Journal of Lie Theory
Volume **25** (2015) 807–813

Stabilisation of the LHS Spectral Sequence for Algebraic Groups

Alison E. Parker and David I. Stewart

Communicated by E. B. Vinberg

Abstract. In this note, we consider the Lyndon–Hochschild–Serre spectral sequence corresponding to the first Frobenius kernel of an algebraic group G and computing the extensions between simple G-modules. We state and discuss a conjecture that $E_2 = E_\infty$ and provide general conditions for low-dimensional terms on the E_2-page to be the same as the corresponding terms on the E_∞-page, i.e. its abutment.
Mathematics Subject Classification 2010: 20G10, 20G05, 18G40.
Key Words and Phrases: Reductive algebraic groups, Lyndon–Hochschild–Serre spectral sequence, positive characteristic, cohomology of simple modules.

1. Introduction

Let G be a simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. Let λ and μ be two dominant weights for G. This paper concerns the representation theory of G and its first Frobenius kernel G_1; we refer to [Jan03] for notation. It is the purpose of this short note to state and provide some evidence towards the following conjecture.

Conjecture. Suppose all G_1-injective hulls have the structure of G-modules, for instance if $p \geq 2h - 2$. Then the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{ij} = \mathrm{Ext}^i_{G/G_1}(k, \mathrm{Ext}^j_{G_1}(L(\lambda), L(\mu))) \Rightarrow \mathrm{Ext}^{i+j}_G(L(\lambda), L(\mu)) \qquad (*)$$

stabilises (i.e. reaches its abutment) at the E_2-page. That is, $E_2^{ij} \cong E_\infty^{ij}$ for all i, j.

Hence

$$\mathrm{Ext}^n_G(L(\lambda), L(\mu)) \cong \bigoplus_{i+j=n} \mathrm{Ext}^i_{G/G_1}(k, \mathrm{Ext}^j_{G_1}(L(\lambda), L(\mu))).$$

Note that it is an open conjecture of Humphreys and Verma that all G_1-injective hulls do indeed have the structure of G-modules.

ISSN 0949–5932 / \$2.50

Let us underline the fact that we are unaware of any occasion where any differential in the spectral sequence (*) is known to be non-zero—even after replacing G with an arbitrary connected algebraic group and replacing $L(\lambda)$ and $L(\mu)$ by arbitrary G-modules. Showing that certain differentials in the spectral sequence are zero has some history; we pick out a few cases. For a large class of naturally occurring modules V and W, it was shown by the first author in [Par07] that when $G = \mathrm{SL}_2$ the spectral sequence does stabilise at the E_2-page. In particular the conjecture is confirmed for the case $G = \mathrm{SL}_2$, with no condition on p. It was shown by Donkin in [Don82] that the differentials $d_{m,1} : E_2^{m,1} \to E_2^{m+2,0}$ are zero, also with no condition on p. Some other special cases involving maps needed to compute second cohomology were considered in work of McNinch [McN02], the second author [Ste10, Ste12], and Ibraev [Ibr11, Ibr12].

Another case in which the conjecture is true is if λ and μ are p-regular restricted weights, $p \geq 2h - 2$ and p is large enough that the Lusztig Character Formula holds. Then [PS13, Theorem 5.3] shows that $\mathrm{Ext}^n_{G_1}(L(\lambda), L(\mu))^{[-1]}$ has a good filtration for each n. Under these circumstances the spectral sequence moreover degenerates to a line; in particular the conjecture is true.

Note that the conjecture is not true if G is replaced by an arbitrary group. See [BF94, §6], [Lea93] and [Sie00] for examples of non-zero differentials.

The main theorem of this paper is a confirmation of the conjecture in a generic sense. Here, the vanishing of differentials of degree much lower than p is guaranteed.

Theorem. *Suppose $p \geq (2r+1)(h-1)$. Then the differentials d_n^{ij} in the spectral sequence (*) are all zero whenever i, j, n satisfy $i \leq r - 1$ and $n \geq 2$, or $j = 0$ and $n \geq 2$, or $j = 1$ and $n \geq 2$.*

In particular,

$$\mathrm{Ext}^i_G(L(\mu), L(\lambda)) \cong \bigoplus_{j=0}^{i} E_2^{i-j,j}$$

for $i \leq r + 1$.

Remark 1.1. In fact, as the proof will show, one may replace $L(\lambda)$ and $L(\mu)$ with modules $L(\lambda_0) \otimes M^{[1]}$ and $L(\mu_0) \otimes N^{[1]}$ respectively, where $\lambda_0, \mu_0 \in X_1(T)$ and M and N are arbitrary finite-dimensional G-modules.

We prove the above theorem by applying techniques from [Par07]. First, we show, in a proposition, that part of a minimal G_1-injective resolution has a compatible G-structure. We then reconstruct the spectral sequence (*) in such a way that the bottom-most complex in the double complex giving the E_0-page contains this part of a minimal G_1-injective resolution. It follows that many maps in the E_0-page are zero. Then some derived couple arguments prove the theorem.

2. Proposition and proof of the theorem

In the proposition below, note that the case $r = 0$ (which is not covered by the proposition) would be a special case of the Humphreys–Verma conjecture. (It is not known if the bound $p \geq 2h-2$ could be reduced to $p \geq h-1$ for G_1-injective hulls to lift to G-modules.)

Proposition. *Let $\mu \in X_1(T)$ and $r \geq 1$. Provided $p \geq (2r+1)(h-1)$, there is a minimal G_1-resolution*

$$0 \to L(\mu) \to I_0 \to I_1 \to \cdots \to I_r \to \cdots$$

such that the sequence up to term I_r has a G-structure.

Proof. We prove *a fortiori* that there is such a sequence of G-modules with I_r having weights $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1(T)$, which satisfy $(\lambda_1, \alpha_0^\vee) \leq (2r+1)(h-1)$.

First, let us treat the case $r = 1$. Set $I_0 = Q_1(\mu)$. The hypotheses imply that $p \geq 2h-2$; thus we know that $Q_1(\mu)$ has the structure of a G-module. The injection $L(\mu) \to Q_1(\mu)$ is then a map of G-modules.

Let $M := Q_1(\mu)/L(\mu)$. Since G is simply-connected, all simple G_1-modules lift to G-modules. Hence, as a G-module, we may write $\operatorname{Soc}_{G_1} M = \bigoplus_{\nu \in X_1(T)} L(\nu) \otimes M_\nu^{[1]}$ where M_ν is some G-module, possibly zero. Set $I_1 = \bigoplus_{\nu \in X_1(T)} Q_1(\nu) \otimes M_\nu^{[1]}$. So $\operatorname{Soc}_{G_1} I_1 = \operatorname{Soc}_{G_1} Q_1(\mu)/L(\mu)$. (It is worth noting that the condition on the weights here is enough to ensure that $\operatorname{Soc}_{G_1} M = \operatorname{Soc}_G M$ but we do not need this fact explicitly.) Thus I_1 is the G_1-injective hull of M, hence if there is a G-map $I_0 \to I_1$, this will be part of a minimal resolution. It remains to show that there is indeed a map $I_0 \to I_1$ of G-modules whose kernel is $L(\mu)$, i.e. a map $I_0/L(\mu) \to I_1$. Note that we do have a map $\operatorname{Soc}_{G_1} M \to I_1$ by construction, so consider the exact sequence

$$\operatorname{Hom}_G(M, I_1) \to \operatorname{Hom}_G(\operatorname{Soc}_{G_1} M, I_1) \to \operatorname{Ext}^1_G(M/\operatorname{Soc}_{G_1} M, I_1). \qquad (*)$$

If we could show that the third term in this sequence is zero then we would have that the first map were surjective, hence the G-map $\operatorname{Soc}_{G_1} M \to I_1$ would lift to a map $M = I_0/L(\mu) \to I_1$ and we would be done.

Now, the space $\operatorname{Ext}^1_G(M/\operatorname{Soc}_{G_1} M, I_1)$ has a filtration by spaces $E = \operatorname{Ext}^1_G(M/\operatorname{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^{[1]})$ over certain weights $\nu = \nu_0 + p\nu_1$. And E can be computed via the 5-term exact sequence of the LHS spectral sequence, of which part is

$$\operatorname{Ext}^1_{G/G_1}(k, \operatorname{Hom}_{G_1}(M/\operatorname{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^{[1]})) \to E$$
$$\to \operatorname{Hom}_{G/G_1}(k, \operatorname{Ext}^1_{G_1}(M/\operatorname{Soc}_{G_1} M, Q_1(\nu_0) \otimes L(\nu_1)^{[1]})).$$

Now the third term here is zero, as $Q_1(\nu_0)$ is injective for G_1, hence, to show $E = 0$, it suffices to show that the first term is zero.

Now by [Jan03, II.11.6(a)], $I_0 = Q_1(\mu)$ has weights $\xi = \xi_0 + p\xi_1$ satisfying $\xi \leq w_0\mu + 2(p-1)\rho$ where w_0 is the longest element in the Weyl group. Since $(\mu, \alpha_0^\vee) \geq 0$, $(w_0\mu, \alpha_0^\vee) \leq 0$ and so

$$p(\xi_1, \alpha_0)^\vee \leq (\xi, \alpha_0^\vee) \leq 2(p-1)(\rho, \alpha_0^\vee) = 2(p-1)(h-1).$$

Thus we conclude that $(\xi_1, \alpha_0^\vee) \le 2(\rho, \alpha_0^\vee) = 2h-2$. Thus the composition factors of I_0 (hence of M) are of the form $L(\xi_0) \otimes L(\xi_1)^{[1]}$. In particular, we have that the weights ν_1 satisfy $(\nu_1, \alpha_0^\vee) \le 2h-2$. Thus $(\xi_1+\rho, \alpha_0^\vee), (\nu_1+\rho, \alpha_0^\vee) \le 3h-3$ and our condition on p implies that both ξ_1 and ν_1 are in the closure of the lowest alcove, $\bar{C}_{\mathbb{Z}}$. So let $L(\xi_0) \otimes L(\xi_1)^{[1]}$ be a composition factor of $M/\operatorname{Soc}_{G_1} M$. We compute:

$$\begin{aligned}&\operatorname{Ext}^1_{G/G_1}(k, \operatorname{Hom}_{G_1}(L(\xi_0) \otimes L(\xi_1)^{[1]}, Q_1(\nu_0) \otimes L(\nu_1)^{[1]})) \\ &\qquad\qquad \cong \operatorname{Ext}^1_G(L(\xi_1), \operatorname{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))^{[-1]} \otimes L(\nu_1))\end{aligned}$$

Now $\operatorname{Hom}_{G_1}(L(\xi_0), Q_1(\nu_0))$ is non-zero, hence equal to k, if and only $\xi_0 = \nu_0$; in that case, the term on the right becomes $\operatorname{Ext}^1_G(L(\xi_1), L(\nu_1))$, and since $\xi_1, \nu_1 \in \bar{C}_{\mathbb{Z}}$, this vanishes by the linkage principle. This concludes the proof in case $r=1$.

Now by induction we may assume that we have a sequence of G-modules

$$0 \to I_0 \to \cdots \to I_{r-2} \xrightarrow{\pi} I_{r-1},$$

which is minimal as an injective G_1-resolution, such that the composition factors of I_{r-1} have high weights λ satisfying $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X_1(T)$ and $(\lambda_1, \alpha_0^\vee) \le (2r-1)(h-1)$. We construct I_r in a similar way to before: As a G-module, write $\operatorname{Soc}_{G_1} I_{r-1}/\pi I_{r-2} = \bigoplus_{\nu \in X_1(T)} L(\nu) \otimes M_\nu^{[1]}$ where M_ν is some G-module and set $I_r = \bigoplus_{\nu \in X_1(T)} Q_1(\nu) \otimes M_\nu^{[1]}$. By hypothesis, a weight ν_1 of M_ν satisfies $(\nu_1, \alpha_0^\vee) \le (2r-1)(h-1)$. By arguing as in the case $r=1$, we see a weight ξ of I_r, say $\xi_0 + p\xi_1$ with $\xi_0 \in X_1(T)$ satisfies $(\xi_1, \alpha_0^\vee) \le (\nu_1, \alpha_0^\vee) + 2(\rho, \alpha_0^\vee) = (2r+1)(h-1)$ as required. Note that I_r is again a G_1-injective hull of $I_{r-1}/\operatorname{im}\pi$ so if we can show there is a G-module map $I_{r-1} \to I_r$ with kernel $\operatorname{im}\pi$, we will be done.

Of course, it is equivalent to produce an injective map from $M := I_{r-1}/\operatorname{im}\pi$ to I_r. By construction we do have an injective map from $\operatorname{Soc}_{G_1} M \to I_r$. Now the same argument as before shows that the third term in the sequence (*) (with I_r replacing I_1) is zero. This completes the proof. ∎

Proof of the theorem. We write $L(\mu) = L(\mu_0) \otimes L(\mu_1)^{[1]}$ using Steinberg's tensor product theorem where $\mu_0 \in X_1(T)$ and $\mu_1 \in X^+$. Using the proposition we have a G-resolution which is also a G_1-injective resolution:

$$0 \to L(\mu_0) \to I_0 \to I_1 \to \cdots \to I_r \to \cdots,$$

where, up to I_r, the resolution is minimal for G_1.

We denote the differentials by $\delta_i : I_i \to I_{i+1}$ and the kernels by $K_i := \ker \delta_i$. Dimension shifting gives us $\operatorname{Ext}^i_{G_1}(L(\lambda_0), L(\mu_0)) \cong \operatorname{Ext}^1_{G_1}(L(\lambda_0), K_{i-1})$. Minimality gives us for $\mu_1 \in X_1(T)$ that $\operatorname{Ext}^i_{G_1}(L(\lambda_0), L(\mu_0)) \cong \operatorname{Hom}_{G_1}(L(\lambda_0), K_i) \cong \operatorname{Hom}_{G_1}(L(\lambda_0), I_i)$ for $i \le r$.

We now have a G-resolution:

$$0 \to L(\mu) \to I_0 \otimes L(\mu_1)^{[1]} \xrightarrow{\partial_0} I_1 \otimes L(\mu_1)^F \xrightarrow{\partial_1} \cdots$$

where $\partial_i = \delta_i \otimes \operatorname{id}$, as tensoring is exact. Also note that such a resolution stays injective as a G_1-resolution as $L(\mu_1)^F$ is trivial as a G_1-module.

Now consider the E_0-page of the LHS spectral sequence that converges to $\mathrm{Ext}^*_G(L(\lambda), L(\mu))$ as constructed in [Par07, §2]

$$E_0^{mn} = \mathrm{Hom}_{G/G_1}(k, \mathrm{Hom}_{G_1}(L(\lambda), I_n \otimes L(\mu_1)^F) \otimes J_m^{[1]})$$

where we have a G-injective resolution of the trivial module:

$$0 \to k \to J_0 \to J_1 \to \cdots$$

and this spectral sequence has E_1 and E_2 page

$$\begin{aligned} E_1^{mn} &= \mathrm{Hom}_{G/G_1}(k, \mathrm{Ext}^n_{G_1}(L(\lambda), L(\mu)) \otimes J_m^{[1]}) \\ E_2^{mn} &= H^m(G/G_1, \mathrm{Ext}^n_{G_1}(L(\lambda), L(\mu))). \end{aligned}$$

Consider the induced maps ∂_m^* in the following complex, which has homology $\mathrm{Ext}^*_{G_1}(L(\lambda), L(\mu))$:

$$\mathrm{Hom}_{G_1}(L(\lambda), I_0 \otimes L(\mu_1)^F) \xrightarrow{\partial_0^*} \mathrm{Hom}_{G_1}(L(\lambda), I_1 \otimes L(\mu_1)^F) \xrightarrow{\partial_1^*} \cdots .$$

Now

$$\begin{aligned} \mathrm{Ext}^m_{G_1}(L(\lambda), L(\mu)) &\cong \mathrm{Ext}^m_{G_1}(L(\lambda_0), L(\mu_0)) \otimes L(\mu_1)^F \otimes L(\lambda_1^*)^F \\ &\cong \mathrm{Hom}_{G_1}(L(\lambda_0), I_m) \otimes L(\mu_1)^F \otimes L(\lambda_1^*)^F \\ &\cong \mathrm{Hom}_{G_1}(L(\lambda), I_m \otimes L(\mu_1)^F) \end{aligned}$$

for $m \leq r$. Thus all the differentials ∂_m^* for $m \leq r$ must be zero.

Now by [Ben98, §3.2, §3.4] we know that the spectral sequence can be constructed using derived couples. We have

$$D_0^{mn} = \bigoplus_{m+n=e+f,\ e\geq m} E_0^{ef}$$

$$E_1^{mn} = H(E_0^{mn}, d_0)$$

$$D_1^{mn} = H(E_0^{mn} \oplus E_0^{m+1,n-1} \oplus \cdots, d_0 + d_1)$$

We define the higher derived couples by taking the derived couple of the previous one. We have an exact diagram of doubly graded k-modules

$$\begin{array}{ccc} D_l & \xrightarrow{i_l} & D_l \\ {\scriptstyle k_l}\nwarrow & & \swarrow{\scriptstyle j_l} \\ & E_l & \end{array}$$

The derived couple (for $l \geq 1$) is defined by

$$\begin{aligned} D_{l+1}^{mn} &= \mathrm{im}\, i_l^{m+1,n-1} \subseteq D_l^{mn} & E_{l+1}^{mn} &= H(E_l^{mn}, d_l) \\ i_{l+1}^{mn} &= i_l^{mn}\Big|_{D_{l+1}} & j_{l+1}^{mn}(i_l^{m+1,n-1}(x)) &= j_l^{m+1,n-1}(x) + \mathrm{im}(d_l) \\ k_{l+1}^{mn}(z + \mathrm{im}(d_l)) &= k_l^{mn}(z) & d_{l+1} &= j_{l+1} \circ k_{l+1} \end{aligned}$$

And the degrees of the maps k, j and d are:

$$\deg(i_n) = (-1, 1), \qquad \deg(j_n) = (n-1, n+1), \qquad \deg(k_n) = (1, 0).$$

Now using [Par07, Lemma 2.1], we have that the map $d_0^{mn} = 0$ implies also that $k_2^{m-1,n+1} = 0$. Thus since $d_0^{mn} = 0$ for $m \le r$ we have $k_2^{mn} = 0$ for $m \le r-1$. Thus all $k_l^{mn} = 0$ for all $l \ge 2$ and $m \le r-1$. As $d_l^{mn} = j_l^{m+1,n} \circ k_l^{mn}$ we also get $d_l^{mn} = 0$ for all $l \ge 2$ and $m \le r-1$.

In other words, as all these differentials are zero on the E_2 page and remain zero, the terms E_2^{mn} with $m \le r-1$ must already be the stable value. That is, $E_\infty^{mn} = E_2^{mn}$ for $m \le r-1$.

This easily gives us that

$$\mathrm{Ext}_G^i(L(\lambda), L(\mu)) = \bigoplus_{j=0}^{i} E_2^{i-j,j}$$

for $i < r$. To get the result for $i = r$, we note that all the terms in the sum

$$\bigoplus_{j=0}^{r} E_\infty^{r-j,j}$$

stabilise at the E_2 page by the above, except, possibly the term E_∞^{r0}. But here clearly $E_\infty^{r0} = E_2^{r0}$ as all incoming differentials are zero by the above, and the leaving differential $d_l^{r,0}$ is always zero as our spectral sequence is first quadrant.

We may similarly argue for $r+1$. We consider

$$\bigoplus_{j=0}^{r+1} E_\infty^{r+1-j,j}.$$

As before all terms except possibly $E_\infty^{r,1}$ and $E_\infty^{r+1,0}$ stabilise at the E_2 page. The same argument as in the previous case gives $E_\infty^{r+1,0} = E_2^{r+1,0}$.

Now note that all incoming differentials to $E_l^{r,1}$ are zero for $l \ge 2$ by the above. We also have that $d_l^{r,1} = 0$ for $l \ge 3$, again since the spectral sequence is first quadrant. So we need only check that $d_2^{r1} = 0$, but this is true using [Don82, Main Theorem]. Thus we also get the result for $r+1$. ■

Acknowledgements. The authors wish to thank Len Scott and Dan Nakano for comments on a previous version of this paper. Also we wish to thank the referee for several helpful suggestions on improving the exposition of this article.

References

[Ben98] Benson, D. J., "Representations and cohomology. II, second ed.," Cambridge Studies in Advanced Mathematics **31**, Cambridge University Press, Cambridge, 1998.

[BF94] Benson, D. J., and M. Feshbach, *On the cohomology of split extensions*, Proc. Amer. Math. Soc. **121** (1994), 687–690.

[Don82] Donkin, S., *On* Ext^1 *for semisimple groups and infinitesimal subgroups*, Math. Proc. Cambridge Philos. Soc. **92** (1982), 231–238.

[Ibr11] Ibraev, S. S., *The second cohomology groups of simple modules for* G_2, Sib. Èlektron. Mat. Izv. **8** (2011), 381–392.

[Ibr12] —, *The second cohomology of simple modules over* $\mathrm{Sp}_4(k)$, Comm. Algebra **40** (2012), 1122–1130.

[Jan03] Jantzen, J. C., “Representations of algebraic groups, second ed.,” Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence, RI, 2003.

[Lea93] Leary, I., *A differential in the Lyndon-Hochschild-Serre spectral sequence*, J. Pure Appl. Algebra **88** (1993), 155–168.

[McN02] McNinch, G. J., *The second cohomology of small irreducible modules for simple algebraic groups*, Pacific J. Math. **204** (2002), 459–472.

[Par07] Parker, A. E., *Higher extensions between modules for* SL_2, Adv. Math. **209** (2007), 381–405.

[PS13] Parshall, B. J., and L. L. Scott, *New graded methods in the homological algebra of semisimple algebraic groups*, arxiv:1304.1461 (2013).

[Sie00] Siegel, S. F., *The cohomology of split extensions of elementary abelian 2-groups and Totaro's example*, J. Pure Appl. Algebra **145** (2000), 183–198.

[Ste10] Stewart, D. I., *The second cohomology of simple* SL_2*-modules*, Proc. Amer. Math. Soc. **138** (2010), 427–434.

[Ste12] —, *The second cohomology of simple* SL_3*-modules*, Comm. Algebra **40** (2012), 4702–4716.

Alison E. Parker
School of Mathematics
University of Leeds
Leeds, LS2 9JT, UK
a.e.parker@leeds.ac.uk

David I. Stewart
University of Cambridge
Cambridge, UK
dis20@cantab.net

Received October 6, 2014
and in final form November 29, 2014