  $*$ -algebra of all uniformly bounded operator fields  $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1, \dots, d}$ , which are operator norm continuous on the subsets  $\Gamma_i \cap C$  for every  $i \in \{1, \dots, d\}$  for which  $\Gamma_i \cap C \neq \emptyset$ . We provide the algebra  $CB(C)$  with the infinity-norm:

$$\|\varphi\|_C := \sup_{\gamma \in C} \|\varphi(\gamma)\|_{\text{op}}.$$

**Definition 3.3.** We say that our  $C^*$ -algebra  $A$  has *norm controlled dual limits* (NCDL for short), if for every  $a \in A$ :

1. The mappings  $\gamma \rightarrow \mathcal{F}(a)(\gamma)$  are norm continuous on the difference sets  $\Gamma_i$ .
2. For any  $i = 0, \dots, d$ , and for any converging sequence contained in  $\Gamma_i$  with limit set outside  $\Gamma_i$ , there exists a properly converging sub-sequence  $\bar{\gamma} = (\gamma_k)_{k \in \mathbb{N}}$ , a  $C > 0$  and for every  $k \in \mathbb{N}$  an involutive linear mapping  $\tilde{\sigma}_{\bar{\gamma}, k} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$ , which is bounded by  $C \|\cdot\|_{S_{i-1}}$ , such that

$$\lim_{k \rightarrow \infty} \|\mathcal{F}(a)(\gamma_k) - \tilde{\sigma}_{\bar{\gamma}, k}(\mathcal{F}(a)|_{S_{i-1}})\|_{\text{op}} = 0.$$

By the condition 1. the restriction of  $\mathcal{F}(a)$  to the limit set  $L(\bar{\gamma})$  is contained in  $CB(S_{i-1})$ .

**Definition 3.4.** Let  $S = S_0 \subset S_d$  be a locally compact topological space of step  $\leq d$ . Choose for every  $i = 1, \dots, d$  a Hilbert space  $\mathcal{H}_i$  and assume that  $\mathcal{H}_0 = \mathbb{C}$ . Let  $B^*(S)$  be the set of all operator fields  $\varphi$  defined over  $S$  such that

1.  $\varphi(\gamma) \in \mathcal{K}(\mathcal{H}_i)$  for every  $\gamma \in \Gamma_i, i = 1, \dots, d$ .
2. The field  $\varphi$  is uniformly bounded, i.e.  $\|\varphi\| := \sup_{\gamma \in \widehat{A}} \|\varphi(\gamma)\|_{\text{op}} < \infty$ .
3. The mappings  $\gamma \rightarrow \varphi(\gamma)$  are norm continuous on the difference sets  $\Gamma_i$ .
4. We have for any sequence  $(\gamma_k)_{k \in \mathbb{N}} \subset S$  going to infinity, that  $\lim_{k \rightarrow \infty} \|\varphi(\gamma_k)\|_{\text{op}} = 0$ .
5. For any  $i = 0, \dots, d$ , and for any converging sequence contained in  $\Gamma_i$  with limit set outside  $\Gamma_i$ , there exists a properly converging sub-sequence  $\bar{\gamma} = (\gamma_k)_{k \in \mathbb{N}}$ , a  $C > 0$  and for every  $k \in \mathbb{N}$  an involutive linear mapping  $\tilde{\sigma}_{\bar{\gamma}, k} : CB(S_{i-1}) \rightarrow \mathcal{B}(\mathcal{H}_i)$ , which is bounded by  $C \|\cdot\|_{S_{i-1}}$ , such that

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma}, k}(\varphi|_{S_{i-1}})\|_{\text{op}} = 0.$$

By the condition 1. the restriction of  $\mathcal{F}(a)$  to the limit set  $L(\bar{\gamma})$  is contained in  $CB(S_{i-1})$ .

**Theorem 3.5.** *Let  $S$  be a locally compact topological space of step  $\leq d$ . Then the subset  $B^*(S)$  of Definition 3.4 is a closed involutive subspace of  $l^\infty(S)$ . Furthermore  $B^*(S)$  is a  $C^*$ -subalgebra of  $l^\infty(S)$  with spectrum  $S$  if and only if either*

1. *all the mappings  $\tilde{\sigma}_{\bar{\gamma},k}$  are almost homomorphisms, i.e.*

$$\lim_{k \rightarrow \infty} \|\tilde{\sigma}_{\bar{\gamma},k}(\varphi \cdot \psi) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi) \cdot \tilde{\sigma}_{\bar{\gamma},k}(\psi)\|_{\text{op}} = 0, \varphi, \psi \in B^*(S),$$

*and the restrictions  $B^*(S)|_{S_i}$  contain the spaces  $C_0(\Gamma_i, \mathcal{H}_i)$ ,  $i = 0, \dots, d$ ,*

2. *or  $B^*(S)$  contains a sub-algebra  $D^*(S)$  whose spectrum is homeomorphic to  $S$ , i.e. such that for every  $s \in S$ , the representation  $\pi_s : \varphi \rightarrow \varphi(s)$ ,  $\varphi \in D^*(S)$  is irreducible and such that the mapping  $S \ni s \rightarrow \pi_s \in \widehat{D^*(S)}$  is a bijection.*

**Proof.** We easily see that the conditions 1. to 4. imply that  $B^*(S)$  is a closed involution-invariant subspace of  $l^\infty(S)$ . For  $i = 0, \dots, d$ , let  $B_i^*$  be the set of all operator fields defined over  $S_i$ , satisfying conditions 1. to 5. on the sets  $S_j$ ,  $j = 1, \dots, i$ . Then obviously for every  $i$  the restriction  $B^*(S)|_{S_i}$  of the space  $B^*(S)$  to  $S_i$  is contained in  $B_i^*$ .

Assume first the condition 1). Then obviously  $B^*(S)$  is also an algebra and hence a  $C^*$ -subalgebra of  $l^\infty(S)$ . Let us show that the spectrum of  $B^*(S)$  can be identified with the space  $S$ . Since  $C_0(\Gamma_i, \mathcal{H}_i)$  is contained in  $B_i^*$  for every  $i$ , it follows that the representations  $\pi_s : \varphi \rightarrow \varphi(s) \in \mathcal{K}(\mathcal{H}_i)$  of  $B^*(S)$  are irreducible. It follows from the choice of  $\mathcal{H}_0$  and the properties of  $B^*(S)$ , that  $B_0^* = B^*(S)|_{S_0} = C_0(S_0)$ . Suppose that for some  $0 < i \leq d$  the spectrum of the algebra  $B_{i-1}^*$  is the space  $S_{i-1}$ . Let  $\pi \in \widehat{B_{i-1}^*}$ . Consider the kernel  $K_{i-1}$  of the restriction mapping  $R_{i-1}$  from  $B_i^*$  into  $l^\infty(S_{i-1})$ . If  $\pi(K_{i-1}) = \{0\}$ , then we can consider  $\pi$  as being a representation of the quotient algebra  $B_i^*/K_{i-1}$ . But the image  $\mathcal{B}^*(S)|_{S_{i-1}}$  of  $R_{i-1}$  is a  $C^*$ -subalgebra of  $B_{i-1}^*$ , the spectrum of  $B_{i-1}^*$  is by assumption the set  $S_{i-1}$  and  $S_{i-1}$  is also contained in the spectrum of the subalgebra  $\mathcal{B}^*(S)|_{S_{i-1}}$ . Hence by the theorem of Stone-Weierstrass the algebras  $B_{i-1}^*$  and  $B^*(S)|_{S_{i-1}}$  coincide. Hence  $\pi$  is an evaluation at a point in  $S_{i-1}$ . If  $\pi(K_{i-1}) \neq \{0\}$ , then we look at the restriction of  $\pi$  to the ideal  $K_{i-1}$ . The elements in  $K_{i-1}$  are operator fields defined on  $S_i$  which are norm continuous, which go to 0 at infinity and by condition 5., for any properly converging sequence  $\bar{\gamma} \subset \Gamma_i$  with limit outside  $\Gamma_i$ , for every  $\varphi \in K_{i-1}$ , we have

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k)\|_{\text{op}} = \lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi|_{S_{i-1}})\|_{\text{op}} = 0.$$

This shows that  $K_{i-1} \subset C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$ . Since by condition 1) we know that  $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) \subset K_{i-1}$  it follows that  $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i)) = K_{i-1}$  and then  $\pi$  is an evaluation at an element  $s \in \Gamma_i$ . Finally the spectrum of  $B_i^*$  is the set  $S_i$ . Therefore

again by the theorem of Stone-Weierstrass the algebras  $B_i^*$  and  $B^*(S)_{|S_i}$  coincide. Let us assume now condition 2). Let  $D_i^* = D^*(S)_{|S_i}$ ,  $i = 0, \dots, d$ . These subspaces are then  $C^*$ -subalgebras of  $l^\infty(S_i)$  for every  $i$ . Furthermore, the evaluation at  $s \in S_i$  defines an irreducible representation of  $D_i^*$  by the property of  $D^*(S)$  and every irreducible representation  $\pi \in \widehat{D_i^*}$  defines an irreducible representation of  $D^*(S)$  and so  $\pi = \pi_s$  for some  $s \in S$ . Taking  $i = 0$ , we see that  $D_0^*$  is a  $C^*$ -subalgebra of  $C_0(S_0)$  by the conditions 1) to 4) and  $S_0$  separates the elements of  $D_0^*$ . Hence the classical Stone-Weierstrass theorem implies that  $D_0^* = C_0(S) = B_0^*$ . Let us assume now that for some  $1 \leq i < d$ ,  $D_{i-1}^* = B_{i-1}^*$ . We shall prove then that  $B_i^*$  is an algebra too. Let  $\varphi, \psi$  be in  $B_i^*$ . Obviously the product  $\varphi \cdot \psi$  satisfies then also the conditions 1. to 4. for  $i$ . We shall show that it satisfies also condition 5) for  $i$ . By assumption, there exists  $a, b \in D_i^*$ , such that  $a_{|S_{i-1}} = \varphi_{|S_{i-1}}$  and  $b_{|S_{i-1}} = \psi_{|S_{i-1}}$ . Then we have for any properly converging sequence  $(\gamma_k)_k \subset \Gamma_i$  with limit set outside  $\Gamma_i$  that:

$$\begin{aligned} & \|\varphi \circ \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi \circ \psi_{|S_{i-1}})\|_{\text{op}} \\ &= \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi_{|S_{i-1}} \circ \psi_{|S_{i-1}})\|_{\text{op}} \\ &= \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(a_{|S_{i-1}} \circ b_{|S_{i-1}})\|_{\text{op}} \\ &\leq \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi_{|S_{i-1}}) \circ \tilde{\sigma}_{\bar{\gamma},k}(\psi_{|S_{i-1}})\|_{\text{op}} \\ &+ \|\tilde{\sigma}_{\bar{\gamma},k}(\varphi_{|S_{i-1}}) \circ \tilde{\sigma}_{\bar{\gamma},k}(\psi_{|S_{i-1}}) - a(\gamma_k) \circ b(\gamma_k)\|_{\text{op}} \\ &+ \|a(\gamma_k) \circ b(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(a_{|S_{i-1}} \circ b_{|S_{i-1}})\|_{\text{op}} \\ &= \|\varphi(\gamma_k) \circ \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi_{|S_{i-1}}) \circ \tilde{\sigma}_{\bar{\gamma},k}(\psi_{|S_{i-1}})\|_{\text{op}} \\ &+ \|\tilde{\sigma}_{\bar{\gamma},k}(a_{|S_{i-1}}) \circ \tilde{\sigma}_{\bar{\gamma},k}(b_{|S_{i-1}}) - a(\gamma_k) \circ b(\gamma_k)\|_{\text{op}} \\ &+ \|(ab)(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(ab_{|S_{i-1}})\|_{\text{op}}. \end{aligned}$$

This shows that  $\lim_{k \rightarrow \infty} \|\varphi \cdot \psi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi \cdot \psi_{|S_{i-1}})\|_{\text{op}} = 0$  and so  $\varphi \cdot \psi \in B_i^*$ . Hence the subspace  $B_i^*$  is a  $C^*$ -sub-algebra of  $l^\infty(S_i)$  containing the algebra  $D_i^*$ . In order to prove that  $B_i^* = D_i^*$ , by Stone-Weierstrass it suffices to show that the spectrum of  $B_i^*$  equals that of  $D_i^*$ , i.e. that every element in  $\widehat{B_i^*}$  is an evaluation at some point in  $S_i$ . Let  $\pi \in \widehat{B_i^*}$ . Consider the kernel  $K_{i-1}$  of  $R_{i-1}$ , the restriction mapping from  $B_i^*$  into  $l^\infty(S_{i-1})$ . If  $\pi(K_{i-1}) = \{0\}$ , then we can consider  $\pi$  as being a representation of the quotient algebra  $B_i^*/K_{i-1}$ . But the image of  $R_{i-1}$  is contained in  $B_{i-1}^*$  and contains  $D_{i-1}^*$  and so by assumption  $R_{i-1}(B_i^*) = D_{i-1}^*$ . Hence  $B_i^*/K_{i-1} \simeq D_{i-1}^*$  and therefore  $\pi$  is an evaluation at a point in  $S_{i-1}$ . If  $\pi(K_{i-1}) \neq \{0\}$ , then we look at the restriction of  $\pi$  to this ideal. The elements in  $K_{i-1}$  are operator fields defined on  $S_i$  which are norm continuous, which go to 0 at infinity and by condition 5., for any properly converging sequence  $\bar{\gamma} \subset \Gamma_i$  with limit outside  $\Gamma_i$ , for every  $\varphi \in K_{i-1}$ ,

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k)\|_{\text{op}} = \lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \tilde{\sigma}_{\bar{\gamma},k}(\varphi_{|S_{i-1}})\|_{\text{op}} = 0.$$

This shows that  $K_{i-1} \subset C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$ . On the other hand the elements of  $D_i^* \cap K_{i-1}$  separate the points of  $\Gamma_i$  (see Proposition 2.11.2 in [Dix77] for details). Therefore by the theorem of Stone-Weierstrass the algebras  $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$  and  $K_{i-1}$  coincide. This tells us that  $\pi$  is an evaluation at a point in  $\Gamma_i$ , since the

spectrum of the algebra  $C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))$  is homeomorphic to  $\Gamma_i$ . We conclude that  $B_i^* = D_i^*$ . If  $B^*(S)$  is a  $C^*$ -algebra, then obviously condition 1) of the theorem is fulfilled and for condition 2) we can take  $D^*(S) = B^*(S)$ . ■

**Corollary 3.6.** *Let  $A$  be a  $C^*$ -algebra with norm controlled dual limits. Then the Fourier transform of  $A$  is the  $C^*$ -algebra  $B^*(\widehat{A})$ .*

#### 4. The list of the nilpotent Lie algebras of dimension $\leq 5$ according to [Nie83].

There are 8 un-decomposable nilpotent non-abelian Lie algebras of dimension  $\leq 5$ : the Heisenberg Lie algebras of dimension 3 and 5,  $\mathfrak{h}_1$  and  $\mathfrak{h}_2$ , the thread-like Lie algebras of dimension 4 and 5,  $\mathfrak{f}_4$  and  $\mathfrak{f}_5$ , the step 2 Lie algebra  $\mathfrak{g}_{5,2}$ , two step 3 Lie algebras,  $\mathfrak{g}_{5,3}$  and  $\mathfrak{g}_{5,4}$  and the step 4 Lie algebra  $\mathfrak{g}_{5,6}$ .

1. The Heisenberg Lie algebras  $\mathfrak{h}_n, n \geq 1$  :

Let  $\mathfrak{h}_n$  be the nilpotent Lie algebra of dimension  $2n+1$  spanned by the basis

$$\mathcal{B} = \{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$$

equipped with the Lie bracket

$$[X_i, Y_j] = \delta_{i,j}Z, i, j = 1, \dots, n.$$

2. The thread-like Lie algebras  $\mathfrak{f}_n, n \geq 4$  :

Let  $\mathfrak{f}_n$  be the nilpotent Lie algebra spanned by the basis  $\mathcal{B} = \{X_n, \dots, X_1\}$  equipped with the Lie brackets:

$$[X_n, X_j] = X_{j-1}, j = n - 1, \dots, 2.$$

3. The step 2 Lie algebra  $\mathfrak{g}_{5,2}$  :

Let  $\mathfrak{g}_{5,2}$  be the nilpotent Lie algebra spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[C, A] = -U, [C, B] = V.$$

This is a semi-direct product of  $\mathbb{R}$  with  $\mathbb{R}^4$ .

4. The step 3 Lie algebras  $\mathfrak{g}_{5,3}$  and  $\mathfrak{g}_{5,4}$ :

- Let  $\mathfrak{g}_{5,3}$  be the nilpotent Lie algebra spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = U, [A, U] = V, [B, C] = V.$$

This is a semi-direct product of  $\mathbb{R}$  with  $\mathfrak{h}_1 \times \mathbb{R}$ .

- Let  $\mathfrak{g}_{5,4}$  be the nilpotent Lie algebra spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = C, [A, C] = U, [B, C] = V.$$

This is a semi-direct product of  $\mathbb{R}$  with  $\mathfrak{h}_1 \times \mathbb{R}$ .

5. The step 4 Lie algebra  $\mathfrak{g}_{5,6}$  :

Let  $\mathfrak{g}_{5,6}$  be the nilpotent Lie algebra spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = C, [A, C] = U, [A, U] = V, [B, C] = V.$$

This is a semi-direct product of  $\mathbb{R}$  with  $\mathfrak{h}_1 \times \mathbb{R}$ .

**5. The  $C^*$ -algebras of  $H_n, n \geq 1$  and  $F_n, n \geq 4$ .**

The  $C^*$ -algebras of the Heisenberg groups  $H_n$  and of the threadlike groups  $F_n$  have been realized as algebras of operator fields in [LT11]. It is shown there that the corresponding  $C^*$ -algebras have the norm-controlled dual limit property and the mappings  $\tilde{\sigma}_{\overline{\mathcal{O}},k}$  are explicitly given.

**6. The  $C^*$ -algebra of the group  $G_{5,2}$ .**

**6.1.** We shall describe in this and in the following sections for our 4 remaining groups the limit sets of properly converging sequences  $\overline{\mathcal{O}} = (\pi_k)_k$  in their dual spaces and we are explicitly constructing the mappings  $\sigma_{\overline{\mathcal{O}},k}$ . In this way it will turn out that the  $C^*$ -algebra of every connected Lie group of dimension  $\leq 5$  has norm controlled dual limits.

**6.2.** Recall that the Lie algebra of  $\mathfrak{g}_{5,2}$  is spanned by the basis  $\mathcal{B}=\{A, B, C, U, V\}$  equipped with the Lie brackets

$$[C, A] = -U, [C, B] = V.$$

The Lie algebra  $\mathfrak{g}_{5,2}$  has a two-dimensional centre  $\mathfrak{z} = \text{span}\{U, V\}$ . The group  $G_{5,2} = \exp(\mathfrak{g}_{5,2})$  can be realized as  $\mathbb{R}^5$  with the Campbell-Baker-Hausdorff multiplication

$$\begin{aligned} & \exp(a, b, c, u, v)\exp(a', b', c', u', v') \\ &= (a + a', b + b', c + c', u + u' + \frac{1}{2}(ac' - a'c), v + v' + \frac{1}{2}(b'c - bc')). \end{aligned}$$

For all  $(a, b, c, u, v) \in \mathfrak{g}_{5,2}, (x, y, t, \mu, \nu) \in \mathfrak{g}_{5,2}^*$  we obtain the following expression for  $Ad^*(a, b, c, u, v)$  :

$$Ad^*(\exp(a, b, c, u, v))(x, y, t, \mu, \nu) = (x - \mu c, y + \nu c, t + \mu a - \nu b, \mu, \nu). \tag{6.1}$$

We give now a description of the co-adjoint orbits:

1. The generic elements  $(\alpha, \beta, \rho, \mu, \nu)$  in  $\mathfrak{g}_{5,2}^*$  are those for which  $r_{\mu,\nu}^2 = r^2 = \mu^2 + \nu^2 \neq 0$ . It follows from equation (6.1) that a generic orbit  $\mathcal{O}$  is determined by 3 parameters  $(\beta, \mu, \nu) \in \mathbb{R}^3, r_{\mu,\nu} \neq 0$ :

$$\mathcal{O} = \mathcal{O}_{\beta,\mu,\nu} = \{xA^* + yB^* + cC^* + \mu U^* + \nu V^*, c \in \mathbb{R}, \nu x + \mu y = \beta r_{\mu,\nu}\}.$$

Let

$$\ell_{\beta,\mu,\nu} = \frac{\beta}{r_{\mu,\nu}}(\nu A^* + \mu B^*) + \mu U^* + \nu V^* \quad (6.2)$$

We take a new basis  $\mathcal{B}_{\mu,\nu}$  of  $\mathfrak{g}_{5,2}$ . For that, letting  $\tilde{\mu} := \frac{\mu}{r_{\mu,\nu}}$ ,  $\tilde{\nu} := \frac{\nu}{r_{\mu,\nu}}$  we put:

$$\mathcal{B}_{\mu,\nu} := \{A_{\mu,\nu} := \tilde{\mu}A - \tilde{\nu}B, B_{\mu,\nu} := \tilde{\nu}A + \tilde{\mu}B, C, U, V\}. \quad (6.3)$$

Let also

$$Z_{\mu,\nu} = \tilde{\mu}U + \tilde{\nu}V, T_{\mu,\nu} = \tilde{\nu}U - \tilde{\mu}V.$$

Then:

$$\begin{aligned} [A_{\mu,\nu}, C] &= Z_{\mu,\nu}, [B_{\mu,\nu}, C] = T_{\mu,\nu} \\ \langle \ell_{\beta,\mu,\nu}, [A_{\mu,\nu}, C] \rangle &= r_{\mu,\nu}. \end{aligned}$$

In this basis for any  $R = xA_{\mu,\nu} + bB_{\mu,\nu} + yC + zZ_{\mu,\nu} + tT_{\mu,\nu}$ , resp  $R' = x'A_{\mu,\nu} + b'B_{\mu,\nu} + y'C + z'Z_{\mu,\nu} + t'T_{\mu,\nu}$  in  $\mathfrak{g}_{5,2}$  we obtain the multiplication:

$$\begin{aligned} \exp(R) \cdot \exp(R') &= (x + x')A_{\mu,\nu} + (b + b')B_{\mu,\nu} + (y + y')C + \\ &+ (z + z' + \frac{1}{2}(xy' - x'y))Z_{\mu,\nu} + (t + t' + \frac{1}{2}(by' - b'y))T_{\mu,\nu}. \end{aligned} \quad (6.4)$$

We see that the vectors  $A_{\mu,\nu}, C, Z_{\mu,\nu}$  span the three dimensional Heisenberg Lie algebra, that  $B_{\mu,\nu}$  is contained in the stabilizer of the linear form  $\ell_{\beta,\mu,\nu}$  and that  $T_{\mu,\nu}$  is contained in the kernel of the restriction of  $\ell_{\beta,\mu,\nu}$  to the centre of  $\mathfrak{g}_{\mu,\nu}$ . In the dual basis  $\mathcal{B}_{\mu,\nu}^*$  of the basis  $\mathcal{B}_{\mu,\nu}$  the orbit  $\mathcal{O}_{\beta,\mu,\nu}$  (for some  $\beta \in \mathbb{R}$ ) of the element

$$\ell_{\beta,\mu,\nu} = \beta B_{\mu,\nu}^* + r_{\mu,\nu} Z_{\mu,\nu}^*,$$

is given by:

$$\mathcal{O}_{\beta,\mu,\nu} = \{aA_{\mu,\nu}^* + \beta B_{\mu,\nu}^* + cC^* + \mu U^* + \nu V^*, a, c \in \mathbb{R}\}. \quad (6.5)$$

The stabilizer of  $\ell_{\beta,\mu,\nu}$  is the set

$$\mathfrak{g}_{5,2}(\ell_{\beta,\mu,\nu}) = \text{span}\{B_{\mu,\nu}, U, V\}.$$

We can take as polarization at  $\ell_{\beta,\mu,\nu}$  the sub-algebra

$$\mathfrak{p} := \text{span}\{A, B, U, V\}, P := \exp(\mathfrak{p}).$$

We denote by  $\Gamma_1^{5,2}$  the orbit space of this layer and we parametrize it by

$$\Gamma_1^{5,2} := \{\ell_{\beta,\mu,\nu} \equiv (\beta, \mu, \nu), \beta \in \mathbb{R}, (\mu, \nu) \in \mathbb{R}^2, r_{\mu,\nu}^2 = \mu^2 + \nu^2 \neq 0\}. \quad (6.6)$$

2. The second layer, denoted by

$$\Gamma_0^{5,2} = (\mathfrak{g}_{5,2}^*/G_{5,2})_{char} \simeq \mathbb{R}^3 \tag{6.7}$$

is the collection of all characters  $\ell_{\alpha,\beta,\rho} = \alpha A^* + \beta B^* + \rho C^*$ ,  $\alpha, \beta, \rho \in \mathbb{R}$ . Their orbits are the one point sets  $\{\ell_{\alpha,\beta,\rho}\}$

**Theorem 6.1.**

1. On the set  $\Gamma_1^{5,2}$  the dual topology is Hausdorff.
2. Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\beta_k, \mu_k, \nu_k})_k$  be a sequence such that  $\lim_{k \rightarrow \infty} r_{\mu_k, \nu_k} = 0$ . Then this sequence has a converging sub-sequence if and only if  $\liminf_k |\beta_k|$  is finite. If  $\overline{\mathcal{O}}$  is properly converging, then, passing to a sub-sequence (also denoted by the same symbol for simplicity of notations) we can assume that  $\lim_{k \rightarrow \infty} \beta_k = \beta$  exists, that the sequences of vectors  $(A_k^* = A_{\mu_k, \nu_k}^*)_k$ , resp.  $(B_k^* = B_{\mu_k, \nu_k}^*)_k$  converges to a  $A_\infty^*$ , resp.  $B_\infty^*$  in  $\mathfrak{g}_{5,2}^*$  and then

$$L(\overline{\mathcal{O}}) = \mathbb{R}A_\infty^* + \beta B_\infty^* + \mathbb{R}C^* \subset \Gamma_0^{5,2}.$$

**Proof.** 1. The point 1) is evident.

2. If  $\liminf_k |\beta_k|$  exists in  $\mathbb{R}$ , then we take a sub-sequence (indexed also by  $(\beta_k)_k$  for simplicity of notation) such that  $\lim_{k \rightarrow \infty} \beta_k = \beta$  exists in  $\mathbb{R}$  and such that the sequences of vectors  $(A_k^* = A_{\mu_k, \nu_k}^*)_k$ , resp.  $(B_k^* = B_{\mu_k, \nu_k}^*)_k$  converges to a  $A_\infty^*$ , resp.  $B_\infty^*$  in  $\mathfrak{g}_{5,2}^*$ . It follows then from the description (6.5) of the coadjoint orbits that the limit set  $L(\overline{\mathcal{O}})$  of the sub-sequence is the set described in the theorem. If  $\liminf_k |\beta_k| = +\infty$ , then  $\lim_{k \rightarrow \infty} \{|\langle \mathcal{O}_k, B_k \rangle|\} = \lim_{k \rightarrow \infty} \{|\beta_k|\} = +\infty$ . Hence  $\overline{\mathcal{O}}$  goes to infinity. ■

**6.3. The unitary dual of  $G_{5,2}$ .**

The spectrum of the group  $G_{5,2}$  can be identified by Kirillov’s orbit theory with the orbit space  $\mathfrak{g}_{5,2}^*/G_{5,2} = \widehat{G}_{5,2} := \Gamma_1^{5,2} \cup \Gamma_0^{5,2}$ . For every  $(\beta, \mu, \nu) \in \mathbb{R} \times (\mathbb{R} \times \mathbb{R})^*$ , we take the irreducible representation  $\pi_{\beta, \mu, \nu} = \text{ind}_P^{G_{5,2}} \chi_{\beta, \mu, \nu}$  which is associated to  $\mathcal{O}_{\beta, \mu, \nu}$ . This representation acts on the Hilbert space  $L^2(\mathbb{R})$  and is given by the formula

$$\begin{aligned} & \pi_{\beta, \mu, \nu}(x, b, y, z, t)\xi(s) \\ = & e^{-2i\pi(\beta b + r_{\mu, \nu} z + r_{\mu, \nu} x s - \frac{r_{\mu, \nu}}{2} xy)} \xi(s - y), \quad s \in \mathbb{R}, \quad \xi \in L^2(\mathbb{R}), \quad (x, b, y, z, t) \in G_{5,2}, \end{aligned}$$

using the coordinates coming from the basis

$$\mathcal{B}^{\mu, \nu} := \{A_{\mu, \nu}, B_{\mu, \nu}, C, Z_{\mu, \nu}, T_{\mu, \nu}\}$$

and formula (6.4). Since  $G_{5,2}$  is nilpotent, by Lie’s theorem every irreducible finite dimensional representation of  $G_{5,2}$  is one-dimensional. Any one-dimensional representation is a unitary character  $\chi_{\alpha, \beta, \rho}$ ,  $(\alpha, \beta, \rho) \in \mathbb{R}^3$ , of  $G_{5,2}$  which is given by

$$\chi_{\alpha, \beta, \rho}(a, b, c, u, v) = e^{-2i\pi(\alpha a + \beta b + \rho c)}, \quad (a, b, c, u, v) \in G_{5,2}.$$

For  $F \in L^1(G_{5,2})$ , let

$$\begin{aligned} \widehat{F}(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(F) &= \int_{G_{5,2}} F(a, b, c, u, v) e^{-2\pi i(\alpha a + \beta b + \rho c)} da db cdudv \\ &= \int_{\mathbb{R}^3} \widehat{F}^{\mu, \nu}(a, b, c, 0, 0) e^{-2\pi i(\alpha a + \beta b + \rho c)} da db dc. \end{aligned}$$

**Definition 6.2.** Define for  $a \in C^*(G_{5,2})$  its Fourier transform  $\mathcal{F}(a) = \widehat{a} \in l^\infty(\widehat{G_{5,2}})$  by

$$\mathcal{F}(a)(\beta, \mu, \nu) := \pi_{\beta, \mu, \nu}(a) \in \mathcal{B}(L^2(\mathbb{R})), \quad (\beta, \mu, \nu) \in \Gamma_1^{5,2}$$

and

$$\mathcal{F}(a)(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(a), \quad \alpha, \beta, \rho \in \mathbb{R}.$$

By Formula (2.1), for all  $F \in L^1(G_{5,2})$ , and  $(\beta, \mu, \nu) \in \Gamma_1^{5,2}$  the operator  $\pi_{\beta, \mu, \nu}(F)$  is a kernel operator with kernel function  $F_{\beta, \mu, \nu}$  given by:

$$F_{\beta, \mu, \nu}(s, c) = \widehat{F}^{\mathfrak{p}}(s - c, cr_{\mu, \nu}A_{\mu, \nu}^* + \beta B_{\mu, \nu}^* + uU^* + \nu V^*), \quad (6.8)$$

where

$$\widehat{F}^{\mathfrak{p}}(c, q) = \int_P F(\exp(cC)p) e^{-2\pi i\langle q, \log p \rangle} dp, \quad c \in \mathbb{R}, q \in \mathfrak{p}^*.$$

The following proposition is a consequence of Formula (6.8).

**Proposition 6.3.** (Proposition 2.3 in [LT11]). For any  $a \in C^*(G_{5,2})$  and  $(\beta, \mu, \nu) \in \Gamma_1^{5,2}$ , the operator  $\pi_{\beta, \mu, \nu}(a)$  is compact, the mapping  $\Gamma_1^{5,2} \rightarrow \mathcal{B}(L^2(\mathbb{R})) : (\beta, \mu, \nu) \mapsto \pi_{\beta, \mu, \nu}(a)$  is norm continuous in  $(\beta, \mu, \nu)$  and tending to 0 for  $r_{\mu, \nu}$  going to infinity. Furthermore the restriction of the operator field  $\mathcal{F}(a)$  to  $\Gamma_0^{5,2}$  is contained in  $C_0(\mathbb{R}^3)$ .

**Definition 6.4.** Let as before for  $\mu^2 + \nu^2 \neq 0$ ,  $A_{\mu, \nu} := \tilde{\mu}A - \tilde{\nu}B$ ,  $B_{\mu, \nu} := \tilde{\nu}A + \tilde{\mu}B$ . Choose a Schwartz-function  $\eta$  in  $\mathcal{S}(\mathbb{R})$  with  $L^2$ -norm equal to 1. For  $(\alpha, \rho) \in \mathbb{R}^2$  we define the function  $\eta_{\mu, \nu}(\alpha, \rho)$  by

$$\eta_{\mu, \nu}(\alpha, \rho)(s) := r_{\mu, \nu}^{\frac{1}{4}} e^{2i\pi s \rho} \eta\left(r_{\mu, \nu}^{\frac{1}{2}}\left(s + \frac{\alpha}{r_{\mu, \nu}}\right)\right), \quad s \in \mathbb{R}. \quad (6.9)$$

**6.4. A  $C^*$ -condition.**

The  $C^*$ -conditions for the group  $G_{5,2}$  can be copied from the corresponding conditions for the Heisenberg groups (see [LT11]).

**Lemma 6.5.** Let  $\xi \in \mathcal{S}(\mathbb{R})$ . Then,

$$\xi = \frac{1}{r_{\mu, \nu}} \int_{\mathbb{R}^2} \langle \xi, \eta_{\mu, \nu}(\alpha, \rho) \rangle \eta_{\mu, \nu}(\alpha, \rho) d\alpha d\rho.$$

**Proof.** The proof is the same as the proof of Lemma 2.8 in [LT11]. ■

**Definition 6.6.** 1. For all  $(\alpha, \rho) \in \mathbb{R}^2$  and  $k \in \mathbb{N}$ , let  $P_{\mu,\nu(\alpha,\rho)}$  be the orthogonal projection onto the one dimensional subspace  $\mathbb{C}\eta_{\mu,\nu}(\alpha, \rho)$ .

2. Define for  $(\beta, \mu, \nu) \in \Gamma_1^{5,2}$  and  $h \in C_0(\mathbb{R}^3)$  the linear operator

$$\sigma_{\beta,\mu,\nu}(h) := \frac{1}{r_{\mu,\nu}} \int_{\mathbb{R}^2} h(\alpha A_{\mu,\nu}^* + \beta B_{\mu,\nu}^* + \rho C^*) P_{\mu,\nu(\alpha,\rho)} d\alpha d\rho \tag{6.10}$$

**Proposition 6.7.** (Proposition 2.11 in [LT11])

1. For every  $(\beta, \mu, \nu) \in \Gamma_1^{5,2}$  and  $h$  in the Schwartz space  $\mathcal{S}(\mathbb{R}^3)$ , the integral (6.10) converges in operator norm.
2.  $\sigma_{\beta,\mu,\nu}(h)$  is compact and  $\|\sigma_{\beta,\mu,\nu}(h)\|_{op} \leq \|h\|_\infty$ ,  $h \in \mathcal{S}(\mathbb{R}^3)$ .
3. The mapping  $\sigma_{\beta,\mu,\nu} : C_0(\mathbb{R}^3) \rightarrow l^\infty(\widehat{G}_{6,3})$  is involutive, i.e.  $\sigma_{\beta,\mu,\nu}(h^*) = \sigma_{\beta,\mu,\nu}(h)^*$ ,  $h \in C^*(\mathbb{R}^3)$ , where by  $\sigma_{\beta,\mu,\nu}$  we denote also the extension of  $\sigma_{\beta,\mu,\nu}$  to  $CB(\mathbb{R}^3)$ .

**Theorem 6.8.** Let  $a \in C^*(G_{5,2})$  and let  $\varphi$  be the operator field  $\varphi = \mathcal{F}(a)$ . Let  $\bar{\gamma} = (\gamma_k = (\beta_k, \mu_k, \nu_k))_k$  be a properly converging sequence in  $\Gamma_1^{5,2}$  having its limit set  $L(\bar{\gamma}) = \mathbb{R}A_\infty^* + \beta B_\infty^* + \mathbb{R}C^*$  in  $\Gamma_0^{5,2}$ . Then the function  $\varphi(\beta) : (\alpha, \rho) \rightarrow \mathcal{F}(a)(\alpha, \rho, \beta)$  is contained  $C_0(\mathbb{R}^2)$  and

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \sigma_{\gamma_k}(\varphi(\beta))\|_{op} = 0. \tag{6.11}$$

**Proof.** The proof is the same as that of Theorem 2.12 in [LT11]. ■

**Remark 6.9.** Proposition 6.3 and Theorem 6.8 imply that  $C^*(G_{5,2})$  has the NCDL property.

### 7. The $C^*$ -algebra of the group $G_{5,3}$ .

Recall that the Lie algebra  $\mathfrak{g}_{5,3}$  is spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = U, [A, U] = V, [B, C] = V.$$

This Lie algebra has a one-dimensional centre  $\mathfrak{z} = \mathbb{R}V$ . The group  $G_{5,3}$  can be realized as  $\mathbb{R}^5$  with the Campbell-Baker-Hausdorff multiplication

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]], \quad X, Y \in \mathfrak{g}_{5,3},$$

$$\begin{aligned} \text{i.e. } & (a, b, c, u, v) \cdot (a', b', c', u', v') \\ &= (a+a', b+b', c+c', u+u' + \frac{ab'}{2} - \frac{a'b}{2}, v+v' + \frac{au'}{2} - \frac{a'u}{2} - \frac{b'c}{2} + \frac{bc'}{2} + \frac{a^2b}{12} + \\ & \quad \frac{a^2b'}{12} - \frac{a'ab}{12} - \frac{aa'b'}{12}). \end{aligned}$$

For all  $(a, b, c, u, v) \in \mathfrak{g}_{5,3}$ ,  $(\alpha, \beta, \rho, \mu, \nu) \in \mathfrak{g}_{5,3}^*$  we obtain the following expression for  $Ad^*(a, b, c, u, v)$  :

$$Ad^*(a, b, c, u, v)(\alpha, \beta, \rho, \mu, \nu) = (\alpha - \mu b - \nu u - \nu \frac{ab}{2}, \beta + \mu a - \nu c + \nu \frac{a^2}{2}, \rho + \nu b, \mu + a\nu, \nu). \tag{7.1}$$

We obtain the following description of the co-adjoint orbits:

1. The generic orbits: They have a non-zero value  $\nu$  on the central element  $V$ . It follows from (7.1) that we can characterize such an orbit  $\mathcal{O}$  by  $\nu \in \mathbb{R}$ . There exists in each generic orbit  $\mathcal{O}_\nu$  a unique element  $\ell_\nu$  which is zero on the vectors  $A, B, C, U$ , i.e.  $\ell_\nu = \nu V^*$  and so

$$\mathcal{O}_\nu = \{(a, b, c, u, \nu), (a, b, c, u) \in \mathbb{R}^4\}.$$

We denote by  $\Gamma_2^{5,3} = (\mathfrak{g}_{5,3}^*/G_{5,3})_{\text{gen}}$  this family of generic co-adjoint orbits, parameterized by the set  $\Gamma_2^{5,3} := \{\ell_\nu = \nu V^* \equiv \nu; \nu \in \mathbb{R}^*\}$ .

2. The second layer is given by the set of linear functionals, which are 0 on  $V$ , but not 0 on  $U$ . We can characterize such an orbit  $\mathcal{O}$  by the pair  $(\rho, \mu) \in \mathbb{R} \times \mathbb{R}^*$  and so

$$\mathcal{O}_{\rho,\mu} = \{(a, b, \rho, \mu, 0), (a, b) \in \mathbb{R}^2\}.$$

We denote by  $\Gamma_1^{5,3}$  this family of co-adjoint orbits, parameterized by the set  $\Gamma_1^{5,3} := \{\ell_{\rho,\mu} = \rho C^* + \mu U^* \equiv (\rho, \mu); (\rho, \mu) \in \mathbb{R} \times \mathbb{R}^*\}$ .

3. The last layer, denoted by  $\Gamma_0^{5,3} = (\mathfrak{g}_{5,3}/G_{5,3})_{\text{char}} \simeq \mathbb{R}^3$  is the collection of all characters  $\ell_{\alpha,\beta,\rho} = \alpha A^* + \beta B^* + \rho C^*$ ,  $\alpha, \beta, \rho \in \mathbb{R}$ . Their orbits are the point set  $\{\ell_{\alpha,\beta,\rho}\}$ .

**Theorem 7.1.**

1. On the set  $\Gamma_2^{5,3}$  ( resp on the set  $\Gamma_1^{5,3}$  ) the dual topology is Hausdorff.
2. Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\nu_k})_k \subset \Gamma_2^{5,3}$  be a sequence, such that  $\lim_{k \rightarrow \infty} \nu_k = 0$ . Then  $\overline{\mathcal{O}}$  is properly converging and  $L(\overline{\mathcal{O}}) = \Gamma_1^{5,3} \cup \Gamma_0^{5,3}$ .
3. Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\rho_k, \mu_k})_k$  be a sequence such that  $\lim_{k \rightarrow \infty} \mu_k = 0$ . If  $\overline{\mathcal{O}}$  has a limit then  $\rho := \lim_{k \rightarrow \infty} \rho_k$  exists in  $\mathbb{R}$ . Conversely, if  $\lim_{k \rightarrow \infty} \rho_k = \rho$  exists, then the sequence  $\overline{\mathcal{O}}$  converges and  $L(\overline{\mathcal{O}}) = \mathbb{R}A^* + \mathbb{R}B^* + \rho C^*$ .

**Proof.** The proof is straightforward, since all the coadjoint orbits are affine linear. ■

**7.1. The Fourier transform for  $C^*(G_{5,3})$ .** The spectrum of the group  $G_{5,3}$  can be identified by Kirillov’s orbit theory with the orbit space

$$\mathfrak{g}_{5,3}^*/G_{5,3} = \widehat{G}_{5,3} = \Gamma_2^{5,3} \cup \Gamma_1^{5,3} \cup \Gamma_0^{5,3}.$$

- Let  $\ell \in \Gamma_2^{5,3}$ , its orbit  $\mathcal{O}_\ell$  is of dimension 4. A polarization at  $\ell_\nu = \nu V^*$  is given by  $\mathfrak{p} = \mathfrak{p}_\nu := \text{span}\{C, U, V\}$ . We realize then  $\pi_\nu$  as  $\pi_\nu := \text{ind}_{P_\nu}^{G_{5,3}} \chi_\nu$ . The Hilbert space  $L^2(G_{5,3}/P_\nu, \chi_\nu)$  is in fact isometric to  $L^2(\mathbb{R}^2)$ . Let  $E: \mathbb{R}^2 \rightarrow G_{5,3}$ ,  $E(a, b) := \exp(aA)\exp(bB)$  and  $S = \exp(\mathbb{R}A)\exp(\mathbb{R}B) = E(\mathbb{R}^2)$ . Then  $G_{5,3} = S.P_\nu$  as topological product and the mapping  $U_E: L^2(G_{5,3}/P_\nu, \chi_\nu) \rightarrow L^2(\mathbb{R}^2)$  defined by  $U_E \xi(t) := \xi(E(t))$ ,  $t \in \mathbb{R}^2$  is unitary. Let us compute for  $\ell \in \mathfrak{p}^*$  the operator  $\pi_\ell(F)$ ,  $F \in L^1(G_{5,3})$  for the representation  $\pi_\ell = U_E \circ \text{ind}_P^{G_{5,3}} \chi_\ell \circ U_E^*$  explicitly. For  $\xi \in L^2(\mathbb{R}^2)$ ,  $E(a', b') = s \in S$  we have  $\pi_\ell(F)\xi(a', b') =$

$$\int_{\mathbb{R}^2} \xi(a, b) \left( \int_P F(E(a', b')(E(a, b)^{-1})(E(a, b)pE(a, b)^{-1})) e^{-2i\pi \langle \ell, p \rangle} dp \right) dadb = \int_{\mathbb{R}^2} \widehat{F}^P(E(a' - a, b' - b), (\text{Ad}^*(E(a, b))\ell)|_{\mathfrak{p}}) e^{-2i\pi \langle (a, b) \cdot \ell, a(b' - b)U + \frac{1}{2}a^2(b' - b)V \rangle} \xi(a, b) dadb.$$

- Let  $\ell = \ell_{\rho, \mu} \in \Gamma_1^{5,3}$ . A polarization at  $\ell$  is given by  $\mathfrak{p}_{\rho, \mu} = \text{span}\{B, C, U, V\}$ . We take  $\pi_{\rho, \mu} := \text{ind}_{P_{\rho, \mu}}^{G_{5,3}} \chi_{\rho, \mu}$ . This representation acts on the Hilbert space  $L^2(G_{5,3}/P_{\rho, \mu}, \chi_{\rho, \mu}) \simeq L^2(\mathbb{R})$  and for  $F \in L^1(G_{5,3})$ ,  $\xi \in L^2(\mathbb{R})$ , we have:

$$\pi_{\rho, \mu}(F)\xi(a') = \int_{\mathbb{R}} \widehat{F}^{P_{\rho, \mu}}(\exp(a' - a)A, a \cdot p_{\rho, \mu}) \xi(a) da,$$

where  $p_{\rho, \mu} = \ell_{\rho, \mu}|_{\mathfrak{p}_{\rho, \mu}}$  and  $a \cdot \ell = \text{Ad}^* \exp(aA)\ell$ .

- Any one-dimensional representation is a unitary character  $\chi_{\alpha, \beta, \rho}$ ,  $(\alpha, \beta, \rho) \in \mathbb{R}^3$ , of  $G_{5,3}$  which is given by

$$\chi_{\alpha, \beta, \rho}(a, b, c, u, v) = e^{-2i\pi(\alpha a + \beta b + \rho c)}, \quad (a, b, c, u, v) \in G_{5,3}.$$

For  $F \in L^1(G_{5,3})$ , let

$$\widehat{F}(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(F) = \int_{G_{5,3}} F(a, b, c, 0, 0) e^{-2i\pi(\alpha a + \beta b + \rho c)} dadbdc, \quad \alpha, \beta, \rho \in \mathbb{R}.$$

**Definition 7.2.** Define for  $a \in C^*(G_{5,3})$  its Fourier transform  $\mathcal{F}(a) \in l^\infty(\widehat{G_{5,3}})$  by

$$\begin{aligned} \widehat{a}(\nu) &= \mathcal{F}(a)(\nu) := \pi_\nu(a) \in \mathcal{K}(L^2(\mathbb{R}^2)), \quad \nu \in \Gamma_2^{5,3}; \\ \widehat{a}(\rho, \mu) &= \mathcal{F}(a)(\rho, \mu) := \pi_{\rho, \mu}(a) \in \mathcal{K}(L^2(\mathbb{R})), \quad (\rho, \mu) \in \Gamma_1^{5,3}; \\ \widehat{a}(\alpha, \beta, \rho) &= \mathcal{F}(a)(\alpha, \beta, \rho) := \chi_{\alpha, \beta, \rho}(a), \quad (\alpha, \beta, \rho) \in \mathbb{R}^3. \end{aligned}$$

### 7.2. The changing of layers condition.

- Passing from  $\Gamma_2^{5,3}$  to  $\Gamma_1^{5,3} \cup \Gamma_0^{5,3}$ .

Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k})_k \subset \Gamma_2^{5,3}$  be a properly converging sequence where  $\ell_k = (0, 0, 0, 0, \nu_k)$ ,  $k \in \mathbb{N}$  such that  $\lim_{k \rightarrow \infty} \nu_k = 0$ . Let  $p_k := (\ell_k)|_{\mathfrak{p}}$ . By Theorem 7.1 the restriction of the limit set  $L(\overline{\mathcal{O}})$  to  $\mathfrak{p}$  is the closed set  $L = L(\mathcal{O})|_{\mathfrak{p}} = \{(\rho, \mu, 0), \rho \in \mathbb{R}, \mu \in \mathbb{R}\}$ .

**Definition 7.3.** For  $k \in \mathbb{N}$  and  $i, j \in \mathbb{Z}$ , let:

$$\begin{aligned} \varepsilon_k &:= |\nu_k|^{\frac{3}{4}}, \\ I_{i,j}^k &:= \{(c, u, \nu_k) \in \mathfrak{p}^*; i\varepsilon_k \leq c < i\varepsilon_k + \varepsilon_k \text{ and } j\varepsilon_k \leq u < j\varepsilon_k + \varepsilon_k\}, \\ U_{i,j}^k &:= \{(x, y) \in \mathbb{R}^2; \text{Ad}^*(\exp(xA + yB))p_k = (xA + yB) \cdot p_k = (x, y) \cdot p_k \in I_{i,j}^k\}. \end{aligned}$$

The sets  $I_{i,j}^k, i, j \in \mathbb{Z}$ , form a partition of  $O_{k|\mathfrak{p}}$ . Let also for  $k \in \mathbb{N}, i, j \in \mathbb{Z}$ :

$$x_j^k := \frac{j\varepsilon_k}{\nu_k}, \quad y_i^k := \frac{i\varepsilon_k}{\nu_k}, \quad g_{i,j}^k = x_j^k A + y_i^k B.$$

Let for  $i, j \in \mathbb{Z}, k \in \mathbb{N}$ :

$$p_{i,j}^k := (i\varepsilon_k, j\varepsilon_k, 0).$$

An easy computation gives:

$$g_{i,j}^k \cdot p_k = (i\varepsilon_k, j\varepsilon_k, \nu_k) = p_{i,j}^k + (0, 0, \nu_k) = p_{i,j}^k + \ell_{\nu_k}. \tag{7.2}$$

**Proposition 7.4.** Let  $K$  be a compact subset of  $\mathbb{R}^2 \simeq G_{5,3}/P$ . For  $k$  large enough we have

$$KU_{i,j}^k \subset \bigcup_{i',j'=-1}^1 U_{i'+i,j'+j}^k =: V_{i,j}^k.$$

**Proof.** We can suppose that  $KP$  is contained in  $[-M, M]^2 P$  for some  $M > 0$ . For  $r = (u, v) \in K \subset G_{5,3}/P$  and  $s = (x, y) \in \mathbb{R}^2$

$$(rs) \cdot p_k = (\nu_k v + \nu_k y, \nu_k x + \nu_k u, \nu_k)$$

and so, for  $k$  large enough such that  $|\nu_k|M < \varepsilon_k$ ,

$$\begin{aligned} &(x, y) \in U_{i,j}^k \\ \Leftrightarrow &(x, y) \cdot p_k \in I_{i,j}^k \\ \Rightarrow &\begin{cases} j\varepsilon_k \leq \nu_k x < j\varepsilon_k + \varepsilon_k, \\ i\varepsilon_k \leq \nu_k y < i\varepsilon_k + \varepsilon_k, \end{cases} \\ \Rightarrow &\begin{cases} (j-1)\varepsilon_k \leq \nu_k x + \nu_k u < (j+1)\varepsilon_k + \varepsilon_k, \\ (i-1)\varepsilon_k \leq \nu_k y + \nu_k v < (i+1)\varepsilon_k + \varepsilon_k. \end{cases} \end{aligned}$$

It follows that  $KU_{i,j}^k \subset \bigcup_{i',j'=-1}^1 U_{i'+i,j'+j}^k$ . ■

**Definition 7.5.** For  $k \in \mathbb{N}^*$  let

$$R^k = \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] \times \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right].$$

**Lemma 7.6.** For  $k \in \mathbb{N}^*$  large enough, for any  $i, j \in \mathbb{Z}^*$  the set  $U_{i,j}^k$  is contained in  $R^k + g_{i,j}^k$ .

**Proof.** Let  $s = (x, y) \in U_{i,j}^k$ . Then:

$$\begin{aligned} & (x, y) \cdot p_k \in I_{i,j}^k \\ \iff & \begin{cases} j\varepsilon_k \leq \nu_k x < j\varepsilon_k + \varepsilon_k \Rightarrow |x - x_j^k| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow x \in \left[-\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|}\right] + x_j^k, \\ i\varepsilon_k \leq \nu_k y < i\varepsilon_k + \varepsilon_k \Rightarrow |y - y_i^k| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow y \in \left[-\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|}\right] + y_i^k. \end{cases} \\ \implies & s \in R^k + g_{i,j}^k, \end{aligned}$$

■

**Lemma 7.7.** For  $k \in \mathbb{N}^*$  large enough, for  $i, j \in \mathbb{Z}^*$  and any  $(x, y) \in U_{i,j}^k$

$$\|(xA + yB) \cdot p_k - ((xA + yB) \cdot (g_{i,j}^k)^{-1}) \cdot p_{i,j}^k\| \leq 3\varepsilon_k.$$

**Proof.** For  $(x, y) \in U_{i,j}^k$  we have  $(x, y) = (x' + x_j^k, y' + y_i^k)$  where  $|\nu_k x'| \leq \varepsilon_k$  and  $|\nu_k y'| \leq \varepsilon_k$ . Therefore by (9.3)

$$\begin{aligned} & \|((x' + x_j^k)A + (y' + y_i^k)B)p_k - (x'A + y'B) \cdot p_{i,j}^k\| \\ &= \|(x', y') \cdot g_{i,j}^k \cdot p_k - (x'y') \cdot p_{i,j}^k\| \\ &= \|(\nu_k y', \nu_k x', \nu_k)\| \\ &= |\nu_k y'| + |\nu_k x'| + |\nu_k| \\ &\leq \varepsilon_k + \varepsilon_k + |\nu_k| \leq 3\varepsilon_k. \end{aligned}$$

■

**Definition 7.8.** For  $\beta, \rho \in \mathbb{R}$  and  $\mu \in \mathbb{R}^*$  let

$$\ell_{\beta,\rho,\mu} = \beta B^* + \rho C^* + \mu U^*, \ell_{\beta,\mu} = \beta B^* + \mu U^*, \text{ and } \ell_{\rho,\mu} = \rho C^* + \mu U^* \in \mathfrak{g}_{5,3}^*.$$

The sub-algebra  $\mathfrak{m} := \mathfrak{p}_{\beta,\mu} = \text{span}\{B, \mathfrak{p}\}$  is a polarization at  $\ell_{\beta,\mu}$  and also at  $\ell_{\beta,\rho,\mu}$  if  $\mu^2 + \rho^2 \neq 0$ . Since both linear functionals are on the same orbit, we have the equivalent representations  $\pi_{\beta,\mu} = \text{ind}_P^{G_{5,3}} \chi_{\ell_{\beta,\mu}} \in \widehat{G}_{5,3}$  and  $\pi_{\beta,\rho,\mu} = \text{ind}_P^{G_{5,3}} \chi_{\ell_{\beta,\rho,\mu}}$ . Let  $u_{\beta,\rho,\mu}$  be the unitary operator which gives the equivalence between both representations. For the 0 functional on  $\mathfrak{m}$ , the representation  $\text{ind}_M^{G_{5,3}} \chi_0$  is a direct integral of the unitary characters of  $G_{5,3}$ . For  $(\mu, \rho) \neq (0, 0)$  we take the direct integral representation

$$\tau_{\rho,\mu} := \left( \int_{\mathbb{R}}^{\oplus} \pi_{\ell_{\beta,\rho,\mu}} d\beta, \int_{\mathbb{R}}^{\oplus} L^2(G_{5,3}/P, \chi_{\ell_{\beta,\rho,\mu}}) d\beta \right) \tag{7.3}$$

resp.

$$\tau_{0,0} := \int_{\mathbb{R}^2}^{\oplus} \chi_{\alpha,\beta} d\beta d\alpha. \tag{7.4}$$

This representation  $\tau_{\rho,\mu}$  is in fact equivalent to the representation  $\sigma_{\rho,\mu} := \text{ind}_P^{G_{5,3}} \chi_{\ell_{\rho,\mu}}$  and a unitary intertwining operator  $U_{\rho,\mu}$  is given by:

$$\begin{aligned} U_{\rho,\mu} : & L^2(G_{5,3}/P, \chi_{\ell_{\rho,\mu}}) \mapsto \int_{\mathbb{R}}^{\oplus} L^2(G_{5,3}/P, \chi_{\ell_{\beta,\rho,\mu}}) d\beta, \\ U_{\rho,\mu}(\xi)(\beta)(g) : &= \int_{\mathbb{R}} \xi(g \exp(sB)) e^{-2i\pi s\beta} ds, g \in G_{5,3}, \beta \in \mathbb{R}. \end{aligned}$$

Hence for every  $a \in C^*(G_{5,3})$

$$\|\sigma_{\rho,\mu}(a)\|_{\text{op}} = \sup_{\beta \in \mathbb{R}} \|\pi_{\ell_{\beta,\rho,\mu}}(a)\|_{\text{op}}. \tag{7.5}$$

**Definition 7.9.**

- Let  $C_{\overline{\mathcal{O}}} = CB(L(\overline{\mathcal{O}}), \mathcal{B}(L^2(\mathbb{R}^2)))$  be the  $C^*$ -algebra of all continuous, uniformly bounded mappings  $\phi : L(\overline{\mathcal{O}}) \mapsto \mathcal{B}(L^2(\mathbb{R}^2))$  from the locally compact space  $L(\overline{\mathcal{O}})$  into the algebra of bounded linear operators  $\mathcal{B}(L^2(\mathbb{R}^2))$  on the Hilbert space  $L^2(\mathbb{R}^2)$ . By Theorem 7.1 we observe that for any  $a \in C^*(G_{5,3})$ , the operator field  $\widehat{a}|_{L(\overline{\mathcal{O}})}$  is contained in  $C_{\overline{\mathcal{O}}}$ . Furthermore, for  $\ell = \rho C^* + \mu U^* \in \mathfrak{g}_{5,3}^*$ , we obtain a representation  $\check{\sigma}_{\rho,\mu} = \check{\sigma}_\ell$  on the Hilbert space  $L^2(\mathbb{R}^2)$  of the algebra  $C_{\overline{\mathcal{O}}}$  defined by:

$$\check{\sigma}_\ell(\phi)\xi := U_{\rho,\mu}^{-1} \left( \int_{\mathbb{R}}^{\oplus} u_{\beta,\rho,\mu}^* \circ \phi(\beta, \mu) \circ u_{\beta,\rho,\mu}(U_{\rho,\mu}(\xi)(\beta)) d\beta \right), \phi \in C_{\overline{\mathcal{O}}}. \tag{7.6}$$

- Define for  $k \in \mathbb{N}$  and  $\phi \in C_{\overline{\mathcal{O}}}$  the linear operators  $\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi)$  and  $\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi)$  by

$$\begin{aligned} \check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &:= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}^*} U_E \circ M_{V_{i,j}^k} \circ \check{\sigma}_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k}(\phi) \circ M_{U_{i,j}^k} \circ U_E^* \\ \check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &= \frac{1}{2}(\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) + (\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi^*))^*) \end{aligned} \tag{7.7}$$

where  $\check{\sigma}_\ell$  for  $\ell = (g_{i,j}^k)^{-1} \cdot p_{i,j}^k$ , is as in Equation (7.6). For  $a \in C^*(G_{5,3})$

$$\check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{a}|_{L(\overline{\mathcal{O}})}) = \left( \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} U_E \circ M_{V_{i,j}^k} \circ \sigma_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k, \mathfrak{p}}(a) \circ M_{U_{i,j}^k} \circ U_E^*, \tag{7.8}$$

where as before  $\sigma_{\ell,\mathfrak{p}} = \text{ind}_P^{G_{5,3}} \chi_\ell, \ell \in \mathfrak{g}^*$ .

**Theorem 7.10.** *Let  $a \in C^*(G_{5,3})$ . Then:*

$$\lim_{k \rightarrow \infty} \|\pi_{\ell_k}(a) - \sigma_{k,\overline{\mathcal{O}}}(\widehat{a}|_{L(\overline{\mathcal{O}})})\|_{\text{op}} = 0.$$

**Proof.** Take first  $F \in L_c^1(G_{5,3})$ . Let us choose a compact subset  $K \subset \mathfrak{p}^*$  and an  $M > 0$  such that the function  $\mathbb{R}^2 \times \mathfrak{p}^* \ni ((x, y), p) \rightarrow \widehat{F}^P(E(x, y), p)$  is supported in  $[-M, M]^2 \times K$ . By Proposition 7.4 we have for  $k$  large enough:

$$\pi_{\ell_k}(F) \circ M_{U_{i,j}^k} = M_{V_{i,j}^k} \circ \pi_{\ell_k}(F) \circ M_{U_{i,j}^k}, \quad i, j \in \mathbb{Z}.$$

Let us write according to relation (2.2):

$$\begin{aligned} &M_{V_{i,j}^k} \circ \pi_{\ell_k}(F) \circ M_{U_{i,j}^k} - M_{V_{i,j}^k} \circ \check{\sigma}_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k}(F) \circ M_{U_{i,j}^k} \\ &= T_{g_{i,j}^k}^{-1} \circ (M_{V_{i,j}^k - g_{i,j}^k} \circ \pi_{g_{i,j}^k \cdot \ell_k}(F) \circ M_{U_{i,j}^k - g_{i,j}^k} - \\ &\quad - M_{V_{i,j}^k - g_{i,j}^k} \circ \sigma_{p_{i,j}^k, \mathfrak{p}}(F) \circ M_{U_{i,j}^k - g_{i,j}^k}) \circ T_{g_{i,j}^k} \end{aligned}$$

The kernel function  $F_k$  of the operator

$$M_{V_{i,j}^k - g_{i,j}^k} \circ \pi_{g_{i,j}^k \cdot \ell_k}(F) \circ M_{U_{i,j}^k - g_{i,j}^k} - M_{V_{i,j}^k - g_{i,j}^k} \circ \sigma_{p_{i,j}^k, \mathfrak{p}}(F) \circ M_{U_{i,j}^k - g_{i,j}^k}$$

is given by

$$F_k(s, t) = 1_{V_{i,j}^k - g_{i,j}^k}(s) 1_{U_{i,j}^k - g_{i,j}^k}(t) \left( \widehat{F}^P(st^{-1}, t \cdot g_{i,j}^k \cdot p_k) - \widehat{F}^P(st^{-1}, t \cdot p_{i,j}^k) \right)$$

Since the function  $(s, p) \rightarrow |\widehat{F}^P(s, p)|^2$  is in  $C_c^\infty(G_{5,3}/P, \mathfrak{p}^*)$  there exists a non-negative continuous function with compact support  $\varphi : G_{5,3}/P \rightarrow \mathbb{R}_+$  such that for any  $q, p \in \mathfrak{p}^*$ ,  $s \in G_{5,3}/P$  :

$$|\widehat{F}^P(s, q) - \widehat{F}^P(s, p)| \leq \varphi(s) \|q - p\|.$$

It follows then from Formula (7.2), Formula (7.4), Lemma 7.6 and Lemma 7.7 that for  $k \in \mathbb{N}$  large enough,  $i, j \in \mathbb{Z}$ ,  $a, b, a', b' \in \mathbb{R}$  :

$$\begin{aligned} & |F_k(E(a', b'), E(a, b))| \\ \leq & \left| \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), (a, b) \cdot g_{i,j}^k \cdot p_k) - \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), p_{i,j}^k) \right| \\ + & \left| \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), (a, b) \cdot g_{i,j}^k \cdot p_k) (|e^{-2\pi i((a,b) \cdot g_{i,j}^k \cdot p_k - p_{i,j}^k, a(b'-b)U + \frac{1}{2}a^2(b'-b)V)}| - 1) \right| \\ \leq & (3\varepsilon_k + r|\langle \nu_k, \frac{1}{2}a^2(b'-b)V \rangle|) \varphi(a' - a, b' - b) \\ \leq & (3\varepsilon_k + r|\frac{\nu_k \varepsilon_k^3}{\nu_k^3}|) \varphi(a' - a, b' - b) \\ \leq & (3\varepsilon_k + r|\nu_k|^{1/4}) \varphi(a' - a, b' - b), \end{aligned}$$

for a constant  $r > 0$  independent of  $k$ . Using now Young's estimate and Proposition 2.4, we see that there exists a constant  $D = D_F$ , such that for  $k$  large enough:

$$\|\pi_{\ell_k}(F) - \check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{F}|_{L(\overline{\mathcal{O}})})\|_{\text{op}} \leq D|\nu_k|^{1/4}.$$

Hence we also have

$$\|\pi_{\ell_k}(F^*) - (\check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{F^*}|_{\overline{\mathcal{O}}}))\|_{\text{op}} \leq D'|\nu_k|^{1/4}$$

for a new constant  $D' > 0$ . Therefore:

$$\begin{aligned} & \|\pi_{\ell_k}(F) - (\check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{F^*}|_{\overline{\mathcal{O}}}))^*\|_{\text{op}} \\ = & \|\pi_{\ell_k}(F^*) - \check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{F^*}|_{\overline{\mathcal{O}}})\|_{\text{op}} \leq D'|\nu_k|^{1/4} \end{aligned}$$

and so

$$\lim_{k \rightarrow \infty} \|\pi_k(F) - \check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{F}|_{\overline{\mathcal{O}}})\|_{\text{op}} = 0.$$

The theorem follows from the fact  $L_c^1(G_{5,3})$  is dense in  $C^*(G_{5,3})$ . ■

\*

- Passing from  $\Gamma_1^{5,3}$  to  $\Gamma_0^{5,3}$ .

**Definition 7.11.** Choose a Schwartz-function  $\eta$  in  $\mathcal{S}(\mathbb{R})$  with  $L^2$ -norm equal to 1. For  $(\alpha, \beta) \in \mathbb{R}^2$  we define the function  $\eta_\mu(\alpha, \beta)$  by

$$\eta_\mu(\alpha, \beta)(s) := |\mu|^{1/4} e^{2i\pi s\alpha} \eta(|\mu|^{1/2}(s + \frac{\beta}{\mu})), \quad s \in \mathbb{R}. \tag{7.9}$$

**Lemma 7.12.** *Let  $\xi \in \mathcal{S}(\mathbb{R})$ . Then,*

$$\xi = \frac{1}{|\mu|} \int_{\mathbb{R}^2} \langle \xi, \eta_\mu(\alpha, \beta) \rangle \eta_\mu(\alpha, \beta) d\alpha d\beta.$$

**Proof.** The proof is the same as the proof of Lemma 2.8 in [LT11]. ■

**Definition 7.13.** 1. For all  $(\alpha, \beta) \in \mathbb{R}^2$  and  $k \in \mathbb{N}$ , let  $P_{\mu(\alpha, \beta)}$  be the orthogonal projection onto the one dimensional subspace  $\mathbb{C}\eta_\mu(\alpha, \beta)$ .

2. Define for  $(\rho, \mu) \in \Gamma_1^{5,3}$  and  $h \in C_0(\mathbb{R}^3)$  a the linear operator

$$\sigma_{\rho, \mu}(h) := \frac{1}{|\mu|} \int_{\mathbb{R}^2} h(\alpha A^* + \beta B^* + \rho C^*) P_{\mu(\alpha, \beta)} d\alpha d\beta. \tag{7.10}$$

**Proposition 7.14.** *(Proposition 2.11 in [LT11])*

1. For every  $(\rho, \mu) \in \Gamma_1^{5,3}$  and  $h \in \mathcal{S}(\mathbb{R}^3)$  the integral (7.10) converges in operator norm.
2.  $\sigma_{\rho, \mu}(h)$  is compact and  $\|\sigma_{\rho, \mu}(h)\|_{op} \leq \|h\|_\infty$ .
3. The mapping  $\sigma_{\rho, \mu} : C_0(\mathbb{R}^3) \rightarrow l^\infty(\widehat{G}_{5,3})$  is involutive, i.e.  $\sigma_{\rho, \mu}(h^*) = \sigma_{\rho, \mu}(h)^*$ ,  $h \in C^*(\mathbb{R}^3)$ , where by  $\sigma_{\rho, \mu}$  we denote also the extension of  $\sigma_{\rho, \mu}$  to  $CB(\mathbb{R}^3)$ .

**Theorem 7.15.** *Let  $a \in C^*(G_{5,3})$  and let  $\varphi$  be the operator field  $\varphi = \mathcal{F}(a)$ . Let  $\bar{\gamma} = (\gamma_k = (\rho_k, \mu_k))_k$  be a properly converging sequence in  $\Gamma_1^{5,3}$  having its limit set  $L(\bar{\gamma}) = \mathbb{R}A^* + \mathbb{R}B^* + \rho C^*$  in  $\Gamma_0^{5,3}$ . Then the function  $\varphi(\rho) : (\alpha, \beta) \rightarrow \mathcal{F}(a)(\alpha, \beta, \rho)$  is contained  $C_0(\mathbb{R}^2)$  and*

$$\lim_{k \rightarrow \infty} \|\varphi(\gamma_k) - \sigma_{\gamma_k}(\varphi(\rho))\|_{op} = 0. \tag{7.11}$$

**Proof.** The proof is the same as that of Theorem 2.12 in [LT11]. ■

**Remark 7.16.** Theorem 7.15 and Theorem 7.10 imply that  $C^*(G_{5,3})$  has the NCDL property.

### 8. The $C^*$ -algebra of the group $G_{5,4}$ .

Recall that the Lie algebra of  $\mathfrak{g}_{5,4}$  is spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = C, [A, C] = U, [B, C] = V.$$

This Lie algebra has a two-dimensional centre  $\mathfrak{z} = \text{span}\{U, V\}$ . The group  $G_{5,4} = \exp(\mathfrak{g}_{5,4})$  can be realized as  $\mathbb{R}^5$  with the Campbell-Baker-Hausdorff multiplication

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]], \quad X, Y \in \mathfrak{g}_{5,4},$$

$$\begin{aligned} \text{i.e. } & (a, b, c, u, v) \cdot (a', b', c', u', v') \\ &= (a+a', b+b', c+c' + \frac{ab'}{2} - \frac{a'b}{2}, u+u' + \frac{ac'}{2} - \frac{a'c}{2} + \frac{a'^2b}{12} + \frac{a^2b'}{12} - \frac{aa'b'}{12} - \frac{a'ab}{12}, \\ & \quad v+v' + \frac{bc'}{2} - \frac{b'c}{2} - \frac{a'b^2}{12} - \frac{ab'^2}{12} + \frac{abb'}{12} + \frac{a'b'b}{12}). \end{aligned}$$

For all  $(a, b, c, u, v) \in \mathfrak{g}_{5,4}$ ,  $(\alpha, \beta, \rho, \mu, \nu) \in \mathfrak{g}_{5,4}^*$  we obtain the following expression for  $Ad^*(a, b, c, u, v)$  :

$$\begin{aligned} & Ad^*(a, b, c, u, v)(\alpha, \beta, \rho, \mu, \nu) \\ &= (\alpha - b\rho - c\mu - \mu\frac{ab}{2} - \nu\frac{b^2}{2}, \beta + \rho a - \nu c + \nu\frac{ab}{2} + \mu\frac{a^2}{2}, \rho + \mu a + \nu b, \mu, \nu). \end{aligned}$$

Let us give the Pukanszky parameterization of the co-adjoint orbits:

The generic elements  $\ell = (\alpha, \beta, \rho, \mu, \nu)$  in  $\mathfrak{g}_{5,4}^*$ , are those with a non-zero value  $r_{\mu,\nu} = \sqrt{\mu^2 + \nu^2}$ . As in (6.3) we take a new basis  $\mathcal{B}_{\mu,\nu}$  of  $\mathfrak{g}_{5,4}$ . For that, letting  $\tilde{\mu} := \frac{\mu}{r_{\mu,\nu}}$ ,  $\tilde{\nu} := \frac{\nu}{r_{\mu,\nu}}$  we put:

$$\mathcal{B}_\ell = \mathcal{B}_{\mu,\nu} := \{A_{\mu,\nu} = A_\ell := \tilde{\mu}A + \tilde{\nu}B, B_{\mu,\nu} = B_\ell := -\tilde{\nu}A + \tilde{\mu}B, C, U, V\}.$$

Let also

$$Z_{\mu,\nu} = \tilde{\mu}U + \tilde{\nu}V, T_{\mu,\nu} = -\tilde{\nu}U + \tilde{\mu}V.$$

Then:

$$[A_{\mu,\nu}, B_{\mu,\nu}] = C, [A_{\mu,\nu}, C] = Z_{\mu,\nu}, [B_{\mu,\nu}, C] = T_{\mu,\nu}.$$

In the dual basis  $\mathcal{B}_\ell^*$  of the basis  $\mathcal{B}_\ell$  the orbit  $\mathcal{O}_\ell$  of the element  $\ell = \ell_{\beta,\mu,\nu} = \beta B_\ell^* + \mu U^* + \nu V^*$  is given by:

$$\mathcal{O}_\ell = \left\{ aA_\ell^* + \left(\beta + \frac{c^2}{2r_{\mu,\nu}}\right)B_\ell^* + cC^* + \mu U^* + \nu V^*, a, c \in \mathbb{R} \right\}. \tag{8.1}$$

The stabilizer  $\mathfrak{g}_{5,4}(\ell)$  of  $\ell$  is the sub-algebra

$$\mathfrak{g}_{5,4}(\ell) = \text{span}\{B_\ell, U, V\}.$$

We denote by  $\Gamma_2^{5,4}$  the orbit space of this family of generic co-adjoint orbits parameterized by the set

$$\Gamma_2^{5,4} := \{\ell_{\beta,\mu,\nu} \equiv (\beta, \mu, \nu), \beta \in \mathbb{R}, (\mu, \nu) \in \mathbb{R}^2 \setminus \{(0, 0)\}\}$$

Since  $G_{5,4}/\exp(\mathfrak{z}) = H_1$  we can decompose the orbit space  $\mathfrak{g}_{5,4}^*/G_{5,4}$ , and hence also the dual space  $\widehat{G}_{5,4}$ , into the disjoint union

$$\mathfrak{g}_{5,4}^*/G_{5,4} = \Gamma_2^{5,4} \cup \Gamma_1^1 \cup \Gamma_0^1,$$

where  $\Gamma_0^1$  and  $\Gamma_1^1$  are the two Pukanszky layers for the Heisenberg group  $H_1$ .

**8.1. Limit sets of properly converging sequences in  $\Gamma_2^{5,4}$ .**

**Theorem 8.1.** *The sequence  $(\mathcal{O}_{\beta_k, \mu_k, \nu_k})_k$  goes to infinity if and only if the real sequence  $(r_k + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k r_k)_k$  goes to infinity, where for  $k \in \mathbb{N}$ ,  $r_k = \sqrt{\mu_k^2 + \nu_k^2}$  and*

$$\varepsilon_k := \begin{cases} 1 & \text{if } \beta_k > 0 \\ 0 & \text{if } \beta_k \leq 0. \end{cases}$$

**Proof.** Suppose that the sequence of orbits  $(\mathcal{O}_{\beta_k, \mu_k, \nu_k})_k$  does not tend to infinity. Then there is a convergent sub-sequence (also indexed by  $k \in \mathbb{N}$  for simplicity) and for any  $k$  an element  $\ell_k = (\beta_k + \frac{c_k^2}{2r_{\mu, \nu}})B_\ell^* + c_k C^* + \mu_k U^* + \nu_k V^* \in \mathcal{O}_k$ , such that  $\lim_k \ell_k = \ell$  exists. Hence  $c_k$  converges to  $c$ ,  $r_k := \sqrt{\mu_k^2 + \nu_k^2}$  to  $r$  and  $(\frac{2\beta_k r_k + c_k^2}{2r_k})$  to  $b$  as  $k$  tends to infinity. Then  $\beta_k r_k \rightarrow br - \frac{c^2}{2}$ . Hence the sequence  $(r_k - \beta_k r_k)_k$  converges. Moreover, if  $\beta_k \geq 0$  for any  $k$ , then  $0 \leq \beta_k \leq \beta_k + \frac{c_k^2}{2r_k} \leq b + 1$  for  $k$  large enough and so the sequence  $(r_k + \beta_k)_k$  is bounded. Conversely, suppose that  $(r_k + \varepsilon_k \beta_k + (\varepsilon_k - 1)\beta_k r_k)$  does not tend to infinity. Then there is a convergent sub-sequence  $(r_k)$  such that  $(\beta_k r_k)$  is also convergent. We may choose a convergent sequence  $(c_k)$  such that  $2\beta_k r_k + c_k^2 = 0$  for all  $k$ . Then  $(0, 0, c_k, \mu_k, \nu_k) \in \mathcal{O}_k$  and the sequence of functionals converges as  $k \rightarrow \infty$ . Hence the sequence of orbits  $(\mathcal{O}_k)$  does not tend to infinity. ■

**Theorem 8.2.** *1. On the set  $\Gamma_2^{5,4}$  the dual topology is Hausdorff.  
2. Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k} = \mathcal{O}_{\beta_k B_k^* + \mu_k U^* + \nu_k V^*})_k$  (where  $B_k := B_{\ell_k}, k \in \mathbb{N}$ ) be a sequence in  $\Gamma_2^{5,4}$  with  $\lim_{k \rightarrow \infty} r_k = \lim_{k \rightarrow \infty} \sqrt{\mu_k^2 + \nu_k^2} = 0$ . We can assume (passing if necessary to a sub-sequence) that the real sequence  $(\tilde{\mu}_k)_k$  (res.  $(\tilde{\nu}_k)_k$ ) converges to  $\tilde{\mu}$  (resp. to  $\tilde{\nu}$ ). Then the sequence of vectors  $(A_k = A_{\ell_k})_k$ , resp.  $(B_k = B_{\ell_k})_k$  converges to the vector  $A_\infty = \tilde{\mu}A + \tilde{\nu}B$  (resp. to  $B_\infty = -\tilde{\nu}A + \tilde{\mu}B$ ).*

- If  $\overline{\mathcal{O}}$  has a limit, then  $d := \lim_{k \rightarrow \infty} (-2\beta_k r_k =: d_k)$  exists and  $d \geq 0$ .
- Suppose now that  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k} = \mathcal{O}_{\beta_k B_k^* + \mu_k U^* + \nu_k V^*})_k$  is a properly converging sequence in  $\Gamma_2^{5,4}$ . If  $\overline{\mathcal{O}}$  admits a limit in  $\Gamma_1^1 \cup \Gamma_0^1$ , then its limit set  $L(\overline{\mathcal{O}})$  is given by:

1. if  $d \neq 0$  then  $L(\overline{\mathcal{O}}) = \{\mathcal{O}_{\sqrt{d}}, \mathcal{O}_{-\sqrt{d}}\}$ .
2. if  $d = 0$ , then the number  $\beta_\infty := \limsup_k \beta_k$  is contained in  $[-\infty, +\infty[$  and

$$L(\overline{\mathcal{O}}) = \begin{cases} \mathbb{R}A_\infty^* + [\beta_\infty, +\infty[B_\infty^* & \text{if } \beta_\infty \in \mathbb{R} \\ \mathbb{R}A_\infty^* + ]-\infty, +\infty[B_\infty^* = \Gamma_0^1 & \text{if } \beta_\infty = -\infty. \end{cases} \tag{8.2}$$

**Proof.** The point 1) is evident. For point 2) we observe the following:

- Let  $\overline{\mathcal{O}}$  be such a sequence in  $\Gamma_2^{5,4}$  having a limit. Let  $\ell$  be a point in a limit orbit of the sequence  $\overline{\mathcal{O}}$  and let

$$m_k = a_k A_k^* + (\beta_k + \frac{c_k^2}{2r_k}) B_k^* + c_k C^* + \mu_k U + \nu_k V^* \in \mathcal{O}_k, k \in \mathbb{N}$$

be such that  $\lim_{k \rightarrow \infty} m_k = \ell$ . Then:

$$\ell(B_\infty) = \lim_{k \rightarrow \infty} m_k(B_k) = \lim_{k \rightarrow \infty} \frac{2\beta_k r_k + m_k(C)^2}{2r_k}.$$

This shows that  $\lim_{k \rightarrow \infty} (2\beta_k r_k + m_k(C)^2) = 0$  since  $\lim_{k \rightarrow \infty} r_k = \lim_{k \rightarrow \infty} \sqrt{\mu_k^2 + \nu_k^2} = 0$ , i.e.  $\lim_{k \rightarrow \infty} (-2\beta_k r_k) = \ell(C)^2 =: d$ .

- 1. Let now  $\overline{\mathcal{O}}$  be properly convergent and suppose that  $d > 0$ . For  $k$  large enough,  $\beta_k > 0$  and then the element  $m_{k,\pm} := \pm\sqrt{-2\beta_k r_k} C^* + \mu_k U^* + \nu_k V^*$  of  $\mathcal{O}_k$  converges to  $\pm\sqrt{d} C^*$ . Hence the orbits  $\mathcal{O}_{\pm\sqrt{d}}$  are contained in  $L(\overline{\mathcal{O}})$ . On the other hand, every other  $\ell$  in the limit set  $L(\overline{\mathcal{O}})$  satisfies the relation  $d = \ell(C)^2$ , which means that  $\ell(C) = \pm\sqrt{d}$ . Hence  $L(\overline{\mathcal{O}}) = \{\mathcal{O}_{\sqrt{d}}, \mathcal{O}_{-\sqrt{d}}\}$ .
- 2. If now  $d = 0$ , we see in a similar manner that the limit of the sequence  $\overline{\mathcal{O}}$  must be characters, i.e. vanish on  $C$ . Choose a sub-sequence (also denoted by  $\overline{\mathcal{O}}$  for simplicity), such that

$$\lim_{k \rightarrow \infty} \beta_k = \beta_\infty.$$

Take any sequence

$$\left( m_k = a_k A_k^* + \left( \frac{2\beta_k r_k + c_k^2}{2r_k} \right) B_k^* + c_k C^* + \mu_k U^* + \nu_k V^* \in \mathcal{O}_k \right)_k$$

which converges to  $\ell = aA^* + bB_\infty^* \in L(\overline{\mathcal{O}})$  for some  $a, b \in \mathbb{R}$ . Since

$$b = \lim_{k \rightarrow \infty} \left( \beta_k + \frac{c_k^2}{2r_k} \right) = \lim_{k \rightarrow \infty} \beta_k + \lim_{k \rightarrow \infty} \frac{c_k^2}{2r_k} \geq \lim_{k \rightarrow \infty} \beta_k = \beta_\infty,$$

it follows that  $\beta_\infty \leq b < +\infty$ . On the other hand, for any  $a \in \mathbb{R}$ ,  $b > \beta_\infty$ , we have  $b \geq \beta_k$  for  $k$  large enough and the sequence

$$\left( m_k = aA_k^* + bB_k^* + \sqrt{(b - \beta_k)2r_k} C^* + \mu_k U^* + \nu_k V^* \in \mathcal{O}_k \right)_k$$

converges to  $aA_\infty^* + bB_\infty^*$ , since  $\lim_{k \rightarrow \infty} \beta_k r_k = 0$ . Hence, since  $L(\overline{\mathcal{O}})$  is closed,

$$L(\overline{\mathcal{O}}) = \mathbb{R}A_\infty^* + [\beta_\infty, +\infty[B_\infty^* \text{ ( resp., } L(\overline{\mathcal{O}}) = \mathbb{R}A_\infty^* + ] - \infty, +\infty[B_\infty^* )]. \blacksquare$$

**8.2. The Fourier transform.** Let  $\mathfrak{p} = \text{span}\{C, U, V\}$ ,  $P := \exp(\mathfrak{p})$ . Let  $\ell \in \Gamma_2^{5,4}$ . The dimension of a generic orbit is 2. A polarization at such an  $\ell = \ell_{\beta, \mu, \nu} = \beta B_\ell^* + \mu U^* + \nu V^*$  is given by  $\mathfrak{p}_\ell = \mathfrak{p}_{\mu, \nu} = \text{span}\{B_\ell, \mathfrak{p}\}$ . Then  $G_{5,4} = \exp(\mathbb{R}A_\ell)P_\ell$  (where  $P_\ell = \exp(\mathfrak{p}_\ell)$ ) as topological products. We take  $\pi_\ell := \text{ind}_{P_\ell}^{G_{5,4}} \chi_\ell$ . Its

Hilbert space is isomorphic to  $L^2(\mathbb{R})$  and for  $g = \exp(sA_\ell)p, s, u \in \mathbb{R}, p \in P_\ell$  and  $\xi \in L^2(\mathbb{R})$  we have

$$\pi_\ell(g)\xi(u) = e^{-2\pi i \langle \exp(u-s)A_\ell \cdot \ell, \log(\ell|_{\mathfrak{p}_\ell}) \rangle} \xi(u-s),$$

and so for  $F \in L^1(G_{5,4})$ :

$$\begin{aligned} \pi_\ell(F)\xi(t) &= \int_{G_{5,4}} F(g)\pi_\ell(g)\xi(t)dg \\ &= \int_{\mathbb{R}} \widehat{F}^{P_\ell}(s-t, s.\ell|_{\mathfrak{p}_\ell})\xi(s)ds. \end{aligned} \tag{8.3}$$

Here

$$\widehat{F}^{P_\ell}(s, q) = \int_{P_\ell} F(\exp(sA_\ell)p)\chi_q(p)dp, q \in \mathfrak{p}_\ell^*, s \in \mathbb{R}.$$

and  $s \cdot q = \exp(sA_{\mu,\nu}) \cdot q, q \in \mathfrak{p}_\ell^*$ .

Let  $L_c^1(G_{5,4})$  be the subspace of  $L^1(G_{5,4})$  consisting of all the  $F$ 's in  $L^1(G_{5,4})$ , for which the partial Fourier transform

$$\widehat{F}^{\mathfrak{p}}(E(a, b), q) := \int_P F(\exp(aA)\exp(bB)p)e^{-2\pi i \langle q, \log p \rangle} dp, a, b \in \mathbb{R}, q \in \mathfrak{p}^*$$

is smooth and has compact support. For  $F \in L_c^1(G_{5,4})$ , the function  $\widehat{F}^{P_\ell}$  is of compact support in  $s \in \mathbb{R}$  and Schwartz in the variable  $q \in \mathfrak{p}_\ell^*$ .

**Definition 8.3.** The Fourier transform  $\widehat{a} = \mathcal{F}(a)$  of an element  $a \in C^*(G_{5,4})$  is defined as the field of bounded linear operators over the dual space of  $G_{5,4}$  given by:

$$\begin{aligned} \widehat{a}(\beta, \mu, \nu) = \mathcal{F}(a)(\beta, \mu, \nu) &:= \pi_{\beta, \mu, \nu}(a) \in \mathcal{K}(L^2(\mathbb{R})), (\beta, \mu, \nu) \in \Gamma_2^{5,4}; \\ \widehat{a}(\rho) = \mathcal{F}(a)(\rho) &:= \pi_\rho(a) \in \mathcal{K}(L^2(\mathbb{R})), \rho \in \Gamma_1^1; \\ \widehat{a}(\alpha, \beta) = \mathcal{F}(a)(\alpha, \beta) &:= \langle \chi_{\alpha, \beta}, a \rangle, \alpha, \beta \in \mathbb{R}, \\ & (= \int_G F(g)e^{-2\pi i \langle \ell_{\alpha, \beta}, \log g \rangle} dg \text{ if } a = F \in L^1(G_{5,4})), . \end{aligned}$$

**8.3. The continuity condition.** We have seen in Theorem 8.2, that the topology of the sub-set  $\Gamma_2^{5,4}$  is Hausdorff. This means for the  $F$ 's in  $C^*(G_{5,4})$ , that the functions  $\pi \rightarrow \|\pi(F)\|_{\text{op}}$  are continuous on this set.

**Theorem 8.4.** *The mapping  $\Gamma_2^{5,4} \rightarrow \mathcal{B}(L^2(\mathbb{R})) : \ell \mapsto \pi_\ell(F)$  is norm-continuous for all  $F \in C^*(G_{5,4})$ .*

**Proof.** This follows at once from Theorem 2.3. ■

**8.4. Passing from  $\Gamma_2^{5,4}$  to  $\Gamma_1^1 \cup \Gamma_0^1$ .**

**Definition 8.5.** Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k = \beta_k B_k^* + \mu_k U^* + \nu_k V^*})$  be a properly converging sequence in  $\Gamma_2^{5,4}$  such that  $\lim_{k \rightarrow \infty} (r_k = \sqrt{\mu_k^2 + \nu_k^2}) = 0$ . We can assume (passing if necessary to a subsequence) that  $\lim_{k \rightarrow \infty} \tilde{\mu}_k = \tilde{\mu}$  and  $\lim_{k \rightarrow \infty} \tilde{\nu}_k =: \tilde{\nu}$  exist too. Let

$$d_k := -2\beta_k r_k > 0 \text{ and } t_k = \sqrt{\frac{-2\beta_k}{r_k}} = \frac{\sqrt{d_k}}{r_k}, k \in \mathbb{N}.$$

By Theorem 8.2 and its notations,  $d = \lim_{k \rightarrow \infty} d_k$  exists. If  $d = 0$ , then  $\beta_\infty := \lim_{k \rightarrow \infty} \beta_k$  exists in  $[-\infty, +\infty[$  and the limit set of  $\overline{\mathcal{O}}$  is the set

$L(\overline{\mathcal{O}}) = \{\mathbb{R}A_{\ell_\infty}^* + [\beta_\infty, +\infty]B_{\ell_\infty}^*\}$ , where  $A_{\ell_\infty}^* = \tilde{\mu}A^* + \tilde{\nu}B^*$  and  $B_{\ell_\infty}^* = -\tilde{\nu}A^* + \tilde{\mu}B^*$ . Otherwise, i.e. if  $d \neq 0$ , the limit set  $L(\overline{\mathcal{O}})$  is given by  $L(\overline{\mathcal{O}}) = \{\sqrt{d}C^*, -\sqrt{d}C^*\}$ . Recall that:

$$\begin{aligned} \exp((t_k + s)A_k) \cdot p_k &= \left( \beta_k + \frac{r_k(s + t_k)^2}{2} \right) B_k^* + ((s + t_k)r_k)C^* + \mu_k U^* + \nu_k V^* \\ &= s \left( r_k \frac{s}{2} + r_k t_k \right) B_k^* + ((s + t_k)r_k)C^* + \mu_k U^* + \nu_k V^*, k \in \mathbb{N}. \end{aligned}$$

•1<sup>st</sup> case if  $d \neq 0$ . We consider first the case where  $d := \lim_{k \rightarrow \infty} d_k \neq 0$ . Let

$$q_k = \sqrt{d_k}C^*, k \in \mathbb{N}^*.$$

Let us compute:

$$\begin{aligned} &\exp((s + \pm t_k)A_k) \cdot p_k - \exp(sA_k) \cdot (\pm q_k) \\ &= \left( \beta_k + \frac{r_k(s + \pm t_k)^2}{2} \pm \sqrt{d_k}s \right) B_k^* + \left( r_k(\pm t_k + s) - \pm \sqrt{d_k} \right) C^* + \mu_k U^* + \nu_k V^* \\ &= \left( \frac{r_k s^2}{2} \right) B_k^* + (r_k s)C^* + \mu_k U^* + \nu_k V^*. \end{aligned}$$

Let  $(R_k)_k$  be a sequence in  $\mathbb{R}_+$  such that  $\lim_{k \rightarrow \infty} R_k^2 r_k = 0, \lim_{k \rightarrow \infty} R_k = +\infty$ . Let for  $k \in \mathbb{N}$ :

$$\begin{aligned} J_k &= [-R_k, R_k], \\ I_{k,\pm} &:= (\pm t_k + J_k). \end{aligned}$$

Our condition on  $R_k$  tells us that  $\lim_{k \rightarrow \infty} \frac{R_k}{t_k} = \lim_{k \rightarrow \infty} \frac{R_k}{\sqrt{\frac{-2\beta_k}{r_k}}} = \lim_k \frac{R_k r_k}{\sqrt{-\beta_k r_k}} = 0$  and so  $I_{k,+} \cap I_{k,-} = \emptyset$  for  $k$  large enough.

**Lemma 8.6.** *Let  $K$  be a compact subset of  $\mathfrak{p}^*$ . Then  $\exp((\mathbb{R} \setminus (I_{k,+} \cup I_{k,-}))A_k) \cdot \ell_{k|\mathfrak{p}} \cap K = \emptyset$  for  $k$  large enough.*

**Proof.** Take  $R > 0$  such that  $K \subset [-R, R]$ . For  $s > 0, s \notin I_{k,+}$  we have either  $s > t_k + R_k$  or  $0 \leq s \leq t_k - R_k$  (for  $k$  large enough): In the first case:

$$\begin{aligned} |\exp(sA_k) \cdot \ell_k(B_k)| &= |((s - t_k) \cdot (t_k \cdot \ell_k))(B_k)| \\ &= \left| (s - t_k) \left( r_k \frac{(s - t_k)}{2} + r_k t_k \right) \right| \\ &\geq R_k \sqrt{d_k} > \frac{R_k \sqrt{d}}{2} \text{ (for } k \text{ large enough).} \end{aligned}$$

In the second case:

$$\begin{aligned} |\exp(sA_k) \cdot \ell_k(B_k)| &= |((s - t_k) \cdot (t_k \cdot \ell_k))(B_k)| \\ &= \left| (s - t_k) \left( r_k \frac{(s - t_k)}{2} + r_k t_k \right) \right| \\ &\geq R_k \left( t_k r_k - \frac{(t_k - s_k) r_k}{2} \right) > \frac{R_k \sqrt{d}}{4} \text{ (for } k \text{ large enough).} \end{aligned}$$

Similarly for  $s < 0, s \notin I_{k,-}$ . This means that for  $k$  large enough,  $\exp(tA_k) \cdot \ell_k \notin K$  for  $t \notin I_{k,\pm}$ . ■

**Definition 8.7.** Let  $C_{\overline{\mathcal{O}}} = CB(L(\overline{\mathcal{O}}), \mathcal{B}(L^2(\mathbb{R})))$  be the  $C^*$ -algebra of all continuous, uniformly bounded mappings  $\phi : L(\overline{\mathcal{O}}) \mapsto \mathcal{B}(L^2(\mathbb{R}))$  from the locally compact space  $L(\overline{\mathcal{O}})$  into the algebra of bounded linear operators  $\mathcal{B}(L^2(\mathbb{R}))$  on the Hilbert space  $L^2(\mathbb{R})$ . Let for  $k \in \mathbb{N}^*$ ,

$$\sigma_{k,\pm} = \text{ind}_{P_k}^{G_{5,4}} \chi_{\exp(\pm t_k A_k) \cdot (\sqrt{d_k} C^*)}, \quad \pi_{\sqrt{d_k} C^*} = \text{ind}_{P_k}^{G_{5,4}} \chi_{\sqrt{d_k} C^*}$$

and let  $u_{k,\pm}$  be the unitary intertwining operator between  $\pi_{\sqrt{d_k} C^*}$  and  $\sigma_{k,\pm}$ .

1. Let for  $\phi \in C_{\overline{\mathcal{O}}}, k \in \mathbb{N}$ ,

$$\begin{aligned} \check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &= \check{\sigma}_k(\phi) \\ &:= M_{I_{k,+}} \circ u_{k,+} \circ \phi(\sqrt{d_k}) \circ u_{k,+}^* \circ M_{I_{k,+}} \\ &\quad + M_{I_{k,-}} \circ u_{k,-} \circ \phi(-\sqrt{d_k}) \circ u_{k,-}^* \circ M_{I_{k,-}} \in \mathcal{B}(L^2(\mathbb{R})), \\ \tilde{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &= \frac{1}{2}(\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) + (\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi))^*). \end{aligned} \tag{8.4}$$

2. For  $a \in C^*(G_{5,4})$  let

$$\sigma_{k,\overline{\mathcal{O}}}(a) := \tilde{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{a}|_{\Gamma_1^1}).$$

**Theorem 8.8.** Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k = \beta_k B_k^* + \mu_k U^* + \nu_k V^*})$  be a properly converging sequence in  $\Gamma_2^{5,4}$  with limit set  $L = \{\mathcal{O}_{\sqrt{d}}, \mathcal{O}_{-\sqrt{d}}\}$  where  $d = -2 \lim \beta_k \sqrt{\mu_k^2 + \nu_k^2} = \lim_{k \rightarrow \infty} (-2\beta_k r_k) > 0, k \in \mathbb{N}$ . Then for every  $a \in C^*(G_{5,4})$ ,

$$\lim_{k \rightarrow \infty} \|\pi_{\ell_k}(a) - \sigma_{k,\overline{\mathcal{O}}}(a)\|_{\text{op}} = 0.$$

**Proof.** Let  $F \in L_c^1(G_{5,4})$ . Let for  $k \in \mathbb{N}^*$

$$\mathfrak{p}_k := \mathfrak{p}_{\ell_k}, P_k = P_{\ell_k} = \exp(\mathbb{R}B_k) \cdot P = \exp(\mathfrak{p}_k).$$

The normal subgroup  $P_k = \exp(\mathfrak{p}_k)$  is a polarization at  $\ell_k$  for every  $k$ . Let for  $k \in \mathbb{N}^*$  :

$$F_k(u, t) := \widehat{F}^{P_k}(\exp(uA_k), \exp(tA_k) \cdot \ell_{k|\mathfrak{p}_k}), u, t \in \mathbb{R}.$$

Then the kernel function  $K_k$  of the linear operator  $\pi_{\ell_k}(F)$  is given by:

$$K_k(s, t) = F_k(s - t, t), s, t \in \mathbb{R}.$$

Since  $F \in L_c^1(G_{5,4})$ , there is an  $M > 0$  such that  $F_k(s - t, t) = 0$ , if  $|s - t| > M$  and together with Lemma 8.6 we have therefore for  $k$  large enough that

$$\pi_{\ell_k}(F) = M_{I_{k,+}} \circ \pi_{\ell_k}(F) \circ M_{I_{k,+}} + M_{I_{k,-}} \circ \pi_{\ell_k}(F) \circ M_{I_{k,-}}.$$

The kernel function  $F_k$  of the operator  $\pi_{\ell_k}(F) - \tilde{\sigma}_{k,\overline{O}}(F)$  is given by

$$\begin{aligned} F_k(s, t) &= 1_{I_{k,+}}(s)1_{I_{k,+}}(t) \left( \widehat{F}^{P_k}(\exp((s-t)A_k), \exp((t+t_k)A_k) \cdot \ell_{k|\mathfrak{p}_k}) \right. \\ &\quad \left. - \widehat{F}^{P_k}(\exp((s-t)A_k), \exp(tA_k) \cdot q_k) \right) \\ &+ 1_{I_{k,-}}(s)1_{I_{k,-}}(t) \left( \widehat{F}^{P_k}(\exp((s-t)A_k), \exp((t-t_k)A_k) \cdot \ell_{k|\mathfrak{p}_k}) \right. \\ &\quad \left. - \widehat{F}^{P_k}(\exp((s-t)A_k), \exp(tA_k) \cdot (-q_k)) \right) \end{aligned}$$

Since  $F \in L_c^1(G_{5,4})$ , there exists a continuous function  $\varphi \geq 0$  on  $G_{5,4}/P_k$  with compact support, such that

$$|\widehat{F}^{P_k}(\exp(sA_k), \ell_{|\mathfrak{p}_k}) - \widehat{F}^{P_k}(\exp(sA_k), \ell'_{|\mathfrak{p}_k})| \leq \varphi(s) \|\ell_{|\mathfrak{p}_k} - \ell'_{|\mathfrak{p}_k}\|, s \in G_{5,4}, \ell, \ell' \in \mathfrak{g}_{5,4}^*.$$

It follows for  $t = v + t_k \in I_{k,+}, s = u + t_k \in I_{k,+}$  that

$$\begin{aligned} &|\widehat{F}^{P_k}(\exp(u-v)A_k, (v+t_k) \cdot \ell_{k|\mathfrak{p}_k}) - \widehat{F}^{P_k}(\exp(u-v)A_k, v \cdot q_k)| \\ &\leq \varphi(u-v) \|\exp((v+t_k)A_k) \cdot p_k - \exp(vA_k) \cdot q_k\| \\ &\leq \left( \left| \frac{r_k v^2}{2} \right| + |r_k v| + |\mu_k| + |\nu_k| \right) \varphi(u-v) \\ &\leq \left( \frac{r_k R_k^2}{2} + r_k R_k + |\mu_k| + |\nu_k| \right) \varphi(u-v), \end{aligned}$$

for  $k$  large enough. Similarly for  $t \in I_{k,-}, s \in I_{k,-}$ . Since  $\lim_{k \rightarrow \infty} r_k R_k^2 = 0$ , Young's inequality implies that

$$\lim_{k \rightarrow \infty} \|\pi_{\ell_k}(F) - \tilde{\sigma}_{k,\overline{O}}(F)\|_{\text{op}} = 0,$$

$L_c^1(G_{5,4})$  being dense in  $C^*(G_{5,4})$  the theorem follows. ■

\*

• $2^{nd}$  case if  $d = 0$ . We suppose now that  $\lim_{k \rightarrow \infty} (d_k = -2\beta_k r_k) = 0$ .

**Definition 8.9.** Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k = \beta_k B_k^* + \mu_k U^* + \nu_k V^*})$  be a properly converging sequence in  $\Gamma_2^{5,4}$ . We can suppose that  $\lim_{k \rightarrow \infty} \beta_k = \beta_\infty$  exists in  $[-\infty, +\infty[$  and that  $\lim_{k \rightarrow \infty} \tilde{\mu}_k = \tilde{\mu}$ ,  $\lim_{k \rightarrow \infty} \tilde{\nu}_k = \tilde{\nu}$ . Let as before  $A_k := \tilde{\mu}_k A + \tilde{\nu}_k B$ ,  $B_k := -\tilde{\nu}_k A + \tilde{\mu}_k B$  and  $A_\ell = \tilde{\mu} A + \tilde{\nu} B$ ,  $B_\ell = -\tilde{\nu} A + \tilde{\mu} B$ . The limit set is given by  $L(\overline{\mathcal{O}}) = \mathbb{R}A_\ell^* + [\beta_\infty, \infty[B_\ell^*$  if  $\beta_\infty \in \mathbb{R}$  otherwise  $L(\overline{\mathcal{O}}) = \Gamma_0^1$ .

1. Let  $(\varepsilon_k)_k \subset \mathbb{R}_+$  be a decreasing sequence converging to 0 such that  $\lim_{k \rightarrow \infty} \frac{\varepsilon_k}{r_k} = +\infty$ ,

2. let

$$t_j^k := j \sqrt{\frac{2\varepsilon_k}{r_k}}, k \in \mathbb{N}, j \in \mathbb{Z}.$$

Then:

$$\begin{aligned} & \exp((t_j^k + s)A_k) \cdot p_k \\ &= \left( \beta_k + r_k \frac{(t_j^k + s)^2}{2} \right) B_k^* + (r_k(t_j^k + s))C^* + \mu_k U^* + \nu_k V^* \\ &= \left( \beta_k + j^2 \varepsilon_k - r_k j \sqrt{\frac{2\varepsilon_k}{r_k}} s + r_k \frac{s^2}{2} \right) B_k^* + \left( r_k j \sqrt{\frac{2\varepsilon_k}{r_k}} + r_k s \right) C^* + \mu_k U^* + \nu_k V^* \\ &= \left( \beta_k + j^2 \varepsilon_k - j \sqrt{2\varepsilon_k r_k} s + r_k \frac{s^2}{2} \right) B_k^* + (j \sqrt{2\varepsilon_k r_k} + r_k s) C^* + \mu_k U^* + \nu_k V^*. \end{aligned}$$

Let

$$p_j^k := \exp(t_j^k A_k) \cdot p_k = (\beta_k + j^2 \varepsilon_k) B_k^* + (j \sqrt{2\varepsilon_k r_k}) C^* + \mu_k U^* + \nu_k V^*, \tag{8.5}$$

$$q_j^k := (\beta_k + j^2 \varepsilon_k) B_k^* + j \sqrt{2\varepsilon_k r_k} C^*.$$

Let  $s \in [t_j^k, t_{j+1}^k[$ ,  $j \in \mathbb{Z}$ . Then for  $k$  large enough:  $\|\exp((t_j^k + s)A_k) \cdot p_k - \exp(sA_k) \cdot q_j^k\|$

$$\begin{aligned} &= \left| r_k \frac{s^2}{2} \right| + |r_k s| + |\mu_k| + |\nu_k| \\ &\leq \left| r_k \frac{(t_{j+1}^k - t_j^k)^2}{2} \right| + |r_k(t_{j+1}^k - t_j^k)| + |\mu_k| + |\nu_k| \\ &\leq \varepsilon_k + \sqrt{2r_k \varepsilon_k} + |\mu_k| + |\nu_k| \\ &< \varepsilon_k^{\frac{1}{2}}. \end{aligned} \tag{8.6}$$

**Definition 8.10.** Let for  $k \in \mathbb{N}$  and  $j \in \mathbb{Z}$ :

$$\begin{aligned} I_{k,j} &:= [t_j^k, t_{j+1}^k[, \\ I_k &:= \bigcup_{j \in J_k} I_{k,j}. \end{aligned}$$

**Lemma 8.11.** For all  $R > 0, j \in \mathbb{Z}$ , we have  $R + I_{k,j} \subset I_{k,j} \cup I_{k,j+1}$ , for  $k$  large enough.

**Proof.** This follows from the fact that  $\lim_{k \rightarrow \infty} (t_{j+1}^k - t_j^k) = \sqrt{\frac{2\varepsilon_k}{r_k}} = \infty$ . ■

**Definition 8.12.**

1. Let for  $k \in \mathbb{N}$  and  $j \in \mathbb{Z}$

$$\sigma_{k,j} := \text{ind}_P^{G_{5,4}} \chi_{\exp(\pm t_k A_k) \cdot q_j^k}$$

and let  $u_{k,j}$  be the unitary intertwining operator between  $\pi_{-j\sqrt{2\varepsilon_k r_k}} C^*$  and  $\sigma_{k,j}$ .

2. Let as before  $C_{\overline{\mathcal{O}}}$  be the  $C^*$ -algebra of all continuous bounded mappings from  $L(\overline{\mathcal{O}})$  into  $\mathcal{B}(L^2(\mathbb{R}))$ . Let for  $\phi \in C_{\overline{\mathcal{O}}}$ :

$$\begin{aligned} \check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &:= \sum_{j \in \mathbb{Z}} (M_{I_{k,j+1}} + M_{I_{k,j}}) \circ u_{k,j} \circ \sigma_{k,j}(\phi) \circ u_{k,j}^* \circ M_{I_{k,j}}, \\ \tilde{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &:= \frac{1}{2}(\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) + (\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi^*))^*). \end{aligned} \tag{8.7}$$

3. For  $a \in C^*(G_{5,4})$  let

$$\sigma_{k,\overline{\mathcal{O}}}(a) = \tilde{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{a}|_{L(\overline{\mathcal{O}})}).$$

The proof of the next proposition is similar to that of Proposition 2.4.

**Proposition 8.13.** The linear mappings  $\tilde{\sigma}_{k,\overline{\mathcal{O}}}, k \in \mathbb{N}$ , are bounded by 2.

**Theorem 8.14.** Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k = \beta_k B_k^* + \mu_k U^* + \nu_k V^*})_k$  be a properly converging sequence in  $\Gamma_2^{5,4}$  with the properties of Definition 8.9. Then for every  $a \in C^*(G_{5,4})$ ,

$$\lim_{k \rightarrow \infty} \|\pi_{\ell_k}(a) - \sigma_{k,\overline{\mathcal{O}}}(a)\|_{\text{op}} = 0.$$

**Proof.** Let  $F \in L_c^1(G_{5,4})$ . Then, for  $k$  large enough, we have by Lemma 8.11 that

$$\pi_{\ell_k}(F) = \sum_{j \in \mathbb{Z}} (M_{I_{k,j}} + M_{I_{k,j+1}}) \circ \pi_k(F) \circ M_{I_{k,j}}.$$

Therefore the kernel function  $F_{k,j}$  of the operator

$$(M_{I_{k,j}} + M_{I_{k,j+1}}) \circ (\pi_k(F) - \tilde{\sigma}_{k,j}(F)) \circ M_{I_{k,j}}$$

is given by:

$$\begin{aligned} F_{k,j}(s, t) &= 1_{I_{k,j} \cup I_{k,j+1}}(u + t_j^k)(\widehat{F}^{P_k}(u - v, (v + t_j^k) \cdot p_k) - \widehat{F}^{P_k}(s - t, v \cdot q_j^k)), \\ &\text{with } s = u + t_j^k, t = v + t_j^k. \end{aligned}$$

Since  $F \in L^1_c(G_{5,4})$  there exists a continuous function  $\varphi \geq 0$  on  $\mathbb{R}$  with compact support, such that

$$|\widehat{F}^{P_k}(s-t, p) - \widehat{F}^{P_k}(s-t, q)| \leq \varphi(s-t)\|p - q\|$$

for every  $k \in \mathbb{N}$  and every  $p, q \in \mathfrak{p}_k^*$ . Hence, by (8.6)

$$\begin{aligned} |F_{k,j}(s, t)| &\leq 1_{I_{k,j} \cup I_{k,j+1}}(u + t_j^k) 1_{I_{k,j}}(v + t_j^k) \\ &\quad \times |\widehat{F}^{P_k}(u - v, (v + t_j^k) \cdot p_k) - \widehat{F}^{P_k}(s - t, v \cdot q_j^k)| \\ &\leq \varphi(u - v)\|(v + t_j^k) \cdot p_k - v \cdot q_j^k\| \\ &\leq \varepsilon_k^{\frac{1}{2}} \varphi(u - v). \end{aligned}$$

It follows now from Young’s inequality and from the properties of the sequence that there exists a constant  $D = D_F > 0$  such that for  $k$  large enough:

$$\|\pi_{\ell_k}(F) - \tilde{\sigma}_{k, \overline{\mathcal{O}}}(F)\|_{\text{op}} \leq D\varepsilon_k^{\frac{1}{2}}. \tag{8.8}$$

Since  $L^1_c(G_{5,4})$  is dense in  $C^*(G_{5,4})$  and since the mappings  $\sigma_{k, \overline{\mathcal{O}}}$  are all bounded in  $k$  by a fixed constant, it follows that relation (8.8) also holds for  $a \in C^*(G_{5,4})$ . ■

**Remark 8.15.** Theorem 8.14 and Theorem 8.8 imply that  $C^*(G_{5,4})$  has the NCDL property.

### 9. The $C^*$ -algebra of the group $G_{5,6}$ .

Recall that the Lie algebra  $\mathfrak{g}_{5,6}$  is spanned by the basis  $\mathcal{B} = \{A, B, C, U, V\}$  equipped with the Lie brackets

$$[A, B] = C, [A, C] = U, [A, U] = V, [B, C] = V.$$

It has a one-dimensional centre  $\mathfrak{z} = \mathbb{R}V$ . The group  $G_{5,3}$  can be realized as  $\mathbb{R}^5$  with the Campbell-Baker-Hausdorff multiplication

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] - \frac{1}{24}[Y, [X, [X, Y]]], \quad X, Y \in \mathfrak{g}_{5,6},$$

$$\text{i.e. } (a, b, c, u, v) \cdot (a', b', c', u', v') =$$

$$\begin{aligned} &(a+a', b+b', c+c' + \frac{ab'}{2} - \frac{a'b}{2}, u+u' + \frac{ac'}{2} - \frac{a'c}{2} + \frac{a^2b'}{12} + \frac{a'^2b}{12} - \frac{a'ab}{12} - \frac{aa'b'}{12}, \\ &v+v' + \frac{au'}{2} - \frac{a'u}{2} + \frac{bc'}{2} - \frac{b'c}{2} + \frac{a'^2c}{12} + \frac{a^2c'}{12} - \frac{ab'^2}{12} - \frac{a'b^2}{12} + \frac{abb'}{12} + \frac{a'b'b}{12} - \frac{aa'c'}{12} \\ &\quad - \frac{a'ac}{12} + \frac{a^2a'b'}{24} - \frac{a'^2ab}{24}). \end{aligned}$$

We use the Euclidean scalar product on  $\mathfrak{g}_{5,6}$  to identify  $\mathfrak{g}_{5,6}^*$  with  $\mathfrak{g}_{5,6} = \mathbb{R}^5$  and we obtain the following expression for  $\text{Ad}^*(a, b, c, u, v)$ :

$$\begin{aligned} &\text{Ad}^*(a, b, c, u, v)(\alpha, \beta, \rho, \mu, \nu) = \\ &(\alpha - \rho b - \mu c - \mu \frac{ab}{2} - \nu u - \nu \frac{b^2}{2} - \nu \frac{ac}{2} - \nu \frac{a^2b}{12}, \beta + \rho a + \mu \frac{a^2}{2} - \nu c + \nu \frac{ab}{2} + \nu \frac{a^3}{12}, \\ &\quad \rho + \mu a + \nu b + \nu \frac{a^2}{2}, \mu + \nu a, \nu). \end{aligned}$$

We give now a description of the co-adjoint orbits:

The generic orbits: if  $\nu \neq 0$ . The orbit  $\mathcal{O}_\nu$  of the element  $\ell_\nu = (0, 0, 0, 0, \nu)$  is given by:

$$\mathcal{O}_\nu = \{(a, b, c, u, \nu), a, b, c, u \in \mathbb{R}\}.$$

The stabilizer of  $\ell_\nu$  is the set  $\mathfrak{g}_{5,6}(\ell_\nu) = \text{span}\{V\}$ , we denote by  $\Gamma_3^{5,6}$  the orbit space of this layer and we parametrize it by

$$\Gamma_3^{5,6} := \{\ell_\nu \equiv \nu, \nu \in \mathbb{R}^*\}. \tag{9.1}$$

Since  $G_{5,6}/V = F_4 = \exp(\mathfrak{f}_4)$  (where  $\mathfrak{f}_4$  the thread-like Lie algebra of dimension 4) we can decompose the orbit  $\mathfrak{g}_{5,6}^*/G_{5,6}$  and hence also the dual space  $\widehat{G}_{5,6}$ , into the disjoint union

$$\mathfrak{g}_{5,6}^*/G_{5,6} = \Gamma_3^{5,6} \cup \Gamma_2^4 \cup \Gamma_1^4 \cup \Gamma_0^4.$$

**Theorem 9.1.** *Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\nu_k})_k \subset \Gamma_3^{5,6}$  be a sequence, such that  $\lim_{k \rightarrow \infty} \nu_k = 0$ . Then  $\overline{\mathcal{O}}$  is properly converging and  $L(\overline{\mathcal{O}}) = \Gamma_2^4 \cup \Gamma_1^4 \cup \Gamma_0^4$ .*

**Proof.** This follows immediately from the fact that a generic orbit  $\mathcal{O}_\nu$  is affine linear and more precisely  $\mathcal{O}_\nu = \nu V^* + V^\perp$ . ■

### 9.1. The Fourier transform.

**Definition 9.2.** For  $\ell = (0, 0, 0, 0, \nu) = \nu V^* \in \Gamma_3^{5,6}$ , the abelian sub-algebra  $\mathfrak{p} = \text{span}\{C, U, V\}$  is a polarization at  $\ell$ . We realize then  $\pi_{\ell, P} = \pi_\ell$  as  $\pi_\ell := \text{ind}_P^{G_{5,6}} \chi_\ell$ . The Hilbert space  $L^2(G_{5,6}/P, \ell)$  is in fact isomorphic to  $L^2(\mathbb{R}^2)$ : let as before  $E: \mathbb{R}^2 \rightarrow G_{5,6}$ ,  $E(a, b) := \exp(aA)\exp(bB)$  and  $S = \exp(\mathbb{R}A)\exp(\mathbb{R}B) = E(\mathbb{R} \times \mathbb{R})$ . Then  $G = S \cdot P$  as topological product and the mapping  $U_E: L^2(G_{5,6}/P, \ell) \rightarrow L^2(\mathbb{R}^2)$  defined by  $U_E \xi(t) := \xi(E(t))$ ,  $t \in \mathbb{R}^2$ , is unitary. We identify now  $\pi_\ell$  with the corresponding representation on  $L^2(\mathbb{R}^2)$ . Let us compute the operator  $\pi_\ell(F)$  for  $F \in C^*(G_{5,6})$  explicitly. Now for  $(a', b'), (a, b) \in \mathbb{R}^2$ ,

$$\begin{aligned} & E(a', b')E(a, b)^{-1} \\ &= \exp(a'A)\exp(b'B)\exp(-bB)\exp(-aA) \\ &= \exp((a' - a)A)\exp(aA)\exp((b' - b)B)\exp(-aA) \\ &= \exp((a' - a)A)\exp((b' - b)B + a(b' - b)C + \frac{1}{2}a^2(b' - b)U + \frac{1}{6}a^3(b - b)V) \\ &= E(a' - a, b' - b)\exp(a(b' - b)C + \frac{1}{2}a^2(b' - b)U + (\frac{1}{6}a^3(b' - b) - a(b' - b)^2)V) \end{aligned}$$

For  $\ell \in \mathfrak{g}^*$ ,  $\xi \in L^2(\mathbb{R}^2)$ ,  $t = (a', b') \in S, p \in P$  we have:

$$\begin{aligned}
 & \pi_\ell(F)\xi(t) \tag{9.2} \\
 &= \int_{G_{5,6}/P} \xi(s) \left( \int_P F(tps^{-1})e^{-2i\pi\langle s,\ell,p \rangle} dp \right) ds \text{ (where } s.\ell = \text{Ad}^*(s).\ell) \\
 &= \int_{G_{5,6}/P} \widehat{F}^P(ts^{-1}, s.\ell|_{\mathfrak{p}})\xi(s)ds \\
 &= \int_{G_{5,6}/P} \widehat{F}^P(a' - a, b' - b, (a, b).p) \\
 &\times e^{-2i\pi(\langle (a,b).\ell, \exp(a(b'-b)C + \frac{1}{2}a^2(b'-b)U + (\frac{1}{8}a^3(b'-b) - a(b'-b)^2)V) \rangle)} \xi(a, b)dadb.
 \end{aligned}$$

**Definition 9.3.** The Fourier transform  $\widehat{a} = \mathcal{F}(a)$  of an element  $a \in C^*(G_{5,6})$  is defined as to the field of bounded linear operators over the dual space of  $G_{5,6}$ . This gives us the set  $\Gamma_0^4 \cup \Gamma_1^4 \cup \Gamma_2^4 \cup \Gamma_3^{5,6}$  and we define for  $a \in C^*(G_{5,6})$  the operator field:

$$\begin{aligned}
 \widehat{a}(\nu) &= \mathcal{F}(a)(\nu) := \pi_\nu(a) \in \mathcal{K}(L^2(\mathbb{R}^2)), \nu \in \Gamma_3^{5,6}; \\
 \widehat{a}(\beta, \mu) &= \mathcal{F}(a)(\beta, \mu) := \pi_{\beta,\mu}(a) \in \mathcal{K}(L^2(\mathbb{R})), (\beta, \mu) \in \Gamma_2^4; \\
 \widehat{a}(\rho) &= \mathcal{F}(a)(\rho) := \pi_\rho(a) \in \mathcal{K}(L^2(\mathbb{R})), \rho \in \Gamma_1^4; \\
 \widehat{a}(\alpha, \beta) &= \mathcal{F}(a)(\alpha, \beta) := \langle \chi_{\alpha,\beta}, a \rangle, \alpha, \beta \in \mathbb{R}^2.
 \end{aligned}$$

**Theorem 9.4.** The mapping  $\Gamma_3^{5,6} \mapsto \mathcal{B}(L^2(\mathbb{R}^2)) : \ell \rightarrow \pi_\ell(F)$  is norm-continuous for all  $F \in C^*(G_{5,6})$ .

**Proof.** The proof is as before a consequence of Theorem 2.3. ■

**9.2. Passing from  $\Gamma_3^{5,6}$  to  $\Gamma_2^4 \cup \Gamma_1^4 \cup \Gamma_0^4$ .**

Let  $\overline{\mathcal{O}} = (\mathcal{O}_{\ell_k})_k \subset \Gamma_3^{5,6}$  be a properly converging sequence where  $\ell_k = (0, 0, 0, 0, \nu_k)$ ,  $k \in \mathbb{N}$  such that  $\lim_{k \rightarrow \infty} \nu_k = 0$ . Let  $p_k := (\ell_k)|_{\mathfrak{p}}$ . By Theorem 9.1 the restriction of the limit set  $L(\overline{\mathcal{O}})$  to  $\mathfrak{p}$  is the closed set  $L = L(\mathcal{O})|_{\mathfrak{p}} = \{(\rho, \mu, 0), \rho \in \mathbb{R}, \mu \in \mathbb{R}\}$ .

**Definition 9.5.** For  $k \in \mathbb{N}, i, j \in \mathbb{Z}$  let:

$$\begin{aligned}
 \varepsilon_k &:= |\nu_k|^{\frac{7}{9}}, \\
 I_{i,j}^k &:= \left\{ (c, u, \nu_k) \in p^*; i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} \leq c - \frac{j^2\varepsilon_k^2}{2\nu_k} < i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} + \varepsilon_k \text{ and } j\varepsilon_k \leq u < j\varepsilon_k + \varepsilon_k \right\}, \\
 U_{i,j}^k &:= \{(x, y) \in \mathbb{R}^2; (xA + yB) \cdot p_k \in I_{i,j}^k\}, j \in \mathbb{Z}.
 \end{aligned}$$

The sets  $I_{i,j}^k, i, j \in \mathbb{Z}$ , form a partition of  $\mathcal{O}_{k|_{\mathfrak{p}}}$ . Choose now a sequence  $R_k \subset \mathbb{R}$ , such that  $\lim_{k \rightarrow \infty} R_k = +\infty, \lim_{k \rightarrow \infty} R_k \varepsilon_k = 0$ . Let also for  $k \in \mathbb{N}, i, j \in \mathbb{Z}$  :

$$x_j^k := \frac{j\varepsilon_k}{\nu_k}, y_{i,j}^k := \frac{(x_j^k)^2}{2} + \frac{i\varepsilon_k}{\nu_k}, g_{i,j}^k = x_j^k A + y_{i,j}^k B.$$

Let for  $i, j \in \mathbb{Z}, k \in \mathbb{N}^*$  :

$$p_{i,j}^k := (i\varepsilon_k, j\varepsilon_k, 0).$$

An easy computation gives:

$$g_{i,j}^k \cdot p_k = (i\varepsilon_k, j\varepsilon_k, \nu_k) = p_{i,j}^k + (0, 0, \nu_k). \tag{9.3}$$

**Proposition 9.6.** *Let  $K$  be a compact subset, for  $k$  large enough*

$$KU_{i,j}^k \subset \bigcup_{i',j'=-1}^1 U_{i'+i,j'+j}^k =: V_{i,j}^k.$$

**Proof.** We can suppose that  $KP$  is contained in  $[-M, M]^2P$  for some  $M > 0$ . For  $r = (u, v) \in KP \subset G_{5,6}/P$  and  $s = (x, y) \in U^k$

$$(rs).p_k = (\nu_k v + \nu_k y + \nu_k x u + \nu_k \frac{x^2}{2} + \nu_k \frac{u^2}{2}, \nu_k x + \nu_k u, \nu_k)$$

and so for  $k$  large enough  $\nu_k|u| < \varepsilon_k, \nu_k|v| \leq \varepsilon_k$ . Hence

$$\begin{aligned} & (x, y) \in U_{i,j}^k \\ \Leftrightarrow & (xA + yB) \cdot p_k \in I_{i,j}^k \\ \Rightarrow & \begin{cases} j\varepsilon_k \leq \nu_k x < j\varepsilon_k + \varepsilon_k, \\ i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} \leq \nu_k y < i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} + \varepsilon_k, \end{cases} \\ \Rightarrow & \begin{cases} (j-1)\varepsilon_k \leq \nu_k x + \nu_k u < (j+1)\varepsilon_k + \varepsilon_k, \\ (i-1)\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} \leq \nu_k y + \nu_k v < (i+1)\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} + \varepsilon_k. \end{cases} \end{aligned}$$

It follows that  $KU_{i,j}^k \subset \bigcup_{i',j'=-1}^1 U_{i'+i,j'+j}^k$ . ■

**Definition 9.7.** For  $k \in \mathbb{N}^*$  Let

$$R^k = \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] \times \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right].$$

**Lemma 9.8.** *For  $k \in \mathbb{N}^*$  large enough, for any  $i, j \in \mathbb{Z}$ , the set  $U_{i,j}^k$  is contained in  $R^k + g_{i,j}^k$ .*

**Proof.** Let  $s = (x, y) \in U_{i,j}^k$  Then:

$$\begin{aligned} & (xA + yB) \cdot p_k \in I_{i,j}^k \\ \Leftrightarrow & \begin{cases} j\varepsilon_k \leq \nu_k x < j\varepsilon_k + \varepsilon_k \Rightarrow |x - x_j^k| \leq \frac{\varepsilon_k}{|\nu_k|} \Rightarrow x \in \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] + x_j^k, \\ i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} \leq \nu_k y < i\varepsilon_k - \frac{j^2\varepsilon_k^2}{2\nu_k} + \varepsilon_k \Rightarrow y \in \left[ -\frac{\varepsilon_k}{|\nu_k|}, \frac{\varepsilon_k}{|\nu_k|} \right] + y_{i,j}^k. \end{cases} \\ \Rightarrow & s \in R^k + g_{i,j}^k. \end{aligned} \quad \blacksquare$$

**Lemma 9.9.** For  $k \in \mathbb{N}^*$  large enough, for  $i, j \in \mathbb{Z}$  and any  $(x, y) \in U_{i,j}^k$

$$\|(xA + yB) \cdot p_k - ((xA + yB) \cdot (g_{i,j}^k)^{-1}) \cdot p_{i,j}^k\| \leq 4\varepsilon_k.$$

**Proof.** For  $(x, y) \in U_{i,j}^k$  we write according to the preceding Lemma:  $(x, y) = (x' + x_j^k, y' + y_{i,j}^k)$  with  $(x', y') \in R^k$ , i.e.  $|\nu_k x'| \leq \varepsilon_k$  and  $|\nu_k y'| \leq \varepsilon_k$ . Therefore by Relation 9.3 and the choice of  $\varepsilon_k = |\nu_k|^{7/9}$ :

$$\begin{aligned} & \|((x' + x_j^k)A + (y' + y_{i,j}^k)B) \cdot p_k - (x'A + y'B) \cdot p_{i,j}^k\| \\ &= \|(xA + yB) \cdot (\nu_k V^*)\| \\ &= \left\| \left( \nu_k y' + \nu_k \frac{x'^2}{2}, \nu_k x', \nu_k \right) \right\| \\ &= \left| \nu_k y' + \nu_k \frac{x'^2}{2} \right| + |\nu_k x'| + |\nu_k| \\ &\leq \varepsilon_k + \frac{\varepsilon_k^2}{2|\nu_k|} + \varepsilon_k + |\nu_k| \\ &\leq 4\varepsilon_k. \end{aligned}$$

■

**Definition 9.10.** For  $(\rho, \mu) \neq (0, 0) \in \mathbb{R}^2, \beta \in \mathbb{R}$ , let

$$\ell_{\beta,\rho,\mu} = \beta B^* + \rho C^* + \mu U^* \text{ and } \ell_{\beta,\mu} = \beta B^* + \mu U^* \in \mathfrak{g}_{5,6}^*.$$

The subalgebra  $\mathfrak{m} := \text{span}\{B, \mathfrak{p}\}$  is a polarization at  $\ell_{\rho,\mu}$  and also at  $\ell_{\beta,\rho,\mu}$ . Since both linear functionals are on the same orbit, we have the equivalent representations  $\pi_{\beta,\mu} = \text{ind}_M^{G_{5,6}} \chi_{\ell_{\beta,\mu}} \in \widehat{G}_{5,6}$  and  $\pi_{\rho,\mu} = \text{ind}_M^{G_{5,6}} \chi_{\ell_{\rho,\mu}}$ . Let  $u_{\beta,\rho,\mu}$  be the unitary operator which gives the equivalence between both representations. For the 0 functional on  $\mathfrak{m}$ , the representation  $\text{ind}_M^{G_{5,6}} \chi_0$  is a direct integral of the unitary characters of  $G_{5,6}$ . For  $(\mu, \rho) \neq (0, 0)$  we take the direct integral representation

$$\tau_{\rho,\mu} := \left( \int_{\mathbb{R}}^{\oplus} \pi_{\ell_{\beta,\rho,\mu}} d\beta, \int_{\mathbb{R}}^{\oplus} L^2(G_{5,6}/P, \chi_{\ell_{\beta,\rho,\mu}}) d\beta \right) \tag{9.4}$$

resp.

$$\tau_{0,0} := \int_{\mathbb{R}^2}^{\oplus} \chi_{\alpha,\beta} d\beta d\alpha. \tag{9.5}$$

This representation  $\tau_{\rho,\mu}$  is in fact equivalent to the representation  $\sigma_{\rho,\mu} := \text{ind}_P^{G_{5,6}} \chi_{\ell_{\rho,\mu}}$  and a unitary intertwining operator  $U_{\beta,\mu}$  is given by:

$$\begin{aligned} U_{\rho,\mu} : L^2(G_{5,6}/P, \chi_{\rho,\mu}) &\mapsto \int_{\mathbb{R}}^{\oplus} L^2(G_{5,6}/P, \chi_{\ell_{\rho,\mu}}) d\beta, \\ U_{\rho,\mu}(\xi)(\beta)(g) : &= \int_{\mathbb{R}} \xi(g \exp(sB)) e^{-2i\pi s\beta} ds, g \in G_{5,6}, \beta \in \mathbb{R}. \end{aligned}$$

Hence for every  $a \in C^*(G_{5,6})$

$$\|\sigma_{\rho,\mu}(a)\|_{\text{op}} = \sup_{\beta \in \mathbb{R}} \|\pi_{\ell_{\beta,\rho,\mu}}(a)\|_{\text{op}}. \tag{9.6}$$

Let  $C_{\overline{\mathcal{O}}} = CB(L(\overline{\mathcal{O}}), \mathcal{B}(L^2(\mathbb{R}^2)))$  be the  $C^*$ -algebra of all continuous, uniformly bounded mappings  $\phi : L(\overline{\mathcal{O}}) \mapsto \mathcal{B}(L^2(\mathbb{R}^2))$  from the locally compact space

$L(\overline{\mathcal{O}})$  into the algebra of bounded linear operators  $\mathcal{B}(L^2(\mathbb{R}^2))$  on the Hilbert space  $L^2(\mathbb{R}^2)$ . For  $\ell \in V^\perp \subset \mathfrak{g}_{5,6}^*$  we obtain a representation  $\tilde{\sigma}_\ell$  on the Hilbert space  $L^2(\mathbb{R}^2)$  of the algebra  $C_{\overline{\mathcal{O}}}$  defined by:

$$\tilde{\sigma}_\ell(\phi)\xi := U_\ell^{-1} \left( \int_{\mathbb{R}}^\oplus u_{\beta,\rho,\mu}^* \circ \phi(\beta, \mu) \circ u_{\beta,\rho,\mu}(U_\ell(\xi)(\beta)) d\beta \right), \phi \in C_{\overline{\mathcal{O}}}. \tag{9.7}$$

**Definition 9.11.** Similarly to Definition 7.6, define for  $k \in \mathbb{N}$  and  $\phi \in C_{\overline{\mathcal{O}}} = CB(L(\overline{\mathcal{O}}))$ : the linear operators  $\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi)$  and  $\tilde{\sigma}_{k,\overline{\mathcal{O}}}(\phi)$  by

$$\begin{aligned} \check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &:= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} U_E \circ M_{V_{i,j}^k} \circ \tilde{\sigma}_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k}(\phi) \circ M_{U_{i,j}^k} \circ U_E^* \\ \tilde{\sigma}_{k,\overline{\mathcal{O}}}(\phi) &= \frac{1}{2}(\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi) + (\check{\sigma}_{k,\overline{\mathcal{O}}}(\phi^*))^*) \end{aligned} \tag{9.8}$$

where  $\tilde{\sigma}_\ell$  for  $\ell = (g_{i,j}^k)^{-1} \cdot p_{i,j}^k$ , is as in Equation (9.7). For  $a \in C^*(G_{5,6})$

$$\check{\sigma}_{k,\overline{\mathcal{O}}}(\widehat{a}|_{L(\overline{\mathcal{O}})}) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} U_E \circ M_{V_{i,j}^k} \circ \sigma_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k, \mathfrak{p}}(a) \circ M_{U_{i,j}^k} \circ U_E^*, \tag{9.9}$$

where as before  $\sigma_{\ell, \mathfrak{p}} = \text{ind}_P^{G_{5,6}} \chi_\ell, \ell \in \mathfrak{g}^*$ .

The proof of the next theorem is similar to that of Theorem 7.10.

**Theorem 9.12.** *Let  $a \in C^*(G_{5,6})$ . Then:*

$$\lim_{k \rightarrow \infty} \|\pi_{\ell_k}(a) - \sigma_{k,\overline{\mathcal{O}}}(a)\|_{\text{op}} = 0.$$

**Proof.** Take first  $F \in L_c^1(G_{5,6})$ . Let us choose a compact subset  $K \subset \mathfrak{p}^*$  and an  $M > 0$  such that the function  $\mathbb{R}^2 \times \mathfrak{p}^* \ni ((x, y), p) \rightarrow \widehat{F}^P(E(x, y), p)$  is supported in  $[-M, M]^2 \times K$ . By Proposition 7.4 we have for  $k$  large enough:

$$\pi_{\ell_k}(F) \circ M_{U_{i,j}^k} = M_{V_{i,j}^k} \circ \pi_{\ell_k}(F) \circ M_{U_{i,j}^k}, \quad i, j \in \mathbb{Z}.$$

Let us write according to relation (2.2):

$$\begin{aligned} &M_{V_{i,j}^k} \circ \pi_{\ell_k}(F) \circ M_{U_{i,j}^k} - M_{V_{i,j}^k} \circ \sigma_{(g_{i,j}^k)^{-1} \cdot p_{i,j}^k, \mathfrak{p}}(F) \circ M_{U_{i,j}^k} \\ &= T_{g_{i,j}^k}^{-1} \circ (M_{V_{i,j}^k - g_{i,j}^k} \circ \pi_{g_{i,j}^k \cdot \ell_k}(F) \circ M_{U_{i,j}^k - g_{i,j}^k} - \\ &\quad - M_{V_{i,j}^k - g_{i,j}^k} \circ \sigma_{p_{i,j}^k, \mathfrak{p}}(F) \circ M_{U_{i,j}^k - g_{i,j}^k}) \circ T_{g_{i,j}^k}. \end{aligned}$$

The kernel function  $F_k$  of the operator

$$M_{V_{i,j}^k - g_{i,j}^k} \circ \pi_{g_{i,j}^k \cdot \ell_k}(F) \circ M_{U_{i,j}^k - g_{i,j}^k} - M_{V_{i,j}^k - g_{i,j}^k} \circ \sigma_{p_{i,j}^k, \mathfrak{p}}(F) \circ M_{U_{i,j}^k - g_{i,j}^k}$$

is given by

$$F_k(s, t) = 1_{V_{i,j}^k - g_{i,j}^k}(s) 1_{U_{i,j}^k - g_{i,j}^k}(t) \left( \widehat{F}^P(st^{-1}, t \cdot g_{i,j}^k \cdot p_k) - \widehat{F}^P(st^{-1}, t \cdot p_{i,j}^k) \right)$$

Since the function  $(s, p) \rightarrow |\widehat{F}^P(s, p)|^2$  is in  $C_c^\infty(G_{5,6}/P, \mathfrak{p}^*)$  there exists a non-negative continuous function with compact support  $\varphi : G_{5,6}/P \rightarrow \mathbb{R}_+$  such that for any  $q, p \in \mathfrak{p}^*$ ,  $s \in G_{5,6}/P$  :

$$|\widehat{F}^P(s, q) - \widehat{F}^P(s, p)| \leq \varphi(s) \|q - p\|.$$

It follows then from Relation (9.2), Lemma (9.6), Lemma 9.8 and Lemma 9.9 that for  $k \in \mathbb{N}$  large enough,  $i, j \in \mathbb{Z}$ ,  $a, b, a', b' \in \mathbb{R}$  :

$$\begin{aligned} & |F_k(E(a', b'), E(a, b))| \\ & \leq \left| \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), (a, b) \cdot g_{i,j}^k \cdot p_k) - \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), p_{i,j}^k) \right| \\ & + \left| \widehat{F}^{\mathfrak{p}}((a' - a, b' - b), (a, b) \cdot g_{i,j}^k \cdot p_k) \right. \\ & \times \left. | (e^{-2i\pi((\ell_{\nu_k}, (a(b'-b)C + (a + \frac{1}{2}a^2(b'-b))U + (\frac{1}{6}a^3(b'-b) - b - a + \frac{1}{2}a^2 - a(b'-b)^2)V)} - 1) \right| \\ & \leq (4\varepsilon_k + |\langle \nu_k, (\frac{1}{6}a^3(b'-b) - b - a + \frac{1}{2}a^2 - a(b'-b)^2)V \rangle|) \varphi(a' - a, b' - b) \\ & \leq (4\varepsilon_k + r \frac{|\nu_k| \varepsilon_k^4}{|\nu_k|^4}) \varphi(a' - a, b' - b) \\ & \leq (4\varepsilon_k + r |\nu_k|^{1/9}) \varphi(a' - a, b' - b), \end{aligned}$$

for some number  $r > 0$  independent of  $k$ . Using now Young’s estimate and Proposition 2.4, we see that there exists a constant  $D = D_F$ , such that for  $k$  large enough:

$$\|\pi_{\ell_k}(F) - \tilde{\sigma}_{k, \overline{O}}(F)\|_{\text{op}} \leq D |\nu_k|^{1/9}.$$

The theorem follows from the fact  $L_c^1(G_{5,6})$  is dense in  $C^*(G_{5,6})$ . ■

### 10. The final result.

We have treated now all simply connected, connected indecomposable Lie groups of dimension  $\leq 5$ . The other simply connected connected groups of dimension  $\leq 5$  are of the form  $G_1 \times \mathbb{R}^d$ , with  $G_1$  indecomposable and  $\dim(G_1) + d \leq 5$ . It is easy to extend our methods to these groups to. We have thus established the following theorem:

**Theorem 10.1.** *The  $C^*$ -algebra of every connected nilpotent Lie group of dimension  $\leq 5$  has norm controlled dual limits.*

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