Journal of Lie Theory
Volume **25** (2015) 775–786

Lie Bialgebra Structures on Not-Finitely Graded Lie Algebras $B(\Gamma)$ of Block Type

Hao Wang, Ying Xu, and Xiaoqing Yue*

Communicated by K. Schmüdgen

Abstract. In this paper, Lie bialgebra structures on a class of not-finitely graded Lie algebras $B(\Gamma)$ of Block type are investigated. By proving the triviality of the first cohomology group of $B(\Gamma)$ with coefficients in its adjoint tensor module, namely, $H^1(B(\Gamma), B(\Gamma) \otimes B(\Gamma)) = 0$, we obtain that all Lie bialgebra structures on $B(\Gamma)$ are triangular coboundary.
Mathematics Subject Classification 2010: 17B10, 17B65, 17B68.
Key Words and Phrases: Lie bialgebras, derivation, cohomology group, Lie algebras of Block type.

1. Introduction

Zassenhaus algebras were first introduced by Block [1] in 1958, which are a class of infinite dimensional simple Lie algebras over a field of characteristic zero. Recently, much attention has been paid to these algebras and generalizations of Block algebras (usually referred to as Lie algebras of Block type), which play important roles in many areas of mathematics and physics, see, for example, [2, 3, 6, 8, 14, 17, 18, 20, 24, 25, 27]. Surprisingly, Block type Lie algebras also appear in integrable system [8]. In this paper, we study Lie bialgebra structures on a class of not-finitely graded Lie algebras $B(\Gamma)$ of Block type (see Definition 1.1 and Proposition 2.1). Our motivations mainly originate from the following.

- Constructions of Lie bialgebras and their quantizations are important approaches to produce new quantum groups. Since the notion of Lie bialgebras was introduced by Drinfeld in 1983 [4, 5], there have appeared several papers on Lie coalgebras or Lie bialgebras (e.g., [7, 10, 11, 12, 13, 15, 22, 23]).
- Not-finitely graded Lie algebras play important roles in mathematical physics. They appear naturally in the theory of Hamiltonian operators, the theory of vertex algebras and their multi-variable analogues (e.g., [26]). They are also important objects in Lie theory, whose structure and representation theories are subjects of studies with more challenge than that of finitely graded Lie algebras.

* This work is supported by NSF grants no.11301130,11101387,11271284,11431010,11371278.

ISSN 0949–5932 / $2.50

- Block type Lie algebras are not only closely related to the Virasoro algebra or the Virasoro-like algebra but also special cases of Lie algebras of Cartan type S and Cartan type H (e.g., [19, 21, 25]). However, we observe that there has been very little research on Lie bialgebras of the Block type algebra except [2, 9].

Let us give the precise definition of our object $B(\Gamma)$ below. For an arbitrary algebraically closed field $\mathbb{F}$ of characteristic zero, let Γ be any additive subgroup of $\mathbb{F}^2$ satisfying

$$\pi_p(\Gamma) \neq 0 \quad \text{for} \quad p = 1, 2, \tag{1}$$

where $\pi_p : \mathbb{F}^2 \to \mathbb{F}$ is the p-th projection, namely, $\pi_p(\alpha) = \alpha_p$ for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{F}^2$. Set

$$J = \mathbb{Z}_+ \times \mathbb{Z}_+, \text{ which is a semigroup.} \tag{2}$$

Then one has the semigroup algebra $\mathcal{A} := \mathbb{F}[\Gamma \times J]$ with basis $\{X^{\alpha,\underline{i}} \mid \alpha \in \Gamma, \underline{i} \in J\}$ and multiplication $X^{\alpha,\underline{i}} X^{\beta,\underline{j}} = X^{\alpha+\beta,\underline{i}+\underline{j}}$. We define two derivations d_p of $\mathcal{A}$ as follows,

$$d_p(X^{\alpha,\underline{i}}) = \alpha_p X^{\alpha,\underline{i}} + i_p X^{\alpha,\underline{i}-1_{[p]}} \quad \text{for} \quad p = 1, 2, \tag{3}$$

where $\alpha = (\alpha_1, \alpha_2)$, $\underline{i} = (i_1, i_2)$, and $1_{[1]} = (1, 0)$, $1_{[2]} = (0, 1)$. Hereafter, we use the convention that if an undefined symbol appears in an expression, we treat it as zero; for instance, $X^{\alpha,-1_{[p]}} = 0$. By [21, 25], $\mathcal{A}$ is a Lie algebra with bracket defined by

$$[b_1, b_2] = d_1(b_1)d_2(b_2) - d_1(b_2)d_2(b_1) + b_1 d_1(b_2) - b_2 d_1(b_1) \quad \text{for} \quad b_1, b_2 \in \mathcal{A}. \tag{4}$$

More precisely, in terms of basis elements, (4) becomes

$$\begin{aligned}[X^{\alpha,\underline{i}}, X^{\beta,\underline{j}}] = {} & (\alpha_1(\beta_2 - 1) - \beta_1(\alpha_2 - 1)) X^{\alpha+\beta,\underline{i}+\underline{j}} \\ & + (i_1(\beta_2 - 1) - j_1(\alpha_2 - 1)) X^{\alpha+\beta,\underline{i}+\underline{j}-1_{[1]}} \\ & + (\alpha_1 j_2 - \beta_1 i_2) X^{\alpha+\beta,\underline{i}+\underline{j}-1_{[2]}} + (i_1 j_2 - j_1 i_2) X^{\alpha+\beta,\underline{i}+\underline{j}-1_{[1]}-1_{[2]}}. \end{aligned} \tag{5}$$

Observe that $c := X^{(0,1),\underline{0}}$ is in the center of the Lie algebra $\mathcal{A}$ (we treat c as zero if $(0, 1) \notin \Gamma$). We take $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{F}c$, and we use the same symbols to denote elements in $\overline{\mathcal{A}}$.

Definition 1.1. The $B(\Gamma)$ with Γ satisfying (1) is the Lie algebra with the underlined space $\overline{\mathcal{A}} \oplus \mathbb{F}\partial_1 \oplus \mathbb{F}\partial_2$ and the brackets (5) together with

$$[\partial_1, \partial_2] = 0 \quad \text{and} \quad [\partial_p, X^{\alpha,\underline{i}}] = \alpha_p X^{\alpha,\underline{i}} \quad \text{for} \quad p = 1, 2. \tag{6}$$

We will simply denote $B(\Gamma)$ as B throughout the paper. Then the main result of the present paper is summarized as follows.

Theorem 1.2. *Let B be the Lie algebra defined in Definition 1.1 with Γ satisfying (1).*

(1) *The first cohomology group of B with coefficients in its adjoint tensor module is trivial, namely, $H^1(B, B \otimes B) = 0$.*

(2) *All Lie bialgebra structures on B are triangular coboundary.*

Finally we would like to remark that as we already indicated above, the algebra B is not finitely graded (Proposition 2.1), while the Block type Lie algebras considered in [2, 9] are finitely graded. Further, due to the fact that Γ may not be finitely generated (as a group), B may not be a finitely generated Lie algebra as well. Thus the classical techniques (such as those in [2, 9]) cannot be directly applied to our situation here. One must employ some new techniques in order to tackle problems associated with not-finitely graded and not-finitely generated Lie algebras (this is also one of our motivations to present our results here). For instance, one of our strategies used in the present paper is to introduce the length of a derivation (cf. Definition 3.9) so that the determination of derivations can be done by induction on the length. We would also like to mention that though the result that all Lie bialgebra structures on B are triangular coboundary is not surprising, and coboundary triangular Lie bialgebras have relatively simple structures, it seems to us that it is still worth paying more attention on them, as one can see from [16] that by considering dual structures of Lie bialgebras, one may expect to obtain some new Lie algebras. This is also our next goal.

2. Preliminaries

First let us recall some concepts. A Lie algebra $\mathcal{L}$ is *finitely graded* if there exists an abelian group G such that $\mathcal{L} = \oplus_{a\in G}\mathcal{L}_{[a]}$ is G-graded satisfying

$$[\mathcal{L}_{[a]}, \mathcal{L}_{[b]}] \subset \mathcal{L}_{[a+b]} \quad \text{and} \quad \dim \mathcal{L}_{[a]} < \infty \quad \text{for } a, b \in G. \tag{7}$$

In this case, we also say, $\mathcal{L}$ is finitely G-graded. If there does not exist any abelian group G such that $\mathcal{L}$ is finitely G-graded, then we say $\mathcal{L}$ is *not-finitely graded.* Then similar to the proof of [3, Theorem 2.1(1)], one immediately obtains

Proposition 2.1. *The Lie algebra B is not finitely graded.*

However, by (5) and (6), one can easily sees that $B = \oplus_{\alpha\in\Gamma}B_\alpha$ is still a Γ-graded Lie algebra with

$$B_\alpha = \text{span}\{X^{\alpha,\underline{i}} \,|\, \underline{i} \in J\} \oplus \delta_{\alpha,0}(\mathbb{F}\partial_1 + \mathbb{F}\partial_2), \tag{8}$$

which is infinite-dimensional for all $\alpha \in \Gamma$.

We briefly recall some notions on Lie bialgebras, for details, we refer readers to, e.g., [15].

Definition 2.2. (1) A *Lie bialgebra* is a triple $(L, [\cdot,\cdot], \delta)$, where L is a vector space, $[\cdot,\cdot] : L \otimes L \to L$ and $\delta : L \to L \otimes L$ are linear maps such that

(i) $(L, [\cdot,\cdot])$ is a Lie algebra;

(ii) (L, δ) is a coalgebra;

(iii) $\delta[x, y] = x \cdot \delta(y) - y \cdot \delta(x)$ for $x, y \in L$, where
$x \cdot (y \otimes z) = [x, y] \otimes z + y \otimes [x, z]$ for $x, y, z \in L$.

(2) A Lie bialgebra $(L, [\cdot,\cdot], \delta)$ is *coboundary* if δ is coboundary in the sense that there exists $r \in L \otimes L$ written as $r = \sum r^{[1]} \otimes r^{[2]}$, such that $\delta(x) = x \cdot r$ for $x \in L$.

(3) A coboundary Lie bialgebra $(L, [\cdot,\cdot], \delta)$ is *triangular* if r satisfies the following *classical Yang-Baxter Equation* (CYBE),

$$C(r) = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \tag{9}$$

where $r_{12} = \sum r^{[1]} \otimes r^{[2]} \otimes 1$, $r_{13} = \sum r^{[1]} \otimes 1 \otimes r^{[2]}$, $r_{23} = \sum 1 \otimes r^{[1]} \otimes r^{[2]}$ are elements in $U(L) \otimes U(L) \otimes U(L)$, and $U(L)$ is the universal enveloping algebra of L.

For simplicity, we denote $V = B \otimes B$ (the adjoint tensor module of B). Then (8) naturally gives a Γ-gradation of V such that V is a Γ-graded B-module, namely,

$$V = \bigoplus_{\alpha\in\Gamma} V_\alpha, \quad V_\alpha = \bigoplus_{a,b\in\Gamma:\, a+b=\alpha} B_a \otimes B_b \;\text{ and }\; B_\alpha \cdot V_\beta \subset V_{\alpha+\beta} \;\text{ for } \alpha, \beta \in \Gamma. \tag{10}$$

Recall that a linear map $\delta : B \to V$ satisfying Definition 2.2(1)(iii) is a *derivation* from B to V. Denote by $\mathrm{Der}(B, V)$ the space of derivations. For any $u \in V$, the linear map $u_{\mathrm{inn}} : B \to V$ defined by $x \mapsto x \cdot u$ for $x \in B$ is clearly a derivation, called an *inner derivation*. Denote by $\mathrm{Inn}(B, V)$ the space of inner derivations. Then it is well known that

$$H^1(B, V) \cong \mathrm{Der}(B, V)/\mathrm{Inn}(B, V), \tag{11}$$

where $H^1(B, V)$ is the *first cohomology group* of the Lie algebra B with coefficients in the B-module V.

3. A technical proposition and proof of Theorem 1.2

The proof of Theorem 1.2 heavily depends on the following technical proposition.

Proposition 3.1. *We have* $\mathrm{Der}(B, V) = \mathrm{Inn}(B, V)$.

The proof of this proposition will be divided into several lemmas. First we will give some notations.

We choose a well-order on Γ compatible with its group structure, i.e., $\alpha > \beta$ implies $\alpha + \gamma > \beta + \gamma$ for any $\gamma \in \Gamma$. We also define a well-order on J, for $\underline{i} = (i_1, i_2), \underline{j} = (j_1, j_2)$,

$$\underline{i} > \underline{j} \iff |\underline{i}| > |\underline{j}|, \text{ or } |\underline{i}| = |\underline{j}| \text{ and } i_1 > j_1,$$

where $|\underline{i}| = i_1 + i_2$. Then we define a well-order on $J \times J$,

$$(\underline{i}, \underline{j}) > (\underline{k}, \underline{l}) \iff |\underline{i}| + |\underline{j}| > |\underline{k}| + |\underline{l}|, \text{ or } |\underline{i}| + |\underline{j}| = |\underline{k}| + |\underline{l}| \text{ and } \underline{i} > \underline{k}, \text{ or}$$
$$|\underline{i}| + |\underline{j}| = |\underline{k}| + |\underline{l}|, \ \underline{i} = \underline{k} \text{ and } \underline{j} > \underline{l}.$$

A derivation $D \in \mathrm{Der}(B,V)$ is called *α-homogeneous* if $D(B_\beta) \subseteq V_{\alpha+\beta}$. Denote by $\mathrm{Der}(B,V)_\alpha$ the space of α-homogeneous derivations. In general we have $\mathrm{Der}(B,V) = \prod_{\alpha\in\Gamma} \mathrm{Der}(B,V)_\alpha$. However for arbitrary given $b \in B$ and $D \in \mathrm{Der}(B,V)_\alpha$, $D(b) = \sum_{\alpha\in\Gamma} D_\alpha(b)$ must be a finite sum.

We give the definition *degree* of the linear generators in B and V (considered as vector spaces) as follows. For elements $X, Y \in \{X^{\alpha,\underline{i}}, \partial_1, \partial_2 \,|\alpha \in \Gamma, \underline{i} \in J\}$, we define in B,

$$\deg(X^{\alpha,\underline{i}}) = (\alpha, \underline{i}) \in (\Gamma, J),\ \deg(\partial_p) = (\underline{0}, -(p,p)) \text{ for } p = 1, 2.$$

We add two elements $-(1,1), -(2,2)$ into J with the order $-(2,2) < -(1,1) < \underline{0}$, and denote the resulting set as $\overline{J}$. Define the order on $\Gamma \times \overline{J}$ by $(\alpha, \underline{i}) > (\beta, \underline{j})$ if

$$\alpha > \beta, \quad \text{or} \quad \alpha = \beta \quad \text{and} \quad \underline{i} > \underline{j}.$$

With this order, the degrees of linear generators in B are totally ordered.

We define in V, $\deg(X \otimes Y) = (\deg(X), \deg(Y))$. A well-order on $\Gamma \times \overline{J} \times \Gamma \times \overline{J}$ is defined by $(\alpha, \underline{i}, \beta, \underline{j}) > (\gamma, \underline{k}, \delta, \underline{l})$ if

$$\begin{aligned}&\alpha + \beta > \gamma + \delta, \text{ or}\\ &\alpha + \beta = \gamma + \delta \text{ and } \alpha > \beta, \text{ or}\\ &\alpha = \gamma,\ \beta = \delta \text{ and } (\underline{i}, \underline{j}) > (\underline{k}, \underline{l}).\end{aligned}$$

With this order, the degrees of linear generators in V are totally ordered.

We give the definition *degree* of arbitrary element $b \in B$: $\deg(b) = \max\{\deg(X^{\alpha,\underline{i}}) \,|\, b = \sum_{\alpha,\underline{i}} \lambda_{\alpha,\underline{i}} X^{\alpha,\underline{i}}, \lambda_{\alpha,\underline{i}} \neq 0\}$; the definition *degree* of arbitrary element $v \in V$ is similar.

We denote by $V^{[\overline{j},\overline{k}]}$ (resp., $V^{(\overline{j},\overline{k})}$) the subspace of V spanned by elements

$$\begin{aligned}&X^{\alpha,\underline{i}} \otimes X^{\beta,\underline{l}}, && \partial_1 \otimes X^{\beta^1,\underline{l}^1}, && X^{\alpha^1,\underline{i}^1} \otimes \partial_1, && \partial_2 \otimes X^{\beta^2,\underline{l}^2}, && X^{\alpha^2,\underline{i}^2} \otimes \partial_2,\\ &\partial_1 \otimes \partial_1, && \partial_1 \otimes \partial_2, && \partial_2 \otimes \partial_1, && \partial_2 \otimes \partial_2,\end{aligned}$$

with $(\underline{i}, \underline{l}) \le (\overline{j}, \overline{k})$, $\underline{i}^1, \underline{i}^2 \le \overline{j}$, $\underline{l}^1, \underline{l}^2 \le \overline{k}$ (resp., $(\underline{i}, \underline{l}) < (\overline{j}, \overline{k})$, $\underline{i}^1, \underline{i}^2 < \overline{j}$, $\underline{l}^1, \underline{l}^2 < \overline{k}$).

Lemma 3.2. *Let $D_\alpha \in \mathrm{Der}(B,V)_\alpha$ with $0 \neq \alpha \in \Gamma$. Then $D_\alpha \in \mathrm{Inn}(B,V)$.*

Proof. Since $\alpha \neq 0$, without lose of generality, we assume $\alpha_1 \neq 0$. Let D_α act on both sides of $[\partial_1, X^{\beta,\underline{j}}] = \beta_1 X^{\beta,\underline{j}}$, we obtain,

$$\partial_1 \cdot D_\alpha(X^{\beta,\underline{j}}) - X^{\beta,\underline{j}} \cdot D_\alpha(\partial_1) = \beta_1 D_\alpha(X^{\beta,\underline{j}}),$$

which implies $D_\alpha(X^{\beta,\underline{j}}) = X^{\beta,\underline{j}} \cdot (\alpha_1^{-1} D_\alpha(\partial_1))$, i.e., $D_\alpha = u_{\mathrm{inn}} \in \mathrm{Inn}(B,V)$ for $u = \alpha_1^{-1} D_\alpha(\partial_1)$. ∎

Now consider $D_{\underline{0}} \in \mathrm{Der}(B,V)_{\underline{0}}$. Let $\alpha = (\alpha_1, \alpha_2) \in \Gamma$, we write $D_{\underline{0}}(X^{\alpha,\underline{0}})$ as

$$\begin{aligned} D_{\underline{0}}(X^{\alpha,\underline{0}}) = & \sum_{(\beta,\underline{j},\underline{k})\in\Gamma\times J\times J} c^{\alpha}_{\beta,\underline{j},\underline{k}} X^{\beta,\underline{j}} \otimes X^{\alpha-\beta,\underline{k}} + \sum_{\underline{k^1}\in J} d^{\alpha}_{\underline{k^1}} \partial_1 \otimes X^{\alpha,\underline{k^1}} + \sum_{\underline{j^1}\in J} e^{\alpha}_{\underline{j^1}} X^{\alpha,\underline{j^1}} \otimes \partial_1 \\ & + \sum_{\underline{k^2}\in J} f^{\alpha}_{\underline{k^2}} \partial_2 \otimes X^{\alpha,\underline{k^2}} + \sum_{\underline{j^2}\in J} g^{\alpha}_{\underline{j^2}} X^{\alpha,\underline{j^2}} \otimes \partial_2 \\ & + \delta_{\alpha,\underline{0}}(a\partial_1\otimes\partial_2 + b\partial_2\otimes\partial_1 + c\partial_1\otimes\partial_1 + d\partial_2\otimes\partial_2). \end{aligned} \tag{12}$$

For an arbitrary fixed $\overline{\alpha} = (\alpha_1, \alpha_2)$ in Γ satisfying $\alpha_1\alpha_2 \neq 0$, there exists minimal $(\underline{J}, \underline{K})$ (which may depend on α) such that all corresponding coefficients are 0 if their indices satisfy $(\underline{j}, \underline{k}) > (\underline{J}, \underline{K})$, or $\underline{k^1}, \underline{k^2} > \underline{J}$, or $\underline{j^1}, \underline{j^2} > \underline{K}$.

In the following, we assume $\alpha = (\alpha_1, \alpha_2) \in \Gamma$ is a fixed element satisfying $\alpha_1\alpha_2 \neq 0$, $\alpha > 0$. Since $X^{i\alpha,\underline{0}}$, $i = 0, 1, 2, -1, -2$ generate $\{X^{\ell\alpha,\underline{0}} \mid \ell \in \mathbb{Z}\}$, we have the following.

Lemma 3.3. *Given an expression of $D_{\underline{0}}(X^{\ell\alpha,\underline{0}})$ as in* (12) *for some $\ell \in \mathbb{Z}$, there exists minimal $(\underline{J}, \underline{K})$, which does not depend on ℓ, such that all corresponding coefficients are* 0 *if their indices satisfy $(\underline{j}, \underline{k}) > (\underline{J}, \underline{K})$, or $\underline{k^1}, \underline{k^2} > \underline{J}$, or $\underline{j^1}, \underline{j^2} > \underline{K}$.*

From (5) we immediately get the following.

Lemma 3.4. *If $[X^{k\alpha,\underline{0}}, aX^{\beta,\underline{j}}] = 0$ for all $k \in \mathbb{Z}_+$, then $a = 0$. In general, if $X \in B$, $[X^{k\alpha,\underline{0}}, X] = 0$ for all $k \in \mathbb{Z}_+$, then $X = 0$.*

Lemma 3.5. *Let $v \in V$. If $X^{k\alpha,\underline{0}} \cdot v = 0$ for all $k \in \mathbb{Z}_+$, then $v = 0$. In particular, if $X^{\beta,\underline{i}} \cdot v = 0$ for all $(\beta, \underline{i}) \in \Gamma \otimes J$, then $v = 0$.*

Proof. Suppose $v \neq 0$. We consider the highest degree summand of v, it must be a nonzero multiple of $X \otimes Y$, in which $X, Y \in \{X^{\beta,\underline{i}}, \partial_1, \partial_2\}$. Since $\deg(\alpha) > 0$, the condition $X^{k\alpha,\underline{0}} \cdot v = 0$ implies $[X^{k\alpha,\underline{0}}, X] \otimes Y = 0$, which then implies $[X^{k\alpha,\underline{0}}, X] = 0$ for $k > 0$. By Lemma 3.4, we get $X = 0$, a contradiction. Hence $v = 0$. ■

Lemma 3.6. $D_{\underline{0}}(\partial_1) = D_{\underline{0}}(\partial_2) = 0$.

Proof. Let $D_{\underline{0}}$ act on both sides of $[\partial_1, X^{\beta,\underline{j}}] = \beta_1 X^{\beta,\underline{j}}$, we obtain $\partial_1 \cdot D_{\underline{0}}(X^{\beta,\underline{j}}) - X^{\beta,\underline{j}} \cdot D_{\underline{0}}(\partial_1) = \beta_1 D_{\underline{0}}(X^{\beta,\underline{j}})$, which implies

$$X^{\beta,\underline{j}} \cdot D_{\underline{0}}(\partial_1) = 0 \quad \text{for all} \quad (\beta, \underline{j}) \in \Gamma \times J. \tag{13}$$

This together with Lemma 3.5 implies $D_{\underline{0}}(\partial_1) = 0$. Similarly, we can obtain $D_{\underline{0}}(\partial_2) = 0$. ■

Lemma 3.7. *We have $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}$.*

Proof. Let $D_{\underline{0}}$ act on both sides of $[X^{\underline{0},\underline{0}}, X^{\beta,\underline{0}}] = \beta_1 X^{\beta,\underline{0}}$, we obtain

$$X^{\underline{0},\underline{0}} \cdot D_{\underline{0}}(X^{\beta,\underline{0}}) - X^{\beta,\underline{0}} \cdot D_{\underline{0}}(X^{\underline{0},\underline{0}}) = \beta_1 D_{\underline{0}}(X^{\beta,\underline{0}}). \tag{14}$$

By Lemma 3.3, $X^{\underline{0},\underline{0}} \cdot D_{\underline{0}}(X^{\beta,\underline{0}}) \equiv \beta_1 D_{\underline{0}}(X^{\beta,\underline{0}}) \mod V^{(\underline{J},\underline{K})}$. This together with (14) gives

$$X^{\beta,\underline{0}} \cdot D_{\underline{0}}(X^{\underline{0},\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})} \text{ for all } \beta \in \Gamma. \tag{15}$$

We claim: $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}$. Suppose not, we give a formal expression of $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \mod V^{(\underline{J},\underline{K})}$: $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \equiv \sum_{k=1}^{n} \lambda_k X^{\alpha_{(k)},\underline{i}_{(k)}} \otimes X^{-\alpha_{(k)},\underline{j}_{(k)}} \mod V^{(\underline{J},\underline{K})}$. In which $\alpha_{(k)} \in \Gamma$, $\underline{i}_{(k)}$, $\underline{j}_{(k)} \in \overline{J}$, $\lambda_k \neq 0$, $(\underline{i}_{(k)}, \underline{j}_{(k)}) \succeq (\underline{J}, \underline{K})$ for all k, $(\alpha_{(k)}, \underline{i}_{(k)}, -\alpha_{(k)}, \underline{j}_{(k)}) > (\alpha_{(k+1)}, \underline{i}_{(k+1)}, -\alpha_{(k+1)}, \underline{j}_{(k+1)})$ for $k = 1, 2, ..., n-1$. Here we assume $(\underline{i}_{(k)}, \underline{j}_{(k)}) = (\underline{J}, \underline{K})$ for all k since if we can prove $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \not\equiv 0 \mod V^{(\underline{L},\underline{M})}$, for any $(\underline{L}, \underline{M}) > (\underline{J}, \underline{K})$, it's trivial to see $D_{\underline{0}}(X^{\underline{0},\underline{0}}) \not\equiv 0 \mod V^{(\underline{J},\underline{K})}$, and such $(\underline{L}, \underline{M})$ can be chosen as the largest one in $(\underline{i}_{(k)}, \underline{j}_{(k)})$ for all k. If $\alpha_{(k)} = (0,1)$ for all k, it is easy to find a $\beta > (0,0)$ such that $\deg([X^{\beta,\underline{0}}, X^{(0,-1),\underline{K}}]) = (\beta - (0,1), \underline{K})$ from (5), which implies $\deg(X^{\beta,\underline{0}} \cdot D_{\underline{0}}(X^{\underline{0},\underline{0}}) \mod V^{(\underline{J},\underline{K})}) = ((0,1), \underline{J}, \beta - (0,1), \underline{K})$, it's a contradiction with (15). Else, suppose m $(1 \leq m \preceq n)$ is the minimal subscript such that $\alpha_{(k)} \neq (0,1)$, it is easy to find a big enough β such that $\deg([X^{\beta,\underline{0}}, X^{\alpha_{(m)},\underline{J}}]) = (\beta + \alpha_{(m)}, \underline{J})$ from (5) and $\beta + \alpha_{(m)} > (0,1)$, which implies $\deg(X^{\beta,\underline{0}} \cdot D_{\underline{0}}(X^{\underline{0},\underline{0}}) \mod V^{(\underline{J},\underline{K})}) = (\beta + \alpha_{(m)}, \underline{J}, -\alpha_{(m)}, \underline{K})$, it's a contradiction with (15). ■

Lemma 3.8. *By replacing $D_{\underline{0}}$ by $D_{\underline{0}} - u_{\mathrm{inn}}$, where $u \in V_{\underline{0}}$ (note that this replacement does not affect the result we already obtain), we can suppose*

$$D_{\underline{0}}(X^{\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})} \textit{ for } \alpha = (\alpha_1, \alpha_2) \textit{ and } \alpha_1\alpha_2 \neq 0.$$

Proof. Given an expression of $D_{\underline{0}}(X^{\alpha,\underline{0}})$ as in (12), we can write $D_{\underline{0}}(X^{\alpha,\underline{0}}) \equiv S_1 + S_2 \mod V^{(\underline{J},\underline{K})}$, where

$$\begin{aligned} , S_1 &= \sum_{\beta\in\Gamma} c_\beta X^{\beta,\underline{J}} \otimes X^{\alpha-\beta,\underline{K}}, \\ S_2 &= d\partial_1 \otimes X^{\alpha,\underline{K}} + eX^{\alpha,\underline{J}} \otimes \partial_1 + f\partial_2 \otimes X^{\alpha,\underline{K}} + gX^{\alpha,\underline{J}} \otimes \partial_2. \end{aligned} \tag{16}$$

We want to prove the following.

Claim 1. By subtracting $D_{\underline{0}}$ by some suitable u_{inn} with $u \in V_{\underline{0}}$, we can assume $D_{\underline{0}}(X^{\alpha,\underline{0}}) \equiv S_1 \mod V^{(\underline{J},\underline{K})}$.

It is sufficient to prove Claim 1 by showing that there exist $v_i \in V_{\underline{0}}$ for $i = 1,\ 2,\ 3,\ 4$, such that

$$X^{\alpha,\underline{0}} \cdot v_i \equiv u_i \mod V^{(\underline{J},\underline{K})},$$

in which

$$u_1 = \partial_1 \otimes X^{\alpha,\underline{K}}, \qquad u_2 = X^{\alpha,\underline{J}} \otimes \partial_1,$$
$$u_3 = \partial_2 \otimes X^{\alpha,\underline{K}}, \qquad u_4 = X^{\alpha,\underline{J}} \otimes \partial_2.$$

Such v_i's are found by explicit construction from (5), we simply give them as follows,

$$v_1 = -\frac{1}{\alpha_1}\partial_1 \otimes X^{\underline{0},\underline{K}} + \frac{1}{2\alpha_1}X^{\alpha,\underline{J}} \otimes X^{-\alpha,\underline{K}}$$
$$v_2 = -\frac{1}{\alpha_1}X^{\underline{0},\underline{J}} \otimes \partial_1 + \frac{1}{2\alpha_1}X^{-\alpha,\underline{J}} \otimes X^{\alpha,\underline{K}}$$
$$v_3 = -\frac{1}{\alpha_1}\partial_2 \otimes X^{\underline{0},\underline{K}} + \frac{\alpha_2}{2{\alpha_1}^2}X^{\alpha,\underline{J}} \otimes X^{-\alpha,\underline{K}}$$
$$v_4 = -\frac{1}{\alpha_1}X^{\underline{0},\underline{J}} \otimes \partial_2 + \frac{\alpha_2}{2{\alpha_1}^2}X^{-\alpha,\underline{J}} \otimes X^{\alpha,\underline{K}}$$

Claim 2. By further replacing $D_{\underline{0}}$ by $D_{\underline{0}} - u_{inn}$, where u has the form $\sum_{\beta\in\Gamma} c_\beta X^{\beta,\underline{0}} \otimes X^{-\beta,\underline{0}}$, we can suppose $D_{\underline{0}}(X^{\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}$ for $\alpha = (\alpha_1, \alpha_2)$ and $\alpha_1\alpha_2 \neq 0$.

To prove the Claim, let $D_{\underline{0}}$ act on both sides of the equation $[X^{\alpha,\underline{0}}, X^{-\alpha,\underline{0}}] = -2\alpha_1 X^{\underline{0},\underline{0}}$, we get,

$$X^{\alpha,\underline{0}} \cdot D_{\underline{0}}(X^{-\alpha,\underline{0}}) \equiv X^{-\alpha,\underline{0}} \cdot D_{\underline{0}}(X^{\alpha,\underline{0}}) \mod V^{(\underline{J},\underline{K})}. \tag{17}$$

We can assume

$$\begin{aligned} D_{\underline{0}}(X^{\alpha,\underline{0}}) &\equiv \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}X^{\beta^i+j\alpha,\underline{J}} \otimes X^{-\beta^i-(j-1)\alpha,\underline{K}} \mod V^{(\underline{J},\underline{K})}, \\ D_{\underline{0}}(X^{-\alpha,\underline{0}}) &\equiv \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}X^{\beta^i+j\alpha,\underline{J}} \otimes X^{-\beta^i-(j+1)\alpha,\underline{K}} \mod V^{(\underline{J},\underline{K})}, \end{aligned} \tag{18}$$

for some $a_{ij}, b_{ij} \in \mathbb{C}$, where, $\beta^1 < \beta^2 < \cdots < \beta^m$, and $\beta^i \not\equiv \beta^j \mod \mathbb{Z}\alpha$ if $i \neq j$.

By the above assumption, we only need to prove the claim in the case that $\beta^i \neq 0$ only for one i. To further simplify our notation, we denote this β^i as β, and rewrite (18) in the following simpler form,

$$\begin{aligned} D_{\underline{0}}(X^{\alpha,\underline{0}}) &\equiv \sum_{i=1}^{m} a_i X^{\beta+i\alpha,\underline{J}} \otimes X^{-\beta-(i-1)\alpha,\underline{K}} \mod V^{(\underline{J},\underline{K})}, \\ D_{\underline{0}}(X^{-\alpha,\underline{0}}) &\equiv \sum_{i=1}^{m} b_i X^{\beta+i\alpha,\underline{J}} \otimes X^{-\beta-(i+1)\alpha,\underline{K}} \mod V^{(\underline{J},\underline{K})}. \end{aligned} \tag{19}$$

Before continuing the arguments in the proof of Claim 2, we give the following notion.

Definition 3.9. Let i (resp., j) be the largest (resp., smallest) integer such that $a_i \neq 0$ (resp., $a_j \neq 0$). We define the *length* of $D_{\underline{0}}$ to be $i - j + 1$.

We assume the length of $D_{\underline{0}}$ is m, i.e., $a_1 a_m \neq 0$. We prove the claim by reducing the length of $D_{\underline{0}}$ as follows. Since $V^{(\underline{J},\underline{K})}$ is invariant under $X^{\alpha,\underline{0}}$

and $X^{-\alpha,\underline{0}}$, using (19) in (17), and comparing the coefficients of the same degree summands on both sides, we can obtain $b_1 b_m \neq 0$, and further,

$$\begin{aligned} &b_m[X^{\alpha,\underline{0}}, X^{\beta+m\alpha,\underline{J}}] \otimes X^{-\beta-(m+1)\alpha,\underline{K}} \equiv 0 \mod V^{(\underline{J},\underline{K})}, \\ &a_1[X^{-\alpha,\underline{0}}, X^{\beta+\alpha,\underline{J}}] \otimes X^{-\beta,\underline{K}} \equiv 0 \mod V^{(\underline{J},\underline{K})}. \end{aligned} \tag{20}$$

Using

$$X^{\alpha,\underline{0}} \cdot X^{\beta+m\alpha,\underline{J}} \otimes X^{-\beta-m\alpha,\underline{K}} \equiv -2\alpha_1 X^{\beta+m\alpha,\underline{J}} \otimes X^{-\beta-(m-1)\alpha,\underline{K}} \mod V^{(\underline{J},\underline{K})}, \tag{21}$$

by replacing $D_{\underline{0}}$ with $D_{\underline{0}} + a_m \frac{1}{2\alpha_1}(X^{\beta+m\alpha,\underline{J}} \otimes X^{-\beta-m\alpha,\underline{K}})_{\text{inn}}$, we can successfully reduce the length of $D_{\underline{0}}$ by at least one. Repeat the above arguments, we obtain the claim, thus the lemma. ∎

From (19) and the above lemma, we immediately get the following conclusion.

Lemma 3.10. $D_{\underline{0}}(X^{-\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}$.

Lemma 3.11. $D_{\underline{0}}(X^{k\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})},\ k \in \mathbb{Z}$.

Proof. We only prove this lemma in case $k > 0$; the case $k < 0$ is similar. Since $X^{\alpha,\underline{0}}$, $X^{2\alpha,\underline{0}}$ generate $\{X^{k\alpha,\underline{0}} | k \in Z, k > 0\}$, it is sufficient to prove

$$D_{\underline{0}}(X^{2\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}.$$

Let $D_{\underline{0}}$ act on both sides of the equation $[X^{-\alpha,\underline{0}}, X^{2\alpha,\underline{0}}] = 3\alpha_1 X^{\alpha,\underline{0}}$, we have

$$X^{-\alpha,\underline{0}} \cdot D_{\underline{0}}(X^{2\alpha,\underline{0}}) \equiv 0 \mod V^{(\underline{J},\underline{K})}. \tag{22}$$

Assume $D_{\underline{0}}(X^{2\alpha,\underline{0}}) \not\equiv 0 \mod V^{(\underline{J},\underline{K})}$. Then we have a nonzero expression of $D_{\underline{0}}(X^{2\alpha,\underline{0}}) \mod V^{(\underline{J},\underline{K})}$ similar to (12). Consider the lowest degree summand of such an expression, it must be a nonzero multiple of $X \otimes Y$ for

$$X \otimes Y \in \{X^{\beta,\underline{J}} \otimes X^{-\beta+2\alpha,\underline{K}}, X^{2\alpha,\underline{J}} \otimes \partial_i, \partial_i \otimes X^{2\alpha,\underline{K}} \,|\, i = 1, 2\}.$$

From (22), we have

$$[X^{-\alpha,\underline{0}}, X] \otimes Y \equiv 0 \mod V^{(\underline{J},\underline{K})}. \tag{23}$$

If one of X, Y is ∂_i for $i = 1, 2$, then one can easily obtain a contradiction from (23). Thus we may suppose $X \otimes Y = X^{\beta,\underline{J}} \otimes X^{-\beta+2\alpha,\underline{K}}$. First assume

$$X \otimes [X^{-\alpha,\underline{0}}, Y] \equiv 0 \mod V^{(\underline{J},\underline{K})}. \tag{24}$$

Then (23) together with (24) implies

$$\begin{aligned} &\alpha_1(\beta_2 - 1) + \beta_1(-\alpha_2 - 1) = 0, \\ &\alpha_1(-\beta_2 + 2\alpha_2 - 1) + (-\beta_1 + 2\alpha_1)(-\alpha_2 - 1) = 0, \end{aligned}$$

which forces $\alpha_1 = 0$. This is a contradiction with the assumption in Lemma 3.8.

Now assume that (24) does not hold. From (22) and by induction on i, we obtain that, for all $i \in \mathbb{Z}_+$, $X^{\beta+i\alpha,\underline{J}} \otimes X^{-\beta-i\alpha+2\alpha,\underline{K}}$ with suitable nonzero coefficients must appeare as summands of the expression of $D_{\underline{0}}(X^{2\alpha,\underline{0}})$, which is impossible since V is a vector space (elements in V can only be a finite sum). Hence $D_{\underline{0}}(X^{k\alpha}) \equiv 0 \mod V^{(\underline{J},\underline{K})},\ k \in \mathbb{Z}$. ∎

From the order on J, we know that, for arbitrary two given elements in J, there are only finite many elements between them. Thus from the above lemmas, by induction on $(\underline{J}, \underline{K})$, we obtainthe following conclusion.

Lemma 3.12. *By subtracting $D_{\underline{0}}$ by finitely many inner derivations in $V_{\underline{0}}$, we can assume $D_{\underline{0}}(X^{k\alpha,\underline{0}}) = 0$ for $k \in \mathbb{Z}$.*

Lemma 3.13. *For arbitrary $\beta \in \Gamma$, we have $D_{\underline{0}}(X^{\beta,\underline{0}}) = 0$.*

Proof. It is sufficient to prove this lemma for $\beta \in \Gamma \setminus \{k\alpha \,|\, k \in \mathbb{Z}\}$. We have the following equation,

$$[X^{k\alpha,\underline{0}}, [X^{l\alpha,\underline{0}}, X^{\beta,\underline{0}}]] = \lambda_{k,l}[X^{(k+l)\alpha,\underline{0}}, X^{\beta,\underline{0}}],$$

where $\lambda_{k,l}$ can be given in an explicit form using (5), since it is not important for our proof, we simply denote it as $\lambda_{k,l}$ for convenience. Let $D_{\underline{0}}$ act on both sides of the above equation, we get,

$$X^{k\alpha,\underline{0}} \cdot X^{l\alpha,\underline{0}} \cdot D_{\underline{0}}(X^{\beta,\underline{0}}) = \lambda_{k,l} X^{(k+l)\alpha,\underline{0}} \cdot D_{\underline{0}}(X^{\beta,\underline{0}}). \tag{25}$$

Assume $D_{\underline{0}}(X^{\beta,\underline{0}}) \neq 0$. Then we have an expression of $D_{\underline{0}}(X^{\beta,\underline{0}})$ as in (12). Consider its highest degree summand, it must be a nonzero multiple of $X \otimes Y$ for

$$X \otimes Y \in \{X^{\gamma,\underline{j}} \otimes X^{-\gamma+\beta,\underline{k}}, X^{\beta,\underline{j}} \otimes \partial_i, \partial_i \otimes X^{\beta,\underline{k}} \,|\, i = 1, 2\}.$$

From (25), we get,

$$[X^{k\alpha,\underline{0}}, X] \otimes [X^{l\alpha,\underline{0}}, Y] = 0 \quad \text{for} \quad k, l \in \mathbb{Z}_+.$$

From Lemma 3.4, we get $X \otimes Y = 0$, a contradiction. ∎

Lemma 3.14. *For arbitrary $\delta \in \Gamma$, we have $D_{\underline{0}}(X^{\delta,1_{[1]}}) = 0 = D_{\underline{0}}(X^{\delta,1_{[2]}})$.*

Proof. We only prove this lemma for $D_{\underline{0}}(X^{\delta,1_{[1]}}) = 0$, the other is similar. We have,

$$[X^{\beta,\underline{0}}, [X^{\gamma,\underline{0}}, X^{\delta,1_{[1]}}]] = \lambda_{\beta,\gamma}[X^{\beta+\gamma,\underline{0}}, X^{\delta,1_{[1]}}] + \mu_{\beta,\gamma} X^{\beta+\gamma+\delta,\underline{0}},$$

where $\lambda_{\beta,\gamma}, \mu_{\beta,\gamma}$ can be explicitly given using (12), since it is not important for our proof, we simply denote them as $\lambda_{\beta,\gamma}, \mu_{\beta,\gamma}$ for convenience. Let $D_{\underline{0}}$ act on both sides of the above equation, we get,

$$X^{\beta,\underline{0}} \cdot X^{\gamma,\underline{0}} \cdot D_{\underline{0}}(X^{\delta,1_{[1]}}) = \lambda_{\beta,\gamma} X^{\beta+\gamma} \cdot D_{\underline{0}}(X^{\delta,1_{[1]}}). \tag{26}$$

Suppose $D_{\underline{0}}(X^{\delta,1_{[1]}}) \neq 0$. Using exactly the same arguments after (25) as in the above lemma, from (26) we can get a contradiction. ∎

Lemma 3.15. $D_{\underline{0}}(B) = 0$.

Proof. It is trivial since B is generated by $X^{\alpha,\underline{0}}$, $X^{\beta,1_{[1]}}$, $X^{\gamma,1_{[2]}}$, ∂_1, ∂_2. ∎

Now we can complete the proof of Theorem 1.2 as follows. Theorem 1.2 (1) follows from (11) and Proposition 3.1, and Theorem 1.2 (2) follows from Theorem 1.2 (1).

References

[1] Block, R., *On torsion-free abelian groups and Lie algebras*, Proc. Amer. Math. Soc. **9** (1958), 613–620.

[2] Cheng, Y., G. Song, and B. Xin, *Lie bialgebra structures on Lie algebras of Block type*, Algebra Colloq. **16** (2009), 677–690.

[3] Chen, Q., J. Han, and Y. Su, *Structures of not-finitely graded Lie algebras related to generalized Virasoro algebras*, Preprint.

[4] Drinfeld, V. G., *Constant quasiclassical solutions of the Yang-Baxter quantum equation*, Soviet Math. Dokl. **28** (1983), 667–671.

[5] —, *Quantum groups*," in: Proceedings of the International Congress of Mathematicians, Vol. 1,2, Amer. Math. Soc., Providence, RI, 1987, 798–820.

[6] Dokovic, D., and K. Zhao, *Derivations, isomorphisms and second cohomology of generalized Block algebras*, Algebra Colloq. **3** (1996), 245–272.

[7] Han, J., J. Li, and Y. Su, *Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra*, J. Math. Phys. **50** (2009), 083504, 12 pp.

[8] Li, C., J. He, and Y. Su, *Block type symmetry of bigraded Toda hierarchy*, J. Math. Phys. **53** (2012), 013517, 24pp.

[9] Li, J., Y. Su , and B. Xin, *Lie bialgebras of a family of Lie algebras of Block type*, Chinese Annals of Mathematics (Ser. B) **29** (2008), 487–500.

[10] Michaelis, W., *A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras*, Adv. Math. **107** (1994), 365–392.

[11] —, *Lie coalgebras*, Adv. Math. **38** (1980), 1–54.

[12] —, *The dual Poincare-Birkhoff-Witt theorem*, Adv. Math. **57** (1985), 93–162.

[13] Ng, S. H., and E. J. Taft, *Classification of the Lie bialgebra structures on the Witt and Virasoro algebras*, J. Pure Appl. Algebra **151** (2000), 67–88.

[14] Osborn, J. M., and K. Zhao, *Infinite-dimensional Lie algebras of generalized Block type*, Proc. Amer. Math. Soc. **127** (1999), 1641–1650.

[15] Song, G., and Y. Su, *Lie bialgebras of generalized Witt type*, Sci. in China (Ser. A) **49** (2006), 533–544.

[16] —, *Dual Lie Bialgebras of Witt and Virasoro Types* (in Chinese, doi: 10.1360/012011-XXX) Sci. Sin. Math. **43** (2013), 1–20,

[17] Su, Y., *Derivations and structure of the Lie algebras of Xu type*, Manuscripta Math. **105** (2001), 483–500.

[18] —, *Structure of Lie superalgebras of Block type related to locally finite derivations*, Comm. Algebra **31** (2003), 1725–1751.

[19] —, *Quasifinite representations of a family of Lie algebras of Block type*, J. Pure Appl. Algebra, **192** (2004), 293–305.

[20] Su, Y., and J. Zhou, *Structure of the Lie algebras with a feature of Block algebras*, Comm. Algebra **30** (2002), 3205–3226.

[21] —, *Structure of the Lie algebras related to those of Block*, Comm. Algebra **30** (2002), 3205–3226.

[22] Taft, E. J., *Witt and Virasoro algebras as Lie bialgebras*, J. Pure Appl. Algebra **87** (1993), 301–312.

[23] Wu, Y., G. Song, and Y. Su, *Lie bialgebras of generalized Virasoro-like type*, Acta Math. Sin. (Engl. Ser.) **22** (2006), 1915–1922.

[24] Xia, C., and R. B. Zhang, *Unitary highest weight modules over Block type Lie algebras $B(q)$*, J. Lie Theory **23** (2013), 159–176.

[25] Xu, X., *Generalizations of Block algebras*, Manuscripta Math. **100** (1999), 489–518.

[26] —, "Introduction to Vertex Operator Superalgebras and Their Modules," Kluwer Academic Publishers, Dordrecht–Boston–London, 1998.

[27] Zhang, H., and K. Zhao, *Representations of the Virasoro-like Lie algebra and its q-analog*, Comm. Algebra **24** (1996), 4361–4372.

Hao Wang
Wu Wen-Tsun Key Laboratory
of Mathematics and
School of Mathematical Sciences
University of Science and
Technology of China
Hefei 230026, China
wangh228@mail.ustc.edu.cn

Ying Xu
Department of Mathematics
Hefei University of Technology
Hefei 230009, Anhui, China
xuying@hfut.edu.cn

Xiaoqing Yue
Department of Mathematics
Tongji University
Shanghai 200092, China
xiaoqingyue@tongji.edu.cn

Received January 21, 2014
and in final form November 20, 2014