Journal of Lie Theory
Volume **25** (2015) 717–732

Classification of a Class of Nonrigid Carnot Groups

Michael R. Hughes, Mihai D. Staic, and Xiangdong Xie*

Communicated by G. Mauceri

Abstract. We classify up to isomorphism a class of nonrigid Carnot groups. As an application we obtain quasiisometric classification of a class of finitely generated nilpotent groups. We also identify all C^2 quasiconformal maps of these nonrigid Carnot groups.
Mathematics Subject Classification 2010: 22E25, 20F65, 30C65.
Key Words and Phrases: Nonrigid Carnot groups, complex Heisenberg product groups.

1. Introduction

Carnot groups (with Carnot metrics) have arisen in many branches of mathematics. They are the simplest sub-Riemannian manifolds, and arise as the tangent cones of general sub-Riemannian manifolds [M]. They are also (one-point complement of) ideal boundaries of certain negatively curved homogeneous manifolds [H]. The rigidity property of Carnot groups has direct consequences for the rigidity property of these negatively curved homogeneous manifolds and certain finitely generated solvable groups. Furthermore, Carnot groups also arise as the asymptotic cones of finitely generated nilpotent groups [G] and are closely related to the quasiisometric classification question of finitely generated nilpotent groups.

Recent progresses suggest that Carnot groups are more rigid than previously thought with respect to quasiconformal maps. On one hand, in [X1] it was proved that all quasiconformal maps on nonrigid Carnot groups (except Euclidean groups, Heisenberg product groups and complex Heisenberg product groups) are biLipschitz. On the other hand, Michael Cowling and Alessandro Ottazzi [CO] have shown that all C^2 quasiconformal maps on rigid Carnot groups are affine maps, so in particular they are biLipschitz. It is known that there are non-biLipschitz quasiconformal maps on Euclidean groups and Heisenberg groups [B]. The general Heisenberg product groups and complex Heisenberg product groups are the least understood in this respect. In [CR], Cowling and Riemann studied

*M. Hughes and X. Xie were partially supported by NSF grant DMS–1265735. M. Staic was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0635, contract nr. 253/5.10.2011.

ISSN 0949–5932 / $2.50

quasiconformal maps of the product of two copies of the Heisenberg group under the assumption that the quasiconformal maps have continuous differentials. In [RR], Reimann and Ricci showed that C^2 contact maps on the complex Heisenberg group must be holomorphic. In this paper we take a closer look at a class of complex Heisenberg product groups.

Let $H^1_{\mathbb{C}}$ be the Heisenberg Lie algebra over $\mathbb{C}$ (that is the three dimensional Lie algebra with generators X, Y and Z and the only nontrivial relation $[X,Y] = Z$). We consider the induced real Lie algebra structure on $H^1_{\mathbb{C}}$ with generators $X_1 = X$, $X_2 = iX$, $Y_1 = Y$, $Y_2 = iY$, $Z_1 = Z$ and $Z_2 = iZ$. The nontrivial relations are: $[X_1, Y_1] = Z_1 = [Y_2, X_2]$ and $[X_1, Y_2] = [X_2, Y_1] = Z_2$. We denote by V_1 the real vector subspace spanned by X_1, X_2, Y_1 and Y_2, and by V_2 the real vector subspace spanned by Z_1 and Z_2. Notice that $H^1_{\mathbb{C}} = V_1 \oplus V_2$.

Suppose that $H^1_{\mathbb{C}}$ and $\widetilde{H}^1_{\mathbb{C}}$ are two copies of the complex Heisenberg Lie algebra, and $f : V_2 \to \widetilde{V}_2$ is an isomorphism of $\mathbb{R}$-vector spaces. Then one can see that $I_f = \{(x, -f(x)) \mid x \in V_2\}$ is an ideal in $H^1_{\mathbb{C}} \oplus \widetilde{H}^1_{\mathbb{C}}$. We denote by $\mathfrak{R}_f$ the quotient Lie algebra $(H^1_{\mathbb{C}} \oplus \widetilde{H}^1_{\mathbb{C}})/I_f$. Notice that $\mathfrak{R}_f = W_1 \oplus W_2$, where $W_1 = V_1 \oplus \widetilde{V}_1$ and $W_2 = (V_2 \oplus \widetilde{V}_2)/I_f$. So $\mathfrak{R}_f$ is a 2-step Carnot Lie algebra. Let R_f denote the connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{R}_f$. Then R_f is a 2-step Carnot group. Furthermore, it is a nonrigid Carnot group, see Section 2. The class of nonrigid Carnot groups we shall study is

$$\big\{R_f | f : V_2 \to \widetilde{V}_2 \text{ is a real vector space isomorphism}\big\}.$$

For $a \in (0, 1]$ we consider $f_a : V_2 \to \widetilde{V}_2$ given by $f_a(Z_1) = Z_1$ and $f_a(Z_2) = aZ_2$. We will denote $\mathfrak{R}_{f_a}$ by $\mathfrak{R}_a$ and R_{f_a} by R_a.

Our first result is a classification of this class of nonrigid Carnot groups up to isomorphism.

Theorem 1.1. *Every Carnot group R_f is isomorphic to R_a for some $a \in (0, 1]$. Furthermore, R_a and R_b ($a, b \in (0, 1]$) are isomorphic if and only if $a = b$.*

A natural question is whether there is a similar classification result for Carnot groups obtained from $m \geq 2$ copies of the n-th ($n \geq 2$) complex Heisenberg group by identifying the second layers through real linear isomorphisms. We believe that the methods in this paper can be useful in proving such a result.

As an application of the above result, we obtain quasiisometric classification of a class of finitely generated nilpotent groups. Let p, q be two relatively prime integers such that $0 < p \leq q$. Define a group $G_{p,q}$ by presentation $G_{p,q} = < S | R >$, where $S = \{x_1, y_1, x_2, y_2, \tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, z_1, z_2\}$ is the set of generators and R is the set of relators consisting of

$$[x_1, y_1] = z_1^{2q}, [x_1, y_2] = z_2^{2p}, [x_2, y_1] = z_2^{2p}, [x_2, y_2] = z_1^{-2q},$$

$$[\tilde{x}_1, \tilde{y}_1] = z_1^{2q}, [\tilde{x}_1, \tilde{y}_2] = z_2^{2q}, [\tilde{x}_2, \tilde{y}_1] = z_2^{2q}, [\tilde{x}_2, \tilde{y}_2] = z_1^{-2q},$$

$$[z_j, x] = e \text{ for } j = 1, 2 \text{ and any } x \in S,$$

$$[x_1, x_2] = e, \ [y_1, y_2] = e, \ [\tilde{x}_1, \tilde{x}_2] = e, \ [\tilde{y}_1, \tilde{y}_2] = e,$$

$$[x, y] = e \text{ for any } x \in \{x_1, x_2, y_1, y_2\};\ y \in \{\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2\}.$$

It is easy to see that $G_{p,q}$ is a nilpotent group. In fact, $G_{p,q}$ is a lattice in $R_{\frac{p}{q}}$, see Lemma 4.1.

Theorem 1.2. *Let $0 < p \leq q$, $0 < p' \leq q'$ be integers such that $(p,q) = (p',q') = 1$. Then $G_{p,q}$ and $G_{p',q'}$ are quasiisometric if and only if $p = p'$ and $q = q'$.*

We also study quasiconformal maps on R_a. It turns out that there are very few C^2 quasiconformal maps on R_a. See Section 2 for the definition of graded isomorphism.

Theorem 1.3. *Every C^2 quasiconformal map $F : R_a \to R_a$ is an affine map; to be more precise, every C^2 quasiconformal map $F : R_a \to R_a$ is the composition of a left translation and a graded isomorphism.*

Conjecturally the above statement holds for all quasiconformal maps, but we are unable to remove the C^2 assumption at this point.

2. Preliminaries

In this Section we collect definitions and results that shall be needed later. We first recall the basic definitions related to Carnot groups in Subsection 2. Then we review the BCH formula (Subsection 2), the definition of quasiconformal maps and Pansu differentiability theorem (Subsection 2).

2.1. Carnot algebras and Carnot groups.

A *Carnot Lie algebra* is a finite dimensional Lie algebra $\mathcal{G}$ over $\mathbb{R}$ together with a direct sum decomposition $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ of non-trivial vector subspaces such that $[V_1, V_i] = V_{i+1}$ for all $1 \leq i \leq r$, where we set $V_{r+1} = \{0\}$. The integer r is called the degree of nilpotency of $\mathcal{G}$. Every Carnot algebra $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ admits a one-parameter family of automorphisms $\lambda_t : \mathcal{G} \to \mathcal{G}$, $t \in (0, \infty)$, where $\lambda_t(x) = t^i x$ for $x \in V_i$. Let $\mathcal{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and $\mathcal{G}' = V_1' \oplus V_2' \oplus \cdots \oplus V_s'$ be two Carnot algebras. A Lie algebra homomorphism $\phi : \mathcal{G} \to \mathcal{G}'$ is graded if ϕ commutes with λ_t for all $t > 0$; that is, if $\phi \circ \lambda_t = \lambda_t \circ \phi$. We observe that $\phi(V_i) \subset V_i'$ for all $1 \leq i \leq r$.

A connected and simply connected nilpotent Lie group is a *Carnot group* if its Lie algebra is a Carnot algebra. Let G be a Carnot group with Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$. The subspace V_1 defines a left invariant distribution $HG \subset TG$ on G. We fix a left invariant inner product on HG. An absolutely continuous curve γ in G whose velocity vector $\gamma'(t)$ is contained in $H_{\gamma(t)}G$ for almost every t is called a horizontal curve. By Chow's theorem ([BR, Theorem 2.4]), any two points of G can be connected by horizontal curves. Let $p, q \in G$, the *Carnot metric* $d_c(p,q)$ between them is defined as the infimum of length of horizontal curves that join p and q.

Since the inner product on HG is left invariant, the Carnot metric on G is also left invariant. Different choices of inner product on HG result in Carnot metrics that are biLipschitz equivalent.

Recall that, for a connected and simply connected nilpotent Lie group G with Lie algebra $\mathcal{G}$, the exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism. Under this identification the Lesbegue measure on $\mathcal{G}$ is a Haar measure on G. Furthermore, the exponential map induces a one-to-one correspondence between Lie subalgebras of $\mathcal{G}$ and connected Lie subgroups of G.

Let G be a Carnot group with Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$. Since $\lambda_t : \mathcal{G} \to \mathcal{G}$ $(t > 0)$ is a Lie algebra automorphism and G is simply connected, there is a unique Lie group automorphism $\Lambda_t : G \to G$ whose differential at the identity is λ_t. For each $t > 0$, Λ_t is a similarity with respect to the Carnot metric: $d(\Lambda_t(p), \Lambda_t(q)) = t\, d(p, q)$ for any two points $p, q \in G$. A Lie group homomorphism $f : G \to G'$ between two Carnot groups is a graded homomorphism if it commutes with Λ_t for all $t > 0$; that is, if $f \circ \Lambda_t = \Lambda_t \circ f$. Notice that, a Lie group homomorphism $f : G \to G'$ between two Carnot groups is graded if and only if the corresponding Lie algebra homomorphism is graded.

A C^2 map $F : U \to V$ between open subsets of Carnot groups is *contact* if the differential sends horizontal subspaces into horizontal subspaces. A Carnot group G is *rigid* if the space of C^2 contact maps is finite dimensional, and *nonrigid* otherwise. A Carnot group G with Lie algebra $\mathcal{G}$ is nonrigid if and only if there exists some nonzero X in the complexified Lie algebra $\mathcal{G} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathrm{ad}(X)$ has rank at most one, see [OW]. In the proof of Theorem 1.1 (see Section 3) it will be shown that $\mathfrak{R}_f \otimes_{\mathbb{R}} \mathbb{C}$ contains elements with rank one. Hence R_f is a nonrigid Carnot group.

2.2. The Baker-Campbell-Hausdorff formula.

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathcal{G}$. The exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism. One can then pull back the group operation from G to get a group structure on $\mathcal{G}$. This group structure can be described by the Baker-Campbell-Hausdorff formula (BCH formula in short), which expresses the product $X * Y$ $(X, Y \in \mathcal{G})$ in terms of the iterated Lie brackets of X and Y. The group operation in G will be denoted by $\cdot$. The pull-back group operation $*$ on $\mathcal{G}$ is defined as follows. For $X, Y \in \mathcal{G}$, define

$$X * Y = \exp^{-1}(\exp X \cdot \exp Y).$$

Then the first few terms of the BCH formula ([CG], page 11) is given by:

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots .$$

Let $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ be a Carnot Lie algebra and $\pi_1 : \mathcal{G} \to V_1$ be the projection onto V_1. Then it is easy to check using the BCH formula that π_1 is a group homomorphism, where the group operation on V_1 is vector addition.

2.3. Quasiconformal maps and Pansu differentiability theorem.

Here we recall the definition of quasiconformal maps and Pansu differentiability theorem.

Let $F : (X_1, d_1) \to (X_2, d_2)$ be a homeomorphism between two metric spaces. For $x \in X_1$ and $t > 0$, define

$$H_F(x,t) = \frac{\sup\{d_2(F(x'), F(x)) | d_1(x', x) \le t\}}{\inf\{d_2(F(x'), F(x)) | d_1(x', x) \ge t\}}.$$

The map F is called λ-quasiconformal if $\limsup_{t\to 0} H_F(x,t) \le \lambda$ for all $x \in X$. We say F is quasiconformal if it is λ-quasiconformal for some $\lambda \ge 1$.

Definition 2.1. Let G and G' be two Carnot groups endowed with Carnot metrics d and d' respectively, and $U \subset G$, $U' \subset G'$ open subsets. A map $F : U \to U'$ is *Pansu differentiable* at $x \in U$ if there exists a graded homomorphism $L : G \to G'$ such that

$$\lim_{y\to x} \frac{d'(F(x)^{-1} \cdot F(y),\, L(x^{-1} \cdot y))}{d(x,y)} = 0.$$

In this case, the graded homomorphism $L : G \to G'$ is called the *Pansu differential* of F at x, and is denoted by $dF(x)$.

The following result (except the terminology) is due to Pansu [P].

Theorem 2.2. *Let G, G' be Carnot groups, and $U \subset G$, $U' \subset G'$ open subsets. Let $F : U \to U'$ be a quasiconformal map. Then F is a.e. Pansu differentiable. Furthermore, at a.e. $x \in U$, the Pansu differential $dF(x) : G \to G'$ is a graded isomorphism.*

In Theorem 2.2 and the proofs below, "a. e." is with respect to the Lesbegue measure on $\mathcal{G} = G$.

3. Classification

The goal of this Section is to prove Theorem 1.1. Since two connected and simply connected Lie groups are isomorphic if and only if their Lie algebras are isomorphic, it suffices to classify the Lie algebras $\mathfrak{N}_f$ up to isomorphism.

Lemma 3.1. *Suppose that $u : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ and $v : \widetilde{H}^1_{\mathbb{C}} \to \widetilde{H}^1_{\mathbb{C}}$ are graded isomorphism of Carnot Lie algebras and $f : V_2 \to \widetilde{V}_2$ is an isomorphism of $\mathbb{R}$-vector spaces. Then there is a graded isomorphism $\mathfrak{N}_f \cong \mathfrak{N}_{vfu}$, where $vfu = (v|_{\widetilde{V}_2}) \circ f \circ (u|_{V_2})$.*

Proof. Define the map $\phi : H^1_{\mathbb{C}} \oplus \widetilde{H}^1_{\mathbb{C}} \to H^1_{\mathbb{C}} \oplus \widetilde{H}^1_{\mathbb{C}}$ by $\phi(x,y) = (u^{-1}(x), v(y))$. One can easily check that ϕ is a graded isomorphism and $\phi(I_f) = I_{vfu}$, which induces a graded isomorphism $\mathfrak{N}_f \cong \mathfrak{N}_{vfu}$. ■

Remark 3.2. If the linear map f is the identity, then $\mathfrak{N}_{id}$ is the five dimensional complex (with dimension ten over the reals) Heisenberg algebra $H^2_{\mathbb{C}}$.

It follows from the Singular Value Decomposition Theorem for matrices (see for example [L], Section 7.4) that for every $A \in GL_2(\mathbb{R})$ there exist U, $V \in O_2(\mathbb{R})$, and $\lambda > 0$, $a \in (0,1]$ such that $A = (\lambda U)DV$ where $D = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$. Next we observe that for any $\lambda > 0$ and any $U \in O_2(\mathbb{R})$, there exists a graded automorphism $u : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ such that $u|_{V_2} : V_2 \to V_2$ has matrix representation λU with respect to the basis $\{Z_1, Z_2\}$. For $\lambda = 1$ and $U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, the corresponding graded automorphism is the complex conjugation $\tau_0 : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ given by $\tau_0(X_1) = X_1$, $\tau_0(X_2) = -X_2$, $\tau_0(Y_1) = Y_1$, $\tau_0(Y_2) = -Y_2$, $\tau_0(Z_1) = Z_1$, $\tau_0(Z_2) = -Z_2$. For $\lambda > 0$ and $U = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, the corresponding graded automorphism $\psi : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ is given by $\psi(X_1) = \lambda(\cos t\, X_1 + \sin t\, X_2)$, $\psi(X_2) = \lambda(-\sin t\, X_1 + \cos t\, X_2)$, $\psi(Y_1) = Y_1$, $\psi(Y_2) = Y_2$, $\psi(Z_1) = \lambda(\cos t\, Z_1 + \sin t\, Z_2)$, $\psi(Z_2) = \lambda(-\sin t\, Z_1 + \cos t\, Z_2)$.

If we combine Lemma 3.1 with the above discussion we get the following proposition:

Proposition 3.3. *For any $\mathbb{R}$-linear isomorphism $f : V_2 \to \widetilde{V}_2$ of vector spaces there exists $a \in (0,1]$ such that $\mathfrak{R}_f \cong \mathfrak{R}_a$.*

We will dedicate the rest of this section to proving that for a, $b \in (0,1]$ we have $\mathfrak{R}_a \cong \mathfrak{R}_b$ if and only if $a = b$.

Notice that $\mathfrak{R}_a$ is a ten dimensional real vector space with basis X_i^a, Y_i^a, $\widetilde{X}_i^a$, $\widetilde{Y}_i^a$, and Z_i^a for $i \in \{1,2\}$. The identification is given by $X_i^a = (X_i, 0)$, $Y_i^a = (Y_i, 0)$, $\widetilde{X}_i^a = (0, \widetilde{X}_i)$, $\widetilde{Y}_i^a = (0, \widetilde{Y}_i)$, $Z_1^a = (Z_1, 0) = (0, \widetilde{Z}_1)$ and $Z_2^a = (a^{-1}Z_2, 0) = (0, \widetilde{Z}_2)$. The only nontrivial brackets are given in Table 1.

	Y_1^a	Y_2^a	$\widetilde{Y}_1^a$	$\widetilde{Y}_2^a$
X_1^a	Z_1^a	aZ_2^a	0	0
X_2^a	aZ_2^a	$-Z_1^a$	0	0
$\widetilde{X}_1^a$	0	0	Z_1^a	Z_2^a
$\widetilde{X}_2^a$	0	0	Z_2^a	$-Z_1^a$

Table 1: Bracket relations in $\mathfrak{R}_a$

We denote by W_1^a the vector subspace of $\mathfrak{R}_a$ spanned by X_i^a, $\widetilde{X}_i^a$, Y_i^a and $\widetilde{Y}_i^a$, $i = 1, 2$, and by W_2^a the vector subspace spanned by Z_1^a, Z_2^a. Notice that $\mathfrak{R}_a = W_1^a \oplus W_2^a$ and $[W_1^a, W_1^a] = W_2^a$. So $\mathfrak{R}_a$ is a 2-step Carnot algebra.

Lemma 3.4. *Let $\mathcal{G} = V_1 \oplus V_2$ and $\widetilde{\mathcal{G}} = \widetilde{V}_1 \oplus \widetilde{V}_2$ be 2-step Carnot Lie algebras. If $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are isomorphic, then there exists a graded isomorphism between them.*

Proof. Let $f : \mathcal{G} \to \widetilde{\mathcal{G}}$ be a Lie algebra isomorphism. Since $[\mathcal{G}, \mathcal{G}] = V_2$ and $[\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}] = \widetilde{V}_2$, we see that $f|_{V_2} : V_2 \to \widetilde{V}_2$ is a linear isomorphism onto $\widetilde{V}_2$. In general, $f(V_1) \not\subset \widetilde{V}_1$. For $i = 1, 2$, let $\pi_i : \widetilde{\mathcal{G}} \to \widetilde{V}_i$ be the projection onto $\widetilde{V}_i$.

Set $f_i := \pi_i \circ (f|_{V_1})$. Then $f|_{V_1} = f_1 + f_2$. Notice that $f_1 : V_1 \to \widetilde{V}_1$ is a linear isomorphism. Now define a linear isomorphism $h : \mathcal{G} \to \widetilde{\mathcal{G}}$ by

$$h(v_1 + v_2) = f_1(v_1) + f(v_2), \text{ for } v_1 \in V_1,\ v_2 \in V_2.$$

Since f is a Lie algebra isomorphism, $f(v_1 + v_2) = h(v_1 + v_2) + f_2(v_1)$ and $[f_2(V_1), \widetilde{\mathcal{G}}] = 0$, it is easy to check that h is a Lie algebra isomorphism. By definition, h is graded. ∎

If $\mathfrak{g}$ is a real Lie algebra, we consider the complex Lie algebra structure on $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ defined by $[T \otimes_{\mathbb{R}} z_1, U \otimes_{\mathbb{R}} z_2] = [T, U] \otimes_{\mathbb{R}} z_1 z_2$. If $\omega : \mathfrak{g} \to \mathfrak{h}$ is a morphism of real Lie algebras, then $\Omega = \omega \otimes_{\mathbb{R}} \mathbb{C} : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$, $\Omega(T \otimes_{\mathbb{R}} z) = \omega(T) \otimes_{\mathbb{R}} z$ is a morphism of complex Lie algebras. Since there is no danger on confusion we will denote $T \otimes_{\mathbb{R}} 1 \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by T and $T \otimes_{\mathbb{R}} z$ by zT. We will denote $\mathfrak{R}_a \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathfrak{C}_a$.

There is a natural conjugation map $- : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, defined by $\overline{T \otimes_{\mathbb{R}} z} = T \otimes_{\mathbb{R}} \overline{z}$. If W is a subset of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ then we define $\overline{W} = \{\overline{w} \mid w \in W\}$. One can check that if $\omega : \mathfrak{g} \to \mathfrak{h}$ is a morphism of real Lie algebras, then $\Omega : \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ has the property $\Omega(\overline{W}) = \overline{\Omega(W)}$.

Recall that $\mathfrak{R}_a = W_1^a \oplus W_2^a$. We have $\mathfrak{C}_a = (W_1^a \otimes_{\mathbb{R}} \mathbb{C}) \oplus (W_2^a \otimes_{\mathbb{R}} \mathbb{C})$. We want to find all $T \in W_1^a \otimes_{\mathbb{R}} \mathbb{C}$ with the property that $ad(T) : \mathfrak{C}_a \to \mathfrak{C}_a$ has rank one. Let $T = a_1 X_1^a + a_2 X_2^a + b_1 \widetilde{X}_1^a + b_2 \widetilde{X}_2^a + c_1 Y_1^a + c_2 Y_2^a + d_1 \widetilde{Y}_1^a + d_2 \widetilde{Y}_2^a \in W_1^a \otimes_{\mathbb{R}} \mathbb{C}$, we have:

$$\begin{aligned}
ad(T)(X_1^a) &= [T, X_1^a] = -c_1 Z_1^a - a c_2 Z_2^a, \\
ad(T)(X_2^a) &= [T, X_2^a] = -a c_1 Z_2^a + c_2 Z_1^a, \\
ad(T)(\widetilde{X}_1^a) &= [T, \widetilde{X}_1^a] = -d_1 Z_1^a - d_2 Z_2^a, \\
ad(T)(\widetilde{X}_2^a) &= [T, \widetilde{X}_2^a] = -d_1 Z_2^a + d_2 Z_1^a, \\
ad(T)(Y_1^a) &= [T, Y_1^a] = a_1 Z_1^a + a a_2 Z_2^a, \\
ad(T)(Y_2^a) &= [T, Y_2^a] = a a_1 Z_2^a - a_2 Z_1^a, \\
ad(T)(\widetilde{Y}_1^a) &= [T, \widetilde{Y}_1^a] = b_1 Z_1^a + b_2 Z_2^a, \\
ad(T)(\widetilde{Y}_2^a) &= [T, \widetilde{Y}_2^a] = b_1 Z_2^a - b_2 Z_1^a.
\end{aligned}$$

If the rank of $ad(T)$ is one then $ad(T)(X_1^a)$ and $ad(T)(X_2^a)$ must be linearly dependent and so there exists $\alpha \in \mathbb{C}$ such that,

$$-c_1 Z_1^a - a c_2 Z_2^a = \alpha(-a c_1 Z_2^a + c_2 Z_1^a)$$

or equivalently $c_1 = -\alpha c_2$ and $c_2 = \alpha c_1$ which implies $c_1 = -\alpha^2 c_1$. If $c_1 \neq 0$ then $\alpha^2 = -1$. When $\alpha = i$ the image of $ad(T)$ contains the vector $-c_1(Z_1^a + iaZ_2^a)$, if $\alpha = -i$ then the image of $ad(T)$ contains the vector $-c_1(Z_1^a - iaZ_2^a)$.

A similar argument shows that if $d_1 \neq 0$ then the image of $ad(T)$ contains either the vector $-d_1(Z_1^a + iZ_2^a)$ or the vector $-d_1(Z_1^a - iZ_2^a)$. Also if $a_1 \neq 0$ then the image of $ad(T)$ contains the element $a_1(Z_1^a + iaZ_2^a)$ or the vector $a_1(Z_1^a - iaZ_2^a)$, and finally when $b_1 \neq 0$ the image of $ad(T)$ contains the vector $b_1(Z_1^a + iZ_2^a)$ or the vector $b_1(Z_1^a - iZ_2^a)$. If we put all the information together we get two different cases depending on whether $a \neq 1$ or $a = 1$.

Let $Rank_1(\mathfrak{C}_a)$ be the set of all elements $T \in W_1^a \otimes_{\mathbb{R}} \mathbb{C}$ with the property that $ad(T)$ has rank one.

Set

$$L_1^a = \{a_1(X_1^a + iX_2^a) + c_1(Y_1^a + iY_2^a) \mid a_1, c_1 \in \mathbb{C}\}$$

and

$$L_2^a = \{b_1(\widetilde{X}_1^a + i\widetilde{X}_2^a) + d_1(\widetilde{Y}_1^a + i\widetilde{Y}_2^a) \mid b_1, d_1 \in \mathbb{C}\}.$$

If $a \neq 1$, then

$$Rank_1(\mathfrak{C}_a) \bigcup \{0\} = L_1^a \bigcup \overline{L_1^a} \bigcup L_2^a \bigcup \overline{L_2^a}.$$

If $a = 1$, then $Rank_1(\mathfrak{C}_1) \bigcup \{0\} = L \bigcup \overline{L}$, where L is the complex linear subspace of $W_1^a \otimes_{\mathbb{R}} \mathbb{C}$ spanned by $X_1^a + iX_2^a$, $\widetilde{X}_1^a + i\widetilde{X}_2^a$, $Y_1^a + iY_2^a$ and $\widetilde{Y}_1^a + i\widetilde{Y}_2^a$.

Proposition 3.5. *If $a \neq 1$, then $\mathfrak{R}_1$ and $\mathfrak{R}_a$ are not isomorphic.*

Proof. Suppose that we have an isomorphism $\omega : \mathfrak{R}_a \to \mathfrak{R}_1$. From Lemma 3.4 we may assume that ω is a graded isomorphism. The induced map $\Omega : \mathfrak{C}_a \to \mathfrak{C}_1$ is also a graded isomorphism and $\Omega(Rank_1(\mathfrak{C}_a)) = Rank_1(\mathfrak{C}_1)$. But this is obviously impossible since from the above discussion we know that $Rank_1(\mathfrak{C}_a)$ is a union of four vector spaces of dimension two, while $Rank_1(\mathfrak{C}_1)$ is a union of two vector spaces of dimension four. ■

Consider now a, $b \in (0,1)$. We want to show that $\mathfrak{R}_a$ and $\mathfrak{R}_b$ are isomorphic only when $a = b$. Suppose that $\omega : \mathfrak{R}_a \to \mathfrak{R}_b$ is an isomorphism of real Lie algebras. From Lemma 3.4 we can assume that ω is a graded isomorphism. Also from the above discussion we have that $\Omega(Rank_1(\mathfrak{C}_a)) = Rank_1(\mathfrak{C}_b)$. Since $Rank_1(\mathfrak{C}_a) \bigcup \{0\} = L_1^a \bigcup \overline{L_1^a} \bigcup L_2^a \bigcup \overline{L_2^a}$ we must have $\Omega(L_1^a) \in \{L_1^b, \overline{L_1^b}, L_2^b, \overline{L_2^b}\}$. Moreover, since $\Omega(\overline{W}) = \overline{\Omega(W)}$ we must have $\{\Omega(L_1^a), \Omega(\overline{L_1^a})\} = \{L_1^b, \overline{L_1^b}\}$ or $\{\Omega(L_1^a), \Omega(\overline{L_1^a})\} = \{L_2^b, \overline{L_2^b}\}$.

We consider the following $\mathbb{R}$-vector spaces $V_1^a = \mathbb{R}X_1^a \oplus \mathbb{R}X_2^a \oplus \mathbb{R}Y_1^a \oplus \mathbb{R}Y_2^a$, $\widetilde{V}_1^a = \mathbb{R}\widetilde{X}_1^a \oplus \mathbb{R}\widetilde{X}_2^a \oplus \mathbb{R}\widetilde{Y}_1^a \oplus \mathbb{R}\widetilde{Y}_2^a$ and $V_2^a = \mathbb{R}Z_1^a \oplus \mathbb{R}Z_2^a$.

Case I: $\{\Omega(L_1^a), \Omega(\overline{L_1^a})\} = \{L_1^b, \overline{L_1^b}\}$. Suppose that $\Omega(L_1^a) = L_1^b$ and $\Omega(\overline{L_1^a}) = \overline{L_1^b}$ (the case $\Omega(L_1^a) = \overline{L_1^b}$ and $\Omega(\overline{L_1^a}) = L_1^b$ is similar). There exist α, $\beta \in \mathbb{C}$ such that

$$\Omega(X_1^a + iX_2^a) = \alpha(X_1^b + iX_2^b) + \beta(Y_1^b + iY_2^b)$$

and

$$\Omega(X_1^a - iX_2^a) = \overline{\alpha}(X_1^b - iX_2^b) + \overline{\beta}(Y_1^b - iY_2^b).$$

This implies that

$$\omega(X_1^a) = \Omega(X_1^a) = Re(\alpha)X_1^b - Im(\alpha)X_2^b + Re(\beta)Y_1^b - Im(\beta)Y_2^b \in V_1^b.$$

A similar computation shows that $\omega(X_2^a)$, $\omega(Y_1^a)$ and $\omega(Y_2^a) \in V_1^b$. This means that $\omega(V_1^a) = V_1^b$. Similarly we can show that $\omega(\widetilde{V}_1^a) = \widetilde{V}_1^b$. In particular we see that

$$\omega : V_1^a \oplus V_2^a \to V_1^b \oplus V_2^b$$

and

$$\omega : \widetilde{V}_1^a \oplus V_2^a \to \widetilde{V}_1^b \oplus V_2^b$$

are graded isomorphisms.

One can see that $\phi_a : V_1^a \oplus V_2^a \to H^1_{\mathbb{C}}$ given by $\phi_a(X_i^a) = X_i$, $\phi_a(Y_i^a) = Y_i$, $\phi_a(Z_1^a) = Z_1$ and $\phi_a(Z_2^a) = a^{-1}Z_2$ is a graded isomorphism. Also $\widetilde{\phi}_a : \widetilde{V}_1^a \oplus V_2^a \to H^1_{\mathbb{C}}$ given by $\widetilde{\phi}_a(\widetilde{X}_i^a) = X_i$, $\widetilde{\phi}_a(\widetilde{Y}_i^a) = Y_i$, $\widetilde{\phi}_a(Z_1^a) = Z_1$ and $\widetilde{\phi}_a(Z_2^a) = Z_2$ is a graded isomorphism.

This means that $\theta = \phi_b \omega \phi_a^{-1}$ and $\widetilde{\theta} = \widetilde{\phi}_b \omega \widetilde{\phi}_a^{-1} : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ are graded isomorphisms. We recall from [S], page 110, the following result.

Lemma 3.6. *If $\theta : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ is a graded isomorphism, then there exist α, $\beta \in \mathbb{R}$, $\alpha^2 + \beta^2 \neq 0$ such that $\theta(Z_1) = \alpha Z_1 + \beta Z_2$ and $\theta(Z_2) = \pm(-\beta Z_1 + \alpha Z_2)$.*

Suppose that $\theta = \phi_b \omega \phi_a^{-1}$ and $\widetilde{\theta} = \widetilde{\phi}_b \omega \widetilde{\phi}_a^{-1}$ correspond to the matrix $\begin{pmatrix} \alpha & \mp\beta \\ \beta & \pm\alpha \end{pmatrix}$ respectively $\begin{pmatrix} \gamma & \mp\delta \\ \delta & \pm\gamma \end{pmatrix}$. Then $\omega_{|V_2^a} = \phi_b^{-1} \theta \phi_a = \widetilde{\phi}_b^{-1} \widetilde{\theta} \widetilde{\phi}_a$ corresponds to the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \alpha & \mp\beta \\ \beta & \pm\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} \gamma & \mp\delta \\ \delta & \pm\gamma \end{pmatrix}.$$

This is equivalent to

$$\begin{pmatrix} \alpha & \mp\beta \\ b\beta & \pm b\alpha \end{pmatrix} = \begin{pmatrix} \gamma & \mp a\delta \\ \delta & \pm a\gamma \end{pmatrix}.$$

It follows that $\alpha = \gamma$, $b\alpha = a\gamma$, $\delta = b\beta$ and $\beta = a\delta$. The last two equalities imply $\beta(ab - 1) = 0$. Hence $\beta = 0$ as $a, b \in (0, 1)$. It follows that $\alpha \neq 0$ and we must have $a = b$.

Case II: $\{\Omega(L_1^a), \Omega(\overline{L_1^a})\} = \{L_2^b, \overline{L_2^b}\}$. Using the same ideas as in the first case, we can show that $\omega : V_1^a \oplus V_2^a \to \widetilde{V}_1^b \oplus V_2^b$ and $\omega : \widetilde{V}_1^a \oplus V_2^a \to V_1^b \oplus V_2^b$ are graded isomorphisms. This implies that $\theta = \widetilde{\phi}_b \omega \phi_a^{-1}$ and $\widetilde{\theta} = \phi_b \omega \widetilde{\phi}_a^{-1} : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ are graded isomorphisms. Again we use Lemma 3.6 to get the following identity:

$$\begin{pmatrix} \alpha & \mp\beta \\ \beta & \pm\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \gamma & \mp\delta \\ \delta & \pm\gamma \end{pmatrix}.$$

In this case we obtain $\alpha = 0$ and $a = b$.

The proof of Theorem 1.1 is now complete.

We extract the following fact from the proof:

Lemma 3.7. *Suppose $a \in (0, 1)$ and $h : \mathfrak{R}_a \to \mathfrak{R}_a$ is a graded isomorphism. Then exactly one of the following holds:*

(1) *$h(V_1^a) = V_1^a$ and $h(\widetilde{V}_1^a) = \widetilde{V}_1^a$; in this case, there exists some $\gamma \neq 0$ such that $h(Z_1^a) = \gamma Z_1^a$ and $h(Z_2^a) = \pm\gamma Z_2^a$;*

(2) *$h(V_1^a) = \widetilde{V}_1^a$ and $h(\widetilde{V}_1^a) = V_1^a$; in this case, there exists some $\delta \neq 0$ such that $h(Z_1^a) = \delta Z_2^a$ and $h(Z_2^a) = \mp\delta Z_1^a$.*

Remark 3.8. If we allow $a \in (0, \infty)$ then one can see that $\mu : \mathfrak{R}_a \cong \mathfrak{R}_{a^{-1}}$ given by $\mu(X_i^a) = \widetilde{X}_i^{a^{-1}}$, $\mu(Y_i^a) = \widetilde{Y}_i^{a^{-1}}$, $\mu(\widetilde{X}_i^a) = X_i^{a^{-1}}$, $\mu(\widetilde{Y}_i^a) = Y_i^{a^{-1}}$, $\mu(Z_1^a) = Z_1^{a^{-1}}$, and $\mu(Z_2^a) = a^{-1} Z_2^{a^{-1}}$ is an isomorphism of Lie algebras.

4. Quasiisometric classification of a class of finitely generated nilpotent groups

In this Section we shall give an application of Theorem 1.1. We shall use Theorem 1.1 to obtain quasiisometric classification of a class of finitely generated nilpotent groups. These finitely generated nilpotent groups are lattices in R_a, where a is rational.

When a is rational, $\mathfrak{R}_a$ is rational; that is, it has a basis with rational structure constants. It is well known that a connected and simply connected nilpotent Lie group has lattices if and only if its Lie algebra is rational. And in this case it is easy to construct lattices. See [CG], Theorem 5.1.8.

Recall that for a connected and simply connected nilpotent Lie group G with Lie algebra $\mathcal{G}$, the exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism. We shall identify R_a and $\mathfrak{R}_a$ via the exponential map. The group operation on $\mathfrak{R}_a$ is given by the BCH formula (see Section 2).

Let $a \in (0, 1]$ be rational. Write $a = p/q$ where p and q are relatively prime integers. The least common multiple of the denominators of the structure constants of the Lie algebra $\mathfrak{R}_a$ with respect to the vector space basis

$$\{X_1, X_2, Y_1, Y_2, \widetilde{X}_1, \widetilde{X}_2, \widetilde{Y}_1, \widetilde{Y}_2, Z_1, Z_2\}$$

is q (here we dropped the superscipt a in the notation for the basis elements). By the proof of [CG], Theorem 5.1.8, the subset $\Gamma_{p,q} \subset \mathfrak{R}_a$ consisting of the integral linear combinations of the following elements:

$$S_0 = \{2qX_1, 2qY_1, 2qX_2, 2qY_2, 2q\widetilde{X}_1, 2q\widetilde{Y}_1, 2q\widetilde{X}_2, 2q\widetilde{Y}_2, 2qZ_1, 2qZ_2\}$$

is a lattice in the Lie group $\mathfrak{R}_a$. One can easily check that this is a subgroup by using the BCH formula (see Section 2). The group $\Gamma_{p,q}$ is generated by S_0 and satisfies the following relations (here $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1, g_2 \in \Gamma_{p,q}$):

$$[2qX_1, 2qY_1] = (2qZ_1)^{2q}, \quad [2qX_1, 2qY_2] = (2qZ_2)^{2p},$$

$$[2qX_2, 2qY_1] = (2qZ_2)^{2p}, \quad [2qX_2, 2qY_2] = (2qZ_1)^{-2q},$$

$$[2q\widetilde{X}_1, 2q\widetilde{Y}_1] = (2qZ_1)^{2q}, \quad [2q\widetilde{X}_1, 2q\widetilde{Y}_2] = (2qZ_2)^{2q},$$

$$[2q\widetilde{X}_2, 2q\widetilde{Y}_1] = (2qZ_2)^{2q}, \quad [2q\widetilde{X}_2, 2q\widetilde{Y}_2] = (2qZ_1)^{-2q},$$

$$[x, 2qZ_j] = e \ \text{ for } j = 1, 2 \ \text{ and any } \ x \in S_0,$$

$$[2qX_1, 2qX_2] = e, \quad [2qY_1, 2qY_2] = e,$$

$$[2q\widetilde{X}_1, 2q\widetilde{X}_2] = e, \quad [2q\widetilde{Y}_1, 2q\widetilde{Y}_2] = e,$$

$$[x, y] = e \text{ for any } \ x \in \{2qX_1, 2qX_2, 2qY_1, 2qY_2\}; \ y \in \{2q\widetilde{X}_1, 2q\widetilde{X}_2, 2q\widetilde{Y}_1, 2q\widetilde{Y}_2\}.$$

We shall show that the set S_0 of generators and the above relators form a presentation of the group $\Gamma_{p,q}$. For this we let $G_{p,q}$ be the group given by the presentation

$$G_{p,q} =< S|R >,$$

where $S = \{x_1, y_1, x_2, y_2, \tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2, z_1, z_2\}$ is the set of generators and R is the set of relators consisting of

$$[x_1, y_1] = z_1^{2q}, [x_1, y_2] = z_2^{2p}, [x_2, y_1] = z_2^{2p}, [x_2, y_2] = z_1^{-2q},$$

$$[\tilde{x}_1, \tilde{y}_1] = z_1^{2q}, [\tilde{x}_1, \tilde{y}_2] = z_2^{2q}, [\tilde{x}_2, \tilde{y}_1] = z_2^{2q}, [\tilde{x}_2, \tilde{y}_2] = z_1^{-2q},$$

$$[z_j, x] = e \ \text{ for } \ j = 1, 2 \text{ and any } x \in S,$$

$$[x_1, x_2] = e, \ [y_1, y_2] = e, \ [\tilde{x}_1, \tilde{x}_2] = e, \ [\tilde{y}_1, \tilde{y}_2] = e,$$

$$[x, y] = e \ \text{ for any } \ x \in \{x_1, x_2, y_1, y_2\}; \ y \in \{\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2\}.$$

Because the relations in R are satisfied by the elements in $\Gamma_{p,q}$, there is a unique group homomorphism $\phi : G_{p,q} \to \Gamma_{p,q}$ such that $\phi(x_i) = 2qX_i, \phi(\tilde{x}_i) = 2q\widetilde{X}_i$, $\phi(y_i) = 2qY_i, \phi(\tilde{y}_i) = 2q\widetilde{Y}_i$, $\phi(z_i) = 2qZ_i$ for $i = 1, 2$.

Lemma 4.1. *The homomorphism ϕ is a group isomorphism.*

Proof. The homomorphism ϕ is clearly surjective. We shall show that it is also injective. Let $g \in G_{p,q}$ be such that $\phi(g) = 0$. The relators in R show that z_1 and z_2 lie in the center of $G_{p,q}$ and that every element (hence g) of $G_{p,q}$ can be written as

$$g = (x_1)^{k_1}(x_2)^{k_2}(y_1)^{l_1}(y_2)^{l_2}(\tilde{x}_1)^{m_1}(\tilde{x}_2)^{m_2}(\tilde{y}_1)^{n_1}(\tilde{y}_2)^{n_2}(z_1)^{p_1}(z_2)^{p_2},$$

where k_j, l_j, m_j, n_j, p_j are integers. Since ϕ and $\pi_1 : \mathfrak{R}_a \to W_1^a$ are homomorphisms, we have

$$\begin{aligned} 0 = \pi_1 \circ \phi(g) = k_1 \cdot (2qX_1) + k_2 \cdot (2qX_2) + l_1 \cdot (2qY_1) + l_2 \cdot (2qY_2) + \\ + m_1 \cdot (2q\tilde{X}_1) + m_2 \cdot (2q\tilde{X}_2) + n_1 \cdot (2q\tilde{Y}_1) + n_2 \cdot (2q\tilde{Y}_2) \in W_1^a. \end{aligned}$$

Since $S_0 \backslash \{2qZ_1, 2qZ_2\}$ is a basis of the vector space W_1^a we must have $k_j = l_j = m_j = n_j = 0$. Hence $g = (z_1)^{p_1}(z_2)^{p_2}$ and $0 = \phi(g) = p_1 \cdot (2qZ_1) + p_2 \cdot (2qZ_2)$. Since Z_1 and Z_2 are linearly independent, we have $p_1 = p_2 = 0$. Hence $g = e$. ■

Proof of Theorem 1.2. Denote $a = p/q$ and $a' = p'/q'$. Suppose $G_{p,q}$ and $G_{p',q'}$ are quasiisometric. By Lemma 4.1, $G_{p,q}$ and $\Gamma_{p,q}$ are isomorphic. So $\Gamma_{p,q}$ and $\Gamma_{p',q'}$ are quasiisometric. Since $\Gamma_{p,q}$ is a cocompact lattice in $\mathfrak{R}_a$ and similarly for $\Gamma_{p',q'}$, we see that $\mathfrak{R}_a$ and $\mathfrak{R}_{a'}$ are quasiisometric. It follows that the asymptotic cones of $\mathfrak{R}_a$ and $\mathfrak{R}_{a'}$ are biLipschitz. Since $\mathfrak{R}_a$ and $\mathfrak{R}_{a'}$ are Carnot groups, their asymptotic cones are themselves. So $\mathfrak{R}_a$ and $\mathfrak{R}_{a'}$ are biLipschitz. Now Pansu's differentiability theorem implies that $\mathfrak{R}_a$ and $\mathfrak{R}_{a'}$ are isomorphic. Now the claim follows from Theorem 1.1. ■

5. C^2 quasiconformal maps

In this Section we identify all the C^2 quasiconformal maps on $\mathfrak{R}_a$. It turns out that all C^2 quasiconformal maps are affine maps. Recall that an affine map of a Carnot group is the composition of a left translation and an automorphism.

We first construct some special graded automorphisms of $\mathfrak{R}_a$. One of them is the conjugation $\tau : \mathfrak{R}_a \to \mathfrak{R}_a$ defined by:

$$\tau(x) = x \;\text{ for }\; x \in \{X_1^a, Y_1^a, \widetilde{X}_1^a, \widetilde{Y}_1^a, Z_1^a\};$$

$$\tau(y) = -y \;\text{ for }\; y \in \{X_2^a, Y_2^a, \widetilde{X}_2^a, \widetilde{Y}_2^a, Z_2^a\}.$$

It is straightforward to check (using Table 1) that τ is a graded automorphism. Also notice that $\tau^2 = \mathrm{id}$.

Next we construct the switch map $s : \mathfrak{R}_a \to \mathfrak{R}_a$, which is a graded automorphism of $\mathfrak{R}_a$ that switch the two linear subspaces $V_1^a := \mathbb{R}X_1^a \oplus \mathbb{R}Y_1^a \oplus \mathbb{R}X_2^a \oplus \mathbb{R}Y_2^a$ and $\widetilde{V}_1^a := \mathbb{R}\widetilde{X}_1^a \oplus \mathbb{R}\widetilde{Y}_1^a \oplus \mathbb{R}\widetilde{X}_2^a \oplus \mathbb{R}\widetilde{Y}_2^a$ of W_1^a.

Lemma 5.1. *There exists a graded automorphism $s : \mathfrak{R}_a \to \mathfrak{R}_a$ such that $s(V_1^a) = \widetilde{V}_1^a$ and $s(\widetilde{V}_1^a) = V_1^a$. Furthermore, s has order 8.*

Proof. Let $s : \mathfrak{R}_a \to \mathfrak{R}_a$ be the linear isomorphism defined by the following formulas:

$$s(X_1^a) = \sqrt[4]{a} \cdot \frac{\sqrt{2}}{2}(\widetilde{X}_1^a + \widetilde{X}_2^a)$$

$$s(X_2^a) = \sqrt[4]{a} \cdot \frac{\sqrt{2}}{2}(-\widetilde{X}_1^a + \widetilde{X}_2^a)$$

$$s(Y_1^a) = \sqrt[4]{a} \cdot \frac{\sqrt{2}}{2}(\widetilde{Y}_1^a + \widetilde{Y}_2^a)$$

$$s(Y_2^a) = \sqrt[4]{a} \cdot \frac{\sqrt{2}}{2}(-\widetilde{Y}_1^a + \widetilde{Y}_2^a)$$

$$s(\widetilde{X}_1^a) = \frac{1}{\sqrt[4]{a}} \cdot \frac{\sqrt{2}}{2}(X_1^a + X_2^a)$$

$$s(\widetilde{X}_2^a) = \frac{1}{\sqrt[4]{a}} \cdot \frac{\sqrt{2}}{2}(-X_1^a + X_2^a)$$

$$s(\widetilde{Y}_1^a) = \frac{1}{\sqrt[4]{a}} \cdot \frac{\sqrt{2}}{2}(Y_1^a + Y_2^a)$$

$$s(\widetilde{Y}_2^a) = \frac{1}{\sqrt[4]{a}} \cdot \frac{\sqrt{2}}{2}(-Y_1^a + Y_2^a)$$

$$s(Z_1^a) = \sqrt{a} \cdot Z_2^a$$

$$s(Z_2^a) = -\frac{1}{\sqrt{a}} \cdot Z_1^a.$$

It is now easy to check (using Table 1) that s is a graded automorphism.

Geometrically, $s|_{V_1^a} : V_1^a \to \widetilde{V}_1^a$ is the obvious identification followed by a rotation of $\pi/4$ and then a dilation by factor $\sqrt[4]{a}$; similarly, $s|_{\widetilde{V}_1^a} : \widetilde{V}_1^a \to V_1^a$ is the obvious identification followed by a rotation of $\pi/4$ and then a dilation by factor $1/\sqrt[4]{a}$. Now, it is easy to see that s has order 8. ■

Next we construct the graded automorphism $R_{\frac{\pi}{2}} : \mathfrak{R}_a \to \mathfrak{R}_a$, which is a rotation by angle $\pi/2$ on W_1^a. We define the linear isomorphism $R_{\frac{\pi}{2}} : \mathfrak{R}_a \to \mathfrak{R}_a$ by:

$$R_{\frac{\pi}{2}}(X_1^a) = X_2^a, \quad R_{\frac{\pi}{2}}(X_2^a) = -X_1^a;$$
$$R_{\frac{\pi}{2}}(Y_1^a) = Y_2^a, \quad R_{\frac{\pi}{2}}(Y_2^a) = -Y_1^a;$$
$$R_{\frac{\pi}{2}}(\widetilde{X}_1^a) = \widetilde{X}_2^a, \quad R_{\frac{\pi}{2}}(\widetilde{X}_2^a) = -\widetilde{X}_1^a;$$
$$R_{\frac{\pi}{2}}(\widetilde{Y}_1^a) = \widetilde{Y}_2^a, \quad R_{\frac{\pi}{2}}(\widetilde{Y}_2^a) = -\widetilde{Y}_1^a;$$
$$R_{\frac{\pi}{2}}(Z_j^a) = -Z_j^a \text{ for } j = 1, 2.$$

Notice that $R_{\frac{\pi}{2}} = s^2$. Hence $R_{\frac{\pi}{2}}$ is a graded isomorphism.

Given any $A, B \in GL(2, \mathbb{C})$ such that $\det(A) = \det(B)$ is positive real, we shall define a graded automorphism $F_{(A,B)} : \mathfrak{R}_a \to \mathfrak{R}_a$. First we notice that V_1^a admits a complex structure $J : V_1^a \to V_1^a$ given by $J := R_{\frac{\pi}{2}}|_{V_1^a}$. We still use $A : V_1^a \to V_1^a$ to denote the complex linear map whose matrix representation with respect to the basis $\{X_1^a, Y_1^a\}$ is A. Similarly we define a complex linear map $B : \widetilde{V}_1^a \to \widetilde{V}_1^a$. Now write $\mathfrak{R}_a = V_1^a \oplus \widetilde{V}_1^a \oplus V_2^a$ and define $F_{(A,B)} : \mathfrak{R}_a \to \mathfrak{R}_a$ by

$$F_{(A,B)}(x, \tilde{x}, z) = (Ax, B\tilde{x}, \det(A)z).$$

It is not hard to see that $F_{(A,B)}$ is a graded automorphism.

When $a = 1$, the group $\mathfrak{R}_a$ is the second complex Heisenberg group. In this case, [X2], Proposition 4.2 says that every C^2 quasiconformal map is an affine map. Our next Theorem is a similar result in the case when $a \in (0, 1)$. Notice that this result implies Theorem 1.3.

Theorem 5.2. *Let $a \in (0,1)$ and $F : \mathfrak{R}_a \to \mathfrak{R}_a$ be a C^2 quasiconformal map. There exist $A, B \in GL(2, \mathbb{C})$ satisfying $\det(A) = \det(B) > 0$, some $p \in \mathfrak{R}_a$ and some $i, j, k \in \{0, 1\}$ such that $F = L_p \circ s^i \circ \tau^j \circ R^k_{\frac{\pi}{2}} \circ F_{(A,B)}$, where $L_p : \mathfrak{R}_a \to \mathfrak{R}_a$, $L_p(x) = p * x$ is the left translation by p.*

Proof. By composing with a left translation we may assume $F(0) = 0$. We shall show that F is a graded automorphism. By Theorem 2.2 and Lemma 3.7 we know that at every point $x \in \mathfrak{R}_a$ where F is Pansu differentiable, the Pansu differential $dF(x) : \mathfrak{R}_a \to \mathfrak{R}_a$ satisfies either $dF(x)(V_1^a) = V_1^a, dF(x)(\widetilde{V}_1^a) = \widetilde{V}_1^a$ or $dF(x)(V_1^a) = \widetilde{V}_1^a, dF(x)(\widetilde{V}_1^a) = V_1^a$. Since F is assumed to be C^2, either $dF(x)(V_1^a) = V_1^a, dF(x)(\widetilde{V}_1^a) = \widetilde{V}_1^a$ always hold or $dF(x)(V_1^a) = \widetilde{V}_1^a, dF(x)(\widetilde{V}_1^a) = V_1^a$ always hold. After possibly composing with the switch map s we may assume $dF(x)(V_1^a) = V_1^a, dF(x)(\widetilde{V}_1^a) = \widetilde{V}_1^a$ always hold. After further composing with the conjugation map τ if necessary, we may assume that for each $x \in \mathfrak{R}_a$, there exists some $0 \neq \gamma(x) \in \mathbb{R}$ such that $dF(x)(Z_j^a) = \gamma(x) Z_j^a$ for $j = 1, 2$. Here we used

Lemma 3.7. Again since F is C^2, either $\gamma(x)$ is always positive or always negative. After composing with the graded automorphism $R_{\frac{\pi}{2}}$ if necessary, we may assume $\gamma(x)$ is always positive.

Set $H = V_1^a \oplus V_2^a$ and $\widetilde{H} = \widetilde{V}_1^a \oplus V_2^a$. These are connected subgroups of $\mathfrak{R}_a$. Since $dF(x)(V_1^a) = V_1^a, dF(x)(\widetilde{V}_1^a) = \widetilde{V}_1^a$ always hold, Proposition 3.4 in [X3] implies that for each $x \in \mathfrak{R}_a$, $F(x * H)$ is a left coset of H and $F(x * \widetilde{H})$ is a left coset of $\widetilde{H}$. Since left cosets of V_2^a are exactly the intersections of left cosets of H and left cosets of $\widetilde{H}$, we see that F must map left cosets of V_2^a to left cosets of V_2^a. In other words, there is some map $F_1 : W_1^a \to W_1^a$ such that $\pi_1 \circ F = F_1 \circ \pi_1$. Furthermore, there exist maps $f : V_1^a \to V_1^a$ and $\tilde{f} : \widetilde{V}_1^a \to \widetilde{V}_1^a$ such that $F_1 = f \oplus \tilde{f}$.

Now consider the restriction $F|_{\widetilde{H}} : \widetilde{H} \to \widetilde{H}$ of F to $\widetilde{H}$. Since F is C^2 quasiconformal, so is $F|_{\widetilde{H}}$. By [X2], Proposition 4.2, there exists some complex linear isomorphism $B : \widetilde{V}_1^a \to \widetilde{V}_1^a$ such that either $F|_{\widetilde{H}}$ has the form $F|_{\widetilde{H}}(y,z) = (By, \det(B)z)$ or $F|_{\widetilde{H}}(y,z) = (B\bar{y}, \det(B)\bar{z})$, where $\bar{y}$ denotes complex conjugation. By the first paragraph, we may assume $\det(B) > 0$ and $F|_{\widetilde{H}}(y,z) = (By, \det(B)z)$. Similarly, $F|_H$ is complex linear or complex anti-linear. Since $H \cap \widetilde{H} = V_2^a$, $F|_H$ must be complex linear and there exists some complex linear isomorphism $A : V_1^a \to V_1^a$ such that $\det(A) = \det(B)$ and $F|_H(x,z) = (Ax, \det(A)z)$. Then $F_1(x,y) = (Ax, By)$ for $(x,y) \in W_1^a = V_1^a \oplus \widetilde{V}_1^a$.

We claim that $F = F_{(A,B)}$. By composing with the inverse of $F_{(A,B)}$ we may assume $A = B = I_2 \in GL(2, \mathbb{C})$. Then $F_1 : W_1^a \to W_1^a$ is the identity map. We need to show that $F = \mathrm{id}_{\mathfrak{R}_a}$. Since $F_1 : W_1^a \to W_1^a$ is the identity map, at every point $x \in \mathfrak{R}_a$, the Pansu differential $dF(x)$ has the property that $dF(x)|_{W_1^a} = \mathrm{id}_{W_1^a}$. It follows that $dF(x) = \mathrm{id}_{\mathfrak{R}_a}$. By Lemma 2.5 in [X2], F must be a left translation. Since $F(0) = 0$, we must have $F = \mathrm{id}_{\mathfrak{R}_a}$. ■

We conclude the paper by briefly describing the group $\mathrm{QC}(\mathfrak{R}_a)$ of all C^2 quasiconformal maps of $\mathfrak{R}_a$. Let G_2 be the subgroup generated by s and τ; let G_3 be the subgroup generated by all maps of the form $F_{A,B}$, where $A, B \in GL(2, \mathbb{C})$ are such that $\det(\mathrm{A}) = \det(\mathrm{B}) > 0$; let G_1 be the subgroup generated by G_2 and G_3. We also denote by $\mathfrak{R}_a$ the subgroup consisting of left translations. Since all elements in G_1 are automorphisms, it is easy to check (using Theorem 5.2) that $\mathrm{QC}(\mathfrak{R}_a) = \mathfrak{R}_a \rtimes G_1$. We already know that $|s| = 8$ and $|\tau| = 2$. One can check the following relations:

$$\tau \circ s \circ \tau^{-1} = s^{-1},$$

$$s \circ F_{A,B} \circ s^{-1} = F_{B,A},$$

$$\tau \circ F_{A,B} \circ \tau^{-1} = F_{\overline{A},\overline{B}},$$

where $\overline{A}$ is the complex conjugation of A. It follows that $G_2 \cong D_8$, $G_3 \lhd G_1$ and $G_1 = G_2 \cdot G_3$. However, G_1 is not the semi-direct product of G_2 and G_3 since $G_2 \cap G_3 = \{id, s^4\} \cong \mathbb{Z}_2$ as one can check.

References

[B] Balogh, Z., *Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group*, J. Anal. Math. **83** (2001), 289–312.

[BR] Bellaiche, A., and J. J. Risler, "Sub-Riemannian Geometry," Progress in Mathematics **144**, Basel 1996.

[CG] Corwin, L., and F. Greenleaf, "Representations of nilpotent Lie groups and their applications, Part I. Basic theory and examples", Cambridge Studies in Advanced Mathematics **18**, Cambridge University Press, Cambridge, 1990.

[CO] Cowling, M., and A. Ottazzi, *Global contact and quasiconformal mappings of Carnot groups*, http://arxiv.org/pdf/1408.1778.pdf

[CR] Cowling, M., and M. Reimann, *Quasiconformal mappings on Carnot groups: three examples*, Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 2003, 111–118.

[G] Gromov, M., *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73.

[H] Heintze, H., *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34.

[L] Lay, D., "Linear Algebra and Its Applications," Pearson, Boston, 2012.

[M] Mitchell, J., *On Carnot-Caratheodory metrics*, J. Differential Geom. **21** (1985), 35–45.

[OW] Ottazzi, A., B. Warhurst, *Contact and 1-quasiconformal maps on Carnot groups*, J. Lie Theory **21** (2011), 787–811.

[P] Pansu, P., *Métriques de Carnot-Caratheodory et quasiisometries des espaces symetriques de rang un*, Ann. of Math. (2) **129** (1989), 1–60.

[RR] Reimann, H.M., F. Ricci, *The complexified Heisenberg group* in: Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, 465-480.

[S] Saal, L., *The automorphism group of a Lie algebra of Heisenberg type*, Rend. Sem. Mat. Univ. Politec. Torino **54** (1996), 101–113.

[X1] Xie, X., *Quasiconformal maps on nonrigid Carnot groups,* , http://front.math.ucdavis.edu/1308.3031.

[X2] —, *Quasiconformal maps on model Filiform groups*, Michigan Mathematical Journal, to appear.

[X3] —, *Quasisymmetric homeomorphisms on reducible Carnot groups*, Pacific J. Math. **265** (2013), 113–122.

Michael R. Hughes
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403, USA
mrhughe@bgsu.edu

Mihai D. Staic
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403, USA
and
Institute of Mathematics
of the Romanian Academy
PO.BOX 1-764, RO-70700 Bucharest, Romania
mstaic@bgsu.edu

Xiangdong Xie
Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403, USA
xiex@bgsu.edu

Received May 31, 2014
and in final form September 16, 2014