Journal of Lie Theory
Volume **25** (2015) 753–774

Lie Semigroups, Homotopy, and Global Extensions of Local Homomorphisms

Eyüp Kizil* **and Jimmie Lawson**

Communicated by K. H. Hofmann

Abstract. For a finite dimensional connected Lie group G with Lie algebra $\mathfrak{g}$, we consider a Lie-generating Lie wedge $\mathbf{W} \subseteq \mathfrak{g}$. If S is a Lie subsemigroup of G with subtangent wedge $\mathbf{W}$ we give sufficient conditions for S to be free on small enough local semigroups $U \cap S$ in the sense that continuous local homomorphisms extend to global ones on S. The constructions involve developing a homotopy theory of $U \cap S$-directed paths. We also consider settings where the free construction leads to a simply connected covering of S.
Mathematics Subject Classification 2010: 22A15, 22E15.
Key Words and Phrases: Lie semigroup, local semigroup, Lie wedge, Lie group, homotopic paths, covering semigroups.

1. Introduction

A subsemigroup of a group is a subset closed under multiplication. For a closed subsemigroup S of a Lie group G containing the identity e, its *subtangent* object $\mathcal{L}(S)$ in the Lie algebra $\mathfrak{g}$ is defined by

$$\mathcal{L}(S) = \{X \in \mathfrak{g} : \exp(tX) \in S \text{ for all } t \geq 0\}.$$

The subtangent object $\mathcal{L}(S)$ is a special type of cone called a Lie wedge; see [4],[5]. Specifically a *Lie wedge* is a closed subset $\mathbf{W}$ of $\mathfrak{g}$ that is closed under addition, scalar multiplication by non-negative scalars, and application of $\exp(\operatorname{ad} X)$ for any $X \in -\mathbf{W} \cap \mathbf{W}$. A *Lie semigroup* is a closed subsemigroup S of a Lie group G containing the identity such that S is infinitesimally generated, i.e., S is the smallest closed subsemigroup of G containing $\exp(\mathcal{L}(S))$ [5]. (An alternative characterization is that a closed subsemigroup S of a Lie group is a Lie semigroup if and only if it is the closure of the semigroup generated by the union of all one-parameter semigroups lying in S.)

Conversely given a Lie wedge $\mathbf{W} \subseteq \mathfrak{g}$, there exist a local subsemigroup $\operatorname{Sg}(\mathbf{W})$ of G with subtangent object $\mathbf{W}$, but not necessarily a corresponding global subsemigroup of G ([4, Chapter V]). It is an open question whether one can define,

* Research partially supported by Fapesp Grant N° 2012/20818-5.

ISSN 0949–5932 / \$2.50

independent of G, such a (global) topological semigroup, whether it has desirable differentiability properties, whether some local semigroup with subtangent object $\mathbf{W}$ can be embedded as an open local subsemigroup, and whether there is a universal such semigroup. Some significant steps were made in regard to this problem in [8], an important reference for our work. The special case that $\mathbf{W}$ is a pointed cone has been studied by W. Weiss in [12], where some of the previously mentioned open problems have been solved for this special case.

Given a local semigroup S, there is a maximal topological semigroup $\hat{S}$ that one can construct from it by taking the free semigroup on the elements of S and dividing out the smallest closed congruence relation that identifies a two-element word with its product for those cases the product is defined and back in S; see Section 2. When one carries out this construction for a small convex neighborhood of 0 in a finite-dimensional real Lie algebra $\mathfrak{g}$ equipped with the Baker-Campbell-Hausdorff multiplication, one obtains the simply connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. (This construction can break down in the infinite-dimensional case.) It has the property that any local homomorphism extends to a global one. An alternative construction is to take any connected Lie group G with Lie algebra $\mathfrak{g}$ and construct $\widetilde{G}$ as the group of homotopy classes of all paths in G with initial point the identity e.

Our main focus in this paper is extending some of these results to the case of Lie semigroups, at least special classes of Lie semigroups. We start with a Lie semigroup S and first identify the class of paths that is appropriate for the semigroup setting, what we call $\mathbf{W}$-directed paths, where $\mathbf{W} = \mathcal{L}(S)$ is the subtangent object of S. Their basic properties are developed in Section 3. Using a restricted homotopy for this class of paths, we obtain in Section 4 a semigroup multiplication induced by concatenation on the set $\Gamma(\mathbf{W})$ of homotopy classes. The endpoint mapping induces a continuous homomorphism of $\Gamma(\mathbf{W})$ onto a dense subsemigroup of S. The semigroup $\Gamma(\mathbf{W})$ has a certain universal property that was worked out in [8].

In Section 5 we consider what we call selections—continuous cross-sections of the homotopy classes. When these exist nice things happen. We are able to show that $\Gamma(\mathbf{W})$ is a topological semigroup and the homomorphism into S is a topological isomorphism, simultaneously an isomorphism and homeomorphism. These are furthermore often topologically isomorphic to the semigroups $\hat{S}$ freely generated by small local subsemigroups of S, which results in continuous local homomorphisms extending to global ones on S. In Section 6 we give some concrete examples in which this theory is applied. In Section 7 we significantly extend the results to certain cases where $\Gamma(\mathbf{W})$ and the free topological semigroups generated by small local subsemigroups are (naturally) topologically isomorphic to the simply connected covering semigroup of the original semigroup S. An important class of such semigroups are the well-known Ol'shanskiĭ semigroups.

2. Local Semigroups

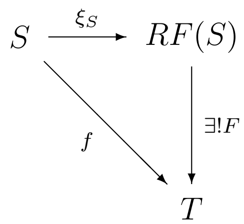
A *topological partial semigroup* consists of a Hausdorff space S equipped with a continuous partial multiplication $m : \mathrm{dom}(m) \subseteq S \times S \to S$ denoted by

$m(x, y) = xy$ that satisfies the implication

$$(x, y), (y, z), (xy, z), (x, yz) \in \mathrm{dom}(m) \Rightarrow (xy)z = x(yz).$$

The category **TPSgp** of topological partial semigroups has as objects topological partial semigroups and as morphisms *continuous partial homomorphisms*, continuous functions $h : S \to T$ satisfying (i) $h \times h(\mathrm{dom}(m_S)) \subseteq \mathrm{dom}(m_T)$ and (ii) $h(st) = h(s)h(t)$ for all $(s, t) \in \mathrm{dom}(m_S)$.

The category **TPSgp** is complete (cf. [2, Section 6.1]) and contains the category **TSgp** of topological semigroups as a full and isomorphism-closed subcategory which is closed under the formation of products and equalizers. Standard arguments ensure the existence of the solution set condition, and hence there exists a left adjoint RF : **TPSgp** $\to$ **TSgp** that sends a topological partial semigroup S to $RF(S)$, the *relatively free topological semigroup over* S. The relatively free semigroup over S can be characterized as a pair $(RF(S), \xi_S)$, where $RF(S)$ is a topological semigroup and $\xi_S : S \to RF(S)$ is a morphism in **TPSgp** such that for any **TPSgp**-morphism $f : S \to T$ into a topological semigroup T, there exists a unique continuous homomorphism $F : RF(S) \to T$ making the following diagram commute:



For the case that S is locally compact and σ-compact, there is a rather straightforward construction yielding $RF(S)$. Let

$$Fr(S) = S \vee S^2 \vee \cdots \vee S^n \vee \cdots$$

be the set of all finite tuples $(s_1, \ldots, s_n)$ for $1 \leq n < \infty$ of members of S. This disjoint union forms a semigroup under the operation of juxtaposition, indeed the free semigroup on the set S. We topologize S by making the union a discrete union (i.e., each S^n is open in $Fr(S)$) and observe that $Fr(S)$ is a locally compact σ-compact Hausdorff space and that the juxtaposition operation is continuous, so that $Fr(S)$ is a topological semigroup. We consider the smallest closed congruence relation $\sim$ on $Fr(S)$ containing all pairs $(s, t) \sim st$ for $s, t, st \in S$. Since $Fr(S)$ is a locally compact, σ-compact topological semigroup, $Fr(S)/\sim$ is a Hausdorff topological semigroup [10], [1]. The map $\xi : S \to Fr(S)/\sim$ sends $s \in S$ to the $\sim$-equivalence class of the singleton tuple s, a continuous map and a morphism in **TPSgp**

A morphism $f : S \to T$ in **TPSgp** into a topological semigroup T extends in the usual way to a homomorphism from the free semigroup $\overline{f} : Fr(S) \to T$ and is continuous since multiplication in T is continuous. The inverse image $(\overline{f} \times \overline{f})^{-1}(\Delta)$ of the (closed) diagonal of T is a closed congruence on $Fr(S)$ identifying each (s, t) with st for $s, t, st \in S$, and hence contains the smallest such, namely $\sim$. There is thus induced the desired continuous homomorphism $F : Fr(S)/\sim \to T$ satisfying $F \circ \xi = f$, so that we may identify $RF(S)$ with $Fr(S)/\sim$.

Our focus will be on the case that G is a connected finite dimensional Lie group with identity e and S is a local semigroup contained in G in the sense that $e \in S$ and there exists a neighborhood V of e such that $st \in S$ for each $s, t \in S \cap V$. A subset S of G is a *local semigroup with respect to* U, an open subset containing e, if $e \in S \subseteq U$ and $st \in S$ whenever $s, t \in S$ and $st \in U$. For any V open containing e, such that $VV \subseteq U$, it will then be the case that $(S \cap V)(S \cap V) \subseteq S$, so that S is also a local semigroup in the previous sense. Note also that if S is a local semigroup with respect to U, then for any V open such that $e \in V \subseteq U$, then $S \cap V$ is a local semigroup with respect to V.

We call a local semigroup S contained in G a *local Lie semigroup* if there exists a Lie wedge $\mathbf{W}$ in the Lie algebra $\mathfrak{g}$ of G and an open set U containing e such that
(i) S is the smallest local semigroup with respect to U that is closed in U and contains $\exp(\mathbf{W} \cap B)$, where B is a BCH-neighborhood of 0 such that exp carries B diffeomorphically onto U, and
(ii) $\mathcal{L}(S) = \mathbf{W}$, where

$$\mathcal{L}(S) = \{X \in \mathfrak{g} : \exists \varepsilon > 0 \text{ such that } \exp(tX) \in S \text{ for } 0 \le t < \varepsilon\}.$$

It is a fundamental theorem of the Lie theory of semigroups that for every finite dimensional Lie group G with Lie wedge $\mathbf{W} \subseteq \mathfrak{g}$, there exists a local Lie group S with subtangent wedge $\mathcal{L}(S) = \mathbf{W}$ [4, Chapter IV]. By a straightforward chasing of the universal diagram, one sees that the question of whether this local semigroup extends to a topological semigroup is equivalent to the question of whether the map $\xi_S : S \to RF(S)$ is a topological embedding.

In light of the preceding considerations it is not surprising that the relatively free semigroup $RF(S)$ can be a useful tool for constructing extensions of topological local semigroups to (global) topological semigroups (for example, W. Weiss [12] uses it as one of his methods to construct global semigroups for local Lie semigroups S that have subtangent set $\mathcal{L}(S)$ a pointed cone). However, our principal goal in this paper is in a different direction. Very little concrete information is available about the explicit structure of $RF(S)$. We develop tools that allow one to start with an S from a certain class of Lie semigroups, take a local semigroup $S \cap U$, where U is an exponential BCH-neighborhood, and show that $RF(S \cap U)$ is in some cases the semigroup S, in others the simply connected covering semigroup of S.

3. Directed Paths

We work throughout assuming that G is a connected finite dimensional Lie group with Lie algebra $\mathfrak{g}$. A *Baker-Campbell-Hausdorff neighborhood*, *BCH-neighborhood* for short, is a symmetric convex open neighborhood B of 0 in $\mathfrak{g}$ such that (i) the exponential mapping $\exp : \mathfrak{g} \to G$ restricted to B is an analytic diffeomorphism, (ii) the Baker-Campbell-Hausdorff multiplication $(X, Y) \to X * Y$ defined from the Baker-Campbell-Hausdorff series is defined on $B \times B$, making it a local group, and (iii) for all $X, Y \in B$, $\exp(X * Y) = \exp(X)\exp(Y)$. It is standard that there exists a basis of BCH-neighborhoods at 0. We call an open neighborhood U of the identity $e \in G$ an *exponential BCH-neighborhood* if it is of the form $\exp B$ for some BCH-neighborhood B.

We start with something more general than a Lie wedge, namely a nonempty subset Ω of $\mathfrak{g}$ that is closed under scalar multiplication by non-negative scalars, i.e., $\mathbb{R}^+\Omega \subseteq \Omega$. We further assume that Ω is *Lie generating* in the sense that $\mathfrak{g}$ is the smallest Lie subalgebra containing Ω. For any exponential BCH-neighborhood $U = \exp B$, we let $\mathrm{Sg}_U(\Omega)$ denote the local semigroup generated by $\exp(\Omega \cap B)$ in U, i.e., the smallest subset of U containing $\exp(\Omega \cap B)$ such that $s, t \in \mathrm{Sg}_U(\Omega)$ and $st \in U$ imply $st \in \mathrm{Sg}_U(\Omega)$. We denote by $\mathrm{Sg}_U(\Omega)^-$ the closure of $\mathrm{Sg}_U(\Omega)$ in U. By continuity of multiplication in G, $\mathrm{Sg}_U(\Omega)^-$ is also a local semigroup with respect to U. We denote by $\mathrm{Sg}(\Omega)$ the semigroup $\langle \exp(\Omega) \rangle_{sg}$ generated by $\exp(\Omega)$, which may be characterized as the smallest subsemigroup of G containing $\exp(\Omega)$, or as the set of all finite products of elements of $\exp(\Omega)$. Its closure $\mathrm{Sg}(\Omega)^-$ in G is the smallest closed subsemigroup of G containing $\exp(\Omega)$. Since $\langle \exp(\mathbb{R}^+X \cap B) \rangle_{sg} = \exp(\mathbb{R}^+X)$ for $X \in \Omega$ and B a BCH-neighborhood, we conclude that $\mathrm{Sg}(\Omega)$ is generated by $\exp(\Omega \cap B)$ for each BCH-neighborhood B.

Remark 3.1. Standard results about Lie generating subsets imply that $\mathrm{Sg}_U(\Omega)$ (and hence $\mathrm{Sg}_U(\Omega)^-$) has dense interior (interior taken in G), in particular, that the identity e of G is in the closure of $\mathrm{int}(\mathrm{Sg}_U(\Omega))$ ([4, Theorem IV.5.18]). Our work will rely heavily on [8], which assumes the setting of local semigroups with dense interior, and the Lie generating hypothesis is needed to ensure the local semigroups $\mathrm{Sg}_U(\Omega)$ that we consider have dense interior.

A basic construction for the universal covering group of a Lie group involves realizing its members as homotopy classes of paths originating from the identity. To generalize this to local semigroups one needs to restrict to a special class of paths tied to the local semigroup. We fix some exponential BCH-neighborhood $U = \exp B$ of e. We recall from [8, Definition 5.2] the notion of a directed path.

Definition 3.2. (i) Let S_U be a local semigroup with respect to U containing e. An *S_U-directed path*, is a continuous function $\alpha : [0, T] \to G$ with $\alpha(0) = e$ that satisfies any of the following three equivalent conditions:

1. Given any $t \in [0, T]$, there exists Q open in $[0, T]$ such that $\alpha(t_2) \in \alpha(t_1)S_U$ for $t_1, t_2 \in Q$ and $t_1 < t_2$.
2. If $t_1 < t_2$ in $[0, T]$ and $\alpha([t_1, t_2]) \subseteq \alpha(t_1)U$, then $\alpha(t_2) \in \alpha(t_1)S_U$.
3. Given any V open, $e \in V \subseteq U$, there exists $\varepsilon > 0$ such that whenever $s, t \in [0, T]$ satisfy $s < t < s + \varepsilon$, then $\alpha(t) \in \alpha(s)(S_U \cap V)$.

Sometimes if S_U is understood, we speak simply of *directed paths*.

Initial sections of one-parameter semigroups are examples of directed paths.

Lemma 3.3. *Suppose for some $X \in \mathfrak{g}$ that $\exp(\mathbb{R}^+X) \cap U \subseteq S_U$, where S_U is a local semigroup with respect to U. Then $\alpha : [0, T] \to G$ defined by $\alpha(t) = \exp(tX)$ is an S_U-directed path.*

Proof. Pick $\varepsilon > 0$ such that $\exp(sX) \in U$, and hence in S_U, for $0 \leq s < 2\varepsilon$. For $t \in [0,T]$, set $Q = (t-\varepsilon, t+\varepsilon) \cap [0,T]$. Then

$$\alpha(t_2) = \exp(t_2X) = \exp(t_1X)\exp((t_2-t_1)X) = \alpha(t_1)\alpha(t_2-t_1) \in \alpha(t_1)S_U$$

for $t_1, t_2 \in Q$, $t_1 < t_2$. ■

Definition 3.4. Given directed paths $\alpha\colon [0,S] \to G$ and $\beta\colon [0,T] \to G$, we define the *concatenation* $\alpha * \beta\colon [0, S+T] \to G$ by

$$\alpha * \beta(t) = \begin{cases} \alpha(t) & \text{for } 0 \leq t \leq T, \\ \alpha(T)\beta(t-T) & \text{for } T \leq t \leq T+S. \end{cases}$$

Remark 3.5. It is straightforward to verify that the concatenation of two directed paths is again a directed path and that the set of directed paths equipped with the concatenation operation is a semigroup ([8, Proposition 5.1]).

Directed paths are a rather extensive class of paths with important robustness properties, including the following convergence property.

Lemma 3.6. *Let S_U be a local semigroup with respect to U that is closed in the relative topology of U. Then a uniform limit of a sequence of S_U-directed paths, all defined on a fixed $[0,T]$, is again an S_U-directed path, and thus the set of S_U-directed paths defined on $[0,T]$ is closed in the topology of uniform convergence.*

Proof. Let $\{\alpha_n\}$ be a sequence of S_U-directed paths uniformly converging to α. Suppose $0 \leq s < t \leq T$ and $\alpha([s,t]) \subseteq \alpha(s)U$. By uniform convergence $(\alpha_n(s))^{-1}(\alpha_n([s,t]) \subseteq U$ for all n large enough, and hence by S_U-directedness $(\alpha_n(s))^{-1}\alpha_n([s,t]) \subseteq S_U$. Using the closedness of S_U in U we conclude that $(\alpha(s))^{-1}\alpha([s,t]) \subseteq S_U$, and hence that α is S_U-directed. It follows that the set of S_U-directed paths is closed in the space of all paths defined on $[0,T]$ equipped with the topology of uniform convergence. ■

In geometric control theory one considers a subset $\Omega \subseteq \mathfrak{g}$ as a *control set* consisting of vector fields on G invariant under left (alternatively right) translation. *Steering functions* $u : [0,T] \to \Omega$ give rise to *trajectories*, solutions on $[0,T]$ of the associated differential equation

$$\dot{x}(t) = u(t)(x(t)), \quad x(0) = e. \tag{1}$$

If $0 = t_0 < t_1 < \cdots < t_n = T$, and $u(t) = X_i \in \Omega$ for $t_{i-1} \leq t < t_i$, then u is called a *piecewise constant control* and has solution on $[0,T]$ given by

$$x(t) = \exp(t_1X_1)\cdots\exp(t_{i-1}X_{i-1})\exp((t-t_{i-1})X_i) \text{ for } t_{i-1} \leq t < t_i.$$

This solution $x(\cdot)$ is the concatenation $\alpha_1 * \cdots * \alpha_n$, where $\alpha_i : [0, t_i - t_{i-1}] \to G$ is given by $\alpha_i(t) = \exp(tX_i)$, and thus by Lemma 3.3 and Remark 3.5 is $\mathrm{Sg}_U(\Omega)$-directed. By the weak-$*$ density of the piecewise constant functions in the bounded measurable controls and the continuous dependence of solutions on control, we conclude from Lemma 3.6 that all solutions of equation (1), i.e., all trajectories of the control system, are $\mathrm{Sg}_U(\Omega)^-$-directed. We have thus derived the following proposition.

Proposition 3.7. *Let $\Omega = \mathbb{R}^+\Omega \subseteq \mathfrak{g}$ be Lie generating, let $U = \exp B$ be an exponential BCH-neighborhood and let $\mathrm{Sg}_U(\Omega)^-$ be the closure in U of the local semigroup with respect to U generated by $\exp(\Omega \cap B)$. Then every trajectory of equation (1) is a directed path with respect to the local semigroup $\mathrm{Sg}_U(\Omega)^-$.*

Remark 3.8. Let $\mathbf{W}$ be a Lie-generating Lie wedge in $\mathfrak{g}$. A path $\alpha : [0,T] \to G$ is called $\mathbf{W}$-*admissible* if it is piecewise smooth and if $\dot{\alpha}(t) \in d\lambda(\alpha(t))(\mathbf{W})$, the left translate of the wedge to $\alpha(t)$; see Definition VI.1.7 of [4]. By Proposition VI.1.13 of [4], α can be uniformly approximated by trajectories with piecewise constant controls, so as in the preceding proposition, one concludes that the $\mathbf{W}$-admissible paths that start at e are directed paths with respect to the local semigroup $\mathrm{Sg}_U(\Omega)^-$.

We see from the preceding proposition and remark that $\mathrm{Sg}_U(\Omega)^-$-directed paths include those paths that have been important in the theory of Lie semigroups.

Local semigroups S_U and S_V are said to *belong to the same germ* if there exists a neighborhood W of e such that $S_U \cap W = S_V \cap W$.

Lemma 3.9. *If local semigroups S_U and S_V belong to the same germ, then a path α is S_U-directed if and only if it is S_V-directed.*

Proof. Suppose an exponential BCH-neighborhood W of e is chosen small enough so that $(S_U \cap W)(S_U \cap W) \subseteq S_U$, $(S_V \cap W)(S_V \cap W) \subseteq S_V$, and $S_U \cap W = S_V \cap W$. If $\alpha : [0,T] \to G$ is S_U-directed, then (by definition) there exists $\varepsilon > 0$ such that whenever $s, t \in [0,T]$ satisfy $s < t < s + \varepsilon$, then $\alpha(t) \in \alpha(s)(S_U \cap W) = \alpha(s)(S_V \cap W)$. Since this holds for all sufficiently small W, we conclude that α is also S_V-directed. Since one may reverse the roles of S_U and S_V, the proof is complete. ∎

Corollary 3.10. *Let $\mathbf{W} \subseteq \mathfrak{g}$ be a Lie-generating Lie wedge. Then there exists an exponential BCH-neighborhood U such that a path $\alpha : [0,T] \to G$ is $(\mathrm{Sg}_U(\mathbf{W}))^-$-directed if and only if it is $(\mathrm{Sg}_V(\mathbf{W}))^-$-directed for every open set V such that $e \in V \subseteq U$.*

Proof. By [4, Theorem IV.8.6] there is an exponential BCH-neighborhood U of e such that the local semigroups $(\mathrm{Sg}_V(\mathbf{W}))^-$ all belong to the same germ as V ranges over all open V such that $e \in V \subseteq U$. The conclusion follows from Lemma 3.9. ∎

The preceding corollary shows that the directed paths depend on the Lie wedge alone, not the particular local semigroup that is chosen.

Definition 3.11. Let $\mathbf{W}$ be a Lie-generating Lie wedge in the Lie algebra $\mathfrak{g}$ of a Lie group G. A path $\alpha : [0,T] \to G$ is called a $\mathbf{W}$-directed path if it is directed with respect to $\mathrm{Sg}_U(\mathbf{W})^-$ for all sufficiently small open neighborhoods U of e.

Remark 3.12. We note that $\mathbf{W}$-admissible paths starting at e are particular examples of $\mathbf{W}$-directed paths by Remark 3.8.

4. Directed Homotopy

We continue in the setting that G is a finite dimensional Lie group with Lie algebra $\mathfrak{g}$ and $\mathbf{W} \subseteq \mathfrak{g}$ is a Lie-generating Lie wedge. We turn now to the homotopy theory of $\mathbf{W}$-directed paths. *For the purposes of homotopy we restrict our attention to* $\mathbf{W}$*-directed paths defined on* $[0,1]$. Any $\mathbf{W}$-directed path $\alpha : [0,T] \to G$ can be converted to such a path by the *normalization* $\alpha_\nu(t) = \alpha(t/T)$. The concatenation of normalized paths is the concatenation of the previous section followed by normalization:

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2, \\ \alpha(1)\beta(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Remark 4.1. For a $\mathbf{W}$-directed path α, define $\varepsilon(\alpha) = \alpha(1)$. The map ε is called the *endpoint map.* Note that the following homomorphic property holds:

$$\varepsilon(\alpha * \beta) = (\alpha * \beta)(1) = \alpha(1) \cdot \beta(1) = \varepsilon(\alpha) \cdot \varepsilon(\beta).$$

Equipping the space of $\mathbf{W}$-directed paths with the topology of uniform convergence, we note that ε is also continuous.

We recall the appropriate notion of homotopy for $\mathbf{W}$-directed paths [8].

Definition 4.2. For some fixed S_U, a local semigroup with respect to U, two S_U-directed paths $\alpha, \beta : [0,1] \to G$ are *homotopic with respect to* S_U if there exists an S_U*-homotopy* $H : [0,1] \times [0,1] \to G$ between them, i.e., a continuous map H satisfying

(i) $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$ for $0 \le t \le 1$;

(ii) the map $\gamma_s : [0,1] \to G$ defined by $\gamma_s(t) = H(t,s)$ is a an S_U-directed path with respect to S_U;

(iii) $H(1,s) = \alpha(1) = \beta(1)$ for $0 \le s \le 1$.

Two $\mathbf{W}$-directed paths are $\mathbf{W}$-*homotopic* if they are homotopic with respect to $\mathrm{Sg}_U(\mathbf{W})^-$ for all small enough open neighborhoods U of e.

The next result gives basic properties of $\mathbf{W}$-homotopy that are corollaries of the theory for local semigroups; see [8]. The proofs are rather straightforward modifications of those for the usual homotopy of paths (see, for example, Sections 1.8 and 1.9 of [11]).

Proposition 4.3. *The relation of* $\mathbf{W}$*-homotopy is an equivalence relation on the set of* $\mathbf{W}$*-directed paths, and the concatenation operation induces a well-defined associative operation on the set of* $\mathbf{W}$*-homotopy classes of* $\mathbf{W}$*-directed paths.*

Remark 4.4. We denote the semigroup of $\mathbf{W}$-homotopy classes of $\mathbf{W}$-directed paths by $\Gamma(\mathbf{W})$. We equip $\Gamma(\mathbf{W})$ with the quotient topology from the space of $\mathbf{W}$-directed paths equipped with the topology of uniform topology. The continuity of the concatenation operation on the space of $\mathbf{W}$-directed paths implies left and right translations maps are continuous, which in turn implies that the induced left and right translations maps on the semigroup $\Gamma(\mathbf{W})$ are continuous. Furthermore, the endpoint map ε induces a continuous homomorphism $\tilde{\varepsilon} : \Gamma(\mathbf{W}) \to G$.

We turn to a "freeness" property of $\Gamma(\mathbf{W})$. Let $\alpha : [0,1] \to G$ be a $\mathrm{Sg}_U(\mathbf{W})^-$-directed path. Let $0 = t_0 < t_1 < \ldots < t_n = 1$ be a U-*fine* partition, one fine enough such that $\alpha([t_{i-1}, t_i]) \subseteq \alpha(t_{i-1})\,\mathrm{Sg}_U(\mathbf{W})^-$ for $1 \le i \le n$. Then $\alpha(t_i) = \alpha(t_{i-1})s_i$ where $s_i \in \mathrm{Sg}_U(\mathbf{W})^-$, i.e., $s_i = (\alpha(t_{i-1}))^{-1}\alpha(t_i)$. If $\psi : \mathrm{Sg}_U(\mathbf{W})^- \to T$ is a morphism in **TPSgp** into a topological semigroup T, we define $\Psi(\alpha)$ to be $\psi(s_1)\cdots\psi(s_n)$. If we take another U-fine partition that refines the first one, then a straightforward computation based on the fact that $\mathrm{Sg}_U(\mathbf{W})^-$ is a local semigroup establishes that calculating $\Psi(\alpha)$ from the second partition yields the same $\Psi(\alpha)$. Hence any two U-fine partitions yield the same result, as one sees by taking a common partition refinement of both. See Lemma 6.1 of [8].

Proposition 4.5. *Let* U *be an exponential BCH-neighborhood small enough so that the* $\mathrm{Sg}_U(\mathbf{W})^-$*-directed paths agree with the* $\mathbf{W}$*-directed paths. For any* $\psi : \mathrm{Sg}_U(\mathbf{W})^- \to T$*, a morphism in* ***TPSgp*** *into a topological semigroup* T*, there exists a unique continuous homomorphism* $\widetilde{\Psi}$ *from* $\Gamma(\mathbf{W})$ *into* T *that sends each* $\mathbf{W}$*-directed path* α *with* $\alpha([0,1]) \subseteq U$ *to* $\psi(\alpha(1)) \in T$*.*

Proof. We first define the map on the collection of $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths as described just before the statement of the proposition. Let α_k be a sequence of $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths converging uniformly to another such path α. Fix some U-fine partition $0 = t_0 < \cdots < t_n = 1$ for α. Then $s_i^k := (\alpha_k(t_{i-1}))^{-1}\alpha_k(t_i)$ converges to $s_i := (\alpha(t_{i-1}))^{-1}\alpha(t_i)$ for each i, so $\Psi(\alpha_k) = \psi(s_1^k)\cdots\psi(s_n^k)$ converges to $\Psi(\alpha) = \psi(s_1)\cdots\psi(s_n)$. Thus the map Ψ is continuous at α, and since α was arbitrary, continuous everywhere.

The fact that that this continuous map from the space of $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths to the topological semigroup T is constant on homotopy classes follows from Theorem 7.2 and its proof in [8], and hence a continuous map $\widetilde{\Psi} : \Gamma(\mathbf{W}) \to T$ is induced. The other assertions follow from [8, Theorem 7.2]. ■

Remark 4.6. (i) The preceding proposition applies to the relative free semigroup $T = RF(\mathrm{Sg}_U(\mathbf{W})^-)$. Thus the morphism $\xi : \mathrm{Sg}_U(\mathbf{W})^- \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ that sends a member $x \in \mathrm{Sg}_U(\mathbf{W})^-$ to the $\sim$-class of the singleton word x extends to a continuous homomorphism $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$.
(ii) If S is a Lie semigroup with subtangent object $\mathbf{W}$, then by definition S is

closed and its intersection with U is a local semigroup, so S contains $\mathrm{Sg}_U(\mathbf{W})^-$. The inclusion $g\colon \mathrm{Sg}_U(\mathbf{W})^- \hookrightarrow S$ induces a continuous homomorphism

$$\gamma\colon RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S,$$

and one verifies directly that the composition $\gamma\circ\Xi$ is the induced endpoint mapping $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ that sends the $\mathbf{W}$-homotopy class $[\alpha]$ to $\alpha(1)$.

The following will be a useful corollary for identifying when the relative free semigroup of a local semigroup agrees with $\Gamma(\mathbf{W})$.

Corollary 4.7. *Let S be a Lie semigroup in G with subtangent wedge $\mathcal{L}(S) = \mathbf{W}$, a Lie-generating Lie wedge, such that the endpoint mapping $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ is a topological isomorphism. Let U be an exponential BCH-neighborhood small enough so that the $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths agree with the $\mathbf{W}$-directed paths. Assume further that for each $x \in \mathrm{Sg}_U(\mathbf{W})^-$, there exists an $\mathrm{Sg}_U(\mathbf{W})^-$-directed path contained in U that has endpoint x. Then the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ and $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$ are topological isomorphisms.*

Proof. By the preceding remark, $\widetilde{\varepsilon} = \gamma\circ\Xi$ and γ and Ξ are continuous homomorphisms. Since Ξ sends the homotopy classes of $\mathbf{W}$-directed paths contained in U to the $\sim$-class of their endpoints (thought of as singleton words), the image of Ξ contains a set of generators of $RF(\mathrm{Sg}_U(\mathbf{W})^-)$, and hence is surjective. Since $\gamma\circ\Xi = \widetilde{\varepsilon}$ is injective by hypothesis, it follows that Ξ is also injective. Thus $\gamma = \widetilde{\varepsilon}\circ\Xi^{-1}$ is also bijective. Since $\gamma^{-1} = \Xi\circ\widetilde{\varepsilon}^{-1}$, it is continuous. Similarly $\Xi^{-1} = \widetilde{\varepsilon}^{-1}\circ\gamma$ is continuous. ■

5. Selections

We continue to work in the setting of a finite dimensional connected Lie group G with Lie algebra $\mathfrak{g}$. We further assume that G contains a Lie semigroup S with Lie-generating subtangent wedge $\mathbf{W}$. We conjecture that if S is simply connected, then $S = \Gamma(\mathbf{W}) = \mathrm{Sg}_U(\mathbf{W})^-$ for a basis of exponential BCH-neighborhoods U. If true, this would mean that continuous homomorphisms defined locally on S extend to global continuous homomorphisms, a well known property for simply connected Lie groups. We show in this section that this is true for semigroups satisfying a stronger hypothesis than simple connectivity, namely those that admit a global selection; see Definition 5.3. *For convenience, we denote by $\mathcal{A}$ the class of $\mathbf{W}$-directed paths.*

A modified form of concatenation will be important in our arguments in this section. It provides a tool for building a homotopy between α and β if we have a continuous family α_s of paths, $0 \le s \le 1$ such that α_s terminates at $\beta(s)$.

Definition 5.1. Let $\alpha, \beta : [0,1] \to S$ be paths. Suppose that $\alpha(1) = \beta(s)$ for some $s > 0$. The *spliced concatenation* (at s) of α and β is given by

$$\alpha *_s \beta(t) = \begin{cases} \alpha(t/s) & \text{for } 0 \le t \le s, \\ \beta(t) & \text{for } s \le t \le 1. \end{cases}$$

Remark 5.2. It is straightforward to check, and runs along the same lines as for the usual concatenation, that the spliced concatenation yields another $\mathbf{W}$-directed path, provided α and β are $\mathbf{W}$-directed.

We come now to the key concept of this section.

Definition 5.3. A *selection* with respect to the family $\mathcal{A}$ is a function $\psi : S \to \mathcal{A}$ satisfying (i) $\psi(x)(1) = x$, (ii) $\psi(e) = e^*$, the constant path with image e, and (iii) the map $(x,t) \mapsto \psi(x)(t)$ from $S \times [0,1]$ to S is continuous.

Theorem 5.4. (*Selection Theorem*) *Let S be a Lie semigroup with Lie-generating subtangent wedge $\mathbf{W}$, and suppose the family $\mathcal{A}$ of $\mathbf{W}$-directed paths admits a selection $\psi : S \to \mathcal{A}$. Then the mapping $\widetilde{\varepsilon}$ that assigns to a $\mathbf{W}$-homotopic class the common endpoint of the class is a topological isomorphism from $\Gamma(\mathbf{W})$ to S.*

Proof. Let $\alpha \in \mathcal{A}$, $\alpha(1) = x$ and let $\beta = \psi(x)$. We define a $\mathbf{W}$-homotopy $H{:}[0,1]\times[0,1]\to S$ between α and β. We define $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$. For $0 < s < 1$, we define γ_s to be the spliced concatenation of α and $\psi(\alpha(s))$, i.e., $\gamma_s(t) = \psi(\alpha(s))(t/s)$ for $0 \le t \le s$ and $\gamma_s(t) = \alpha(t)$ for $s \le t \le 1$. We set $H(t,s) = \gamma_s(t)$. By Remark 5.2 $\gamma_s \in \mathcal{A}$.

We check that H is continuous. We consider first H restricted to $\{(t,s) : s \le t\}$. It follows by definition that $H(t,s) = \alpha(t)$ on this set and is hence continuous on it. Next consider $\{(t,s) : t \le s\}$. On this set $H(t,s) = \gamma_s(t) = \psi(\alpha(s))(t/s)$ for $s \ne 0$, and is thus continuous by (iii) of Definition 5.3. For the case $s = 0$, then also $t = 0$, and continuity at this point follows from the assumption that $\psi(e) = e^*$, the constant path with value e, and the uniform continuity of convergence on $[0,1]$ of $\psi(x)$ to $\psi(e) = e^*$, which follows from (iii) of Definition 5.3. Thus H is continuous since it is continuous on each of two closed sets whose union is $[0,1] \times [0,1]$.

We conclude that H is a $\mathbf{W}$-homotopy between α and $\beta = \psi(x)$, where $x = \alpha(1)$. It follows that any two $\mathbf{W}$-directed paths with the same endpoint x are $\mathbf{W}$-homotopic to $\beta = \psi(x)$, and hence $\mathbf{W}$-homotopic to each other. Thus the map $\widetilde{\varepsilon}$ that sends a member of $\Gamma(\mathbf{W})$ to the common endpoint of its representatives is injective. It is also surjective since any $x \in S$ is the image of the $\mathbf{W}$-homotopy class of $\psi(x)$ and continuous by Remark 4.4. The inverse map that sends x to the $\mathbf{W}$-homotopy class of $\psi(x)$ is the composition of two continuous maps, hence continuous. ■

Corollary 5.5. *Let S be a Lie semigroup with Lie-generating subtangent wedge $\mathbf{W}$, and suppose the family $\mathcal{A}$ of $\mathbf{W}$-directed paths admits a selection $\psi : S \to \mathcal{A}$. Suppose further that U is an exponential BCH-neighborhood small enough so that the $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths agree with the $\mathbf{W}$-directed paths and that for each $x \in \mathrm{Sg}_U(\mathbf{W})^-$, there exists an $\mathrm{Sg}_U(\mathbf{W})^-$-directed path contained in U that has endpoint x. Then the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ and $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$ are topological isomorphisms.*

Proof. The corollary is an immediate consequence of the preceding theorem and Corollary 4.7. ■

Remark 5.6. Under the conditions of the preceding theorem, if S has a basis of exponential BCH neighborhooods U satisfying for each $x \in \mathrm{Sg}_U(\mathbf{W})^-$, there exists an $\mathrm{Sg}_U(\mathbf{W})^-$-directed path contained in U that has endpoint x, then any local continuous homomorphism h into a topological semigroup T has a unique continuous homomorphic local extension $H : S \to T$ that agrees with h on some neighborhood of e in S. Indeed restrict the homomorphism to a smaller of the just mentioned local semigroups, extend it to $RF(\mathrm{Sg}_U(\mathbf{W})^-)$ by its "freeness" property, and note the identification with S by the previous corollary. Since S is a Lie semigroup, the semigroup generated by $\mathrm{Sg}_U(\mathbf{W})^-$ is dense in S, which insures the uniqueness of the continuous homomorphic extension.

The principal techniques of the preceding proof can be carried out in certain settings where only regional, not global selections exist.

Definition 5.7. A *partial selection* with respect to the family $\mathcal{A}$ of $\mathbf{W}$-directed paths for the Lie semigroup S is a function $\psi : T \to \mathcal{A}$, where T is a subset of S containing e satisfying (i) $\psi(x)(1) = x$ for all $x \in T$, (ii) $\psi(e) = e^*$, the constant path with image e, and (iii) the map $(x,t) \mapsto \psi(x)(t)$ from $T \times [0,1]$ to S is continuous.

Corollary 5.8. *Let S be a Lie semigroup with identity e, let T be a subset containing e, and suppose that there exists a partial selection $\psi : T \to \mathcal{A}$. Then any $\mathbf{W}$-directed path α with image contained in T is $\mathbf{W}$-homotopic to $\psi(\alpha(1))$ and any two $\mathbf{W}$-directed paths that are contained in T and have the same endpoint are $\mathbf{W}$-homotopic in S.*

Proof. For $\alpha \in \mathcal{A}$ such that $\alpha([0,1]) \subseteq T$, we may define a $\mathbf{W}$-homotopy H between α and $\beta = \psi(\alpha(1))$ using spliced concatenation in the same manner as that given in the first part of the proof of Theorem 5.4. It follows that α and β are $\mathbf{W}$-homotopic (in S). Similarly if $\gamma \in \mathcal{A}$ has image lying in T and has $\gamma(1) = \alpha(1)$, then γ is $\mathbf{W}$-homotopic to β. Hence α and γ are $\mathbf{W}$-homotopic. ■

There are variants of the ideas of locally path connected and semilocally simply connected that are important in the setting of $\mathbf{W}$-directed paths.

Definition 5.9. Let G be a connected Lie group and $\mathbf{W}$ a generating Lie wedge in its Lie algebra $\mathfrak{g}$.
(i) We say G is *locally $\mathbf{W}$-directed path connected at e* if there is a basis of neighborhoods U of e such that each point of $\mathrm{Sg}_U(\mathbf{W})^-$ is the terminal point for an $\mathbf{W}$-directed path lying entirely in U.
(ii) We say G is *semilocally simply connected at e* (with respect to $\mathbf{W}$) if the basis of neighborhoods of part (i) further satisfies the condition that any two $\mathbf{W}$-directed paths lying in U with the same endpoints are $\mathbf{W}$-homotopic in S.

We close this section with the following open problem.

Problem 1. If S is a Lie subsemigroup of a Lie group G, is there some neighborhood of the identity e in S for which properties (i) and (ii) of the preceding definition hold. Is there some neighborhood that admits a partial selection?

6. Some Examples

A semigroup is called *uniquely divisible* if each element has a unique n-th root for each $n > 1$. If S is a Lie semigroup with subtangent wedge $\mathbf{W}$ and the exponential function $\exp : \mathbf{W} \to S$ is a homeomorphism onto S, then S is uniquely divisible. In this case the semigroup S has an obvious selection, namely we define $\psi(s) = \alpha_s$, where $\alpha_s(t) = \exp(t\log(s))$ for $0 \leq t \leq 1$.

Proposition 6.1. *Let S be a Lie semigroup in G and let $\mathcal{L}(S) = \mathbf{W}$, a generating Lie wedge. If $\exp : \mathbf{W} \to S$ is a homeomorphism, then for any exponential BCH-neighborhood $U = \exp B$, the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$, $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$, and $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ are topological isomorphisms. Furthermore, continuous locally defined homomorphisms on S may be (uniquely) extended to continuous global ones.*

Proof. By Theorem 5.4 and the comments before the statement of this proposition, the map $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ is a topological isomorphism. For $x \in U = \exp B$ with $x = \exp X$, $X \in B$, the image of the $\mathbf{W}$-directed $\psi(x)$ is $\exp([0,1])X)$, which is contained in S. Since $B \cap \mathbf{W}$ is closed in B, its image under exp is closed in U, i.e., $\mathrm{Sg}_U(\mathbf{W})^- = U \cap S = U \cap \exp \mathbf{W}$. Thus Corollary 5.5 applies and yields all desired topological isomorphisms. The last assertion follows from Remark 5.6. ■

Remark 6.2. The preceding results apply almost trivially in the setting of simply connected abelian groups, where $\mathbf{W}$ may be chosen to be any closed wedge in $\mathfrak{g}$ with interior, and G (up to isomorphism) may be chosen to be $\mathfrak{g}$ with the identity the exponential function.

6.1. Subsemigroups of Almost-Abelian Groups.

The notion of an *almost abelian* Lie group and the basic theory of its Lie semigroups is laid out on pages 93–95 of Section II.2 and in Example V.4.13 of [4]. Any such group can be realized up to an isomorphism of Lie groups as a matrix group of the following form:

$$G = \left\{ \begin{pmatrix} tI_n & v \\ 0 & 1 \end{pmatrix} \middle| 0 < t,\, v \in \mathbb{R}^n \right\}, \qquad \mathfrak{g} = \left\{ \begin{pmatrix} tI_n & v \\ 0 & 0 \end{pmatrix} \middle| t \in \mathbb{R},\, v \in \mathbb{R}^n \right\},$$

where I_n is the $n \times n$-identity matrix. The standard matrix exponential function is a diffeomorphism from $\mathfrak{g}$ to G. For every wedge $\mathbf{W}$ (a closed subset of $\mathfrak{g}$ that is closed under addition and multiplication by non-negative scalars), the image $\exp \mathbf{W}$ is a Lie subsemigroup of G that is uniquely divisible. Hence by Proposition 6.1 for any exponential BCH-neighborhood $U = \exp B$, $\mathrm{Sg}_U(\mathbf{W}) = \mathrm{Sg}_U(\mathbf{W})^- =$

$\exp(B \cap W)$, and the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$, $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$, and $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ are topological isomorphisms. Furthermore, continuous locally defined homomorphisms on S may be (uniquely) extended to continuous global ones.

6.2. An Invariant Semigroup of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$.

Let $\mathrm{sl}(2,\mathbb{R})$ denote the Lie algebra of real 2×2-matrices of trace 0, the Lie algebra of the special linear group $\mathrm{Sl}(2,\mathbb{R})$. We introduce the normalized trace $\tau(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \frac{1}{2}(a+d)$ and a normalized Cartan-Killing form

$$k : \mathrm{sl}(2,\mathbb{R}) \times \mathrm{sl}(2,\mathbb{R}) \to \mathbb{R} \text{ defined by } k(X,Y) = \tau(XY).$$

The set of points X such that $k(X,X) \le 0$ is a double cone, called the *standard double cone.* The set $\mathbb{R}U$, where $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, forms the axis of symmetry of this cone. Let $\mathcal{K}$ denote that part of the cone containing the ray $\mathbb{R}^+U$. We denote the conal boundary of $\mathcal{K}$, namely $\{X \in \mathcal{K} : k(X,X) = 0\}$, by $\mathcal{N}$.

The simply connected covering $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ of $\mathrm{Sl}(2,\mathbb{R})$ also has Lie algebra $\mathrm{sl}(2,\mathbb{R})$. A geometric realization of $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ is given in [4] in the subsection "Semigroups of semisimple Lie groups," pages 414–434, a realization obtained by assigning a coordinate system to $\mathrm{sl}(2,\mathbb{R})$ of the form $hH + tT + uU$, $h,t,u \in \mathbb{R}$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The HT-plane is taken as the horizontal plane and the line through U as the vertical axis. The $\widetilde{\mathrm{Sl}}(2,\mathbb{R})$ group operation $\circ$ on $\mathrm{sl}(2,\mathbb{R})$ is defined in such a way that the 0-matrix is the identity and the exponential function $\mathrm{Exp}:\mathrm{sl}(2,\mathbb{R}) \to \widetilde{\mathrm{Sl}}(2,\mathbb{R}) (= \mathrm{sl}(2,\mathbb{R}))$ is the identity when restricted to $\mathbb{R}U$ and to the HT-plane. It is shown in the just cited [4] that while $\mathrm{Exp}(\mathcal{K})$ is not a subsemigroup of $(\mathrm{sl}(2,\mathbb{R}),\circ)$, the set $S = \mathrm{Exp}(\mathcal{K})\,\mathrm{Exp}(\mathcal{K})$ is a closed Lie subsemigroup with $\mathcal{L}(S) = \mathcal{K}$, a semigroup which is invariant under all inner automorphisms.

The exponential image of a ray in $\mathcal{N}$ that is in the intersection of $\mathcal{N}$ with the HU-plane is an increasing concave-down curve that approaches the horizontal line through $(\pi/2)U$ in the HU-plane as an asymptote (see [4, Figure 17]). The image $\mathrm{Exp}(\mathcal{N})$ is the surface of revolution obtained by revolving this curve around the U-axis and the semigroup S is the closed region above this surface. By [4, Theorem V.4.40] Exp restricted to $\mathcal{N}$ is a homeomorphism, and then from [4, Theorem V.4.37] every member of S has a unique representation of the form $\mathrm{Exp}(X) \circ rU$, where $X \in \mathcal{N}$ and $0 \le r$. This unique factorization then determines a selection for S given by $\psi(\mathrm{Exp}(X) \circ rU) = \alpha * \beta$, where $\alpha(t) = \mathrm{Exp}(tX)$ and $\beta(t) = trU$ for $0 \le t \le 1$. Near the identity all members of S lie on one-parameter semigroups, so the hypotheses and hence conclusions of Corollary 5.5 are satisfied and yield all desired topological isomorphisms.

6.3. The Heisenberg Beak.

Let G be the 3-dimensional Heisenberg Lie group, usually represented as the multiplicative group of all real 3×3 matrices of the form

$$(a,b,c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c \in \mathbb{R}.$$

Its Lie algebra $\mathfrak{g}$ can be represented as the real 3×3 matrices of the form

$$(x,y,z) = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}, \qquad x,y,z \in \mathbb{R}.$$

Taking (x,y,z) to be each of the three unit vectors yields the standard basis $\{X^1, X^2, X^3\}$ of $\mathfrak{g} = \mathbb{R}X^1 + \mathbb{R}X^2 + \mathbb{R}X^3$, which has only one nonvanishing Lie bracket, namely $[X^1, X^2] = X^3$. Since all triple Lie bracket products are 0, the Campbell-Baker-Hausdorff multiplication is globally defined and reduces to

$$Z * Y = Z + Y + \frac{1}{2}[Z,Y].$$

We consider the semigroup S, called the *Heisenberg beak*, generated in the Lie group $(G,\cdot)$ by the pointed cone, and hence Lie wedge, $\mathbf{W} = \{(s,t,0) : s,t \geq 0\}$. As worked in [5] in Section 2.1, the semigroup S is given by

$$S = \{(a,b,c) \in G : 0 \leq a,b \text{ and } 0 \leq c \leq ab\}. \tag{2}$$

and has subtangent object $\mathcal{L}(S) = \mathbf{W}$. We may view S as the region in the first octant bounded above by the surface $z = xy$ and below by the xy-plane. The one-parameter semigroups $\sigma(t) = \exp t(1,0,0) = (t,0,0)$ and $\tau(t) = \exp t(0,1,0) = (0,t,0)$ generate S. In fact, if $(a,b,c) \in S$ and $b > 0$ then

$$(a,b,c) = \sigma(\frac{c}{b}) \cdot \tau(b) \cdot \sigma(a - \frac{c}{b}), \tag{3}$$

where "$\cdot$" is defined as

$$(a_1,a_2,a_3) \cdot (b_1,b_2,b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3 + a_1 b_2). \tag{4}$$

If $b = 0$, then $c = 0$ and hence in this case $(a,b,c) = \sigma(a)$. Alternatively for $(a,b,c) \in S$ and $a > 0$ we may write

$$(a,b,c) = \tau(b - \frac{c}{a}) \cdot \sigma(a) \cdot \tau(\frac{c}{a}), \tag{5}$$

We denote the identity $(0,0,0)$ of S (and G) by $\mathbf{0}$.

Our goal is to establish that $\Gamma(\mathbf{W})$ is topologically isomorphic to S under the induced endpoint mapping. The ideal way to do this would be to find a selection function for the class of $\mathbf{W}$-directed paths, but we have had to resort to partial selection functions instead. For this purpose we consider the two subsets T_1 and T_2 of S given by $T_1 = \{(a,b,c) \in S : a > 0\} \cup \{\mathbf{0}\}$ and $T_2 = \{(a,b,c) \in S : b > 0\} \cup \{\mathbf{0}\}$. We note that both T_1 and T_2 are subsemigroups of S.

To obtain a selection on T_2 we use formula (3) to build our path for the case $b > 0$. We start by following σ over the time interval $[0, c/b]$ arriving at $\sigma(c/b)$. We then travel from $\sigma(c/b)$ to $\sigma(c/b)\cdot\tau(b)$ by following $\sigma(c/b)\cdot\tau(t)$ over the interval $[0,b]$. Finally we follow $\sigma(b/c)\cdot\tau(b)\cdot\sigma(t)$ for $0 \le t \le a-(c/b)$. If we concatenate these to a map over $[0,T]$, where $T = (c/b)+b+a-(c/b) = a+b$ and reparameterize to the interval $[0,1]$ by dividing by T, we obtain a $\mathbf{W}$-directed path from $(0,0,0)$ to (a,b,c), and the selection $\psi(a,b,c)$ is defined to be this path. As $(a,b,c) \to (0,0,0)$, also $c/b \to 0$ and $a-(c/b) \to 0$ since $0 \le c \le ab$. From this one sees directly that in the topology of uniform convergence $\psi(a,b,c) \to \mathbf{0}^*$ as $(a,b,c) \to (0,0,0)$, where $\mathbf{0}^*$ is the constant path with value $\mathbf{0}$. Thus defining $\psi(\mathbf{0}) = \mathbf{0}^*$, extends ψ to a partial selection on T_2. In a completely analogous way we use formula (5) to define a partial selection ξ on T_1.

Suppose now that $\beta \ne \mathbf{0}^*$ is a $\mathbf{W}$-directed path with $\beta(1) = (a,b,c)$. Since S lies in the first octant and the group multiplication is addition in the first two coordinates (equation (4)), it follows that any $\mathbf{W}$-directed path is nondecreasing in the first two coordinates. Hence the image of β cannot touch both the X^1-axis and the X^2-axis except at $\mathbf{0}$. It follows that the image of β must lie in T_1 or T_2. By Corollary 5.8 β is $\mathbf{W}$-homotopic to $\xi(a,b,c)$ or $\psi(a,b,c)$ resp.

We show that two $\mathbf{W}$-directed paths with the same endpoint are $\mathbf{W}$-homotopic. We omit the trivial case that the endpoint is $\mathbf{0}$, since $\mathbf{0}^*$ is the only possible path in this case. There are then two cases to consider.

Case 1: Suppose $a, b > 0$. Then (a,b,c) lies in both T_1 and T_2, and hence both $\xi(a,b,c)$ and $\psi(a,b,c)$ are defined. If $c = 0$ or $c = ab$, then from their definitions $\xi(a,b,c) = \psi(a,b,c)$, so by the previous paragraph every $\mathbf{W}$-directed path ending at (a,b,c) is $\mathbf{W}$-homotopic to $\xi(a,b,c) = \psi(a,b,c)$, and thus they are all homotopic to one another. If $0 < c < ab$, then from its definition $\psi(a,b,c)$ lies in T_1 and $\xi(a,b,c)$ lies in T_2. By Corollary 5.8 applied to T_2, $\xi(a,b,c)$ is $\mathbf{W}$-homotopic to $\psi(a,b,c)$, and since any β with endpoint (a,b,c) is $\mathbf{W}$-homotopic to one of them, they are all $\mathbf{W}$-homotopic.

Case 2: Suppose $a = 0$ or $b = 0$, say $b = 0$. Then the image of β must be contained in T_1 (actually contained in $\mathbb{R}^+X^1$). We apply Corollary 5.8 to conclude that all such β are $\mathbf{W}$-homotopic.

We conclude from the preceding arguments and Remark 4.4 that the induced endpoint map $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ is a continuous isomorphism. In the reverse direction we can take any point (a,b,c) in T_1 and send it to the $\mathbf{W}$-homotopy class $[\xi(a,b,c)]$, which as a composition of continuous maps is continuous on T_1. Similarly $(a,b,c) \mapsto [\psi(a,b,c)]$ is continuous on T_2. Note that both maps are inverse to $\widetilde{\varepsilon}$, so the two maps agree on $T_1 \cap T_2$. Since each of T_1 and T_2 are neighborhoods of each of their points except $\mathbf{0}$, we obtain the continuity of $\widetilde{\varepsilon}^{-1}$ except at $\mathbf{0}$ and the continuity there follows from the continuity on T_1 and T_2. Thus $\widetilde{\varepsilon} : \Gamma(\mathbf{W}) \to S$ is a topological isomorphism.

Planes perpendicular to the horizontal X^1X^2-plane through $(t,0,0)$ and $(0,t,0)$ separate S, and the component C containing $\mathbf{0}$ is an open (in S) component and a local semigroup generated by $\mathbf{W} \cap C$. Picking U open such that $U \cap S = C$, we have $C = U \cap S = \mathrm{Sg}_U(\mathbf{W}) = \mathrm{Sg}_U(\mathbf{W})^-$. For each $(a,b,c) \in C$ the paths $\psi(a,b,c)$ and $\xi(a,b,c)$, if they exist, are contained in

C. We conclude that the conditions of Corollary 4.7 are satisfied, so that the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ and $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$ are topological isomorphisms and thus continuous local homomorphisms from S into a topological semigroup T may be (uniquely) extended to continuous homomorphisms on S.

7. Homotopy Equivalence and Simply Connected Coverings

In the previous sections we have identified Lie semigroups which are the relative free extensions of their (appropriately chosen) local subsemigroups. In this section we consider settings where we may identify the relative free extensions of local semigroups with the simply connected covering semigroup. Our approach is to identify settings where the inclusion of the space of directed paths into the space of all paths induces a bijection of the respective homotopy classes of paths.

As usual, we fix a connected Lie group G with Lie algebra $\mathfrak{g}$ and a generating Lie wedge $\mathbf{W} \subseteq \mathfrak{g}$. We further suppose that $\mathbf{W}$ is global in the sense that there exists a Lie semigroup S with $\mathcal{L}(S) = \mathbf{W}$. We consider two homotopy structures on S. The first is the standard one for pointed spaces where homotopy is taken with respect to the class $\mathcal{P}$ of all paths, all continuous maps $\alpha : [0,1] \to S$ such that $\alpha(0) = e$. It is known in this case that the Lie semigroup S is path connected, locally path connected, and semi-locally simply connected [5, Proposition 3.13], hence has a simply connected covering $\widetilde{S}$. A standard construction for $\widetilde{S}$ is to take the points of $\widetilde{S}$ to be all homotopy classes of members of $\mathcal{P}$, equipped with an appropriate topology. The covering map $p : \widetilde{S} \to S$ is the map induced by the endpoint map. It is further known that concatenation induces a continuous semigroup operation on $\widetilde{S}$, and the covering map p is a continuous homomorphism [5, Theorem 3.14].

The second homotopy structure on S is that introduced in this paper, where homotopy is taken with respect to the class $\mathcal{A}$ of $\mathbf{W}$-directed paths. In this case we obtain the previously introduced semigroup $\Gamma(\mathbf{W})$ as the space of homotopy classes of members of $\mathcal{A}$.

Definition 7.1. We say that the families $\mathcal{P}$ and $\mathcal{A}$ are *homotopically equivalent* if the inclusion map $j : \mathcal{A} \hookrightarrow \mathcal{P}$ induces a bijection on the homotopy classes. This may be interpreted as a bijection $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$.

Lemma 7.2. *The inclusion j of the homotopy structure of $\mathcal{A}$ of $\mathbf{W}$-directed paths in the homotopy structure $\mathcal{P}$ of all paths in S originating from e induces a continuous homomorphism $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$, which is a continuous isomorphism when the homotopy structures are homotopically equivalent.*

Proof. It is known (and reasonably straightforward to prove, although a published proof is hard to find) that the topology on the simply connected covering space, as typically defined, agrees with the quotient topology from the space of paths equipped with the compact open topology; see [11, Section 2.5]. The inclusion $j : \mathcal{A} \to \mathcal{P}$ is continuous when the respective path spaces are equipped with the compact open topology, so induces a continuous map on the quotients

$j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$. The map j is a also a homomorphism for concatenation, so induces a homomorphism. The last assertion follows since j^* is a bijection in this case. ■

Lemma 7.3. *Suppose that the Lie semigroup S satisfies:*

(i) Two $\mathbf{W}$-directed paths with the same endpoint are $\mathbf{W}$-homotopic if they are homotopic,

(ii) Every path in S emanating from e is homotopic to a $\mathbf{W}$-directed path.

Then the families $\mathcal{P}$ and $\mathcal{A}$ are homotopically equivalent and $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$ is a continuous isomorphism.

Proof. Item (i) ensures that j^* is injective, while item (ii) yields surjectivity. That it is a continuous isomorphism follows from the preceding lemma. ■

Theorem 7.4. *Let S be a Lie subsemigroup of the Lie group G, let $\mathbf{W} = \mathcal{L}(S)$ be a generating Lie wedge, and assume that $H = H(S)$, the group of units of S, is a closed connected Lie subgroup. If there exists $e \in T \subseteq S$ satisfying*

(i) there exists a partial selection $\psi : T \to \mathcal{A}$, the set of $\mathbf{W}$-directed paths, and

(ii) the mapping $\mu : H \times T \to S$ defined by $(g, x) \mapsto gx$ is a homeomorphism,

then the collection $\mathcal{A}$ of $\mathbf{W}$-directed paths is homotopically equivalent to the collection $\mathcal{P}$ of all paths in S with initial point e. Furthermore, the map $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$ is a topological isomorphism.

Proof. We first show that the conditions of Lemma 7.3 are satisfied, from which we may conclude that $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$ is a continuous isomorphism. Suppose that α and β are $\mathbf{W}$-directed paths with $\alpha(1) = \beta(1)$ that are homotopic. Let $\phi : S \to H \times T$ be the homeomorphism inverse to μ. Let $\pi_T : S \to T$ and $\pi_H : S \to H$ be defined by first applying ϕ and then projecting onto S resp. H. We define $\alpha_T = \pi_T\alpha : [0, 1] \to T$ and $\alpha_H = \pi_H\alpha : [0, 1] \to H$. Analogously one has the paths $\beta_T = \pi_T\beta$ and $\beta_H = \pi_H\beta$. Let $I = [0, 1]$ and let $F : I \times I \to S$ be the given homotopy between α and β, where $F(t, 0) = \alpha(t)$ and $F(t, 1) = \beta(t)$ for $0 \le t \le 1$. It then follows that $\alpha_H \simeq \beta_H$ in H via the homotopy $\pi_H \circ F$. Note that all paths contained in the group H starting from e are $\mathbf{W}$-directed paths since as a connected Lie group, any open neighborhood of e in H generates H. Thus α_H and β_H are also $\mathbf{W}$-homotopic paths.

Let $x = \alpha_T(1) = \beta_T(1)$. Then $\psi(x)$ is a $\mathbf{W}$-directed path from e to x. Define a $\mathbf{W}$-directed path γ by

$$\gamma(t) = \begin{cases} \psi(x)(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ (\psi(x)(1))\, \alpha_H(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Note that γ is a $\mathbf{W}$-directed path, since it is the concatenation of two $\mathbf{W}$-directed paths.

We define a $\mathbf{W}$-homotopy $\Phi : I \times I \to S$ between α and γ by

$$\Phi(t,s) = \begin{cases} \psi\left(\alpha_T(s)\right)\left(\frac{2t}{s}\right) & \text{for } 0 \le t \le \frac{1}{2}s, \\ [\psi\left(\alpha_T(s)\right)(1)]\alpha_H(2t-s) & \text{for } \frac{1}{2}s \le t \le s, \\ \alpha(t) & \text{for } s \le t \le 1. \end{cases}$$

The homotopy at the s-level, $\Phi(\cdot, s)$ first traces out the selection $\psi(\alpha_T(s))$ scaled over the interval $[0, s/2]$, then left multiplies the path α_H scaled to $[s/2, s]$ by $\psi(\alpha_T(s))(1) = \alpha_T(s)$, and finally follows the original path α from $t = s$ to $t = 1$. Note that the middle expression evaluated at $t = s$ yields

$$[\psi\left(\alpha_T(s)\right)(1)]\alpha_H(2s-s) = \alpha_T(s)\alpha_H(s) = \alpha(s).$$

Since each segment is $\mathbf{W}$-directed, it follows that $\Phi(\cdot, s)$ is also.

The verification of continuity for Φ on each of the three sets $0 \le t \le (1/2)s$, $(1/2)s \le t \le s$ and $s \le t \le 1$ is straightforward from the three properties satisfied by a partial selection (Definition 5.7), particularly property (iii), although continuity of Φ at $(0,0)$ depends heavily on property (ii) of that definition as well. From the Pasting Lemma one concludes that Φ is continuous on all of $I \times I$.

In a completely similar manner one shows that β is $\mathbf{W}$-homotopic to

$$\lambda(t) := \begin{cases} \psi(x)(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \left(\psi(x)(1)\right)\beta_H(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Since α_H and β_H are $\mathbf{W}$-homotopic, it follows that

$$\alpha \simeq_{\mathbf{W}} \gamma = \psi(x) * \alpha_H \simeq_{\mathbf{W}} \psi(x) * \beta_H = \lambda \simeq_{\mathbf{W}} \beta,$$

so α is $\mathbf{W}$-homotopic to β.

Now let $\alpha : [0,1] \to S$ be any path such that $\alpha(0) = e$, and set $x = \alpha_T(1)$. Define the path γ as previously:

$$\gamma(t) = \begin{cases} \psi(x)(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \left(\psi(x)(1)\right)\alpha_H(2t-1) & \text{for } \frac{1}{2} \le t \le 1. \end{cases}$$

Then γ is the concatenation of two $\mathbf{W}$-directed paths, hence $\mathbf{W}$-directed. As in the earlier part of the proof one obtains a homotopy H between α and γ, no longer necessarily a $\mathbf{W}$-homotopy since α is not assumed $\mathbf{W}$-directed.

Thus the two conditions of Lemma 7.2 are satisfied and we conclude that $j^* : \Gamma(\mathbf{W}) \to \widetilde{S}$ is a continuous isomorphism. Let $\alpha_n \to \alpha$ in the compact-open topology of $\mathcal{P}$, the set of all continuous paths from $[0,1]$ to S with initial point e. Then $\gamma_n = \psi((\alpha_n)_T(1)) * (\alpha_n)_H \to \psi(\alpha_T(1)) * \alpha_H = \gamma$ and by the preceding paragraph every γ_n and γ are $\mathbf{W}$-directed and $\gamma_n \simeq \alpha_n$, $\gamma \simeq \alpha$. Since $\Gamma(\mathbf{W})$ has the quotient topology, we have the convergence of $\mathbf{W}$-homotopy classes $[\gamma_n] \to [\gamma]$. By construction $j^*([\gamma_n]) = [\alpha_n]$ and $j^*([\gamma]) = [\alpha]$, so

$$(j^*)^{-1}([\alpha_n]) = [\gamma_n] \to [\gamma] = (j*)^{-1}([\alpha]),$$

so $(j^*)^{-1}$ is continuous. ∎

The following corollary is an immediate result of the preceding theorem and Corollary 4.7.

Corollary 7.5. *Assume the hypotheses and setting of the previous theorem. Let U be an exponential BCH-neighborhood small enough so that the $\mathrm{Sg}_U(\mathbf{W})^-$-directed paths agree with the $\mathbf{W}$-directed paths. Assume further that for each $x \in \mathrm{Sg}_U(\mathbf{W})^-$, there exists an $\mathrm{Sg}_U(\mathbf{W})^-$-directed path contained in U that has end-point x. Then the maps $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ and $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$ are topological isomorphisms.*

Example 7.6. Let G be a connected Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, $\mathfrak{g}_- \cap \mathfrak{g}_+ = \{0\}$ such that the following conditions are satisfied:

(i) $[\mathfrak{g}_+, \mathfrak{g}_\pm] \subseteq \mathfrak{g}_\pm$, $[\mathfrak{g}_-, \mathfrak{g}_-] \subseteq \mathfrak{g}_+$;

(ii) The connected analytic subgroup H with Lie algebra $\mathfrak{g}_+$ is closed;

(iii) There exists a cone $\mathbf{C} \subseteq \mathfrak{g}_-$ invariant under the adjoint action of H satisfying $\mathbf{C} - \mathbf{C} = \mathfrak{g}_-$.

If $S := (\exp \mathbf{C})H$ is a closed semigroup of G such that the mapping $(X, h) \mapsto \exp(X)h : \mathbf{C} \times H \to S$ is a homeomorphism, then we call S an *Ol'shanskiĭ semigroup*. Note in this case $\mathbf{W} := \mathcal{L}(S) = \mathbf{C} + \mathfrak{g}_+$ [7].

Such an Ol'shanskiĭ semigroup satisfies all the hypotheses of both Theorem 7.4 and Corollary 7.5. Indeed we define the selection on $\exp \mathbf{C}$ by $\psi(x) = \alpha_x$, where $\alpha_x(t) = \exp(t \log x)$ for $0 \le t \le 1$, and we conclude from Theorem 7.4 that $j^* : \Gamma(\mathbf{W}) \to S$ is a continuous isomorphism.

Furthermore, since $d\exp(0) = 1_{\mathfrak{g}}$ and $\mathfrak{g}$ is the direct sum of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, there exist convex open neighborhoods B_+ and B_- of 0 in $\mathfrak{g}_+$ and $\mathfrak{g}_-$ resp., such that the map $\mu : B_+ \times B_- \to G$ defined by $\mu(X, Y) = \exp(X)\exp(Y)$ is a homeomorphism onto an open neighborhood U of e. By hypothesis there exists V open containing $e = \exp(0)\exp(0)$, which may be chosen as a subset of U, such that $\gamma(V \cap S) \subseteq (\mathbf{C} \cap B_-) \times B_+$, where γ is the inverse of the homeomorphism $(X, h) \mapsto \exp(X)h : \mathbf{C} \times H \to S$ followed by $1_{\mathfrak{g}_-} \times \log$. Pick convex open neighborhoods $A_+ \subseteq B_+$ and $A_- \subseteq B_-$ of 0 in $\mathfrak{g}_+$ and $\mathfrak{g}_-$ resp. such that the open set $W = \mu(A_+ \times A_-) \subseteq V$. We consider $S_W = S \cap W$. By definition it is a closed local semigroup with respect to W. Let $s \in S_W$. Since $s \in V \cap S$, $s = \exp(X)h$, where $X \in \mathbf{C} \cap B_-$ and $h = \exp Y$ for some $Y \in B_+ \subseteq \mathfrak{g}_+$; in particular $s \in \mathrm{Sg}_W(\mathbf{W})$. Also by definition of W, $s = \exp(X_1)\exp(Y_1)$ for some $X_1 \in A_+$, $X_2 \in A_-$. Since μ is a bijection on $B_- \times B_+$, we conclude that $X = X_1$ and $Y = Y_1$, so $X \in A_- \cap C$ and $Y \in A_+$. Setting $\alpha(t) = \exp(tX)$ and $\beta(t) = \exp(tY)$ for $0 \le t \le 1$ yields two $\mathbf{W}$-directed paths contained in W. Their concatenation $\alpha * \beta$ is thus also a $\mathbf{W}$-directed path running from e to s and is contained in W by its definition. From all this we conclude that $S_W = \mathrm{Sg}_W(\mathbf{W}) = \mathrm{Sg}_W(\mathbf{W})^-$ and that each point in S_W lies on a $\mathbf{W}$-directed path lying in W. From Corollary 7.5 we conclude that $\Xi : \Gamma(\mathbf{W}) \to RF(\mathrm{Sg}_U(\mathbf{W})^-)$ and $\gamma : RF(\mathrm{Sg}_U(\mathbf{W})^-) \to S$ are topological isomorphisms.

8. Future Directions

As future work we want to extend ideas and methods from this paper, particularly those of Sections 3 and 4, to geometric control theory, control theory where the states are points on a smooth manifold M, the control set is a subset Ω of smooth vector fields of $V^{\infty}(M)$, and the admissible controls are more general classes of control functions than the piecewise constant functions. We would assume the system is forward complete in the sense that for any admissible control function $u(\cdot)$ defined on $[0,T]$ and any initial point $x_0 \in M$, there would exist a (unique) absolutely continuous trajectory $x(\cdot)$ defined on $[0,T]$ such that $\dot{x}(t) = u(t)x(t)$ a.e. One can directly extend the definition of homotopy in Section 3 to this setting and our program would be to study properties of the homotopy, uncover situations where one might identify the space of homotopy classes with a universal orbit for the given control system, or, as in this paper, with the orbit of some point $x_0 \in M$. We would anticipate close connections of significant parts of this program with ideas developed in [3] and [6]

References

[1] Carruth, J. H., J. A. Hildebrant, R. J. Koch, "The Theory of Topological Semigroups, I," Marcel Dekker, New York, 1983.

[2] —, "The Theory of Topological Semigroups, II," Marcel Dekker, New York, 1986.

[3] Colonius, F., E. Kizil and L. A. B. San Martin, *Covering space for monotonic homotopy of trajectories of control systems*, J. of Differential Equations **216** (2005), 324–353.

[4] Hilgert, J., K. H. Hofmann, and J. D. Lawson, "Lie Groups, Convex Cones, and Semigroups," Oxford Press, Oxford, 1989.

[5] Hilgert, J., and K. H. Neeb, "Lie Semigroups and their Applications," Lecture Notes in Math. **1552**, Springer-Verlag, 1993.

[6] Kizil, E., *Universal covering of driftless control systems*, J. Dyn. Control Syst. **14** (2008), 453–464.

[7] Lawson, J., *Polar and Ol'shanskiĭ decompositions*, J. reine angew. Math. **448** (1994), 191–219.

[8] —, *Free local semigroup constructions*, Monatsh. Math. **121** (1996), 309–333.

[9] —, *Universal Objects in Lie Semigroup Theory*, In: J. Hilgert, J.D. Lawson, K.-H. Neeb, E. B. Vinberg, Eds, "Positivity in Lie Theory," De Gruyter Expositions in Mathematics **20** (1995), 121–132.

[10] Lawson, J., and B. Madison, *Quotients of k-semigroups*, Semigroup Forum **9** (1974), 1–18.

[11] Spanier, E., "Algebraic Topology," McGraw-Hill, New York, 1966.

[12] Weiss, W., *Local Lie semigroups and open embeddings into global topological semigroups*, Indag. Math. (N.S.) **2** (1991), 115–138.

Eyüp Kizil
Instituto de Ciências Matemáticas
e de Computação
Universidade de São Paulo
Cx. Postal 668, CEP: 13.560-970
São Carlos-SP, Brasil
kizil@icmc.usp.br

Jimmie Lawson
Department of Mathematics
Louisiana State University
Baton Rouge, LA 70803, USA
lawson@math.lsu.edu

Received April 7, 2014
and in final form November 14, 2014